NON-SIMPLE BLOW-UP FOR SINGULAR LIOUVILLE EQUATIONS IN UNIT BALL

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Abstract. We investigate the existence of blowing-up solutions to the following singular Liouville problem

$$-\Delta u = \lambda V_\lambda(|x|)e^u - 4\pi N \boldsymbol{\delta}_0 \ \text{ in } B_1, \quad u = 0 \ \text{ on } \partial B_1,$$

where B_1 is the unit ball in $\mathbb{R}^2$ centered at the origin, $V_\lambda(|x|)$ is a positive smooth potential, N is a positive integer ($N \geq 1$). Here $\boldsymbol{\delta}_0$ defines the Dirac measure with pole at 0, and $\lambda > 0$ is a small parameter. If the potential $V_\lambda(|x|)$ satisfies some suitable assumptions in terms of the first $2(N+1)$ derivatives at 0, then we find a solution which exhibits a non-simple blow-up profile as $\lambda \to 0^+$.

Mathematics Subject Classification 2010: 35J20, 35J57, 35J61
Keywords: singular Liouville equation, non-simple blow-up, finite-dimensional reduction

1. Introduction

Given Ω a smooth and bounded domain in $\mathbb{R}^2$ containing the origin, consider the following Liouville equation with Dirac mass measure

$$\begin{cases} -\Delta u = \lambda V_\lambda(x)e^u - 4\pi N \boldsymbol{\delta}_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here λ is a positive small parameter, V_λ is a positive and uniformly (with respect to λ) bounded potential, $\boldsymbol{\delta}_0$ denotes Dirac mass supported at 0 and N is a positive integer.

It is well known that solutions to singular Liouville equations of the type (1.1) exhibit blow-up behaviour. Consequently, problem (1.1) and its variants find many applications in mathematics and science. In particular, singular Liouville equations arise in the study of vortices in a planar model of Euler flows (see [12]). In vortex theory the interest in constructing blowing-up solutions is related to relevant physical properties, in particular the presence of vortices with a strongly localised electromagnetic field.

Due to the large amount of applications in which blow-up phenomenon occurs, there has been an increasing need for the development of the analysis of blow-up solutions for singular Liouville equations. The asymptotic behavior of a family of blowing up solutions u_k can be referred to the papers [6], [8], [18], [19], [21], [23] for the regular problem, i.e. when $N = 0$. An extension to the singular case $N > 0$ is contained in [2]-[4]. If a blowup point p is either a regular point or a "non-quantized" singular source, the asymptotic behavior of u_k around p is well understood (see [2, 4, 7, 8, 15, 17, 31, 32]). As a matter of fact, u_k satisfies the spherical Harnack inequality around 0, which implies that, after scaling, the sequence u_k behaves as a single bubble around the maximum point. However, if p happens to be a quantized singular source, the so-called "non-simple" blowup phenomenon does happen (see [16, 27, 28, 29]), which is equivalent to stating that u_k violates the spherical Harnack inequality around p. The study of non-simple blowup solutions, whether or not

The research of T. D'Aprile is partially supported by the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome "Tor Vergata", CUP E83C18000100006. The research of J. Wei is partially supported by NSERC of Canada. The research of Lei Zhang is partially supported by a Simons Foundation Collaboration Grant.

the blowup point has to be a critical point of coefficient functions, has been a major challenge for Liouville equations and its research has intrigued people for years. Recently significant progress has been made by Kuo-Lin, Bartolucci-Tarantello and other authors ([5, 10, 16, 27, 28, 29]. In [27] and [28] Harnack inequalities and up to second order vanishing conditions for non-simple blow-ups are obtained. The "non-simple blowup " assumption in these vanishing theorems is essential, without it Wu [30] constructed blowup solutions with non-vanishing coefficient functions. When nonsimple blowup solutions occur, it is established in [3] and [16] that there are $N+1$ local maximum points evenly distributed on $\mathbb{S}^1$ after scaling according to their magnitude.

The case $N \in \mathbb{N}$ is more difficult to treat, and at the same time the most relevant to physical applications. Indeed, in vortex theory the number N represents vortex multiplicity, so that in that context the most interesting case is precisely when it is a positive integer. The difference between the case $N \in \mathbb{N}$ and $N \notin \mathbb{N}$ is also analytically essential. Indeed, as usual in problems involving concentration phenomena like (1.1), after suitable rescaling of the blowing-up around a concentration point one sees a limiting equation which, in this case, takes the form of the planar singular Liouville equation:

$$-\Delta U = e^U - 4\pi N \delta_0, \quad \int_{\mathbb{R}^2} e^U dx < \infty;$$

only if $N \in \mathbb{N}$ the above limiting equation admits non-radial solutions around 0 since the family of all solutions extends to one carrying an extra parameter (see [22]). This suggests that if $N \in \mathbb{N}$ and the blow-up point happens to be the singular source, then solutions of (1.1) may exhibit non-simple blow-up phenomenon.

So, from analytical viewpoints the study of non-simple blowup solutions is far more challenging than simple blowup solutions, but the impact of this study may be even more significant because they represent certain situations in the blowup analysis of systems of Liouville equations. Indeed, if local maxima of blowup solutions in a system tend to one point, the profile of solutions can be described by a Liouville equation with quantized singular source. For all this reasons, it is desirable to know exactly when non-simple blowup phenomenon happens.

However, the question on the existence of non-simple blowing-up solutions to (1.1) concentrating at 0 is far from being completely settled. A first definite answer is provided by [11] which rules out the non-simple phenomenon for (1.1) if the potential V is constant: more precisely it is established that there is no non-simple blowup sequence for (1.1) with $V = const.$, even if we are in the presence of multiples singularities $\sum_i N_i \boldsymbol{\delta}_{p_i}$. Apart from this, many open problems are still usolved and only specific cases have been addressed: in [10] the construction of solutions exhibiting a non simple blow-up profile at 0 is carried out for equation (1.1) with $V \equiv 1$ provided that Ω is the unit ball and the weight of the source is a positive number $N = N_\lambda$ close an integer N from the right side. On the other hand, in[12], for any fixed positive integer N, it is proved the existence of a solution to (1.1) with $V \equiv 1$, where $\boldsymbol{\delta}_0$ is replaced by $\boldsymbol{\delta}_{p_\lambda}$ for a suitable $p_\lambda \in \Omega$, with $N+1$ blowing up points at the vertices of a sufficiently tiny regular polygon centered in p_λ; moreover the location of p_λ is determined by the geometry of the domain in a $\lambda-$dependent way and does not seem possible to be prescribed arbitrarily. To our knowledge, the existence of non-simple blow-up phenomenon for (1.1) for a fixed V and a fixed N independent of λ is still open, even in the case of the ball: the only example is constructed in [9] for a special class of potentials of the form $V(|x|^{N+1})$.

In this paper we investigate the existence of non-simple blow-up solutions when Ω is the unit ball B_1 centered at the origin and the potential V_λ is radially symmetric:

$$\begin{cases} -\Delta u = \lambda V_\lambda(|x|) e^u - 4\pi N \delta_0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{1.2}$$

Now we state the hypotheses on V_λ that will be used throughout the paper. First we assume some uniform estimates with respect to λ:

(A1) $\inf_{B_1} V_\lambda(|x|) > c > 0$ for a positive constant c independent of λ and, without loss of generality, we may assume $V_\lambda(0) = 1$.

(A2) $V_\lambda(|x|)$ is of class C^1 in the closed unit ball $\overline{B}_1$ and is of class C^{2N+4} in a neighbourhood U of 0; moreover

$$\|V_\lambda\|_{C^1(\overline{B}_1)},\ \|V_\lambda\|_{C^{2N+4}(U)} \le C$$

for a positive constant C independent of λ.

Furthermore we postulate a crucial vanishing condition on the second and the higher derivatives of V_λ at 0 up to the order $2N$, and finally we require an upper bound on the $2(N+1)$ derivative[1]:

(A3) $V_\lambda''(0) > 0$ for all $\lambda > 0$, $\lambda^{\frac{N}{N+1}} = o(V_\lambda''(0))$, $V_\lambda''(0) = o(\lambda^{\frac{N}{N+3}})$.

(A4) the following holds:

$$|V_\lambda^{2i}(0)| \le C V_\lambda''(0) \quad \forall i = 1, \dots, N;$$

$$V_\lambda^{2(N+1)}(0) < 2(2N+2)! - c$$

for positive constants c, C independent of λ.

In the following $G(x,y)$ is the Green's function of $-\Delta$ over Ω under Dirichlet boundary conditions and $H(x,y)$ denotes its regular part:

$$H(x,y) := G(x,y) - \frac{1}{2\pi}\log\frac{1}{|x-y|}.$$

When Ω is the unit ball B_1 we have the explicit formula for H:

$$H(x,y) = \frac{1}{2\pi}\log\left(|x|\left|y - \frac{x}{|x|^2}\right|\right), \quad x, y \in B_1. \tag{1.3}$$

Theorem 1.1. *Assume that hypotheses* (A1) – (A2) – (A3) – (A4) *hold. Then, for λ sufficiently small the problem* (1.2) *has a family of solutions u_λ blowing up at the origin as $\lambda \to 0^+$:*

$$\lambda e^{u_\lambda} \to 8\pi(N+1)\boldsymbol{\delta}_0 \quad \textit{in the measure sense.}$$

More precisely there exist $\mu = \mu(\lambda) > 0$ and $b = b(\lambda) \in B_1$ in a neighborhood of 0 *such that u_λ satisfies*

$$u_\lambda + 4\pi N G(x,0) = -2\log\left(\mu^{2(N+1)} + |x^{N+1} - b|^2\right) + 8\pi H(x^{N+1}, b) + o(1)$$

in H^1-sense, where[2]

$$\mu^{2(N+1)} \sim \lambda, \quad b \sim (V_\lambda''(0))^{\frac{N+1}{2N}}. \tag{1.4}$$

In particular, by (A3), $\mu^{N+1} = o(b)$.

The solution in Theorem 1.1 reveals a non-simple blow-up profile: indeed, denoting by $\beta_0, \dots, \beta_N$ the $N+1$ complex roots of b, since the rate of convergence $\beta_i \to 0$ is lower than the speed of the concentration parameter $\mu \to 0$ (see estimate (1.4)), u_λ develops a branch of $N+1$ local maximum points concentrating at 0 which are arranged as *satellites* close to the vertices β_i of a regular $(N+1)$-polygon. The analysis shows that the configuration of the limiting local maxima is determined by the interaction of two crucial aspects: the rate of convergence $V_\lambda''(0) \to 0^+$ and the boundary effect, represented by the Robin function $H(x,x)$. On the other hand, the existence of this kind

[1] We use the notation $a_\lambda = o(b_\lambda)$ to denote quantities which in the limit $\lambda \to 0^+$ verify $\frac{a_\lambda}{b_\lambda} \to 0$.

[2] We use the notation $\sim$ to denote quantities which in the limit $\lambda \to 0^+$ are of the same order.

of non-simple blow-up is unknown if $V''_\lambda(0)$ is uniformly bounded from below away from 0 or if $V''_\lambda(0) \le 0$.

The proofs use singular perturbation methods which combine the variational approach with a Lyapunov-Schmidt type procedure. Roughly speaking, the first step consists in the construction of an approximate solution, which should turn out to be precise enough. In view of the expected asymptotic behavior, the shape of such approximate solution will resemble, after the change of variables $x \mapsto x^{1/(N+1)}$, a *bubble* of the form (2.6) with a suitable choice of the parameter $\delta = \delta(\lambda, b)$. Then we look for a solution to (1.2) in a small neighborhood of the first approximation. As quite standard in singular perturbation theory, a crucial ingredient is the nondegeneracy of the explicit family of solutions of the limiting Liouville problem (2.5), in the sense that all bounded elements in the kernel of the linearization correspond to variations along the parameters of the family, as established in [1]. This allows us to study the invertibility of the linearized operator associated to the problem (1.2) under suitable orthogonality conditions. Next we introduce an intermediate problem and a fixed point argument will provide a solution for an auxiliary equation, which turns out to be solvable for any choice of b. Finally we test the auxiliary equation on the elements of the kernel of the linearized operator and we find out that, in order to find an *exact* solution of (1.2), the location of the maximum points, which is detected by the parameter b, should be a zero for a reduced finite dimensional map. We point out that the two scales of concentration of $b \to 0^+$ and $V''_\lambda(0) \to 0^+$ appear coupled at almost every step of the proof, so if $V''_\lambda(0) \le 0$ the method breaks down since we are unable to catch a nontrivial zero b for the reduced problem.

Let us comment on the assumption (A3) where a suitable vanishing rate for $V''_\lambda(0)$ is required: the lower bound on $V''_\lambda(0)$ assures that a non symmetric scenarios occurs for equation (2.4) since it distinguishes the blowing-up from the radially symmetric one (see estimate (1.4)); whereas the upper bound on $V''_\lambda(0)$ guarantees that the non simple blow up solutions can still be accurately approximated by global solutions by allowing an a priori estimate for the error which turns out to be is sufficiently small.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results, notation, and the definition of the approximating solution. Moreover, a more general version of Theorems 1.1 is stated there (see Theorem 2.1). In Section 3 we sketch the solvability of the linearized problem by referring to [13] and [14] for the proof. The error up to which the approximating solution solves problem (1.2) is estimated in Section 4. Section 5 considers the solvability of an auxiliary problem by a contraction argument. In Section 6 we complete the proof of Theorem 1.1. In Appendix A we collect some results, most of them well-known, which are usually referred to throughout the paper.

NOTATION: In our estimates throughout the paper, we will frequently denote by $C > 0$, $c > 0$ fixed constants, that may change from line to line, but are always independent of the variables under consideration.

2. Preliminaries and statement of the main results

We are going to provide an equivalent formulation of problem (1.2) and Theorem 1.1. Indeed, let us set

$$\alpha := N + 1 \ge 2$$

and let us observe that, setting v the regular part of u, namely

$$v = u + 4\pi(\alpha - 1)G(x, 0) = u + 2(\alpha - 1)\log\frac{1}{|x|}, \tag{2.1}$$

problem (1.2) is then equivalent to solving the following boundary value problem

$$\begin{cases} -\Delta v = \lambda |x|^{2(\alpha-1)} V_\lambda(|x|) e^v & \text{in } B_1 \\ v = 0 & \text{on } \partial B_1 \end{cases}. \tag{2.2}$$

Here G and H are the Green's function and its regular part as defined in the introduction.

In what follows, we identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 + ix_2 \in \mathbb{C}$ and we denote by $x\,y$ the multiplication of the complex numbers x, y and, analogously, by x^α the power of the complex number x.

Since the solutions considered in the paper are $\frac{2\pi}{\alpha}$-symmetric, we can rewrite problem (2.2) as a regular Liouville problem: more precisely, denoting by $x^{\frac{1}{\alpha}}$ the complex α-roots of x, the change of variables

$$w(x) = v\big(x^{\frac{1}{\alpha}}\big) \tag{2.3}$$

transforms problem (2.2) into a (regular) Liouville problem of the form

$$\begin{cases} -\Delta w = \dfrac{\lambda}{\alpha^2} V_\lambda\big(|x|^{\frac{1}{\alpha}}\big) e^w & \text{in } B_1 \\ w = 0 & \text{on } \partial B_1 \end{cases}. \tag{2.4}$$

Theorem 1.1 will be a consequence of a more general result concerning Liouville-type problems. In order to provide such a result, we now give a construction of a suitable approximate solution for (2.4). We can associate to (2.4) a limiting problem of Liouville type which will play a crucial role in the construction of blowing up solutions as $\lambda \to 0^+$:

$$-\Delta W = e^W \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{W(x)} dx < +\infty. \tag{2.5}$$

All solutions of this problem are given, in complex notation, by the three-parameter family of functions

$$W_{\delta,b}(x) := \log \frac{8\delta^2}{(\delta^2 + |x-b|^2)^2} \quad \delta > 0,\, b \in \mathbb{C}. \tag{2.6}$$

The following quantization property holds:

$$\int_{\mathbb{R}^2} e^{W_{\delta,b}(x)} dx = 8\pi. \tag{2.7}$$

Thanks to the radial symmetry of the problem, up to a rotation of the coordinates it is not restrictive to assume our bubble centered on the positive x_1-axis, which corresponds to consider $b := (b, 0)$ with $b > 0$. Therefore, in the following we agree that

$$W_\lambda(x) = W_{\delta,b}(x), \quad \delta,\, b > 0,$$

where the value $\delta = \delta(\lambda, b)$ is defined by

$$\delta^2 := \frac{\lambda}{8\alpha^2} V_\lambda(b^{\frac{1}{\alpha}}) e^{8\pi H(b,b)} = \frac{\lambda}{8\alpha^2} V_\lambda(b^{\frac{1}{\alpha}})(1-b^2)^4. \tag{2.8}$$

We point out that the diagonal $H(b, b)$ appearing in (2.8) is called the Robin function of the domain and in the case of the ball it takes the form

$$H(x,x) = \frac{1}{2\pi} \log(1 - |x|^2), \quad x \in B_1$$

according to (1.3). To obtain a better first approximation, we need to modify the function W_λ in order to satisfy the zero boundary condition. Precisely, we consider the projection PW_λ onto the space $H^1_0(B_1)$, where the projection $P : H^1(\mathbb{R}^N) \to H^1_0(B_1)$ is defined as the unique solution of the problem

$$\Delta Pv = \Delta v \quad \text{in } B_1, \qquad Pv = 0 \quad \text{on } \partial B_1.$$

We recall that the regular part $H(x,b)$ of the Green function, defined in (1.3), is harmonic in B_1 and satisfies $H(x,b) = \frac{1}{2\pi}\log|x-b|$ for $x \in \partial B_1$; a straightforward computation gives that for any $x \in \partial B_1$

$$\begin{aligned} PW_\lambda - W_\lambda + \log\left(8\delta^2\right) - 8\pi H(x,b) &= -W_\lambda + \log\left(8\delta^2\right) - 4\log|x-b| \\ &= 2\log\left(1 + \frac{\delta^2}{|x-b|^2}\right) = O(\delta^2) \end{aligned}$$

with uniform estimate for $x \in \partial B_1$ and $b > 0$ in a small neighborhood of 0. Since the expression $PW_\lambda - W_\lambda + \log\left(8\delta^2\right) - 8\pi H(x,b)$ is harmonic in B_1, then the maximum principle applies and implies the following asymptotic expansion

$$\begin{aligned} PW_\lambda =& W_\lambda - \log\left(8\delta^2\right) + 8\pi H(x,b) + O(\delta^2) \\ =& -2\log\left(\delta^2 + |x-b|^2\right) + 8\pi H(x,b) + O(\delta^2) \end{aligned} \tag{2.9}$$

uniformly for $x \in \overline{B}_1$ and $b > 0$ in a small neighborhood of 0.

We point out that, in order to simplify the notation, in our estimates throughout the paper we will describe the asymptotic behaviors of quantities under considerations in terms of $\delta = \delta(\lambda, b)$ instead of the parameter λ of the equation. Clearly according to (2.8) δ has the same rate as $\lambda^{\frac{1}{2}}$, so at each step we can easily pass to the analogous asymptotic in terms of λ: for instance, in (2.9) the error term "$O(\delta^2)$" can be equivalently replaced by "$O(\lambda)$".

We shall look for a solution to (2.4) in a small neighborhood of the first approximation, namely a solution of the form

$$w_\lambda = PW_\lambda + \phi_\lambda,$$

where the rest term ϕ_λ is small in $H^1_0(B_1)$-norm. Motivated by the symmetric setting of our problem, we consider the subspaces $H^1_{0,s}(B_1)$ made up of functions which are symmetric with respect to the x_2-axis:

$$H^1_{0,s}(B_1) = \{v \in H^1_0(B_1) \,|\, w(\bar{x}) = w(x)\}.$$

Here, using the complex notation, $\bar{x}$ is the conjugate of x: $\bar{x} = (x_1, -x_2)$. Clearly, $PW_\lambda \in H^1_{0,s}(B_1)$ if $b > 0$ and we shall look for a rest term ϕ_λ in $H^1_{0,s}(B_1)$.

In order to state the main theorem for problem (2.4), let us reformulate the four assumptions (A1) – (A4) in an equivalent way according to the new framework in terms of α instead of N with the vanishing estimates written in terms the new parameter δ in the place of λ:

(A1)* $\inf_{B_1} V_\lambda(|x|) > c > 0$ for a positive constant c independent of δ and, without loss of generality, we may assume $V_\lambda(0) = 1$.

(A2)* $V_\lambda(|x|)$ is of class C^1 in the closed unit ball $\overline{B}_1$ and is of class $C^{2\alpha+2}$ in a neighbourhood U of 0; moreover

$$\|V_\lambda\|_{C^1(\overline{B}_1)},\ \|V_\lambda\|_{C^{2\alpha+2}(U)} \le C$$

for a positive constant C independent of δ.

(A3)* $V''_\lambda(0) > 0$, $\delta^{\frac{2(\alpha-1)}{\alpha}} = o(V''_\lambda(0))$, $V''_\lambda(0) = o(\delta^{\frac{2(\alpha-1)}{\alpha+2}})$.

(A4)* the following holds:

$$|V^{2i}_\lambda(0)| \le C|V''_\lambda(0)| \quad \forall i = 1, \ldots, \alpha - 1$$

$$V^{2\alpha}_\lambda(0) < 2(2\alpha)! - c$$

for positive constants c, C independent of δ.

A non symmetric blow-up occurs for problem (2.4), as stated in the next theorem; more precisely, we provide a solution which develops a bubble centered at a point $(b, 0) = b > 0$; and since the rate of convergence $b \to 0^+$ is lower than the speed of the concentration parameter $\delta \to 0^+$ (see estimate (2.10)), the blowing up turns out to be non symmetric in the first approximation.

Theorem 2.1. *Assume that hypotheses* $(A1)^* - (A4)^*$ *hold. Then, for* λ *sufficiently small the problem* (2.4) *has a family of solutions* w_λ *satisfying*

$$w_\lambda = -2\log\left(\delta^2 + |x - b_\lambda|^2\right) + 8\pi H(x, b_\lambda) + o(1)$$

in H^1*-sense, where*

$$b_\lambda \sim (V''_\lambda(0))^{\frac{\alpha}{2(\alpha-1)}}. \tag{2.10}$$

In particular, by $(A3)^*$, $\delta = o(b)$.

In the remaining part of this paper we will prove Theorems 2.1 and at the end of Section 6 we shall see how Theorems 1.1 follows quite directly as a corollary.

We end this section by setting notation and basic well-known facts to be used in the rest of the paper. Given D a bounded domain, we denote by $\|\cdot\|$ and $\|\cdot\|_p$ the norms in the space $H_0^1(D)$ and $L^p(D)$, respectively, namely

$$\|u\| := \|u\|_{H_0^1(D)}, \qquad \|u\|_p := \|u\|_{L^p(D)} \quad \forall u \in H_0^1(D).$$

Next we recall the well-known Moser-Trudinger inequality ([20, 26]):

Lemma 2.2. *There exists* $C > 0$ *such that for any bounded domain* D *in* $\mathbb{R}^2$

$$\int_D e^{\frac{4\pi u^2}{\|u\|^2}} dy \le C|D| \quad \forall u \in H_0^1(D),$$

where $|D|$ *stands for the measure of the domain* D*. In particular, for any* $q \ge 1$

$$\|e^u\|_q \le C^{\frac{1}{q}} |D|^{\frac{1}{q}} e^{\frac{q}{16\pi}\|u\|^2} \quad \forall u \in H_0^1(D).$$

As commented in the introduction, our proof uses the singular perturbation methods. For that, the nondegeneracy of the functions that we use to build our approximating solution is essential. Next proposition is devoted to the nondegeneracy of the finite mass solutions of the Liouville equation (see [1] for the proof).

Proposition 2.3. *Assume that* $\xi \in \mathbb{R}^2$ *and* $\phi : \mathbb{R}^2 \to \mathbb{R}$ *solves the problem*

$$-\Delta\phi = \frac{8}{(1 + |z - \xi|^2)^2}\phi \ \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla\phi(z)|^2 dz < +\infty. \tag{2.11}$$

Then there exist c_0, c_1, $c_2 \in \mathbb{R}$ *such that*

$$\phi(z) = c_0 Z_0 + c_1 Z_1 + c_2 Z_2,$$

$$Z_0(z) := \frac{1 - |z - \xi|^2}{1 + |z - \xi|^2}, \quad Z_1(z) := \frac{\mathrm{Re}\,(z - \xi)}{1 + |z - \xi|^2}, \quad Z_2(z) := \frac{\mathrm{Im}(z - \xi)}{1 + |z - \xi|^2}.$$

3. Analysis of the linearized operator

According to Proposition 2.3, by the change of variable $x = \delta z$, we immediately get that all solutions ψ of

$$-\Delta\psi = \frac{8\delta^2}{(\delta^2 + |x - b|^2)^2}\psi = e^{W_\lambda}\psi \quad \text{in} \quad \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} |\nabla\psi(x)|^2 dx < +\infty.$$

are linear combinations of the functions

$$Z^0_{\delta,b}(x)=\frac{\delta^2-|x-b|^2}{\delta^2+|x-b|^2},\ Z^1_{\delta,b}(x)=\frac{\delta\mathrm{Re}\,(x-b)}{\delta^2+|x-b|^2},\ Z^2_{\delta,b}(x)=\frac{\delta\mathrm{Im}(x-b)}{\delta^2+|x-b|^2}.$$

We introduce the projections $PZ^j_{\delta,b}$ onto $H^1_0(B_1)$. It is immediate that

$$PZ^0_{\delta,b}(x)=Z^0_{\delta,b}(x)+1+O\left(\delta^2\right)=\frac{2\delta^2}{\delta^2+|x-b|^2}+O(\delta^2) \tag{3.1}$$

and

$$PZ^j_{\delta,b}(x)=Z^j_{\delta,b}(x)+O(\delta) \text{ for } j=1,2 \tag{3.2}$$

uniformly with respect to $x\in\overline{B}_1$ and $b>0$ in a small neighborhood of 0. Clearly $PZ^0_\lambda, PZ^1_\lambda$ are symmetric with respect to the x_2-axis, i.e. $PZ^0_\lambda, PZ^1_\lambda\in H^1_{0,s}(B_1)$.

We agree that $Z^j_\lambda:=Z^j_{\delta,b}$ for any $j=0,1,2$, where δ is defined in terms of λ and b according to (2.8). Let us consider the following linear problem: given $h\in H^1_{0,s}(B_1)$, find a function $\phi\in H^1_{0,s}(B_1)$, and a constant $c_1\in\mathbb{R}$ satisfying

$$\begin{cases} -\Delta\phi-\dfrac{\lambda}{\alpha^2}V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)e^{PW_\lambda}\phi=\Delta h+c_1Z^1_\lambda e^{W_\lambda} \\ \displaystyle\int_{B_1}\nabla\phi\nabla PZ^1_\lambda=0 \end{cases}. \tag{3.3}$$

In order to solve problem (3.3), we need to establish an a priori estimate. For the proof we refer to [13] (Proposition 3.1) or [14] (Proposition 3.1).

Proposition 3.1. *There exist $\lambda_0>0$ and $C>0$ such that for any $\lambda\in(0,\lambda_0)$, any $b>0$ in a small neighborhood of 0 and any $h\in H^1_{0,s}(B_1)$, if $(\phi,c_1)\in H^1_{0,s}(B_1)\times\mathbb{R}$ solves (3.3), then the following holds*

$$\|\phi\|\le C|\log\delta|\|h\|.$$

For any $p>1$, let

$$i^*_p:L^p(B_1)\to H^1_0(B_1) \tag{3.4}$$

be the adjoint operator of the embedding $i_p: H^1_0(B_1)\hookrightarrow L^{\frac{p}{p-1}}(B_1)$, i.e. $u=i^*_p(v)$ if and only if $-\Delta u=v$ in B_1, $u=0$ on ∂B_1. We point out that i^*_p is a continuous mapping, namely

$$\|i^*_p(v)\|\le c_p\|v\|_p, \text{ for any } v\in L^p(B_1), \tag{3.5}$$

for some constant c_p which depends on p. Next let us set

$$K:=\mathrm{span}\left\{PZ^1_\lambda\right\}$$

and

$$K^\perp:=\left\{\phi\in H^1_{0,s}(B_1)\ :\ \int_{B_1}\nabla\phi\nabla PZ^1_\lambda dx=0\right\}$$

and denote by

$$\Pi:H^1_{0,s}(B_1)\to K,\qquad \Pi^\perp:H^1_{0,s}(B_1)\to K^\perp$$

the corresponding projections. Let $L:K^\perp\to K^\perp$ be the linear operator defined by

$$L(\phi):=\frac{1}{\alpha^2}\Pi^\perp\Big(i^*_p\big(\lambda V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)e^{PW_\lambda}\phi\big)\Big)-\phi. \tag{3.6}$$

Notice that problem (3.3) reduces to

$$L(\phi)=\Pi^\perp h,\quad \phi\in K^\perp.$$

As a consequence of Proposition 3.1 we derive the invertibility of L.

Proposition 3.2. *For any $p>1$ there exist $\lambda_0>0$ and $C>0$ such that for any $\lambda\in(0,\lambda_0)$, any $b>0$ in a small neighborhood of 0 and any $h\in K^\perp$ there is a unique solution $\phi\in K^\perp$ to the problem*

$$L(\phi)=h.$$

In particular, L is invertible; moreover,

$$\|L^{-1}\|\le C|\log\delta|.$$

Proof. Observe that the operator $\phi\mapsto\Pi^\perp\big(i_p^*(\lambda V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)e^{PW_\lambda}\phi)\big)$ is a compact operator in $K^\perp$. Let us consider the case $h=0$, and take $\phi\in K^\perp$ with $L(\phi)=0$. In other words, ϕ solves the system (3.3) with $h=0$ for some $c_1\in\mathbb{R}$. Proposition 3.1 implies $\phi\equiv0$. Then, Fredholm's alternative implies the existence and uniqueness result.

Once we have existence, the norm estimate follows directly from Proposition 3.1. □

4. Estimate of the error term

The goal of this section is to provide an estimate of the error up to which the approximate solution PW_λ solves problem (2.4). First of all, we perform the following estimates.

Lemma 4.1. *Let $\gamma=0,1,2$ and $p>1$ be fixed. The following holds:*

$$\||x-b|^\gamma e^{W_\lambda}\|_p\le C\delta^\gamma\delta^{-2\frac{p-1}{p}},\quad \||x-b|^\gamma\lambda e^{PW_\lambda}\|_p\le C\delta^\gamma\delta^{-2\frac{p-1}{p}}\tag{4.1}$$

uniformly for $b>0$ in a small neighborhood of 0.

Proof. We compute

$$\||x-b|^\gamma e^{W_\lambda}\|_p^p=8^p\delta^{2p}\int_{B_1}\frac{|x-b|^{\gamma p}}{(\delta^2+|x-b|^2)^{2p}}dx\le 8^p\delta^{\gamma p-2(p-1)}\int_{\mathbb{R}^2}\frac{|z|^{\gamma p}}{(1+|z|^2)^{2p}}dz.$$

Taking into account that the last integral is finite for $\gamma=0,1,2$ and $p>1$ we deduce the first part of (4.1). To prove the second part it is sufficient to observe that by (2.9) and by the choice of δ in (2.8) we derive

$$\lambda e^{PW_\lambda}=\frac{\lambda}{8\delta^2}e^{W_\lambda+O(1)}=e^{W_\lambda}(1+O(1)).\tag{4.2}$$

□

Lemma 4.2. *Assume that hypotheses* (A1)* − (A4)*. *Then the following holds*

$$\begin{aligned}\frac{V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)}{V_\lambda(b^{\frac{1}{\alpha}})}&=1+\frac{2}{\alpha V_\lambda(b^{\frac{1}{\alpha}})}\sum_{i=1}^{\alpha}\frac{i}{(2i)!}V_\lambda^{2i}(0)|b|^{\frac{2i-\alpha}{\alpha}}\mathrm{Re}\,(x-b)\\&\quad+\sum_{i=1}^{\alpha}O\Big(V_\lambda^i(0)|b|^{2\frac{i-\alpha}{\alpha}}|x-b|^2\Big)+O(|x-b|^{2\frac{\alpha+1}{\alpha}})+O(b^{2\frac{\alpha+1}{\alpha}})\end{aligned}$$

uniformly for $x\in\overline{B}_1$ and $b>0$ in a small neighborhood of 0.

Proof. According to assumption (A2)* we have

$$V_\lambda(|x|)=1+\sum_{i=1}^{\alpha}\frac{1}{(2i)!}V_\lambda^{2i}(0)|x|^{2i}+O(|x|^{2\alpha+2})$$

uniformly for $x \in B_1$. Using that for $i = 1, \dots, \alpha$ we have $(1+2\mathrm{Re}\, y+|y|^2)^{\frac{i}{\alpha}} = 1+\frac{2i}{\alpha}\mathrm{Re}\, y+O(|y|^2)$ uniformly for $y \in \mathbb{R}^2$, we get

$$
\begin{aligned}
|x|^{\frac{2i}{\alpha}} &= \Big(|x-b+b|^2\Big)^{\frac{i}{\alpha}} = \Big(|x-b|^2+|b|^2+2b\mathrm{Re}\,(x-b)\Big)^{\frac{i}{\alpha}} \\
&= |b|^{\frac{2i}{\alpha}}\bigg(1+2\frac{\mathrm{Re}\,(x-b)}{b}+\frac{|x-b|^2}{b^2}\bigg)^{\frac{i}{\alpha}} \\
&= |b|^{\frac{2i}{\alpha}}\bigg(1+\frac{2i}{\alpha}\frac{\mathrm{Re}\,(x-b)}{b}+O\Big(\frac{|x-b|^2}{b^2}\Big)\bigg) \\
&= |b|^{\frac{2i}{\alpha}}+\frac{2i}{\alpha}|b|^{\frac{2i-\alpha}{\alpha}}\mathrm{Re}\,(x-b)+O\Big(b^{2\frac{i-\alpha}{\alpha}}|x-b|^2\Big)
\end{aligned}
$$

uniformly for $x \in B_1$ and $b > 0$ in a small neighborhood of 0. Then we obtain

$$
\begin{aligned}
V_\lambda(|x|^{\frac{1}{\alpha}}) &= 1+\sum_{i=1}^{\alpha}\frac{1}{(2i)!}V_\lambda^{2i}(0)|x|^{\frac{2i}{\alpha}}+O(|x|^{2\frac{\alpha+1}{\alpha}}) \\
&= 1+\sum_{i=1}^{\alpha}\frac{1}{(2i)!}V_\lambda^{2i}(0)|b|^{\frac{2i}{\alpha}}+\frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{i}{(2i)!}V_\lambda^{2i}(0)|b|^{\frac{2i-\alpha}{\alpha}}\mathrm{Re}\,(x-b) \\
&\quad+\sum_{i=1}^{\alpha}O\Big(V_\lambda^{2i}(0)|b|^{2\frac{i-\alpha}{\alpha}}|x-b|^2\Big)+O(b^{2\frac{\alpha+1}{\alpha}})+O\big(|x-b|^{2\frac{\alpha+1}{\alpha}}\big)
\end{aligned}
$$

uniformly for $x \in B_1$ and $b > 0$ in a small neighborhood of 0.

Finally, using that $V_\lambda(b^{\frac{1}{\alpha}}) = 1+\sum_{i=1}^{\alpha}\frac{1}{(2i)!}V_\lambda^{2i}(0)|b|^{\frac{2i}{\alpha}}+O(b^{2\frac{\alpha+1}{\alpha}})$ we deduce the thesis.

□

Let us fix $M > 0$ a sufficiently large number to be chosen later and set I_λ the interval

$$
I_\lambda := \bigg[\frac{1}{M}(V_\lambda''(0))^{\frac{\alpha}{2(\alpha-1)}}, M(V_\lambda''(0))^{\frac{\alpha}{2(\alpha-1)}}\bigg]. \tag{4.3}
$$

Now we are in the position to provide the error estimate for $b \in I_\lambda$.

Proposition 4.3. *Assume that hypotheses* (A1)* − (A4)* *hold and define*

$$
R_\lambda := -\Delta PW_\lambda-\frac{\lambda}{\alpha^2}V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)e^{PW_\lambda} = e^{W_\lambda}-\frac{\lambda}{\alpha^2}V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)e^{PW_\lambda}.
$$

Then the following holds

$$
\begin{aligned}
R_\lambda &= e^{W_\lambda}\bigg(4b-\frac{2}{\alpha V_\lambda(b^{\frac{1}{\alpha}})}\sum_{i=1}^{\alpha}\frac{i}{(2i)!}V_\lambda^{i}(0)|b|^{\frac{2i-\alpha}{\alpha}}\bigg)\mathrm{Re}\,(x-b) \\
&\quad+e^{W_\lambda}\bigg(\sum_{i=1}^{\alpha}O\Big(V_\lambda^{i}(0)|b|^{2\frac{i-\alpha}{\alpha}}|x-b|^2\Big)+o(\delta b)+O(|x-b|^2\bigg)
\end{aligned} \tag{4.4}
$$

uniformly for $x \in B_1$ and $b \in I_\lambda$. Moreover for any $p > 1$

$$
\|R_\lambda\|_p \le Cb\delta^{1-2\frac{p-1}{p}}
$$

uniformly for $b \in I_\lambda$.

Proof. By (2.9) and the choice of δ in (2.8) we derive

$$
\begin{aligned}
\frac{\lambda}{\alpha^2} V_\lambda(|x|^{\frac{1}{\alpha}}) e^{PW_\lambda} &= \frac{\lambda}{8\alpha^2\delta^2} V_\lambda(|x|^{\frac{1}{\alpha}}) e^{W_\lambda + 8\pi H(x,b) + O(\delta^2)} \\
&= \frac{V_\lambda(|x|^{\frac{1}{\alpha}})}{V(b^{\frac{1}{\alpha}})} e^{W_\lambda} e^{8\pi(H(x,b)-H(b,b))+O(\delta^2)} \\
&= \frac{V_\lambda(|x|^{\frac{1}{\alpha}})}{V(b^{\frac{1}{\alpha}})} e^{W_\lambda} e^{8\pi(H(x,b)-H(b,b))}\big(1+O(\delta^2)\big).
\end{aligned} \tag{4.5}
$$

Using the expression of H given in (1.3) we compute

$$
\begin{aligned}
e^{8\pi(H(x,b)-H(b,b))} = e^{4\log(b|x-\frac{1}{b}|)-4\log(1-b^2)} &= \frac{|b(x-b)-(1-b^2)|^4}{(1-b^2)^4} \\
&= 1 - 4b\frac{\mathrm{Re}\,(x-b)}{1-b^2} + O\big(|x-b|^2\big) \\
&= 1 - 4b\mathrm{Re}\,(x-b) + O(b^3) + O\big(|x-b|^2\big).
\end{aligned}
$$

Then (4.5) becomes

$$
\begin{aligned}
\frac{\lambda}{\alpha^2} V_\lambda(|x|^{\frac{1}{\alpha}}) e^{PW_\lambda} = &\frac{V_\lambda(|x|^{\frac{1}{\alpha}})}{V(b^{\frac{1}{\alpha}})} e^{W_\lambda} - 4b\frac{V_\lambda(|x|^{\frac{1}{\alpha}})}{V(b^{\frac{1}{\alpha}})} e^{W_\lambda}\mathrm{Re}\,(x-b) \\
&+ e^{W_\lambda}\Big(O(b^3)+O(\delta^2)+O\big(|x-b|^2\big)\Big).
\end{aligned} \tag{4.6}
$$

Using the expansion provided by Lemma 4.2 into (4.6), we have

$$
\begin{aligned}
\frac{\lambda}{\alpha^2} V_\lambda(|x|^{\frac{1}{\alpha}}) e^{PW_\lambda} = e^{W_\lambda} + e^{W_\lambda}&\left(\frac{2}{\alpha V_\lambda(b^{\frac{1}{\alpha}})}\sum_{i=1}^{\alpha}\frac{i}{(2i)!}V_\lambda^{2i}(0)|b|^{\frac{2i-\alpha}{\alpha}} - 4b\right)\mathrm{Re}\,(x-b) \\
+ e^{W_\lambda}&\left(\sum_{i=1}^{\alpha} O\Big(V_\lambda^{2i}(0)|b|^{2\frac{i-\alpha}{\alpha}}|x-b|^2\Big) + O(|x-b|^2) + O(\delta^2) + O(b^{2\frac{\alpha+1}{\alpha}})\right)
\end{aligned}
$$

and (4.4) follows using that $\delta = o(b)$ and $b^{2\frac{\alpha+1}{\alpha}} = o(\delta b)$ uniformly for $b \in I_\lambda$ according to assumption (A3)*. So, by applying Lemma 4.1 we get for any $p > 1$

$$
\|R_\lambda\|_p \le C\Big(\delta\sum_{i=1}^{\alpha}|V_\lambda^{2i}(0)||b|^{\frac{2i-\alpha}{\alpha}} + \delta^2\sum_{i=1}^{\alpha}|V_\lambda^{2i}(0)||b|^{2\frac{i-\alpha}{\alpha}} + b\delta + \delta^2\Big)\delta^{-2\frac{p-1}{p}}.
$$

Using again that $\delta = o(b)$ if $b \in I_\lambda$, we also get $\delta^2|b|^{2\frac{i-\alpha}{\alpha}} \le \delta|b|^{\frac{2i-\alpha}{\alpha}}$ for $b \in I_\lambda$; so the above L^p estimate becomes

$$
\|R_\lambda\|_p \le C\Big(\delta\sum_{i=1}^{\alpha}|V_\lambda^{2i}(0)||b|^{\frac{2i-\alpha}{\alpha}} + b\delta\Big)\delta^{-2\frac{p-1}{p}}.
$$

Finally, according to (A4)*, for every $i = 1, \dots, \alpha$ we get

$$
|V_\lambda^{2i}(0)||b|^{\frac{2i-\alpha}{\alpha}} \le C|V_\lambda''(0)||b|^{\frac{2i-\alpha}{\alpha}} \le CV_\lambda''(0)|b|^{\frac{2-\alpha}{\alpha}} \le Cb \quad \forall b \in I_\lambda
$$

and last part of the thesis follows. □

5. The nonlinear problem: a contraction argument

In order to solve (2.4), let us consider the following intermediate problem:

$$\begin{cases} -\Delta(PW_\lambda+\phi)-\dfrac{\lambda}{\alpha^2}V_\lambda(|x|^{\frac{1}{\alpha}})e^{PW_\lambda+\phi}=c_1Z^1_\lambda e^{W_\lambda}, \\ \phi\in H^1_{0,s}(B_1), \quad \displaystyle\int_{B_1}\nabla\phi\nabla PZ^1_\lambda dx=0. \end{cases} \tag{5.1}$$

Then it is convenient to solve as a first step the problem for ϕ as a function of b.

Let us rewrite problem (5.1) in a more convenient way. In what follows we denote by $N: H^1_{0,s}(B_1)\to K^\perp$ the nonlinear operator

$$N(\phi)=\Pi^\perp\left(i^*_p\bigg(\frac{\lambda}{\alpha^2}V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)e^{PW_\lambda}(e^\phi-1-\phi)\bigg)\right).$$

Therefore problem (5.1) turns out to be equivalent to the problem

$$L(\phi)+N(\phi)=\tilde{R}, \quad \phi\in K^\perp \tag{5.2}$$

where, recalling Lemma 4.1,

$$\tilde{R}=\Pi^\perp\left(i^*_p(R_\lambda)\right)=\Pi^\perp\left(PW_\lambda-i^*_p\bigg(\frac{\lambda}{\alpha^2}V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)e^{PW_\lambda}\bigg)\right).$$

We need the following auxiliary lemma.

Lemma 5.1. *For any $p>1$ and any $\phi_1,\phi_2\in H^1_0(B_1)$ with $\|\phi\|_1$, $\|\phi_2\|<1$ the following holds*

$$\|e^{\phi_1}-\phi_1-e^{\phi_2}+\phi_2\|_p\le C(\|\phi_1\|+\|\phi_2\|)\|\phi_1-\phi_2\|, \tag{5.3}$$

$$\|N(\phi_1)-N(\phi_2)\|\le C\delta^{-2\frac{p^2-1}{p^2}}(\|\phi_1\|+\|\phi_2\|)\|\phi_1-\phi_2\| \tag{5.4}$$

uniformly for $b>0$ in a small neighborhood of 0.

Proof. A straightforward computation gives that the inequality $|e^a-a-e^b+b|\le e^{|a|+|b|}(|a|+|b|)|a-b|$ holds for all $a,b\in\mathbb{R}$. Then, by applying Hölder's inequality with $\frac{1}{q}+\frac{1}{r}+\frac{1}{t}=1$, we derive

$$\|e^{\phi_1}-\phi_1-e^{\phi_2}+\phi_2\|_p\le C\|e^{|\phi_1|+|\phi_2|}\|_{pq}(\|\phi_1\|_{pr}+\|\phi_2\|_{pr})\|\phi_1-\phi_2\|_{pt}$$

and (5.3) follows by using Lemma 2.2 and the continuity of the embeddings $H^1_0(B_1)\subset L^{pr}(B_1)$ and $H^1_0(B_1)\subset L^{pt}(B_1)$. Let us prove (5.4). According to (3.5) we get

$$\|N(\phi_1)-N(\phi_2)\|\le C\|\lambda V_\lambda(|x|^{\frac{1}{\alpha}})e^{PW_\lambda}(e^{\phi_1}-\phi_1-e^{\phi_2}+\phi_2)\|_p,$$

and by Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$, we derive

$$\begin{aligned}\|N(\phi_1)-N(\phi_2)\|&\le C\|\lambda V_\lambda(|x|^{\frac{1}{\alpha}})e^{PW_\lambda}\|_{p^2}\|e^{\phi_1}-\phi_1-e^{\phi_2}+\phi_2|\|_{pq}\\ &\le C\|\lambda V_\lambda(|x|^{\frac{1}{\alpha}})e^{PW_\lambda}\|_{p^2}(\|\phi_1\|+\|\phi_2\|)\|\phi_1-\phi_2\|\end{aligned}$$

by (5.3),

and the conclusion follows by Lemma 4.1. □

Problem (5.1) or, equivalently, problem (5.2) turns out to be solvable for any choice of point b in the interval I_λ, provided that λ is sufficiently small. Indeed we have the following result.

Proposition 5.2. *Assume* (A1)* $-$ (A4)* *hold and let* $\varepsilon > 0$ *be a fixed small number. Then there exists* $\lambda_0 > 0$ *such that for any* $\lambda \in (0, \lambda_0)$ *and any* $b \in I_\lambda$ *there is a unique* $\phi_\lambda = \phi_{\lambda,b} \in K^\perp$ *satisfying* (5.1) *for some* $c_1 \in \mathbb{R}$ *and*

$$\|\phi_\lambda\| \le \delta^{1-\varepsilon} b. \tag{5.5}$$

Moreover the map $b \in I_\lambda \mapsto \phi_{\lambda,b} \in H_0^1(B_1)$ *is continuous.*

Proof. Since problem (5.2) is equivalent to problem (5.1), we will show that problem (5.2) can be solved via a contraction mapping argument. Indeed, in virtue of Proposition 3.2, let us introduce the map

$$T := L^{-1}(\tilde{R} - N(\phi)), \quad \phi \in K^\perp.$$

Let us fix $p > 1$ sufficiently close to 1. By (3.5) and Proposition 4.3, we get

$$\|\tilde{R}\| \le Cb\delta^{1-\frac{\varepsilon}{2}}. \tag{5.6}$$

Next, by (5.4),

$$\|N(\phi_1) - N(\phi_2)\| \le C\delta^{-\frac{\varepsilon}{2}}(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\| \quad \forall \phi_1, \phi_2 \in H_0^1(B_1), \|\phi_1\|, \|\phi_2\| < 1. \tag{5.7}$$

In particular, by taking $\phi_2 = 0$,

$$\|N(\phi)\| \le C\delta^{-\frac{\varepsilon}{2}}\|\phi\|^2 \quad \forall \phi \in H_0^1(B_1), \|\phi\| < 1. \tag{5.8}$$

We claim that T is a contraction map over the ball

$$\mathcal{B} := \left\{ \phi \in K^\perp \,\middle|\, \|\phi\| \le b\delta^{1-\varepsilon} \right\}$$

provided that λ is small enough. Indeed, combining Proposition 3.2, (5.6), (5.7), (5.8), for any $\phi \in \mathcal{B}$ we have

$$\|T(\phi)\| \le C|\log\delta|\big(\|\tilde{R}\| + \|N(\phi)\|\big) \le C|\log\delta|\big(b\delta^{1-\frac{\varepsilon}{2}} + b^2\delta^{2-2\varepsilon}\big) < b\delta^{1-\varepsilon}.$$

Similarly, for any $\phi_1, \phi_2 \in \mathcal{B}$

$$\|T(\phi_1) - T(\phi_2)\| \le C|\log\delta|\|N(\phi_1) - N(\phi_2)\| \le C\delta^{-\frac{\varepsilon}{2}}|\log\delta|(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\| \le \frac{1}{2}\|\phi_1 - \phi_2\|.$$

Uniqueness of solutions implies continuous dependence of $\phi_\lambda = \phi_{\lambda,b}$ on b. ☐

6. Proof of Theorems 1.1 and Theorem 2.1

During this section we assume that the crucial assumption (A1)* $-$ (A4)* of Theorem 2.1 hold.

After problem (5.1) has been solved according to Proposition 5.2, then we find a solution to the original problem (2.4) if $b \in I_\lambda$ is such that

$$c_1 = 0.$$

Let us find the condition satisfied by b in order to get c_1 equal to zero.

Proof of Theorem 2.1. We multiply the equation in (5.1) by PZ_λ^1 and integrate over B_1:

$$\begin{aligned}\int_{B_1} \nabla(PW_\lambda + \phi_\lambda)\nabla PZ_\lambda^1 dx - \frac{\lambda}{\alpha^2}\int_{B_1} V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)e^{PW_\lambda + \phi_\lambda} PZ_\lambda^1 dx \\ = c_1 \int_{B_1} Z_\lambda^1 e^{W_\lambda} PZ_\lambda^1 dx.\end{aligned} \tag{6.1}$$

The object is now to expand each integral of the above identity and analyze the leading term. In the remaining part of the section all the estimates hold uniformly for $b \in I_\lambda$, without further notice.

Let us begin by observing that the orthogonality in (5.1) gives

$$\int_{B_1} \nabla\phi_\lambda \nabla PZ_\lambda^1 dx = \int_{B_1} e^{W_\lambda}\phi_\lambda Z_\lambda^1 dx = 0 \tag{6.2}$$

and, by (3.2),

$$\int_{B_1} Z_\lambda^1 e^{W_\lambda} PZ_\lambda^1 dx = \int_{B_1} e^{W_\lambda}(Z_\lambda^1)^2 dx + o(1) = 8\int_{\mathbb{R}^2} \frac{z_1^2}{(1+|z|^2)^4}dz + o(1) = \frac{2}{3}\pi + o(1). \tag{6.3}$$

Using the definition of R_λ in Lemma 4.3, (6.2) and (6.3), the above identity(6.1) becomes

$$\int_{B_1} R_\lambda PZ_\lambda^1 dx - \frac{\lambda}{\alpha^2}\int_{B_1} V_\lambda(|x|^{\frac{1}{\alpha}})e^{PW_\lambda}(e^{\phi_\lambda}-1)PZ_\lambda^1 dx \quad = \frac{2}{3}\pi c_1 + o(c_1). \tag{6.4}$$

Let us first estimate the term containing the function ϕ_λ: recalling (6.2)

$$\begin{aligned} \frac{\lambda}{\alpha^2}\int_{B_1} V_\lambda(|x|^{\frac{1}{\alpha}})e^{PW_\lambda}(e^{\phi_\lambda}-1)PZ_\lambda^1 dx &= \int_{B_1} R_\lambda(1-e^{\phi_\lambda})PZ_\lambda^1 dx \\ &+ \int_{B_1} e^{W_\lambda}(e^{\phi_\lambda}-1-\phi_\lambda)PZ_\lambda^1 dx \\ &+ \int_{B_1} e^{W_\lambda}\phi_\lambda(PZ_\lambda^1 - Z_\lambda^1)dx. \end{aligned} \tag{6.5}$$

Now, let us fix $\varepsilon > 0$ sufficiently small and $p > 1$ sufficiently close to 1. Next let $1 < q < \infty$ be such that $\frac{1}{p}+\frac{1}{q} = 1$. Then, (5.3) with $\phi_2 = 0$ and Proposition 5.2 give

$$\|e^{\phi_\lambda} - 1 - \phi_\lambda\|_q \le C\|\phi_\lambda\|^2 \le Cb^2\delta^{2-2\varepsilon}$$

and, consequently,

$$\|e^{\phi_\lambda} - 1\|_q \le C\|\phi_\lambda\| \le Cb\delta^{1-\varepsilon}. \tag{6.6}$$

Therefore, Lemma 4.1 implies

$$\begin{aligned} \int_{B_1} e^{W_\lambda}(e^{\phi_\lambda}-1-\phi_\lambda)PZ_\lambda^1 dx &= O(\|e^{W_\lambda}(e^{\phi_\lambda}-1-\phi_\lambda)\|_1) = O(\|e^{W_\lambda}\|_p\|e^{\phi_\lambda}-1-\phi_\lambda\|_q) \\ &= O\Big(b^2\delta^{2-2\frac{p-1}{p}-2\varepsilon}\Big) \end{aligned} \tag{6.7}$$

Now, by Lemma 4.3

$$\begin{aligned} \int_{B_1} R_\lambda(1-e^{\phi_\lambda})PZ_\lambda^1 dx &= O\big(\big\|R_\lambda(e^{\phi_\lambda}-1)\big\|_1\big) = O\big(\big\|R_\lambda\|_p\|e^{\phi_\lambda}-1\|_q\big) \\ &= O\Big(b^2\delta^{2-2\frac{p-1}{p}-\varepsilon}\Big). \end{aligned} \tag{6.8}$$

Finally by (3.2) and Lemma 4.1

$$\int_{B_1} e^{W_\lambda}\phi_\lambda(PZ_\lambda^1 - Z_\lambda^1)dx = O\Big(\delta\|e^{W_\lambda}\|_p\|\phi_\lambda\|\big) = O(b\delta^{2-2\frac{p-1}{p}-\varepsilon}\Big). \tag{6.9}$$

By inserting (6.7)-(6.8)-(6.9) into (6.5), we obtain

$$\lambda\int_{B_1} V_\lambda\big(|x|^{\frac{1}{\alpha}}\big)e^{PW_\lambda}(e^{\phi_\lambda}-1)PZ_\lambda^1 dx = o(b\delta). \tag{6.10}$$

Next, by (4.4), using Lemma A.2, we get

$$\int_{B_1} R_\lambda P Z^1_\lambda dy = 2\pi\delta\Bigg(4b - \frac{2}{\alpha V_\lambda(b^{\frac{1}{\alpha}})}\sum_{i=1}^{\alpha}\frac{i}{(2i)!}V_\lambda^{2i}(0)|b|^{\frac{2i-\alpha}{\alpha}}\Bigg) + O\Bigg(\delta^2\sum_{i=1}^{\alpha}|V_\lambda^{2i}(0)||b|^{2\frac{i-\alpha}{\alpha}}\Bigg) + O(\delta^2) + o(\delta b).$$

Since $V_\lambda(b^{\frac{1}{\alpha}}) = 1 + o(1)$ and $\delta = o(b)$ uniformly for $b \in I_\lambda$, by inserting the above identity and (6.10) into (6.4) we deduce

$$\begin{aligned} &2\pi\delta\Bigg(4b(1+o(1)) - \frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{i}{(2i)!}V_\lambda^{2i}(0)|b|^{\frac{2i-\alpha}{\alpha}}(1+o(1))\Bigg) + O\Bigg(\delta^2\sum_{i=1}^{\alpha}|V_\lambda^{2i}(0)||b|^{2\frac{i-\alpha}{\alpha}}\Bigg) \\ &= \frac{2}{3}\pi c_1 + o(c_1). \end{aligned} \tag{6.11}$$

Using again that $\delta = o(b)$ for $b \in I_\lambda$, we derive $\delta^2|b|^{2\frac{i-\alpha}{\alpha}} = o(\delta|b|^{\frac{2i-\alpha}{\alpha}})$ for $i = 1, \dots, \alpha$. So, (6.11) can be rewritten as

$$2\pi\delta\Bigg(4b(1+o(1)) - \frac{2}{\alpha}\sum_{i=1}^{\alpha}\frac{i}{(2i)!}V_\lambda^{2i}(0)|b|^{\frac{2i-\alpha}{\alpha}}(1+o(1))\Bigg) = \frac{2}{3}\pi c_1 + o(c_1). \tag{6.12}$$

Now, let us set $b_+ = M(V''_\lambda(0))^{\frac{\alpha}{2(\alpha-1)}}$. Taking into account of assumption (A4)* we have

$$V_\lambda^{2i}(0)b_+^{\frac{2i-\alpha}{\alpha}} = M^{\frac{2i-\alpha}{\alpha}}V_\lambda^{2i}(0)|V''(0)|^{\frac{2i-\alpha}{2(\alpha-1)}} \le CM^{\frac{2i-\alpha}{\alpha}}|V''_\lambda(0)|^{\frac{\alpha}{2(\alpha-1)}} \quad \forall i = 1, \dots, \alpha-1$$

with C independent of M. By evaluating the left hand side of (6.12) at b_+ we get

$$2\pi\delta(V''_\lambda(0))^{\frac{\alpha}{2(\alpha-1)}}\Bigg(4M(1+o(1)) - \sum_{i=1}^{\alpha-1}O(M^{\frac{2i-\alpha}{\alpha}}) - MV_\lambda^{2\alpha}(0)\frac{2(1+o(1))}{(2\alpha)!}\Bigg).$$

Since $\frac{2i-\alpha}{\alpha} < 1$ for $i = 1, \dots, \alpha-1$, taking into account of assumption (A4)*, we get that the left hand side of (6.12) evaluated at b_+ is positive provided that M is large enough.

Next let us set $b_- = \frac{1}{M}(V''_\lambda(0))^{\frac{\alpha}{2(\alpha-1)}}$. Then again by assumption (A4)*

$$V_\lambda^{2i}(0)b_-^{\frac{2i-\alpha}{\alpha}} = M^{\frac{\alpha-2i}{\alpha}}V_\lambda^{2i}(0)|V''(0)|^{\frac{2i-\alpha}{2(\alpha-1)}} \le CM^{\frac{\alpha-2i}{\alpha}}|V''_\lambda(0)|^{\frac{\alpha}{2(\alpha-1)}} \quad \forall i = 2, \dots, \alpha$$

where C is independent of M. The left hand side of (6.12) evaluated at b_- becomes

$$2\pi\delta(V''(0))^{\frac{\alpha}{2(\alpha-1)}}\Bigg(\frac{4}{M}(1+o(1)) - \frac{1}{\alpha}M^{\frac{\alpha-2}{\alpha}}(1+o(1)) - \sum_{i=2}^{\alpha}O(M^{\frac{\alpha-2i}{\alpha}})\Bigg)$$

which is negative for M large enough.

The continuity of the map $b \mapsto \phi_\lambda = \phi_{\lambda,b}$ implies that the right hand side of (6.12) has a zero b_λ in I_λ, so

$$b_\lambda \sim (V''_\lambda(0))^{\frac{\alpha}{2(\alpha-1)}}.$$

For such b_λ we get that (6.12) has only the trivial solution $c_1 = 0$. That concludes the proof of Theorem 2.1.

Remark 6.1. *We point put that, if $V''_\lambda(0) \ge c > 0$, (6.12) takes the form*

$$2\pi\delta\Bigg(4b - \frac{2}{(2i)!}\frac{V''_\lambda(0)}{\alpha}|b|^{\frac{2-\alpha}{\alpha}}\Bigg) + h.o.t. = \frac{2}{3}\pi c_1 + o(c_1).$$

Consequently we only obtain a zero $b \geq c > 0$ for the leading term on the left hand side, so such zero does not correspond to a non simple blow-up solution.

6.1. **Proof of Theorems 1.1.** Theorem 2.1 provides a solution to the problem (2.4) of the form

$$w_\lambda = PW_\lambda + \phi_\lambda$$

where $\phi_\lambda = \phi_{\lambda,b_\lambda} \in H_0^1(B_1)$ satisfies (5.5) and $b = b_\lambda$ satisfies (1.4).

Moreover, using (6.6), by Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ we get

$$\begin{aligned} \lambda \| V_\lambda(|y|^{\frac{1}{\alpha}})(e^{w_\lambda} - e^{PW_\lambda})\|_1 &= \lambda \| e^{PW_\lambda}(e^{\phi_\lambda} - 1)\|_1 \\ &\leq \lambda \| e^{PW_\lambda}\|_p \| e^{\phi_\lambda} - 1\|_q \\ &= O(\delta^{1-2\frac{p-1}{p}-\varepsilon}) = o(1), \end{aligned}$$

if p is chosen sufficiently close to 1 and ε sufficiently close to 0. Similarly, by Proposition 4.3,

$$\Big\| \frac{\lambda}{\alpha^2} V_\lambda(|y|^{\frac{1}{\alpha}}) e^{PW_\lambda} - e^{W_\lambda} \Big\|_1 = \|R_\lambda\|_1 = O(\delta^{1-2\frac{p-1}{p}}) = o(1).$$

Therefore

$$\Big\| \frac{\lambda}{\alpha^2} V_\lambda(|y|^{\frac{1}{\alpha}}) e^{w_\lambda} - e^{W_\lambda} \Big\|_1 = o(1).$$

Clearly, by (2.1) and (2.3),

$$u_\lambda(x) = w_\lambda(x^\alpha) - 4\pi(\alpha-1)G(x,0) = w_\lambda(x^\alpha) - 2(\alpha-1)\log\frac{1}{|x|}$$

solves equation (1.1) and

$$\begin{aligned} \|\lambda V_\lambda(|x|)e^{u_\lambda(x)} - \alpha^2|x|^{2(\alpha-1)}e^{W_\lambda(x^\alpha)}\|_1 &= \alpha^2 \Big\| \frac{\lambda}{\alpha^2}|x|^{2(\alpha-1)}V_\lambda(|x|)e^{w_\lambda(x^\alpha)} - |x|^{2(\alpha-1)}e^{W_\lambda(x^\alpha)} \Big\|_1 \\ &= \alpha \Big\| \frac{\lambda}{\alpha^2} V_\lambda(|y|^{\frac{1}{\alpha}}) e^{w_\lambda(x)} - e^{W_\lambda(x)} \Big\|_1 = o(1) \end{aligned}$$

by Lemma A.3. Hence, recalling (2.7) and Lemma A.3,

$$\begin{aligned} \lambda \int_{B_1} V_\lambda(|x|)e^{u_\lambda}dx &= \alpha^2 \int_{\mathbb{R}^2} |x|^{2(\alpha-1)}V_\lambda(|x|)e^{W_\lambda(x^\alpha)}dx + o(1) \\ &= \alpha \int_{\mathbb{R}^2} V_\lambda(|y|^{\frac{1}{\alpha}})e^{W_\lambda(y)}dy + o(1) = 8\pi\alpha + o(1). \end{aligned}$$

Similarly for every neighborhood U of 0

$$\lambda \int_U V_\lambda(|x|)e^{u_\lambda}dx \to 8\pi\alpha.$$

Theorem 1.1 is thus completely proved by setting $\mu^\alpha = \delta$

Appendix A

In this appendix we derive some crucial integral estimates which arise in the asymptotic expansion of the energy of approximate solution PW_λ.

Lemma A.1. *The following holds:*

$$\int_{B_1} e^{W_\lambda}|x-b|dx = O(\delta), \qquad \int_{B_1} e^{W_\lambda}|x-b|^2 dx = 16\pi\delta^2|\log\delta| + O(\delta^2)$$

uniformly for $b > 0$ in a small neighborhood of 0.

Proof. We compute

$$\int_{B_1} e^{W_\lambda}|x-b|dx \le 8\delta\int_{\mathbb{R}^2}\frac{1}{(1+|z-\delta^{-1}b|^2)^2}|z-\delta^{-1}b|dz = 8\delta\int_{\mathbb{R}^2}\frac{|z|}{(1+|z|^2)^2}dz$$

and the first estimate follows. It remains to show the second estimate: to this aim observe that $B(b,1-b)\subset B(0,1)\subset B(b,1+b)$, so we compute

$$\begin{aligned}\int_{B_1} e^{W_\lambda}|x-b|^2dx &= 8\int_{B_1}\frac{\delta^2|x-b|^2}{(\delta^2+|x-b|^2)^2}dx\\
&= 8\int_{B(b,1-b)}\frac{\delta^2|x-b|^2}{(\delta^2+|x-b|^2)^2}dx + O\Big(\int_{B(b,1+b)\setminus B(b,1-b)}\frac{\delta^2|x-b|^2}{(\delta^2+|x-b|^2)^2}dx\Big)\\
&= 8\int_{B(0,1-b)}\frac{\delta^2|x|^2}{(\delta^2+|x|^2)^3}dx + O\Big(\int_{B(0,1+b)\setminus B(0,1-b)}\frac{\delta^2|x|^2}{(\delta^2+|x|^2)^2}dx\Big)\\
&= 8\delta^2\int_{|z|\le\frac{1-b}{\delta}}\frac{|z|^2}{(1+|z|^2)^2}dz + O\Big(\delta^2\int_{\frac{1-b}{\delta}\le|z|\le\frac{1+b}{\delta}}\frac{1}{|z|^2}dz\Big)\\
&= 8\delta^2\int_{|z|\le\frac{1-b}{\delta}}\frac{1}{1+|z|^2}dz + O(\delta^2)\\
&= 16\pi\delta^2|\log\delta| + O(\delta^2).\end{aligned}$$

□

Since the key part in the proof of Theorem 2.1 relies in testing the equation (5.1) with PZ^1_λ in order to catch the leading terms, a crucial step consists in the evaluation of some integral estimates, as provided by the following lemma.

Lemma A.2. *The following holds:*

$$\int_{B_1} e^{W_\lambda}PZ^1_\lambda \mathrm{Re}\,(x-b)dx = 2\pi\delta + O(\delta^2),\qquad \int_{B_1} e^{W_\lambda}|PZ^1_\lambda||x-b|^2dx = O(\delta^2),$$

uniformly for $b>0$ *in a small neighborhood of* 0.

Proof. We compute

$$\begin{aligned}\int_{B_1} e^{W_\lambda}Z^1_\lambda\mathrm{Re}\,(x-b)dx &= 8\delta\int_{\frac{B_1}{\delta}}\frac{1}{(1+|z-\delta^{-1}b|^2)^3}(\mathrm{Re}\,(z-\delta^{-1}b))^2dz\\
&= 8\delta\int_{\mathbb{R}^2}\frac{1}{(1+|z-\delta^{-1}b|^2)^3}(\mathrm{Re}\,(z-\delta^{-1}b))^2dz + O(\delta^2)\\
&= 8\delta\int_{\mathbb{R}^2}\frac{z_1^2}{(1+|z|^2)^3}dz + O(\delta^2)\\
&= 2\pi\delta + O(\delta^2)\end{aligned}$$

where we have used the identity $\int_{\mathbb{R}^2}\frac{(z_1)^2}{(1+|z|^2)^3} = \frac12\int_{\mathbb{R}^2}\frac{|z|^2}{(1+|z|^2)^3} = \frac{\pi}{4}$. Similarly,

$$\int_{B_1} e^{W_\lambda}|Z^1_\lambda||x-b|^2dx \le 8\delta^2\int_{\mathbb{R}^2}\frac{\delta|x-b|^3}{(\delta^2+|x-b|^2)^3}dx = 8\delta^2\int_{\mathbb{R}^2}\frac{|z|^3}{(1+|z|^2)^3}dz \le C\delta^2.$$

Taking into account that $PZ^1_\lambda = Z^1_\lambda + O(\delta)$ by (3.2), using Lemma A.1 the above integral estimates give the thesis. □

Finally we deduce some integral identities associated to the change of variable $x\mapsto x^\alpha$ which appears frequently when dealing with α-symmetric functions.

Lemma A.3. *For any α-symmetric function $f \in L^1(B_1)$ we have that $|x|^{2(\alpha-1)}f(x^\alpha) \in L^1(B_1)$ and*

$$\int_{B_1} |x|^{2(\alpha-1)} f(x^\alpha)dx = \frac{1}{\alpha}\int_{B_1} f(y)dy. \tag{A.1}$$

Proof. It is sufficient to prove the thesis for a smooth function f. Using the polar coordinates (ρ, θ) and then applying the change of variables $(\rho', \theta') = (\rho^\alpha, \alpha\theta)$

$$\begin{aligned}\int_{B_1} |x|^{2(\alpha-1)} f(x^\alpha)dx &= \int d\rho \int_0^{2\pi} \rho^{2\alpha-1} f(\rho^\alpha e^{\mathrm{i}\alpha\theta})d\theta \\ &= \frac{1}{\alpha^2}\int d\rho' \int_0^{2\alpha\pi} \rho' f(\rho' e^{\mathrm{i}\theta'})d\theta' \\ &= \frac{1}{\alpha}\int d\rho' \int_0^{2\pi} \rho' |f(\rho' e^{\mathrm{i}\theta'})|^2 d\theta' \\ &= \frac{1}{\alpha}\int_{B_1} f(y)dy.\end{aligned}$$

□

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