On the Location and Profile of Spike-Layer Solutions to Singularly Perturbed Semilinear Dirichlet Problems

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1. Introduction

In this paper, we shall study the following singularly perturbed elliptic problem

$$(1.1)\qquad \begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega\,, \\ u > 0 & \text{in } \Omega \quad \text{and } u = 0 \quad \text{on } \partial\Omega\,, \end{cases}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, Ω is a bounded smooth domain in $\mathbb{R}^n$, $\varepsilon > 0$ is a constant, and the exponent p satisfies $1 < p < \frac{n+2}{n-2}$ for $n \geqq 3$ and $1 < p < \infty$ for $n = 2$. We are especially interested in the properties of solutions of (1.1) as ε tends to 0. In particular, we shall establish the existence of a "spike-layer" solution, and determine the location of the peak as well as the profile of the spike.

The corresponding Neumann problem

$$(1.2)\qquad \begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega\,, \\ u > 0 & \text{in } \Omega \quad \text{and } \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,, \end{cases}$$

where ν denotes the unit outer normal to $\partial\Omega$, has been studied extensively; see [12], [14], [15], and, [16]. We also refer to [12] and the references therein for some background of the model (1.2). In [14] and [15], Ni and Takagi showed that for every ε sufficiently small, (1.2) has a *least-energy* solution which possesses a single spike-layer with its unique peak locating on the boundary $\partial\Omega$. Moreover, this unique peak must be situated near the "most curved" part of $\partial\Omega$, i.e., where the boundary mean curvature assumes its maximum, if ε is sufficiently small.

The existence of a least-energy solution of (1.1) can be handled in exactly the same way as in [14]. More precisely, we first define the *energy* as follows:

$$(1.3)\qquad J_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2) - \frac{1}{p+1}\int_\Omega u_+^{p+1}$$

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where $u_+ = \max\{u, 0\}$, for $u \in \mathbb{H}_0^1(\Omega)$. The well-known *Mountain-Pass Lemma* implies that

$$(1.4) \qquad c_\varepsilon = \inf_{h\in\Gamma} \max_{0\leqq t\leqq 1} J_\varepsilon(h(t))$$

is a positive critical value of J_ε, where Γ is the set of all continuous paths joining the origin and a fixed nonzero element e in $\mathbb{H}_0^1(\Omega)$ with $e \geqq 0$ and $J_\varepsilon(e) = 0$. Similarly as in [14] we shall see that c_ε is the least positive critical value and any corresponding critical point u_ε (i.e., $J_\varepsilon'(u_\varepsilon) = 0$ and $J_\varepsilon(u_\varepsilon) = c_\varepsilon$) is a solution of (1.1) and is called a *least-energy* solution.

The purpose of this paper is to study the properties of the solution u_ε, especially when ε is small. Among other things we shall prove that *u_ε has only one (local) maximum over $\overline{\Omega}$, and it is achieved at exactly one point $P_\varepsilon \in \Omega$. Furthermore, we shall show that u_ε tends to 0 as $\varepsilon \to 0$ except at its peak P_ε, thereby exhibiting a single spike-layer, and $d(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega)$ as $\varepsilon \to 0$ where d denotes the distance function. Finally, the asymptotic profile (in ε) of u_ε is obtained which gives a detailed description of u_ε for ε sufficiently small.* It seems interesting to note that, in contrast to the Neumann problem, our result here implies that the peak of the spike-layer for the Dirichlet problem (1.1) must be situated near the "most-centered" part of the domain Ω.

Problems (1.1) and (1.2) can of course be viewed as singular perturbation problems. Due to the exponentially small error terms in the expansions of the solution u_ε, however, traditional techniques in singular perturbations do not seem to apply. Our approach is based on an asymptotic formula for the smallest energy c_ε. To obtain such an expansion, we need to combine the methods developed earlier in [14], [15] and [17]. It seems very interesting to note that while in the Neumann problem (1.2) the boundary mean curvature appears in the second term, i.e., the dominating correction term, of the expansion and is of the *algebraic* order ε^{n+1}, the dominating correction term in the expansion for c_ε in the Dirichlet case (1.1) involves the quantity $d(P_\varepsilon, \partial\Omega)$ and is of *transcendental* order $\exp(-1/\varepsilon)$ which makes the Dirichlet case (1.1) even more delicate. We should also remark that the distance function involved here is actually obtained through a limiting process, namely, the so-called *vanishing viscosity method.*

As a historical remark, we note that in treating nonlinear "autonomous" equations (i.e., no space dependence appears in the coefficients of the equations) in singular perturbations, although there has been some work on (1.1), e.g., [4], it seems that there had been little progress in locating the peaks of the spike-layers until the recent papers [12], [14], and [15]. In those papers, the "energy" method was devised to handle singularly perturbed semilinear "autonomous" Neumann problems as described above. Developing the ideas in [12], [14], and [15] further, our present paper seems to be the first one that succeeds in locating the spike-layers for singularly perturbed semilinear "autonomous" Dirichlet problems. It is perhaps expected that somehow the geometry of the domain would play a decisive role in locating the peaks; it seems extremely interesting, however, to see exactly how the geometry determines the locations of the spike-layers.

This paper is organized as follows. We state our main results — Theorems 2.2 and 2.3 — in the next section. The proof of part (i) of Theorem 2.2 and some preliminaries are given in Section 3. In Section 4 we discuss the viscosity limit of projections and then in Section 5 we derive an upper bound for c_ε. Section 6 contains the asymptotic expansion for c_ε and the proofs of our main results. Finally, the proofs of three important but technical lemmas are included in Section 7.

In closing this section, we remark that throughout this entire paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small.

2. Statements of Main Results

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the following semilinear Dirichlet problem

$$\begin{cases} \varepsilon^2\Delta u - u + f(u) = 0 & \text{in } \Omega\,, \\ u > 0 & \text{in } \Omega \quad \text{and } u = 0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

where $\varepsilon > 0$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^n$. The function $f : \mathbb{R} \to \mathbb{R}$ is of class $C^{1+\sigma}(\mathbb{R})$ with $0 < \sigma < 1$ and satisfies the following conditions:

(f1) $f(t) \equiv 0$ for $t \leqq 0$ and $f(t) \to +\infty$ as $t \to \infty$.

(f2) For $t \geqq 0$, f admits the decomposition in $C^{1+\sigma}(\mathbb{R})$

$$f(t) = f_1(t) - f_2(t)$$

where (i) $f_1(t) \geqq 0$ and $f_2(t) \geqq 0$ with $f_1(0) = f_1'(0) = 0$, whence it follows that $f_2(0) = f_2'(0) = 0$ by (f1); and (ii) there is a $q \geqq 1$ such that $f_1(t)/t^q$ is nondecreasing in $t > 0$, whereas $f_2(t)/t^q$ is nonincreasing in $t > 0$, and in case $q = 1$ we require further that the above monotonicity condition for $f_1(t)/t$ is strict.

(f3) $f(t) = O(t^p)$ as $t \to +\infty$ where $1 < p < \frac{n+2}{n-2}$ if $n \geqq 3$ and $1 < p < \infty$ if $n = 2$.

(f4) There exists a constant $\theta \in (0, \frac{1}{2})$ such that $F(t) \leqq \theta t f(t)$ for $t \geqq 0$, in which

$$F(t) = \int_0^t f(s)\,ds\,. \tag{2.2}$$

To state the next (and last) condition, we need some preparations. Consider the problem in the whole space

$$\begin{cases} \Delta w - w + f(w) = 0 & \text{and } w > 0 & \text{in } \mathbb{R}^n\,, \\ w(0) = \max_{z\in\mathbb{R}^n} w(z) & \text{and } w(z) \to 0 & \text{as } |z| \to +\infty\,. \end{cases} \tag{2.3}$$

It is known that (see [9]) any solution to (2.3) needs to be spherically symmetric about the origin and strictly decreasing in $r = |z|$. A solution w to (2.3) is said to be *nondegenerate* if the linearized operator

$$L = \Delta - 1 + f'(w) \tag{2.4}$$

on $L^2(\mathbb{R}^n)$ with domain $W^{2,2}(\mathbb{R}^n)$ has a bounded inverse when it is restricted to the subspace $L_r^2(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) | u(z) = u(|z|)\}$.

Now condition (f5) is stated as follows:

(f5) Problem (2.3) has a unique solution w, and it is nondegenerate. The unique solution in (f5) will be denoted by w in the rest of this paper.

We note that the function

$$f(t) = t^p - at^q \qquad \text{for } t \geqq 0$$

with a constant $a \geqq 0$ satisfies all the assumptions (f1)–(f4) if $1 < q < p < \frac{n+2}{n-2}$. Furthermore, there is a unique solution w to problem (2.3); see [6] and [11]. The nondegeneracy condition (f5) can be derived from the uniqueness argument; see Appendix C in [15].

Associated with (2.1) is the functional $J_\varepsilon : \mathbb{H}_0^1(\Omega) \to \mathbb{R}$ defined by

$$J_\varepsilon(v) = \frac{1}{2}\int_\Omega (\varepsilon^2 |\nabla v|^2 + v^2) - \int_\Omega F(v)\, dx\,. \tag{2.5}$$

We call $J_\varepsilon(v)$ the *energy* of v. Let

$$c_\varepsilon = \inf_{h \in \Gamma} \max_{0 \leqq t \leqq 1} J_\varepsilon(h(t)) \tag{2.6}$$

where

$$\Gamma = \left\{h \in C([0,1]; \mathbb{H}_0^1(\Omega)) | h(0) = 0, h(1) = e\right\}$$

and $e \not\equiv 0$ is a nonnegative function in $\mathbb{H}_0^1(\Omega)$ with $J_\varepsilon(e) = 0$. We also define

$$M[v] := \sup_{t \geqq 0} J_\varepsilon(tv) \qquad \text{for } v \in \mathbb{H}_0^1(\Omega)\,.$$

Then we have

PROPOSITION 2.1. (i) *c_ε is a positive critical value of J_ε.*

(ii) $c_\varepsilon = \inf \left\{M[v] | v \in \mathbb{H}_0^1(\Omega), v \not\equiv 0, \text{and } v \geqq 0 \text{ in } \Omega\right\}$. *In particular, c_ε is independent of the choice of e.*

Proof: (i) is a direct application of the well-known *Mountain-Pass Lemma* due to Ambrosetti and Rabinowitz; see [3]. The proof of (ii) is identical to that of Lemma 3.1 in [14]; see Appendix B in [15].

From the above proposition, we see that c_ε is the *least positive critical value* of J_ε. Therefore, we call a critical point u_ε of J_ε with $J_\varepsilon(u_\varepsilon) = c_\varepsilon$ a *least-energy* *solution* to (2.1) (or *mountain-pass solution*). Note that, by standard elliptic regularity estimates and the Maximum Principle, critical points of J_ε are classical solutions to (2.1); see, e.g., page 9 in [12].

We now come to the shape of u_ε and the location of the peak.

THEOREM 2.2. *Let u_ε be a least-energy solution to* (2.1). *Then, for ε sufficiently small, we have*

(i) *u_ε has at most one local maximum and it is achieved at exactly one point P_ε in Ω. Moreover, $u_\varepsilon(\cdot+P_\varepsilon) \to 0$ in $C^1_{\mathrm{loc}}(\Omega-P_\varepsilon\backslash\{0\})$ where $\Omega-P_\varepsilon = \{x-P_\varepsilon | x \in \Omega\}$.*

(ii) *$d(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P,\partial\Omega)$ as $\varepsilon \to 0$.*

Next, we state a theorem about the asymptotic profile of u_ε. First, we introduce some notation.

Let w be the unique solution of (2.3) and

$$I(w) := \frac{1}{2}\int_{\mathbb{R}^n} (|\nabla w|^2 + w^2) - \int_{\mathbb{R}^n} F(w) \,. \tag{2.7}$$

Let Q be a point such that $d(Q,\partial\Omega) = \max_{P\in\Omega} d(P,\partial\Omega)$ and P_ε be the peak of u_ε defined in Theorem 2.2. We let

$$\Omega_\varepsilon = \{z \in \mathbb{R}^n | Q + \varepsilon z \in \Omega\} \,, \qquad \tilde{\Omega}_\varepsilon = \{y \in \mathbb{R}^n | P_\varepsilon + \varepsilon y \in \Omega\} \,. \tag{2.8}$$

For any smooth bounded domain U in $\mathbb{R}^n$, we define $\mathscr{P}_U w$ to be the unique solution of

$$\begin{cases} \Delta u - u + f(w) = 0 & \text{in } U \,, \\ u = 0 & \text{on } \partial U \,. \end{cases} \tag{2.9}$$

Then by the Maximum Principle, $\mathscr{P}_U w(y) < w(y)$ for all $y \in U$.

Set

$$\begin{array}{rclrcl} \varphi_\varepsilon &=& w - \mathscr{P}_{\Omega_\varepsilon} w \,, & \tilde{\varphi}_\varepsilon &=& w - \mathscr{P}_{\tilde{\Omega}_\varepsilon} w \\ \psi_\varepsilon(x) &=& -\varepsilon \log \varphi_\varepsilon((x-Q)/\varepsilon) \,, & \tilde{\psi}_\varepsilon(x) &=& -\varepsilon \log \tilde{\varphi}_\varepsilon((x-P_\varepsilon)/\varepsilon) \\ V_\varepsilon &=& e^{\beta\psi_\varepsilon(Q)}\varphi_\varepsilon \,, & \tilde{V}_\varepsilon &=& e^{\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}\tilde{\varphi}_\varepsilon \end{array}$$

where $\beta = 1/\varepsilon$. It will be proved in Lemma 4.6 below that for every sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k_l} \to 0$ such that $\tilde{V}_{\varepsilon_{k_l}} \to \tilde{V}_0$ on every compact set of $\mathbb{R}^n$, where $\tilde{V}_0$ is a solution of

$$\begin{cases} \Delta u - u = 0 & \text{in } \mathbb{R}^n \,, \\ u > 0 & \text{in } \mathbb{R}^n \end{cases} \quad \text{and } u(0) = 1 \,. \tag{2.10}$$

We are now ready to state our second main result.

THEOREM 2.3. (i) *For ε sufficiently small, we have*

$$c_\varepsilon = \varepsilon^n \left\{ I(w) + \gamma e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)} + o\left(e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}\right) \right\}$$

where γ is the constant defined in (4.30) *below and $\tilde{\psi}_\varepsilon(P_\varepsilon) \to 2d(Q, \partial\Omega)$ as $\varepsilon \to 0$.*

(ii) *For every sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k_l} \to 0$ such that*

$$u_{\varepsilon_{k_l}}(x) = \mathscr{P}_{\tilde{\Omega}_{\varepsilon_{k_l}}} w(y) + e^{-\beta_{k_l}\tilde{\psi}_{\varepsilon_{k_l}}(P_{\varepsilon_{k_l}})}\phi_{\varepsilon_{k_l}}(y)$$

where $x = P_{\varepsilon_{k_l}} + \varepsilon_{k_l} y$, $\beta_{k_l} = 1/\varepsilon_{k_l}$, $\|e^{-\mu|y|}(\phi_{\varepsilon_{k_l}} - \phi_0)\|_{L^\infty(\tilde{\Omega}_{\varepsilon_{k_l}})} \to 0$, ϕ_0 is a solution of

$$L\phi_0 - f'(w)\tilde{V}_0 = 0 \qquad in\ \mathbb{R}^n \tag{2.11}$$

and $1 - \sigma < \mu < 1$.

3. Proof of Theorem 2.2 (i) and Preliminaries

Let u_ε be a least-energy solution to (2.1), i.e., $J_\varepsilon(u_\varepsilon) = c_\varepsilon$. In this section, we shall prove part (i) of Theorem 2.2 and derive some properties of u_ε which will be used later.

We begin with some lemmas.

LEMMA 3.1. *For ε sufficiently small, $c_\varepsilon \leqq \varepsilon^n(I(w) + o(1))$, where w is the unique solution of* (2.3) *and $I(w)$ is defined by* (2.7).

LEMMA 3.2. *The following statements hold:*

(i) $\sup_{x\in\overline{\Omega}} u_\varepsilon(x) \leqq C$,

(ii) $m_q\varepsilon^n \leqq \int_\Omega u_\varepsilon^q \leqq M_q\varepsilon^n$, *if* $1 \leqq q < \infty$,

where C, m_q, M_q are positive constants and are independent of ε for $\varepsilon < 1$.

LEMMA 3.3. *There is a $\bar{u} > 0$ such that if u_ε attains a local maximum at $x_0 \in \overline{\Omega}$, then $u_\varepsilon(x_0) \geqq \bar{u}$. Moreover, there exist constants η_0, r_0 independent of x_0 and ε such that for $\varepsilon < \varepsilon_0$ and $B_{r_0\varepsilon}(x_0) \subset \Omega$, then*

$$u_\varepsilon(x) \geqq \eta_0 , \qquad for\ x \in B_{\frac{r_0\varepsilon}{2}}(x_0) . \tag{3.1}$$

Proofs of Lemmas 3.1–3.3: By using Proposition 2.1 and a test function $\chi(x)w((x - Q)/\varepsilon)$ where $\chi(x) \in C_0^\infty(\Omega)$ is a positive cut-off function at a neighborhood of Q, we obtain Lemma 3.1. Lemma 3.2 follows from exactly the same arguments used in the proof of Lemma 2.3 and Corollary 2.1 in [12]. Finally, $u_\varepsilon(x_0) \geqq \bar{u}$ is an easy consequence of the equation (2.1), see Appendix B (c) in [15], and (3.1) is a standard interior Harnack inequality; see Theorem 8.20 in [10].

We now prove part (i) of Theorem 2.2. We shall follow the strategy used in the proof of Theorem 2.1 in [14].

Let P_ε be a point at which u_ε achieves a local maximum. Then $u_\varepsilon(P_\varepsilon) \geqq \bar{u}$.

Step 1. We first prove that

$$\rho_\varepsilon := d(P_\varepsilon, \partial\Omega)/\varepsilon \to +\infty \qquad \text{as } \varepsilon \to 0 . \tag{3.2}$$

Suppose on the contrary that there exists a sequence of $\varepsilon_k \to 0$ and positive constant C^* such that

$$d(P_\varepsilon, \partial\Omega) \leqq C^* \varepsilon , \qquad \text{for } \varepsilon = \varepsilon_k . \tag{3.3}$$

By passing to a subsequence, we may assume that $P_{\varepsilon_k} \to P_0 \in \partial\Omega$. Following the idea in [14], we introduce a diffeomorphism which straightens the boundary portion near the point $P_0 \in \partial\Omega$. Through translation and rotation of the coordinate system, we may assume that P_0 is the origin and the inner normal to $\partial\Omega$ at P_0 is pointing in the direction of the positive x_n-axis. Then there exists a smooth function $\omega_{P_0}(x'), x' = (x_1, \ldots, x_{n-1})$ defined for $|x'|$ sufficiently small such that (i) $\omega_{P_0}(0) = 0$ and $\nabla\omega_{P_0}(0) = 0$; and (ii) $\partial\Omega \cap \mathcal{N} = \{(x', x_n) | x_n = \omega_{P_0}(x')\}$ and $\Omega \cap \mathcal{N} = \{(x', x_n) | x_n > \omega_{P_0}(x')\}$, where $\mathcal{N}$ is a neighborhood of P_0. For $y \in \mathbb{R}^n$ near 0, we define a mapping $x = \mathcal{G}_{P_0}(y) = (\mathcal{G}_{P_0,1}(y), \ldots, \mathcal{G}_{P_0,n}(y))$ by

$$\mathcal{G}_{P_0,j}(y) = \begin{cases} y_j - y_n \frac{\partial\omega_{P_0}}{\partial x_j}(y') , & \text{for } j = 1, \ldots, n-1 , \\ y_n + \omega_{P_0}(y') , & \text{for } j = n . \end{cases} \tag{3.4}$$

Since $\nabla\omega_{P_0}(0) = 0$, the differential map $D\mathcal{G}_{P_0}$ of $\mathcal{G}_{P_0}$ satisfies $D\mathcal{G}_{P_0}(0) = I$, the identity map. Thus $\mathcal{G}_{P_0}$ has the inverse mapping $y = \mathcal{G}_{P_0}^{-1}(x)$ for $|x| < \delta'$. We write $\mathcal{G}_{P_0}^{-1}(x)$ as $\mathcal{F}_{P_0}(x) = (\mathcal{F}_{P_0,1}(x), \ldots, \mathcal{F}_{P_0,n}(x))$. We assume now that $\mathcal{G}_{P_0}$ is defined in an open set including the closed ball $\overline{B}_{3\kappa}$, where $\kappa > 0$ is a small constant. For simplicity, we shall suppress the indices P_0 and k in the rest of the proof.

Set $Q_\varepsilon = \mathcal{F}(P_\varepsilon) \in B_\kappa^+ := \{x \in B_\kappa | x_n > 0\}$ and $v_\varepsilon(y) := u_\varepsilon(\mathcal{G}(y))$ for $y \in \overline{B}_{2\kappa}^+$. If we write $Q_\varepsilon = (q'_\varepsilon, \alpha_\varepsilon\varepsilon)$ with $q'_\varepsilon \in \mathbb{R}^{n-1}$ and $\alpha_\varepsilon > 0$, then (3.3) implies that α_ε is bounded. We now define $w_\varepsilon(z) := v_\varepsilon(Q_\varepsilon + \varepsilon z)$ for $z = (z_1, \ldots, z_n) \in \overline{B}_{\kappa/\varepsilon} \cap \{z_n \geqq -\alpha_\varepsilon\}$ and $w_\varepsilon(z) := 0$ for $z \in \overline{B}_{\kappa/\varepsilon} \cap \{z_n < -\alpha_\varepsilon\}$.

It is easily seen that

$$w_\varepsilon \in C^2 \left(\overline{B}_{\kappa/\varepsilon} \cap \{z_n \geqq -\alpha_\varepsilon\}\right) .$$

Moreover, w_ε satisfies

$$\sum_{i,j=1}^n a_{ij}^\varepsilon \frac{\partial^2 w}{\partial z_i \partial z_j} + \varepsilon \sum_{j=1}^n b_j^\varepsilon \frac{\partial w_\varepsilon}{\partial z_j} - w_\varepsilon + f(w_\varepsilon) = 0 \tag{3.5}$$

in $\overline{B_{\kappa/\varepsilon}} \cap \{z_n > -\alpha_\varepsilon\}$, where

$$\begin{aligned} a_{ij}^\varepsilon(z) &= \textstyle\sum_{l=1}^n \frac{\partial\mathcal{F}_i}{\partial x_l}(\mathcal{G}(Q_\varepsilon + \varepsilon z)) \frac{\partial\mathcal{F}_j}{\partial x_l}(\mathcal{G}(Q_\varepsilon + \varepsilon z)) , \qquad & 1 \leqq i , \ j \leqq n , \\ b_j(z) &:= (\Delta\mathcal{F}_j)(\mathcal{G}(Q_\varepsilon + \varepsilon z)) , & j = 1, \ldots, n . \end{aligned}$$

By Lemma 3.2 (ii), we see that

$$\int_{B_{\kappa/\varepsilon}} w_\varepsilon^s(z)dz \leqq C_s$$

with C_s independent of ε.

Therefore, we may argue as Step 1 of Theorem 2.1 in [14] to obtain a convergent subsequence, which we denote again by w_ε, such that

$$w_\varepsilon \to w_0 \qquad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^n_{\alpha,+})$$

where $\alpha_\varepsilon \to \alpha \geqq 0$, $\mathbb{R}^n_{\alpha,+} = \{y | y_n > -\alpha\}$ and $w_0 \in C^2_{\mathrm{loc}}(\mathbb{R}^n_{\alpha,+}) \cap W^{2,s}(\mathbb{R}^n_{\alpha,+})$.

The limit w_0 satisfies

$$\sum_{i,j=1}^n a_{ij}(0)\frac{\partial^2 w_0}{\partial z_i \partial z_j} - w_0 + f(w_0) = 0 \tag{3.6}$$

in $\mathbb{R}^n_{\alpha,+}$ since $Q_\varepsilon \to 0$ as $k \to \infty$. In view of $D\mathscr{F}(0) = [D\mathscr{G}(0)]^{-1} = I$, one obtains that w_0 satisfies

$$\begin{cases} \Delta u - u + f(u) = 0 & \text{in } \mathbb{R}^n_{\alpha,+} , \\ u > 0 & \text{in } \mathbb{R}^n_{\alpha,+} \quad \text{and } u = 0 \quad \text{on} \quad \partial\mathbb{R}^n_{\alpha,+} . \end{cases} \tag{3.7}$$

Note that $w_0 \in W^{2,s}(\mathbb{R}^n_{\alpha,+})$; see Step 1 in the proof of Theorem 2.1 in [14] for details. Then, by Theorem 1.1 in [7], we conclude that $w_0 \equiv 0$. But $w_\varepsilon(0) = u_\varepsilon(P_\varepsilon) \geqq \bar{u}$ and $w_\varepsilon(0) \to w_0(0) = 0$, a contradiction.

Step 2. We now prove that u_ε has at most one local maximum point. By Step 1, we can use Harnack inequality near P_ε. By making some minor modifications, we see that our assertion follows from the arguments in Step 3 of the proof of Theorem 2.1 in [14].

Step 3. Finally the fact that $u_\varepsilon(\cdot + P_\varepsilon) \to 0$ in $C^1_{\mathrm{loc}}(\Omega - P_\varepsilon\backslash\{0\})$ follows from Proposition 3.4 (ii) below and standard elliptic regularity estimates.

Next, we begin to study the profile of u_ε.

PROPOSITION 3.4. *Let $\tilde{v}_\varepsilon(y) = u_\varepsilon(P_\varepsilon + \varepsilon y)$, then the following statements hold.*

(i) *For any $\eta > 0$, there exist positive constants ε_0 and k_0, such that, for all $0 < \varepsilon < \varepsilon_0$, we have $B_{2k_0\varepsilon}(P_\varepsilon) \subset \Omega$ and $\|\tilde{v}_\varepsilon - w\|_{C^2(B_{k_0}(0))} < \eta$, where w is the unique solution of* (2.3).

(ii) *For any $0 < \delta < 1$ there is a constant C such that*

$$\tilde{v}_\varepsilon(y) \leqq Ce^{-(1-\delta)|y|} \qquad \text{for } y \in \tilde{\Omega}_\varepsilon . \tag{3.8}$$

(iii) $\|\tilde{v}_\varepsilon - w\|_{L^s(\tilde{\Omega}_\varepsilon)} \to 0$ *for all* $1 \leqq s \leqq \infty$ *as* $\varepsilon \to 0$.

Proof: Repeating the arguments in Step 1 of the proof of Theorem 2.1 in [14], we see that $\tilde{v}_\varepsilon \to w$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $\varepsilon \to 0$, where w is the unique solution of (2.3). This proves (i).

We now prove (ii). By a result in [9],

$$w(r) \leqq C_0 e^{-r}\,, \qquad \text{for } r \geqq 0\,. \tag{3.9}$$

For any $\eta > 0$, set

$$R := \log(C_0/\eta) \tag{3.10}$$

so that $\eta = C_0\exp(-R)$. Then by part (i) there is a $\varepsilon_0 > 0$ such that

$$\|\tilde{v}_\varepsilon - w\|_{C^2(\bar{B}_{2R}(0))} \leqq \eta \tag{3.11}$$

if $0 < \varepsilon < \varepsilon_0$. Thus

$$\tilde{v}_\varepsilon(y) \leqq w(y) + \eta \leqq C_0 e^{-R} + \eta = 2\eta \tag{3.12}$$

for $|y| = R$. Now, setting $\Omega_\varepsilon^{(i)} = B_{R\varepsilon}(P_\varepsilon)$, $\Omega_\varepsilon^{(o)} = \Omega\backslash\Omega_\varepsilon^{(i)}$, $\tilde{\Omega}_\varepsilon^{(i)} = B_R(0)$ and $\tilde{\Omega}_\varepsilon^{(o)} = \tilde{\Omega}_\varepsilon\backslash B_R(0)$, we have $u_\varepsilon(x) \leqq 2\eta$ for $x \in \partial\Omega_\varepsilon^{(i)}$ for $\varepsilon \leqq \varepsilon_0$. By Theorem 2.2 (i), the set $\{x \in \Omega | u_\varepsilon(x) > 2\eta\}$ has only one connected component. Consequently

$$u_\varepsilon(x) \leqq 2\eta \qquad \text{in } \Omega_\varepsilon^{(o)}\,. \tag{3.13}$$

Now we choose η such that $1 - \frac{f(\lambda)}{\lambda} > 1 - \delta$ for $\lambda < 2\eta$. Then $\tilde{v}_\varepsilon$ satisfies

$$\begin{cases} \Delta\tilde{v}_\varepsilon - \left(1 - \frac{f(\tilde{v}_\varepsilon)}{\tilde{v}_\varepsilon}\right)\tilde{v}_\varepsilon = 0 & \text{in } \tilde{\Omega}_\varepsilon^{(o)}\,, \\ \tilde{v}_\varepsilon|_{\partial B_R(0)} \leqq 2\eta\,, & \\ \tilde{v}_\varepsilon = 0 & \text{on } \partial\tilde{\Omega}_\varepsilon\,. \end{cases} \tag{3.14}$$

Observe that $1 - \frac{f(\tilde{v}_\varepsilon)}{\tilde{v}_\varepsilon} > 1 - \delta$ in $\tilde{\Omega}_\varepsilon^{(o)}$.

Let $G(y,z)$ be the Green's function for $-\Delta + 1$ on $\mathbb{R}^n$, i.e.,

$$G(y,z) = C_n|y - z|^{-(n-2)/2}K_{(n-2)/2}(|y - z|) \tag{3.15}$$

where C_n is a positive constant depending only on n and $K_m(z)$ is the modified Bessel function of order m; see, e.g., Appendix C in [9]. Let $G_0(|y|) = G(y,0)$ and $\bar{v}(y) = \frac{2\eta G_0(\sqrt{1-\delta}|y|)}{G_0(\sqrt{1-\delta}R)}$. Then $\bar{v}(y)$ satisfies

$$\begin{cases} \Delta\bar{v} - (1 - \delta)\bar{v} = 0\,, & \\ \bar{v} = 2\eta & \text{on } \partial B_R(0)\,. \end{cases} \tag{3.16}$$

By the Maximum Principle on $\tilde{\Omega}_\varepsilon^{(o)}$, we have

$$\tilde{v}_\varepsilon(y) \leqq \bar{v}(y) \qquad \text{on } \tilde{\Omega}_\varepsilon^{(o)}\,.$$

But on $\tilde{\Omega}_\varepsilon^{(i)}$, $\tilde{v}_\varepsilon(y) \leqq C$. Hence

$$\tilde{v}_\varepsilon(y) \leqq Ce^{-(1-\delta)|y|}, \qquad \text{for all } \; y \in \tilde{\Omega}_\varepsilon .$$

Finally we prove (iii). For any $\eta > 0$, by the exponential decay of $\tilde{v}_\varepsilon$ and w, there is an R large such that $\|\tilde{v}_\varepsilon - w\|_{L^s(\tilde{\Omega}_\varepsilon \setminus B_R(0))} \leqq \eta/2$. On the other hand, since $\tilde{v}_\varepsilon \to w$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $\varepsilon \to 0$, there is $\varepsilon_R > 0$ such that for all $\varepsilon < \varepsilon_R$, $\|\tilde{v}_\varepsilon - w\|_{L^s(\tilde{\Omega}_\varepsilon \cap B_R(0))} \leqq \eta/2$. Therefore $\|\tilde{v}_\varepsilon - w\|_{L^s(\tilde{\Omega}_\varepsilon)} \leqq \eta$.

4. Viscosity Limit of Projections

In this section, we shall use the so-called *vanishing viscosity method* to derive some properties of $\mathcal{P}_{\Omega_\varepsilon} w$ and $\mathcal{P}_{\tilde{\Omega}_\varepsilon} w$ defined in Section 2. We first study $\mathcal{P}_{\Omega_\varepsilon} w$. Recall that $\varphi_\varepsilon = w - \mathcal{P}_{\Omega_\varepsilon} w$, $\psi_\varepsilon(x) = -\varepsilon \log \varphi_\varepsilon((x-Q)/\varepsilon), V_\varepsilon = e^{\beta\psi_\varepsilon(Q)}\varphi_\varepsilon$. Thus ψ_ε satisfies

$$\begin{cases} \varepsilon\Delta\psi_\varepsilon - |\nabla\psi_\varepsilon|^2 + 1 = 0 & \text{in } \Omega , \\ \psi_\varepsilon(x) = -\varepsilon \log w((x-Q)/\varepsilon) & \text{on } \partial\Omega , \end{cases} \tag{4.1}$$

and V_ε satisfies

$$\begin{cases} \Delta V_\varepsilon - V_\varepsilon = 0 & \text{in } \Omega_\varepsilon , \\ V_\varepsilon(0) = 1 . \end{cases} \tag{4.2}$$

We need the following results about w from Theorem 2 in [9].

LEMMA 4.1. *The following results hold:*

$$\lim_{|y| \to \infty} |y|^{\frac{n-1}{2}} e^{|y|} w(|y|) = \lambda_0 > 0 , \tag{4.3}$$

$$\lim_{|y| \to \infty} \frac{w'(|y|)}{w(|y|)} = -1 . \tag{4.4}$$

It is immediately seen that on $\partial\Omega$,

$$\psi_\varepsilon(x) = |x-Q| + \frac{n-1}{2}\varepsilon \log \frac{|x-Q|}{\varepsilon} - \varepsilon \log(\lambda_0 + o(1)) . \tag{4.5}$$

In order to study the properties of ψ_ε, we first investigate a closely related problem which is slightly simpler.

LEMMA 4.2. *For ε sufficiently small, there is a unique solution ψ^ε of equation*

$$\begin{cases} \varepsilon\Delta\psi - |\nabla\psi|^2 + 1 = 0 & \textit{in } \Omega , \\ \psi(x) = |x-Q| & \textit{on } \partial\Omega . \end{cases} \tag{4.6}$$

Moreover, there exist two positive constants C_1 and C_2, such that

$$\|\psi^\varepsilon\|_{L^\infty(\Omega)} \leqq C_1 , \qquad \|\nabla\psi^\varepsilon\|_{L^\infty(\Omega)} \leqq C_2 . \tag{4.7}$$

Proof: First, we observe that 0 is a subsolution of (4.6) in Ω. On the other hand, it is easy to see from (4.5) that for ε sufficiently small ψ_ε, the solution of (4.1), is a supersolution to (4.6) and $\psi_\varepsilon > 0$ in Ω by the Maximum Principle. From Theorem 1 in [2], we conclude that there is a solution ψ^ε to (4.6) such that $0 < \psi^\varepsilon < \psi_\varepsilon$. To obtain an upper bound of ψ^ε, we first choose a vector X_0 such that $|X_0| > 1$ and a number b large such that $g(x) = \langle x, X_0\rangle + b > |x - Q|$ on $\partial\Omega$. Then by computation,

$$\begin{cases} \varepsilon\Delta g - |\nabla g|^2 + 1 = 1 - |X_0|^2 < 0 & \text{in } \Omega , \\ g(x) > |x - Q| & \text{on } \partial\Omega . \end{cases} \tag{4.8}$$

By comparison, we have

$$g(x) > \psi^\varepsilon(x) \qquad \text{on } \partial\Omega ,$$

which proves that $\|\psi^\varepsilon\|_{L^\infty(\Omega)} \leqq C_1$.

The uniqueness of ψ^ε follows from the usual Maximum Principle.

We next prove that $\|\nabla\psi^\varepsilon\|_{L^\infty(\Omega)} \leqq C_2$. We first show that $\|\nabla\psi^\varepsilon\|_{L^\infty(\partial\Omega)} \leqq C_2$. We prove this by a barrier method.

In fact, we choose δ small and ρ large such that the distance function $d(x) := d(x, \partial\Omega)$ is C^2 in $\Omega_\delta := \{x \in \Omega | d(x) < \delta\}$ and $\rho\delta > C_1$. Then, considering the functions

$$\psi^\varepsilon_- = |x - Q| , \qquad \psi^\varepsilon_+ = |x - Q| + \rho d(x) , \tag{4.9}$$

we observe that

$$\varepsilon\Delta\psi^\varepsilon_- - |\nabla\psi^\varepsilon_-|^2 + 1 = (\varepsilon(n-1))/|x - Q| \tag{4.10}$$

for $x \neq Q$ and that $\psi^\varepsilon \geqq C(\varepsilon) > 0$. Hence if we take $\varepsilon \leqq \varepsilon_0$ and $\delta(\varepsilon)$ small it is easy to see that ψ^ε_- is a subsolution on $\Omega\backslash B_{\delta(\varepsilon)}(Q)$. Therefore it is a subsolution on Ω_δ and $\psi^\varepsilon_- \leqq \psi^\varepsilon$ on $\overline{\Omega_\delta}$. In fact, $C(\varepsilon)$ here can be chosen independent of $\varepsilon > 0$, since Q is fixed. The above argument, however, works even when Q depends on ε which will be the case in the proof of part (iii) of Lemma 4.6.

On the other hand, we have

$$\begin{aligned} &\varepsilon\Delta\psi^\varepsilon_+ - |\nabla\psi^\varepsilon_+|^2 + 1 \\ &= \varepsilon(\Delta\psi^\varepsilon_- + \rho\Delta d) - |\nabla\psi^\varepsilon_- + \rho\nabla d|^2 + 1 \\ &= -\rho^2|\nabla d|^2 - 2\rho\nabla\psi^\varepsilon_- \cdot \nabla d + (\varepsilon(n-1))/|x - Q| + \varepsilon\rho\Delta d . \end{aligned} \tag{4.11}$$

Note that

$$|\nabla d| = 1 \quad \text{on } \partial\Omega , \qquad |\Delta d| \leqq C \quad \text{in } \Omega_\delta .$$

Hence, if we choose ρ large, we have,

$$\begin{cases} \varepsilon\Delta\psi_+^\varepsilon - |\nabla\psi_+^\varepsilon|^2 + 1 < 0 & \text{in } \Omega_\delta\,, \\ \psi_+^\varepsilon > \psi^\varepsilon & \text{on } \partial\Omega_\delta\,. \end{cases} \tag{4.12}$$

By comparison, we conclude

$$\psi_+^\varepsilon > \psi^\varepsilon \qquad \text{on } \Omega_\delta\,.$$

Therefore, $\psi_-^\varepsilon < \psi^\varepsilon < \psi_+^\varepsilon$ on Ω_δ. Thus, $|\psi^\varepsilon - \psi_-^\varepsilon| < \rho d(x)$ on Ω_δ. Since $\psi^\varepsilon = \psi_-^\varepsilon$ on $\partial\Omega$, it follows that $\|\nabla\psi^\varepsilon\|_{L^\infty(\partial\Omega)} \leqq C_2$.

Finally, simple computation shows

$$\Delta(|\nabla\psi^\varepsilon|^2) - \tfrac{2}{\varepsilon}\nabla\psi^\varepsilon\cdot\nabla(|\nabla\psi^\varepsilon|^2) \geqq 0 \qquad \text{in } \Omega\,.$$

Hence by the Maximum Principle, $|\nabla\psi^\varepsilon|^2 \leqq C$ in Ω.

Next, we need to analyze the limit of ψ^ε as $\varepsilon \to 0$. It turns out that the limit is a viscosity solution.

LEMMA 4.3. *Let ψ^ε be the solution of* (4.6), *then ψ^ε converges, as $\varepsilon \to 0$, uniformly to a function $\psi_0 \in W^{1,\infty}(\Omega)$ which can be explicitly written as*

$$\psi_0(x) = \inf_{P\in\partial\Omega}(|P-Q| + L(P,x))\,, \tag{4.13}$$

where $L(x,y)$ denotes the infimum of T such that there exists $\xi(s)\in C^{0,1}([0,T],\overline{\Omega})$ with $\xi(0)=x, \xi(T)=y$, and $|\frac{d\xi}{ds}| \leqq 1$ almost everywhere in $[0,T]$.

Note that ψ_0 is a viscosity solution of the Hamilton-Jacobi equation: $|\nabla u| = 1$ in Ω; see, e.g., [13]. In order to continue our presentation of the main ideas in this section, we postpone the proof of Lemma 4.3 to Appendix A in Section 7.

LEMMA 4.4. (i) *$\psi_\varepsilon \to \psi_0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$ where ψ_0 is given by* (4.13) *above. In particular, $\psi_0(Q) = 2d(Q,\partial\Omega)$.*

(ii) *For every sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k_l} \to 0$ such that $V_{\varepsilon_{k_l}} \to V_0$ uniformly on every compact set of $\mathbb{R}^n$, where V_0 is a positive solution of*

$$\begin{cases} \Delta u - u = 0 & \text{in } \mathbb{R}^n\,, \\ u > 0 & \text{in } \mathbb{R}^n \text{ and } u(0) = 1\,. \end{cases} \tag{4.14}$$

Moreover, for any $\sigma_1 > 0$, $\sup_{z\in\overline{\Omega}_{\varepsilon_{k_l}}} e^{-(1+\sigma_1)|z|}|V_{\varepsilon_{k_l}}(z) - V_0(z)| \to 0$ as $\varepsilon_{k_l} \to 0$.

Remark. We have partial results on the uniqueness of V_0 which we will report elsewhere.

Proof: (i) Let ψ^ε be the unique solution of (4.6). It follows from (4.5) and the Maximum Principle that

$$\|\psi_\varepsilon - \psi^\varepsilon\|_{L^\infty(\overline{\Omega})} \leqq \|\psi_\varepsilon - \psi^\varepsilon\|_{L^\infty(\partial\Omega)} \leqq C|\varepsilon \log \varepsilon| \ . \tag{4.15}$$

Therefore by Lemma 4.3, we see that $\psi_\varepsilon \to \psi_0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$. Note that Q is a point in Ω such that $d(Q, \partial\Omega) = \max_{P\in\overline{\Omega}} d(P, \partial\Omega)$, hence, $L(Q, P) \geqq d(Q, \partial\Omega)$ and $|P - Q| \geqq d(Q, \partial\Omega)$ for all $P \in \partial\Omega$. So, $\psi_0(Q) = 2d(Q, \partial\Omega)$.

(ii) Note that for $\varepsilon = \varepsilon_k$, V_ε satisfies

$$\begin{cases} \Delta V_\varepsilon - V_\varepsilon = 0 & \text{in } \Omega_\varepsilon \ , \\ V_\varepsilon(0) = 1 \ . \end{cases} \tag{4.16}$$

Now, for any given $\sigma_1 > 0$, we let $\delta_1 = \frac{\sigma_1}{10}$ and χ_1 be a cut-off function such that $\chi_1(r) \equiv 1$ for $r \leqq 1 - \delta_1$ and $\chi_1(r) \equiv 0$ for $r > 1 - \delta_1/2$. Setting $\tau_\varepsilon^1(x) = \Delta\psi_\varepsilon(x)$ and $\tau_\varepsilon(x) = \Delta\psi_\varepsilon(x)\chi_1(|x - Q|/d(Q, \partial\Omega))$, we claim that

$$\max_{x\in\Omega} \tau_\varepsilon(x) \leqq \frac{C}{d(Q, \partial\Omega)} \tag{4.17}$$

where C may depend on δ_1 but is independent of ε.

Once (4.17) is proved, it implies that

$$\Delta\psi_\varepsilon(x) \leqq \frac{C}{d(Q, \partial\Omega)} \qquad \text{for } x \in B_{(1-\delta_1)d(Q,\partial\Omega)}(Q) \ .$$

Hence

$$|\nabla\psi_\varepsilon(x)|^2 = 1 + \varepsilon\Delta\psi_\varepsilon(x) \leqq 1 + C\frac{\varepsilon}{d(Q, \partial\Omega)} \leqq 1 + \frac{\sigma_1}{2}$$

for $x \in B_{(1-\delta_1)d(Q,\partial\Omega)}(Q)$ and $\varepsilon < \varepsilon_0$. It then follows that

$$\begin{aligned} V_\varepsilon(z) &= e^{\beta(\psi_\varepsilon(Q)-\psi_\varepsilon(x))} = e^{-\beta\nabla\psi_\varepsilon(\bar{x})(x-Q)} \\ &= e^{-\nabla\psi_\varepsilon(\bar{x})z} \leqq e^{(1+\frac{\sigma_1}{2})|z|} \end{aligned}$$

for $z \in B_{(1-\delta_1)d(Q,\partial\Omega)/\varepsilon}(0)$. On the other hand, if $(1 - \delta_1)d(Q, \partial\Omega)/\varepsilon \leqq |z|$, i.e., $|x - Q| \geqq (1 - \delta_1)d(Q, \partial\Omega)$, we observe that by (4.5) $\psi_\varepsilon(x) \geqq |x - Q|$ on $\partial\Omega$ for ε sufficiently small and thus the function $|x - Q|$ is a subsolution of (4.1) on $\Omega\backslash B_{\delta_1(\varepsilon)d(Q,\partial\Omega)}(Q)$ for some $\delta_1(\varepsilon) > 0$ sufficiently small, by (4.10) and the arguments immediately following it in the proof of Lemma 4.2. Hence, we have $\psi_\varepsilon(x) \geqq |x - Q|$ for $x \in \overline{\Omega}\backslash B_{(1-\delta_1)d(Q,\partial\Omega)}(Q)$. Therefore, $\varphi_\varepsilon(z) \leqq e^{-|z|}$ for $z \in \overline{\Omega}_\varepsilon\backslash B_{(1-\delta_1)d(Q,\partial\Omega)/\varepsilon}(0)$. By part (i) above, it follows that

$$\begin{aligned} V_\varepsilon(z) &= e^{\beta\psi_\varepsilon(Q)}\varphi_\varepsilon(z) \leqq e^{\beta\psi_\varepsilon(Q)-|z|} \\ &\leqq e^{(2+\delta_1)d(Q,\partial\Omega)/\varepsilon-|z|} \leqq e^{(1+\delta_1+\delta_1)\beta d(Q,\partial\Omega)} \\ &\leqq e^{[(1+2\delta_1)/(1-\delta_1)]|z|} \leqq Ce^{(1+\frac{\sigma_1}{2})|z|} \ . \end{aligned}$$

In summary, we have, for all $z \in \Omega_\varepsilon$,

$$V_\varepsilon(z) \leqq Ce^{(1+\frac{\sigma_1}{2})|z|} . \tag{4.18}$$

Hence taking a diagonal process and passing to a subsequence $\varepsilon_{k_l} \to 0$, we have that, for $\varepsilon = \varepsilon_{k_l}$, $V_\varepsilon(z) \to V_0(z)$ uniformly on any compact set of $\mathbb{R}^n$ and $V_0(z)$ is a solution of (4.14). Moreover, $\sup_{z\in\bar{\Omega}_{\varepsilon_{k_l}}} e^{-(1+\sigma_1)|z|}|V_{\varepsilon_{k_l}}(z) - V_0(z)| \to 0$ as $\varepsilon_{k_l} \to 0$.

So it remains to prove (4.17). In fact, τ_ε^1 satisfies

$$-\varepsilon\Delta\tau_\varepsilon^1 + 2\nabla\psi_\varepsilon \cdot \nabla\tau_\varepsilon^1 + 2|\nabla^2\psi_\varepsilon|^2 = 0 \qquad \text{in } \Omega . \tag{4.19}$$

Since $|\nabla^2\psi_\varepsilon|^2 \equiv \sum_{i,j=1}^n \left(\frac{\partial^2\psi_\varepsilon}{\partial x_i \partial x_j}\right)^2 \geqq C_1|\tau_\varepsilon^1|^2$ for some constant C_1, we see that

$$C_1|\tau_\varepsilon^1|^2 - \varepsilon\Delta\tau_\varepsilon^1 + 2\nabla\psi_\varepsilon \cdot \nabla\tau_\varepsilon^1 \leqq 0 . \tag{4.20}$$

Multiplying (4.20) by $\chi_1^2(|x - Q|/d(Q, \partial\Omega))$, we compute

$$\begin{aligned} C_1|\tau_\varepsilon|^2 - \varepsilon\chi_1\Delta\tau_\varepsilon &+ 2\chi_1\nabla\psi_\varepsilon \cdot \nabla\tau_\varepsilon + 2\varepsilon\nabla\tau_\varepsilon \cdot \nabla\chi_1 \\ &- 2(\nabla\psi_\varepsilon \cdot \nabla\chi_1)\tau_\varepsilon + \varepsilon(\Delta\chi_1 - 2\frac{|\nabla\chi_1|^2}{\chi_1})\tau_\varepsilon \leqq 0 . \end{aligned} \tag{4.21}$$

Note that in (4.21) and the rest of the proof of (4.17), for simplicity, we always write χ_1 for $\chi_1(|x - Q|/d(Q, \partial\Omega))$ while the argument in all other functions is x and the differentiations are taken with respect to x.

Now, let $\tau_\varepsilon(x_0) = \max_{x\in\Omega}\tau_\varepsilon(x)$. If $\tau_\varepsilon(x_0) \leqq 0$, (4.17) certainly holds. If $\tau_\varepsilon(x_0) > 0$, then we have

$$C_1|\tau_\varepsilon|^2 \leqq 2(\nabla\psi_\varepsilon \cdot \nabla\chi_1)\tau_\varepsilon - \varepsilon\left(\Delta\chi_1 - 2\frac{|\nabla\chi_1|^2}{\chi_1}\right)\tau_\varepsilon \qquad \text{at } x_0 .$$

Observe that

$$\begin{aligned} \nabla\psi_\varepsilon \cdot \nabla\chi_1 &\leqq |\nabla\psi_\varepsilon|\,|\nabla\chi_1| = \sqrt{|\nabla\psi_\varepsilon|^2|\nabla\chi_1|^2} \\ &= \sqrt{(1 + \varepsilon\Delta\psi_\varepsilon)|\nabla\chi_1|^2} = \sqrt{|\nabla\chi_1|^2 + \varepsilon\frac{|\nabla\chi_1|^2}{\chi_1}\tau_\varepsilon} \\ &\leqq C\sqrt{1 + \varepsilon\tau_\varepsilon}/d(Q, \partial\Omega) \leqq C(1 + \varepsilon\tau_\varepsilon)/d(Q, \partial\Omega) \\ &\leqq C/d(Q, \partial\Omega) + C\varepsilon\tau_\varepsilon/d(Q, \partial\Omega) . \end{aligned}$$

Note that $|\Delta\chi_1 - 2\frac{|\nabla\chi_1|^2}{\chi_1}| \leqq C/d^2(Q, \partial\Omega)$. Therefore, by (4.21), we have at x_0,

$$\begin{aligned} C_1\tau_\varepsilon^2 &\leqq (C\varepsilon/d^2(Q, \partial\Omega))\tau_\varepsilon + (C/d(Q, \partial\Omega))\tau_\varepsilon + (C\varepsilon/d(Q, \partial\Omega))\tau_\varepsilon^2 \\ &\leqq (C/d(Q, \partial\Omega))\tau_\varepsilon + (C\varepsilon/d(Q, \partial\Omega))\tau_\varepsilon^2 . \end{aligned} \tag{4.22}$$

Hence if we choose $\varepsilon < \varepsilon_0$ such that $C\varepsilon/d(Q, \partial\Omega) \leqq \frac{1}{2}C_1$, then we have

$$\tau_\varepsilon(x_0) \leqq C/d(Q, \partial\Omega) ,$$

and (4.17) is established.

We now study $\mathscr{P}_{\tilde{\Omega}_\varepsilon} w$. Recall that

$$\tilde{\varphi}_\varepsilon = w - \mathscr{P}_{\tilde{\Omega}_\varepsilon} w, \tilde{\psi}_\varepsilon(x) = -\varepsilon \log \tilde{\varphi}_\varepsilon((x - P_\varepsilon)/\varepsilon), \tilde{V}_\varepsilon = e^{\beta\psi_\varepsilon(P_\varepsilon)}\tilde{\varphi}_\varepsilon .$$

Similarly as before, $\tilde{\psi}_\varepsilon$ satisfies

$$\begin{cases} \varepsilon\Delta\tilde{\psi}_\varepsilon - |\nabla\tilde{\psi}_\varepsilon|^2 + 1 = 0 & \text{in } \Omega , \\ \tilde{\psi}_\varepsilon(x) = -\varepsilon \log w((x - P_\varepsilon)/\varepsilon) & \text{on } \partial\Omega , \end{cases} \tag{4.23}$$

and $\tilde{V}_\varepsilon$ satisfies

$$\begin{cases} \Delta\tilde{V}_\varepsilon - \tilde{V}_\varepsilon = 0 & \text{in } \tilde{\Omega}_\varepsilon , \\ \tilde{V}_\varepsilon(0) = 1 . \end{cases} \tag{4.24}$$

LEMMA 4.5. *It holds that*

$$\lim_{\varepsilon\to 0} \frac{\tilde{\psi}_\varepsilon(x)}{|x - P_\varepsilon|} = 1 \quad \textit{uniformly for} \quad x \in \partial\Omega . \tag{4.25}$$

Proof: Note that on $\partial\Omega$, $|x - P_\varepsilon| \geqq d(P_\varepsilon, \partial\Omega)$, i.e., $\frac{|x-P_\varepsilon|}{\varepsilon} \geqq \rho_\varepsilon$. Since by (3.2) $\rho_\varepsilon \to \infty$ as $\varepsilon \to 0$, (4.3) implies that

$$\tilde{\psi}_\varepsilon(x) = |x - P_\varepsilon| + \frac{n-1}{2}\varepsilon \log \frac{|x - P_\varepsilon|}{\varepsilon} - \varepsilon \log(\lambda_0 + o(1))$$

uniformly for $x \in \partial\Omega$. Hence

$$\frac{\tilde{\psi}_\varepsilon(x)}{|x - P_\varepsilon|} = 1 + \frac{n-1}{2} \frac{\log \frac{|x-P_\varepsilon|}{\varepsilon}}{\frac{|x-P_\varepsilon|}{\varepsilon}} - \frac{\log(\lambda_0 + o(1))}{\frac{|x-P_\varepsilon|}{\varepsilon}} \to 1$$

uniformly for $x \in \partial\Omega$ as $\varepsilon \to 0$.

We now have an estimate for $\tilde{\psi}_\varepsilon(x)$.

LEMMA 4.6. (i) *There exists a positive constant C such that $\|\tilde{\psi}_\varepsilon\|_{L^\infty(\Omega)} \leqq C$.* (ii) *For any $\sigma_0 > 0$, there is an ε_0 such that for any $\varepsilon < \varepsilon_0$,*

$$\tilde{\psi}_\varepsilon(P_\varepsilon) \leqq (2 + \sigma_0)d(P_\varepsilon, \partial\Omega) . \tag{4.26}$$

(iii) *For every sequence* $\varepsilon_k \to 0$, *there is a subsequence* $\varepsilon_{k_l} \to 0$ *such that* $\tilde{V}_{\varepsilon_{k_l}} \to \tilde{V}_0$ *uniformly on every compact set of* $\mathbb{R}^n$, *where* $\tilde{V}_0$ *is a positive solution of*

$$\begin{cases} \Delta u - u = 0 & \text{in } \mathbb{R}^n\,, \\ u > 0 & \text{in } \mathbb{R}^n \quad \text{and } u(0) = 1\,. \end{cases} \tag{4.27}$$

Moreover, for any $\sigma_2 > 0, \sup_{y\in\bar{\Omega}_{\varepsilon_{k_l}}} e^{-((1+\sigma_2)|y|)}|\tilde{V}_{\varepsilon_{k_l}}(y) - \tilde{V}_0(y)| \to 0$ *as* $\varepsilon_{k_l} \to 0$.

Proof: (i) The proof of (i) is almost identical to that of its counterpart in Lemma 4.2 and is thus omitted.

(ii) Assume that $d(P_\varepsilon, \partial\Omega) = |P_\varepsilon - \overline{P_\varepsilon}|$ where $\overline{P_\varepsilon} \in \partial\Omega$. Let y_ε be the point on the ray $\overrightarrow{P_\varepsilon \overline{P_\varepsilon}}$ such that $|P_\varepsilon - y_\varepsilon| = (1+\eta)|P_\varepsilon - \overline{P_\varepsilon}|$, where $\eta < \min\{1, \sigma_0/10\}$ is so small that $B_{r_0}(y_\varepsilon) \subset \Omega^c$ and $\overline{B_{r_0}(y_\varepsilon)} \cap \bar{\Omega} = \{\overline{P_\varepsilon}\}$ for $r_0 = \eta|P_\varepsilon - \overline{P_\varepsilon}|$.

Setting $w_\varepsilon(x) = (1+2\eta)(|P_\varepsilon - \overline{P_\varepsilon}| + |y_\varepsilon - x|)$, we see that, on $\partial\Omega$

$$\begin{aligned} \tilde{\psi}_\varepsilon(x) &\leqq (1+\eta/2)|x - P_\varepsilon| \\ &< (1+\eta/2)(|P_\varepsilon - y_\varepsilon| + |y_\varepsilon - x|) \\ &= (1+\eta/2)(|P_\varepsilon - \overline{P_\varepsilon}| + |y_\varepsilon - x| + \eta|P_\varepsilon - \overline{P_\varepsilon}|) \\ &< w_\varepsilon(x) \end{aligned}$$

for ε sufficiently small, since $(1+\frac{\eta}{2})(1+\eta) < 1+2\eta$. Moreover, $w_\varepsilon(x) \in C^2(\bar{\Omega})$ and

$$\begin{aligned} \nabla w_\varepsilon(x) &= (1+2\eta)\frac{x - y_\varepsilon}{|x - y_\varepsilon|}\,, \\ |\Delta w_\varepsilon(x)| &\leqq \frac{C}{|y_\varepsilon - x|} \leqq \frac{C}{(\eta|P_\varepsilon - \overline{P_\varepsilon}|)}\,. \end{aligned}$$

Thus for ε sufficiently small,

$$\varepsilon\Delta w_\varepsilon - |\nabla w_\varepsilon|^2 + 1 \leqq \frac{C\varepsilon}{\eta|P_\varepsilon - \overline{P_\varepsilon}|} - (1+\eta)^2 + 1 < 0$$

since $\varepsilon/|P_\varepsilon - \overline{P_\varepsilon}| = 1/\rho_\varepsilon \to 0$ as $\varepsilon \to 0$. Therefore by the Maximum Principle, we conclude that

$$\tilde{\psi}_\varepsilon(x) \leqq w_\varepsilon(x) \qquad \text{for } x \in \Omega\,.$$

Hence

$$\begin{aligned} \tilde{\psi}_\varepsilon(P_\varepsilon) &\leqq (1+2\eta)(|P_\varepsilon - \overline{P_\varepsilon}| + |y_\varepsilon - P_\varepsilon|) \\ &= (1+2\eta)(2+\eta)(|P_\varepsilon - \overline{P_\varepsilon}|) \\ &< (2+\sigma_0)(|P_\varepsilon - \overline{P_\varepsilon}|)\,. \end{aligned}$$

(iii) To prove part (iii), we just need to show that

$$\tilde{V}_\varepsilon(y) \leqq Ce^{(1+\frac{\sigma_2}{2})|y|} \tag{4.28}$$

for $\varepsilon < \varepsilon_0$. We note that the proof of Lemma 4.4 (ii) also works in this case with minor modifications. (Indeed, we just need to change $d(Q, \partial\Omega)$ to $d(P_\varepsilon, \partial\Omega)$ and observe that $\varepsilon/d(P_\varepsilon, \partial\Omega) \to 0$ as $\varepsilon \to 0$). We omit the proof.

Finally we conclude this section by proving a property of V_0 and $\tilde{V}_0$ which is important in the asymptotic expansion of c_ε.

LEMMA 4.7. *Let w be defined in Section 2 and V be an arbitrary solution of*

$$\begin{cases} \Delta u - u = 0 & \text{in } \mathbb{R}^n\,, \\ u > 0 & \text{in } \mathbb{R}^n \quad \text{and } u(0) = 1\,. \end{cases} \tag{4.29}$$

Then we have

$$2\gamma := \int_{\mathbb{R}^n} f(w)V_* = \int_{\mathbb{R}^n} f(w)V > 0 \tag{4.30}$$

where $V_*(r)$ is the unique positive radial solution of (4.29).

Proof: First, we prove that $\int_{\mathbb{R}^n} f(w)V$ is independent of the choice of V and is equal to $\int_{\mathbb{R}^n} f(w)V_*$. In fact, by Theorem 1 of [5], we have

$$V(x) = \int_{|\lambda|=1} e^{\lambda\cdot x} d\mu(\lambda)\,, \qquad V_*(r) = \frac{1}{|S^{n-1}|}\int_{|\lambda|=1} e^{r\zeta\cdot\lambda} dA(\lambda) \tag{4.31}$$

where $|S^{n-1}|$ is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$, $d\mu(\lambda)$ is a positive measure on S^{n-1}, $\mu(S^{n-1}) = 1$, $dA(\lambda)$ denotes the usual element of the surface area on S^{n-1}, and $\zeta \in S^{n-1}$. Observe that the second integral in (4.31) is independent of ζ. Writing $x = r\zeta$ with $r = |x|$ and $\zeta \in S^{n-1}$, we see that

$$\begin{aligned} &\int_{\mathbb{R}^n} f(w)V \\ &= \int_0^\infty \int_{|\lambda|=1}\int_{|\zeta|=1} e^{r\zeta\cdot\lambda}\, d\mu(\zeta)\, dA(\lambda) f(w(r)) r^{n-1}\, dr \\ &= \int_0^\infty \int_{|\zeta|=1}\int_{|\lambda|=1} e^{r\zeta\cdot\lambda}\, dA(\lambda)\, d\mu(\zeta) f(w(r)) r^{n-1}\, dr \\ &= |S^{n-1}| \int_0^\infty \int_{|\zeta|=1} d\mu(\zeta) f(w(r)) V_*(r) r^{n-1}\, dr \\ &= |S^{n-1}| \int_0^\infty f(w(r)) V_*(r) r^{n-1}\, dr = \int_{R^n} f(w)V_* \end{aligned}$$

since $V(0) = \int_{|\zeta|=1} d\mu(\zeta) = 1$.

Now, we prove that $\int_{\mathbb{R}^n} f(w)V_* > 0$. Letting R be a fixed positive number and $\overline{w}$ be the unique solution of the equation

$$\begin{cases} \Delta w - w + w^p = 0 & \text{and } w > 0 \quad \text{in } \mathbb{R}^n\,, \\ w(0) = \max_{z\in\mathbb{R}^n} w(z) & \text{and } w(z) \to 0 \quad \text{as } |z| \to +\infty\,, \end{cases} \tag{4.32}$$

we observe that

$$
\begin{aligned}
\int_{B_R(0)} f(w)V_* &= \int_{B_R(0)} (w-\Delta w)V_* = \int_{B_R(0)} [(\Delta V_*)w - (\Delta w)V_*] \\
&= \int_{\partial B_R(0)} \left[\frac{\partial V_*}{\partial \nu}w - \frac{\partial w}{\partial \nu}V_*\right] \qquad (4.33)\\
&= \left[w(R)V_*'(R) - w'(R)V_*(R)\right] R^{n-1}|S^{n-1}| \, .
\end{aligned}
$$

Similarly,

$$
\int_{B_R(0)} \overline{w}^p V_* = \left[\overline{w}(R)V_*'(R) - \overline{w}'(R)V_*(R)\right] R^{n-1}|S^{n-1}| \, . \tag{4.34}
$$

By Lemma 4.1, we see that there is a constant $\overline{C} > 0$ such that

$$
\lim_{R\to\infty} \frac{w(R)}{\overline{w}(R)} = \overline{C}\, , \qquad \lim_{R\to\infty} \frac{w'(R)}{\overline{w}'(R)} = \overline{C}\, .
$$

Hence, for R large we have

$$
w(R) > \frac{\overline{C}}{2}\overline{w}(R)\, , \qquad -w'(R) > -\frac{\overline{C}}{2}\overline{w}'(R)\, . \tag{4.35}
$$

From (4.33), (4.34), and (4.35) it follows that

$$
\int_{B_R(0)} f(w)V_* > \frac{\overline{C}}{2}\int_{B_R(0)} \overline{w}^p V_*
$$

for R large since $V_*' \geqq 0$. Letting $R\to\infty$, we obtain

$$
\int_{\mathbb{R}^n} f(w)V_* \geqq \frac{\overline{C}}{2}\int_{\mathbb{R}^n} \overline{w}^p V_* > 0\, .
$$

5. An Upper Bound for c_ε

The purpose of this section is to obtain the following upper bound for c_ε which improves Lemma 3.1. Recall first that $d(Q,\partial\Omega) = \max_{P\in\Omega} d(P,\partial\Omega)$ and $\beta = \frac{1}{\varepsilon}$.

PROPOSITION 5.1. *For ε sufficiently small, we have*

$$
c_\varepsilon \leqq \varepsilon^n \left\{ I(w) + \gamma e^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right)\right\}, \tag{5.1}
$$

where c_ε is given by (2.6) *and γ is defined by* (4.30).

We shall establish (5.1) by choosing an appropriate *test function* in the variational characterization of c_ε. First, setting $u_\varepsilon^*(x) = \mathcal{P}_{\Omega_\varepsilon} w(z), x = Q + \varepsilon z$ and $h_\varepsilon(t) := J_\varepsilon(tu_\varepsilon^*)$ for $t \geqq 0$, we have the following two lemmas.

LEMMA 5.2. *For ε sufficiently small and $0 < t < \infty$, we have*

$$\text{(i)} \int_\Omega (\varepsilon^2 |\nabla u_\varepsilon^*|^2 + (u_\varepsilon^*)^2)$$

$$= \varepsilon^n \left\{ \int_{R^n} wf(w) - 2\gamma e^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right) \right\} \tag{5.2}$$

$$\text{(ii)} \int_\Omega F(u_\varepsilon^*)\, dx$$

$$= \varepsilon^n \left\{ \int_{R^n} F(w) - 2\gamma e^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right) \right\} \tag{5.3}$$

$$\text{(iii)} \int_\Omega u_\varepsilon^* f(tu_\varepsilon^*)$$

$$= \varepsilon^n \left\{ \int_{\mathbb{R}^n} wf(tw) - \gamma(t) e^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right) \right\}, \tag{5.4}$$

where $\gamma(t) = \int_{\mathbb{R}^n} (twf'(tw) + f(tw))V_*$ and γ is defined in (4.30).

Proof: Since $u_\varepsilon^*(x) = \mathcal{P}_{\Omega_\varepsilon} w(z) = w(z) - e^{-\beta\psi_\varepsilon(Q)} V_\varepsilon(z)$, we see that

$$\begin{aligned} \int_\Omega (\varepsilon^2 |\nabla u_\varepsilon^*|^2 + (u_\varepsilon^*)^2) &= \varepsilon^n \int_{\Omega_\varepsilon} (|\nabla \mathcal{P}_{\Omega_\varepsilon} w|^2 + |\mathcal{P}_{\Omega_\varepsilon} w|^2)\, dz \\ &= \varepsilon^n \int_{\Omega_\varepsilon} (\mathcal{P}_{\Omega_\varepsilon} w) f(w)\, dz \\ &= \varepsilon^n \int_{\Omega_\varepsilon} \left[wf(w) - e^{-\beta\psi_\varepsilon(Q)} V_\varepsilon f(w) \right] dz\,. \end{aligned} \tag{5.5}$$

Note that by (4.18),

$$\int_{\Omega_\varepsilon} V_\varepsilon |f(w)|\, dz \leqq C \int_{\Omega_\varepsilon} e^{(1+\frac{\sigma_1}{2})|z|} |f(w)|\, dz \leqq C \int_{\mathbb{R}^n} e^{(1+\frac{\sigma_1}{2})|z|} |f(w)|\, dz \leqq C\,,$$

if we choose $\sigma_1 < \sigma$. By Lemma 4.4, for any sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k_l} \to 0$ such that $V_{\varepsilon_{k_l}} \to V_0$ for some solution V_0 of (4.14). By Lebesgue's Dominated Convergence Theorem and Lemma 4.7,

$$\int_{\Omega_{\varepsilon_{k_l}}} V_{\varepsilon_{k_l}} f(w) \to \int_{\mathbb{R}^n} V_0 f(w) = 2\gamma\,.$$

It follows that

$$\int_{\Omega_\varepsilon} V_\varepsilon f(w) \to 2\gamma \qquad \text{as } \varepsilon \to 0\,. \tag{5.6}$$

We also observe that

$$\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w|f(w)| \leqq Ce^{-\beta(2+\sigma)d(Q,\partial\Omega)} = o\left(e^{-\beta\psi_\varepsilon(Q)}\right). \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.5), we obtain (5.2).

To prove (5.3), we first note that by the mean-value theorem

$$F(\mathcal{P}_{\Omega_\varepsilon}w) = F(w) - f(w_1)e^{-\beta\psi_\varepsilon(Q)}V_\varepsilon \tag{5.8}$$

where $w \geqq w_1 \geqq \mathcal{P}_{\Omega_\varepsilon}w$. Thus

$$\int_{\Omega_\varepsilon} |f(w_1)|V_\varepsilon \leqq C\int_{\Omega_\varepsilon} w^{1+\sigma}V_\varepsilon \leqq C$$

and again Lemma 4.4, the Lebesgue Dominated Convergence Theorem, and Lemma 4.7 guarantee that

$$\int_{\Omega_\varepsilon} f(w_1)V_\varepsilon \to 2\gamma \qquad \text{as } \varepsilon \to 0\,, \tag{5.9}$$

since $w_1(z) \to w(z)$ (because $\mathcal{P}_{\Omega_\varepsilon}w(z) \to w(z)$) for $z \in \Omega_\varepsilon$. Now (5.3) follows immediately from (5.8) and (5.9) as before.

It remains to prove (5.4). Note now that by the same argument in Lemma 4.7, we see that for any solution V of (4.29), $\int_{\mathbb{R}^n}(twf'(tw) + f(tw))V$ is independent of V and is equal to $\gamma(t)$. Then (5.4) can be proved in a similar way as before and thus the proof is omitted.

LEMMA 5.3. *For each $\varepsilon > 0$ sufficiently small, h_ε attains a unique positive maximum at $t = t_0(\varepsilon) > 0$ and*

$$t_0(\varepsilon) = 1 + ae^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right) \tag{5.10}$$

for some constant a.

Proof: This proof is essentially the same as that of (3.16) in [14]; for the sake of completeness, however, we include a sketch here.

We first note that $h_\varepsilon(t)$ has a unique positive maximum. This is clear when $q = 1$. Indeed,

$$h_\varepsilon'(t) = t\int_\Omega (\varepsilon^2|\nabla u_\varepsilon^*|^2 + (u_\varepsilon^*)^2)\,dx - \int_\Omega u_\varepsilon^* f(tu_\varepsilon^*)\,dx \tag{5.11}$$

which implies that $h_\varepsilon'(t_0) = 0$ for some $t_0 > 0$ if and only if

$$\int_\Omega (\varepsilon^2|\nabla u_\varepsilon^*|^2 + (u_\varepsilon^*)^2)\,dx = t_0^{-1}\int_\Omega u_\varepsilon^* f(t_0u_\varepsilon^*)\,dx\,.$$

Since the right-hand side is equal to

$$\int_\Omega (u_\varepsilon^*)^2 \frac{f(t_0 u_\varepsilon^*)}{t_0 u_\varepsilon^*}\, dx$$

and since this is strictly increasing in $t_0 > 0$ by virtue of (ii) of (f2), we find that there exists exactly one $t_0 > 0$ such that $h_\varepsilon'(t_0) = 0$. Hence $t_0(\varepsilon)$ is unique. In the case $q > 1$, we can use the same arguments of Lemma B.1 in [15].

To derive the asymptotic behavior of $t_0(\varepsilon)$ as $\varepsilon \to 0$, we define a function

$$\sigma(\varepsilon, t) := \varepsilon^{-n} h_\varepsilon'(t)$$

for $\varepsilon > 0, t \geqq 0$. By Lemma 5.2, we have

$$\begin{aligned}\sigma(\varepsilon, t) \quad &= \quad \int_{\mathbb{R}^n} [t(|\nabla w|^2 + w^2) - wf(tw)] \\ &\qquad - (2t\gamma - \gamma(t))e^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right)\end{aligned}$$

as $\varepsilon \to 0$, where the term $o\left(e^{-\beta\psi_\varepsilon(Q)}\right)$ is uniform in t on each compact interval. Now let $\tilde{\sigma}(\delta, t) = \sigma(\varepsilon, t)$ where $\delta = e^{-\beta\psi_\varepsilon(Q)}$. Then

$$\tilde{\sigma}(\delta, t) = \int_{\mathbb{R}^n} \left[t\left(|\nabla w|^2 + w^2\right) - wf(tw)\right] - (2t\gamma - \gamma(t))\, \delta + o(\delta)$$

as $\delta \to 0$, where the term $o(\delta)$ is uniform in t on each compact interval. From this it follows that $\tilde{\sigma}(\delta, t)$ can be extended up to $\delta = 0$ as a continuously differentiable function on $[0, \delta_*) \times [0, \infty)$ with $\delta_* > 0$ sufficiently small. Note that $\tilde{\sigma}(0, 1) = 0$. Next we compute $\tilde{\sigma}_t(0, t)$:

$$\begin{aligned}\tilde{\sigma}_t(0, t)|_{t=1} \quad &= \quad \int_{\mathbb{R}^n} \left[\left(|\nabla w|^2 + w^2\right) - w^2 f'(w)\right] \\ &= \quad \int_{\mathbb{R}^n} w\left[f(w) - wf'(w)\right] \; .\end{aligned}$$

By Lemma B.1 in Appendix B of [15] we have $\tilde{\sigma}_t(0, t)|_{t=1} < 0$. The Implicit Function Theorem now yields the existence of a C^1 function $t(\delta)$ defined for $\delta \in [0, \delta_{**})$ with $\delta_{**} > 0$ sufficiently small such that $\tilde{\sigma}(\delta, t(\delta)) \equiv 0$ and $t(0) = 1$. Letting $t(\delta) = t_0(\varepsilon)$, we finish our proof.

Proof of Proposition 5.1: By Proposition 2.1, we have

$$\begin{aligned}c_\varepsilon \quad &\leqq \quad J_\varepsilon(t_0(\varepsilon)u_\varepsilon^*) \\ &= \quad \frac{1}{2}t_0^2(\varepsilon)\int_\Omega (\varepsilon^2|\nabla u_\varepsilon^*|^2 + (u_\varepsilon^*)^2)\, dx - \int_\Omega F(t_0(\varepsilon)u_\varepsilon^*)\, dx \\ &= \quad \varepsilon^n\left\{\frac{1}{2}t_0^2(\varepsilon)\int_{\Omega_\varepsilon} (|\nabla \mathscr{P}_{\Omega_\varepsilon} w(z)|^2 + |\mathscr{P}_{\Omega_\varepsilon} w(z)|^2)\, dz - \int_{\Omega_\varepsilon} F(t_0(\varepsilon)\mathscr{P}_{\Omega_\varepsilon} w(z))\, dz\right\} \\ &= \quad \varepsilon^n\{I_1 - I_2\}\end{aligned}$$

where I_1 and I_2 are defined by the last equality. Using (5.2) and (5.10), we have

$$
\begin{aligned}
I_1 &= \left(\frac{1}{2} + ae^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right)\right) \\
&\quad \times \left(\int_{\mathbb{R}^n} wf(w) - 2\gamma e^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right)\right) \qquad (5.12)\\
&= \int_{\mathbb{R}^n} \frac{1}{2} wf(w) - \gamma e^{-\beta\psi_\varepsilon(Q)} + ae^{-\beta\psi_\varepsilon(Q)} \int_{\mathbb{R}^n} wf(w) + o\left(e^{-\beta\psi_\varepsilon(Q)}\right).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&F(t_0(\varepsilon)\mathcal{P}_{\Omega_\varepsilon} w) - F(w) \\
&\quad = f(w_2)\left[t_0(\varepsilon)\mathcal{P}_{\Omega_\varepsilon} w - w\right] \\
&\quad = f(w_2)\left[\left(1 + ae^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right)\right)\mathcal{P}_{\Omega_\varepsilon} w - w\right] \\
&\quad = f(w_2)\left[\mathcal{P}_{\Omega_\varepsilon} w - w + ae^{-\beta\psi_\varepsilon(Q)}\mathcal{P}_{\Omega_\varepsilon} w + o\left(e^{-\beta\psi_\varepsilon(Q)}\right)\mathcal{P}_{\Omega_\varepsilon} w\right] \\
&\quad = -e^{-\beta\psi_\varepsilon(Q)} f(w_2) V_\varepsilon + ae^{-\beta\psi_\varepsilon(Q)} f(w_2)\mathcal{P}_{\Omega_\varepsilon} w + o\left(e^{-\beta\psi_\varepsilon(Q)}\right) f(w_2)\mathcal{P}_{\Omega_\varepsilon} w
\end{aligned}
$$

for some w_2 lying between w and $t_0(\varepsilon)\mathcal{P}_{\Omega_\varepsilon} w$. Then, the same argument as in the proof of Lemma 5.2 shows that as $\varepsilon \to 0$

$$\int_{\Omega_\varepsilon} f(w_2) V_\varepsilon \to 2\gamma$$

and

$$\int_{\Omega_\varepsilon} f(w_2)\mathcal{P}_{\Omega_\varepsilon} w \to \int_{\mathbb{R}^n} wf(w).$$

Hence it follows that

$$I_2 = \int_{\mathbb{R}^n} F(w) + ae^{-\beta\psi_\varepsilon(Q)} \int_{\mathbb{R}^n} wf(w) - 2\gamma e^{-\beta\psi_\varepsilon(Q)} + o\left(e^{-\beta\psi_\varepsilon(Q)}\right). \qquad (5.13)$$

Putting (5.12) and (5.13) together, we obtain (5.1).

6. Asymptotic Formula for c_ε and the Proofs of Theorem 2.2 (ii) and Theorem 2.3

In this section, we shall derive an asymptotic formula for c_ε and prove Theorem 2.2 (ii) and Theorem 2.3. To this end, we define ϕ_ε by

$$\tilde{v_\varepsilon}(y) = \mathcal{P}_{\tilde{\Omega}_\varepsilon} w(y) + e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}\phi_\varepsilon(y)$$

where $y \in \tilde{\Omega}_\varepsilon$. Recall that $\tilde{v_\varepsilon}(y) = u_\varepsilon(P_\varepsilon + \varepsilon y)$, then ϕ_ε satisfies

$$
\begin{cases}
L\phi_\varepsilon + F_\varepsilon(\phi_\varepsilon) = 0 & \text{in } \tilde{\Omega}_\varepsilon\,, \\
\phi_\varepsilon = 0 & \text{on } \partial\tilde{\Omega}_\varepsilon\,,
\end{cases} \qquad (6.1)
$$

where

$$F_\varepsilon(\phi_\varepsilon) = e^{\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}\left\{f(\tilde{v}_\varepsilon) - f(w) - e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}f'(w)\phi_\varepsilon\right\}.$$

Observe that

$$F_\varepsilon(\phi_\varepsilon) + f'(w)\tilde{V}_\varepsilon = e^{\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}\left\{f(\tilde{v}_\varepsilon) - f(w) + e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}f'(w)(\tilde{V}_\varepsilon - \phi_\varepsilon)\right\}.$$

Since $\tilde{v}_\varepsilon = w - e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}(\tilde{V}_\varepsilon - \phi_\varepsilon)$ and $|f'(t) - f'(s)| \leqq C|t-s|^\sigma$ for bounded s and t, the following estimate is an easy consequence of the mean-value theorem.

LEMMA 6.1. *For ε sufficiently small, we have*

$$|F_\varepsilon(\phi_\varepsilon) + f'(w)\tilde{V}_\varepsilon| \leqq C|\tilde{v}_\varepsilon - w|^\sigma|\tilde{V}_\varepsilon - \phi_\varepsilon|\,. \tag{6.2}$$

Our next result is crucial in deriving the asymptotic expansion for c_ε. For given $1-\sigma < \mu < 1$, we shall fix $0 < \sigma_1 = \sigma_2 < \frac{1}{4}\sigma$ (in Lemmas 4.4 and 4.6) such that $1 - \sigma + \sigma_1 + \sigma_2 < \mu < 1$ in the rest of this section unless otherwise specified.

PROPOSITION 6.2. (i) *For $s > n$, we have $\|e^{-\mu|y|}\phi_\varepsilon\|_{W^{2,s}(\tilde{\Omega}_\varepsilon)} \leqq C(s)$.*

(ii) *For every sequence $\varepsilon_k \to 0$, there is a subsequence ε_{k_l} and a solution $\tilde{V}_0$ of* (4.27) *such that $\|e^{-\mu|y|}(\phi_{\varepsilon_{k_l}} - \phi_0)\|_{L^\infty(B_{(1-\delta_2)\rho_{\varepsilon_{k_l}}})} \to 0$ as $\varepsilon_{k_l} \to 0$ where $\delta_2 = \sigma_2/10$ and ϕ_0 is a solution of*

$$L\phi_0 - f'(w)\tilde{V}_0 = 0 \qquad in\ \mathbb{R}^n\,. \tag{6.3}$$

Furthermore, $e^{-\mu|y|}\phi_0 \in W^{2,s}(\mathbb{R}^n)$ for $s > 1$.

Assuming Proposition 6.2, we proceed to obtain an asymptotic expansion for c_ε, which improves our upper bound in Proposition 5.1.

PROPOSITION 6.3. *For ε sufficiently small, we have*

$$c_\varepsilon = \varepsilon^n\left\{I(w) + \gamma e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)} + o\left(e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}\right)\right\} \tag{6.4}$$

where c_ε is given by (2.6) *and γ is defined by* (4.30).

Proof: Since u_ε satisfies equation (2.1), we have

$$\begin{aligned} c_\varepsilon &= \frac{1}{2}\int_\Omega(\varepsilon^2|\nabla u_\varepsilon|^2 + u_\varepsilon^2) - \int_\Omega F(u_\varepsilon) \\ &= \frac{1}{2}\int_\Omega u_\varepsilon f(u_\varepsilon) - \int_\Omega F(u_\varepsilon) \\ &= \varepsilon^n\left\{\frac{1}{2}\int_{\tilde{\Omega}_\varepsilon}\tilde{v}_\varepsilon f(\tilde{v}_\varepsilon) - \int_{\tilde{\Omega}_\varepsilon}F(\tilde{v}_\varepsilon)\right\} \\ &= \varepsilon^n\{I_3 - I_4\}\,, \end{aligned}$$

where I_3 and I_4 are defined by the last equality.

We begin with the estimate of I_3. Since $||e^{-\mu|y|}\phi_\varepsilon||_{W^{2,s}(\tilde{\Omega}_\varepsilon)} \leqq C(s)$ for $s > n$ by Proposition 6.2, by the Sobolev Imbedding Theorem (see p. 107, Lemma 5.15, in [1]), we have

$$\phi_\varepsilon(y) \leqq Ce^{\mu|y|}$$

for ε sufficiently small (note that the constant C is still independent of $\varepsilon \leqq \varepsilon_0$). Observe that by mean-value theorem,

$$\tilde{v}_\varepsilon f(\tilde{v}_\varepsilon) = wf(w) + e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}(f(w_3) + w_3 f'(w_3))(\phi_\varepsilon - \tilde{V}_\varepsilon) \tag{6.5}$$

where w_3 lies between w and $\tilde{v}_\varepsilon$. Thus by (3.8) with the choice $\delta = \frac{\sigma}{20}$

$$\begin{aligned}
&\int_{\tilde{\Omega}_\varepsilon} |(f(w_3) + w_3 f'(w_3))(\phi_\varepsilon - \tilde{V}_\varepsilon)| \\
&\leqq C\int_{\tilde{\Omega}_\varepsilon} w_3^{1+\sigma}|\tilde{V}_\varepsilon - \phi_\varepsilon| \\
&\leqq C\int_{\tilde{\Omega}_\varepsilon} e^{-(1-\delta)(1+\sigma)|y|}\left(e^{(1+\sigma_2)|y|} + e^{\mu|y|}\right) \\
&\leqq C
\end{aligned}$$

for $\varepsilon \leqq \varepsilon_0$. By Lemma 4.6, for any sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k_l} \to 0$ and a solution $\tilde{V}_0$ of (4.27) such that $\tilde{V}_{k_l} \to \tilde{V}_0$. Then by Proposition 6.2 and Lebesgue's Dominated Convergence Theorem,

$$\int_{\tilde{\Omega}_{\varepsilon_{k_l}}} (w_3 f'(w_3) + f(w_3))(\phi_{\varepsilon_{k_l}} - \tilde{V}_{\varepsilon_{k_l}})\,dy \to \int_{\mathbb{R}^n} (wf'(w) + f(w))(\phi_0 - \tilde{V}_0)\,dy$$

as $\varepsilon_{k_l} \to 0$. We note also that by Lemma 4.6 (ii) with $\sigma_0 < \sigma$,

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus\tilde{\Omega}_\varepsilon} w|f(w)|\,dy \\
&\leqq Ce^{-\beta(2+\sigma_0)d(P_\varepsilon,\partial\Omega)}\int_{\mathbb{R}^n\setminus\tilde{\Omega}_\varepsilon} e^{-(\sigma-\sigma_0)|y|}\,dy = o\left(e^{-\beta\tilde{\psi}_\varepsilon(P_\varepsilon)}\right).
\end{aligned} \tag{6.6}$$

Observing that ϕ_0 satisfies equation (6.3), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} wf'(w)\tilde{V}_0 &= \int_{\mathbb{R}^n} (\Delta\phi_0 - \phi_0 + f'(w)\phi_0)w \\
&= \int_{\mathbb{R}^n} (wf'(w) - f(w))\phi_0
\end{aligned}$$

since $e^{-\mu|y|}\phi_0 \in W^{2,s}(\mathbb{R}^n)$. Hence

$$I_3 = \frac{1}{2}\int_{\tilde{\Omega}_{\varepsilon_{k_l}}} \left\{wf(w) - e^{-\beta_{k_l}\tilde{\psi}_{\varepsilon_{k_l}}(P_{\varepsilon_{k_l}})}(f(w_3) + w_3 f'(w_3))\left(\tilde{V}_{\varepsilon_{k_l}} - \phi_{\varepsilon_{k_l}}\right)\right\}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^n} wf(w) \\
(6.7)\qquad &\quad - e^{-\beta_{k_l}\tilde{\psi}_{\varepsilon_{k_l}}(P_{\varepsilon_{k_l}})}\int_{\mathbb{R}^n}\frac{1}{2}[f(w)+wf'(w)](\tilde{V}_0-\phi_0)+o\left(e^{-\beta_{k_l}\tilde{\psi}_{\varepsilon_{k_l}}(P_{\varepsilon_{k_l}})}\right) \\
&= \frac{1}{2}\int_{\mathbb{R}^n} wf(w) \\
&\quad + e^{-\beta_{k_l}\tilde{\psi}_{\varepsilon_{k_l}}(P_{\varepsilon_{k_l}})}\int_{\mathbb{R}^n}\left[f(w)\phi_0-\frac{1}{2}f(w)\tilde{V}_0\right]+o\left(e^{-\beta_{k_l}\tilde{\psi}_{\varepsilon_{k_l}}(P_{\varepsilon_{k_l}})}\right).
\end{aligned}
$$

I_4 may be handled in a similar fashion and we have

$$
(6.8)\qquad I_4=\int_{\mathbb{R}^n}F(w)+e^{-\beta_{k_l}\tilde{\psi}_{\varepsilon_{k_l}}(P_{\varepsilon_{k_l}})}\int_{\mathbb{R}^n}f(w)(\phi_0-\tilde{V}_0)+o\left(e^{-\beta_{k_l}\tilde{\psi}_{\varepsilon_{k_l}}(P_{\varepsilon_{k_l}})}\right).
$$

From (6.7), (6.8), and (4.30), we see that (6.4) holds for $\varepsilon=\varepsilon_{k_l}$. Since the sequence ε_k is arbitrary, it is easy to see that (6.4) is true for ε sufficiently small.

Proof of Theorem 2.2 (ii) and Theorem 2.3: By Theorem 2.2 (i) and (3.2), u_ε has at most one local maximum P_ε and $\rho_\varepsilon=d(P_\varepsilon,\partial\Omega)/\varepsilon\to\infty$. Since Lemma 4.7 guarantees that $\gamma>0$, from Propositions 5.1 and 6.3 we deduce easily that

$$
\tilde{\psi}_\varepsilon(P_\varepsilon)\geqq\psi_\varepsilon(Q)+o(1)\,.
$$

Then, by Lemma 4.6 (ii), for any σ_0 there is ε_0 such that

$$
(6.9)\qquad (2+\sigma_0)d(P_\varepsilon,\partial\Omega)\geqq\tilde{\psi}_\varepsilon(P_\varepsilon)\geqq\psi_\varepsilon(Q)+o(1)
$$

for all $\varepsilon<\varepsilon_0$. Since $\psi_\varepsilon(Q)\to 2d(Q,\partial\Omega)$ as $\varepsilon\to 0$, we see that

$$
(2+\sigma_0)\underline{\lim}_{\varepsilon\to 0}d(P_\varepsilon,\partial\Omega)\geqq 2d(Q,\partial\Omega)
$$

which in turn implies that $d(P_\varepsilon,\partial\Omega)\to d(Q,\partial\Omega)$ by the choice of Q. This establishes part (ii) of Theorem 2.2.

Theorem 2.3 follows from Proposition 6.3, (6.9), part (ii) of Theorem 2.2, and Proposition 6.2.

The rest of this section is devoted to the proof of Proposition 6.2. We need some preparations before we go into the proof.

Letting U be a bounded smooth domain in $\mathbb{R}^n$, we define, for $0\leqq\mu<1$,

$$
\|u\|_{W^{k,s}_\mu(U)}=\|e^{-\mu<y>}u\|_{W^{k,s}(U)}
$$

where $<y>=(1+|y|^2)^{\frac{1}{2}}$. As usual, when $k=0$, we denote $W^{0,s}_\mu(U)$ as $L^s_\mu(U)$.

The following two lemmas play a basic role in our estimates.

LEMMA 6.4. (i) *Let $s>1$, $0\leqq\mu<1$ and u be the solution of*

$$
(6.10)\qquad \begin{cases}\Delta u-u+\overline{f}=0 & \text{in } \tilde{\Omega}_\varepsilon\,,\\ u=0 & \text{on } \partial\tilde{\Omega}_\varepsilon\,.\end{cases}
$$

Then

$$||u||_{W^{2,s}_\mu(\tilde\Omega_\varepsilon)} \leqq C\left(||\tilde f||_{L^s_\mu(\tilde\Omega_\varepsilon)} + ||\tilde f||_{L^2_\mu(\tilde\Omega_\varepsilon)}\right), \tag{6.11}$$

where C is a constant independent of $\varepsilon \leqq \varepsilon_0$.

(ii) *Let u be a solution of*

$$\Delta u - u + \tilde f = 0 \qquad in\ \mathbb{R}^n$$

with $\|\tilde f\|_{L^s_\mu(\mathbb{R}^n)} < \infty$ *and* $\|\tilde f\|_{L^2_\mu(\mathbb{R}^n)} < \infty$. *Then*

$$\|u\|_{W^{2,s}_\mu(\mathbb{R}^n)} \leqq C\left(\|\tilde f\|_{L^s_\mu(\mathbb{R}^n)} + \|\tilde f\|_{L^2_\mu(\mathbb{R}^n)}\right). \tag{6.12}$$

(iii) *For every function* $\tilde K \in C^2(\overline{\tilde\Omega}_\varepsilon)$, *there exists an extension* $K \in C^2_0(\mathbb{R}^n)$ *with*

$$\|K\|_{W^{2,s}(\mathbb{R}^n)} \leqq C\|\tilde K\|_{W^{2,s}(\tilde\Omega_\varepsilon)},$$

where $s > 1$ *and C is independent of* $\tilde K$ *and* $\varepsilon \leqq 1$.

LEMMA 6.5. *If the domain of L is* $W^{2,s}_\mu(\mathbb{R}^n)$ *where* $s > n/2, 0 \leqq \mu < 1$, *then* $\ker(L) = X = \operatorname{Span}\{e_1, \ldots, e_n\}$ *where* $e_i = \frac{\partial w}{\partial x_i}, i = 1, 2, \ldots, n$.

The proofs of Lemmas 6.4 and 6.5 are included in Appendix B and Appendix C.

We now explain the plan of the proof of Proposition 6.2. Our proof will follow the idea in [15] and [17]. We first prove that $||\phi_\varepsilon||_{L^s_\mu(\tilde\Omega_\varepsilon)}$ is bounded for $s > n$. Then by Lemma 4.6, for every sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k_l} \to 0$ and a solution $\tilde V_0$ of (4.27) such that $\tilde V_{\varepsilon_{k_l}} \to \tilde V_0$. Letting $\overline{\phi}_{\varepsilon_{k_l}} = \chi(|y|/\rho_{\varepsilon_{k_l}})\phi_0(y)$ where $\chi(r) = 1$ when $r \leqq 1 - \delta_2$ and $\chi(r) = 0$ when $r > 1 - \delta_2/2$, we show that $\|\phi_{\varepsilon_{k_l}} - \overline{\phi}_{\varepsilon_{k_l}}\|_{W^{2,s}_\mu(\tilde\Omega_{\varepsilon_{k_l}})} = o(1)$, which, by the Sobolev Imbedding Theorem, proves Proposition 6.2.

We begin with the following useful estimates.

LEMMA 6.6. *For every sequence* $\varepsilon_k \to 0$, *there is a subsequence* $\varepsilon_{k_l} \to 0$ *and a solution* $\tilde V_0$ *of* (4.27) *such that for* $2 < s \leqq \infty$

$$\|F_{\varepsilon_{k_l}}(\phi_{\varepsilon_{k_l}}) + f'(w)\tilde V_0\|_{L^s_\mu(\tilde\Omega_{\varepsilon_{k_l}})} \leqq C\left(o(1)||\phi_{\varepsilon_{k_l}}||_{L^s_\mu(\tilde\Omega_{\varepsilon_{k_l}})} + o(1)\right), \tag{6.13}$$

$$\|F_{\varepsilon_{k_l}}(\phi_{\varepsilon_{k_l}}) + f'(w)\tilde V_0\|_{L^2_\mu(\tilde\Omega_{\varepsilon_{k_l}})} \leqq C\left(o(1)||\phi_{\varepsilon_{k_l}}||_{L^s_\mu(\tilde\Omega_{\varepsilon_{k_l}})} + o(1)\right). \tag{6.14}$$

Proof: By Lemma 4.6, for every sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k_l} \to 0$ and a solution $\tilde V_0$ of (4.27) such that $\tilde V_{\varepsilon_{k_l}} \to \tilde V_0$. By Lemma 6.1, we have

$$\begin{aligned} e^{-\mu<y>}|F_{\varepsilon_{k_l}}(\phi_{\varepsilon_{k_l}}) + f'(w)\tilde V_0| &\leqq Ce^{-\mu<y>}|f'(w)||\tilde V_0 - \tilde V_{\varepsilon_{k_l}}| \\ &\quad + Ce^{-\mu<y>}|\tilde v_{\varepsilon_{k_l}} - w|^\sigma|\phi_{\varepsilon_{k_l}} - \tilde V_{\varepsilon_{k_l}}|. \end{aligned}$$

Since $|\tilde{V}_{\varepsilon_{k_l}} - \tilde{V}_0| = o(1)e^{(1+\sigma_2)|y|}$ by Lemma 4.6 (iii), it follows that

$$
\begin{aligned}
(6.15)\qquad & e^{-\mu<y>}|F_{\varepsilon_{k_l}}(\phi_{\varepsilon_{k_l}}) + f'(w)\tilde{V}_0| \\
& \leqq o(1)e^{(-\mu-\sigma+(1+\sigma_2))<y>} + Ce^{-\mu<y>}|\tilde{v}_{\varepsilon_{k_l}} - w|^\sigma |\phi_{\varepsilon_{k_l}} - \tilde{V}_{\varepsilon_{k_l}}| \,.
\end{aligned}
$$

Therefore, from Proposition 3.4 we conclude, for $2 < s < \infty$,

$$
\begin{aligned}
&\|F_{\varepsilon_{k_l}}(\phi_{\varepsilon_{k_l}}) + f'(w)\tilde{V}_0\|_{L^s_\mu(\tilde{\Omega}_{\varepsilon_{k_l}})} \\
\leqq\;& o(1) + C\|\phi_{\varepsilon_{k_l}}\|_{L^s_\mu(\tilde{\Omega}_{\varepsilon_{k_l}})}\|\tilde{v}_{\varepsilon_{k_l}} - w\|_{L^\infty(\tilde{\Omega}_{\varepsilon_{k_l}})} + C\|\tilde{V}_{\varepsilon_{k_l}}(\tilde{v}_{\varepsilon_{k_l}} - w)^\sigma\|_{L^s_\mu(\tilde{\Omega}_{\varepsilon_{k_l}})} \\
=\;& o(1) + o(1)\|\phi_{\varepsilon_{k_l}}\|_{L^s_\mu(\tilde{\Omega}_{\varepsilon_{k_l}})} \,,
\end{aligned}
$$

since $\|\tilde{V}_{\varepsilon_{k_l}}(\tilde{v}_{\varepsilon_{k_l}} - w)^\sigma\|_{L^s_\mu(\tilde{\Omega}_{\varepsilon_{k_l}})} = o(1)$ by Lebesgue's Dominated Convergence Theorem. Similarly,

$$
\begin{aligned}
&\|F_{\varepsilon_{k_l}}\left(\phi_{\varepsilon_{k_l}}\right) + f'(w)\tilde{V}_0\|_{L^2_\mu(\tilde{\Omega}_{\varepsilon_{k_l}})} \\
\leqq\;& o(1) + C\|\phi_{\varepsilon_{k_l}}\|_{L^s_\mu(\tilde{\Omega}_{\varepsilon_{k_l}})}\|(\tilde{v}_{\varepsilon_{k_l}} - w)^\sigma\|_{L^{2s/(s-2)}(\tilde{\Omega}_{\varepsilon_{k_l}})} \\
&+ C\|\tilde{V}_{\varepsilon_{k_l}}(\tilde{v}_{\varepsilon_{k_l}} - w)^\sigma\|_{L^2_\mu(\tilde{\Omega}_{\varepsilon_{k_l}})} \\
=\;& o(1) + o(1)\|\phi_{\varepsilon_{k_l}}\|_{L^s_\mu(\tilde{\Omega}_{\varepsilon_{k_l}})} \,.
\end{aligned}
$$

The case $s = \infty$ can be handled in a similar manner.

LEMMA 6.7. *Let $n < s < \infty$. Then $\|\phi_\varepsilon\|_{L^s_\mu(\tilde{\Omega}_\varepsilon)} \leqq C(s)$.*

Proof: We prove this lemma by contradiction. Suppose on the contrary that there exists a sequence of $\varepsilon_j \to 0$ such that $\|\phi_{\varepsilon_j}\|_{L^s_\mu(\tilde{\Omega}_{\varepsilon_j})} \to \infty$.

Let $M_j = \|\phi_{\varepsilon_j}\|_{L^s_\mu(\tilde{\Omega}_{\varepsilon_j})}, g_j = \phi_{\varepsilon_j}/M_j$. For simplicity, we denote $\tilde{\Omega}_{\varepsilon_j}$ as $\tilde{\Omega}_j$, ϕ_{ε_j} as ϕ_j, etc. Then g_j satisfies

$$
(6.16)\qquad \begin{cases} \Delta g_j - g_j + f'(w)g_j + F_j(\phi_j)/M_j = 0 & \text{in } \tilde{\Omega}_j \,, \\ g_j = 0 & \text{on } \partial\tilde{\Omega}_j \,. \end{cases}
$$

We divide our proof into the following steps:

Step 1. We show that $\|g_j\|_{W^{2,s}_\mu(\tilde{\Omega}_{\varepsilon_j})}$ is bounded.

Step 2. We extend g_j to $\mathbb{R}^n$ and prove that $g_j \to 0$ weakly in $W^{2,s}_\mu(\mathbb{R}^n)$.

Step 3. We prove that $\|g_j\|_{W^{2,s}_\mu(\mathbb{R}^n)} = o(1)$, which gives a contradiction (because $\|g_j\|_{L^s_\mu(\tilde{\Omega}_j)} = 1$).

Now we begin to prove Step 1. In fact, by Lemma 6.4 (i),

$$
\begin{aligned}
\|g_j\|_{W_\mu^{2,s}(\tilde\Omega_j)} \leqq\ & C\Big(\|f'(w)g_j\|_{L_\mu^s(\tilde\Omega_j)} + \|f'(w)g_j\|_{L_\mu^2(\tilde\Omega_j)} \\
& + \|F_j(\phi_j)/M_j\|_{L_\mu^s(\tilde\Omega_j)} + \|F_j(\phi_j)/M_j\|_{L_\mu^2(\tilde\Omega_j)}\Big) .
\end{aligned}
\tag{6.17}
$$

Since

$$
\begin{aligned}
\|f'(w)g_j\|_{L_\mu^s(\tilde\Omega_j)} &\leqq C\|g_j\|_{L_\mu^s(\tilde\Omega_j)} , \\
\|f'(w)g_j\|_{L_\mu^2(\tilde\Omega_j)} &\leqq C\|w^\sigma\|_{L^{2s/(s-2)}(\tilde\Omega_j)}\|g_j\|_{L_\mu^s(\tilde\Omega_j)} \leqq C\|g_j\|_{L_\mu^s(\tilde\Omega_j)} ,
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\frac{F_j(\phi_j)}{M_j}\right\|_{L^s(\tilde\Omega_j)} &\leqq C\frac{\|f'(w)\tilde V_0\|_{L_\mu^s(\tilde\Omega_j)}}{M_j} + o(1)\|g_j\|_{L_\mu^s(\tilde\Omega_j)} + o(1) = o(1) , \\
\left\|\frac{F_j(\phi_j)}{M_j}\right\|_{L^2(\tilde\Omega_j)} &\leqq C\frac{\|f'(w)\tilde V_0\|_{L_\mu^2(\tilde\Omega_j)}}{M_j} + o(1)\|g_j\|_{L_\mu^s(\tilde\Omega_j)} + o(1) = o(1)
\end{aligned}
$$

by Lemma 6.6, we obtain

$$
\|g_j\|_{W_\mu^{2,s}(\tilde\Omega_j)} \leqq C .
$$

Now, from Lemma 6.4 (iii), we can extend g_j to a C^2 function with compact support in $\mathbb{R}^n$ (still denoted by g_j) in such a way that $||g_j||_{W_\mu^{2,s}(\mathbb{R}^n)} \leqq C$, where the constant C is independent of j. Thus we conclude that, first,

$$
||g_j||_{L_\mu^\infty(\mathbb{R}^n)} \leqq C \tag{6.18}
$$

by the Sobolev Imbedding Theorem, and that there exists a function $g_0 \in W_\mu^{2,s}(\mathbb{R}^n)$ such that (by passing to a subsequence if necessary) $g_j \to g_0$ weakly in $W_\mu^{2,s}(\mathbb{R}^n)$ and $g_j \to g_0$ in $C^1_{\text{loc}}(\mathbb{R}^n)$.

To finish the second step, we just need to show that $g_0 = 0$. To this end, we estimate $||F_j(\phi_j)/M_j||_{L_\mu^\infty(\tilde\Omega_j)}$. Note that in Lemma 6.6, we now take $s = \infty$. Then, as before, we have $||F_j(\phi_j)/M_j||_{L_\mu^\infty(\tilde\Omega_j)} \to 0$ as $j \to \infty$ by (6.18). Therefore $F_j(\phi_j)/M_j \to 0$ on every compact set of $\mathbb{R}^n$. Hence g_0 is a weak (thus classical) solution of the following equation

$$
\begin{cases}
Lg_0 = \Delta g_0 - g_0 + f'(w)g_0 = 0 & \text{in } \mathbb{R}^n , \\
g_0 \in W_\mu^{2,s}(\mathbb{R}^n) , & n < s .
\end{cases}
\tag{6.19}
$$

By Lemma 6.5, $g_0 \in X$. That is,

$$
g_0 = \sum_{i=1}^n a_i e_i
$$

for some constants $a_i, i = 1, 2, \ldots, n$.

But note that by definition, $\tilde{v}_j(y) = w(y) + e^{-\beta_j \tilde{\psi}_j(P_j)}(\phi_j - \tilde{V}_j)$. Hence,

$$0 = \nabla \tilde{v}_j(0) = \nabla w(0) + e^{-\beta_j \tilde{\psi}_j(P_j)}(\nabla \phi_j(0) - \nabla \tilde{V}_j(0))$$

which implies that $\nabla \phi_j(0) = \nabla \tilde{V}_j(0)$. Thus

$$\nabla g_j(0) = \frac{\nabla \tilde{V}_j(0)}{M_j} \to 0$$

as $j \to \infty$ since $\tilde{V}_j$ is bounded in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ by (4.28) and standard elliptic regularity estimates. Therefore,

$$\nabla g_0(0) = \sum_{i=1}^{n} a_i \nabla e_i(0) = 0 \, .$$

Observing that $\nabla e_1(0), \dots, \nabla e_n(0)$ are linearly independent, we conclude that $a_i = 0, i = 1, 2, \dots, n$. Hence $g_0 = 0$ and $g_j \to 0$ weakly in $W^{2,s}_\mu(\mathbb{R}^n)$, which completes Step 2.

We now show that $||g_j||_{W^{2,s}_\mu(\tilde{\Omega}_j)} = o(1)$. Similarly to Step 1, we have (6.17) and

$$\begin{aligned} \|F_j(\phi_j)/M_j\|_{L^s_\mu(\tilde{\Omega}_j)} &= o(1) \, , \\ \|F_j(\phi_j)/M_j\|_{L^2_\mu(\tilde{\Omega}_j)} &= o(1) \, . \end{aligned}$$

For the other two terms in (6.17), we estimate as follows:

$$\begin{aligned} \|f'(w) g_j\|^s_{L^s_\mu(\tilde{\Omega}_j)} &\leqq C \int_{\tilde{\Omega}_j \cap B^c_R} e^{-s\mu<y>} w^{s\sigma} g^s_j + C \int_{\tilde{\Omega}_j \cap B_R} e^{-s\mu<y>} w^{s\sigma} g^s_j \\ &\leqq C e^{-s\sigma R} \|g_j\|_{L^s_\mu(\tilde{\Omega}_j)} + C \int_{\tilde{\Omega}_j \cap B_R} e^{-s\mu<y>} w^{s\sigma} g^s_j \qquad (6.20) \\ &\leqq C e^{-s\sigma R} + C \int_{\tilde{\Omega}_j \cap B_R} g^s_j \, , \end{aligned}$$

$$\begin{aligned} \|f'(w) g_j\|^2_{L^2_\mu(\tilde{\Omega}_j)} &\leqq C \int_{\tilde{\Omega}_j \cap B^c_R} e^{-2\mu<y>} w^{2\sigma} g^2_j + C \int_{\tilde{\Omega}_j \cap B_R} e^{-2\mu<y>} w^{2\sigma} g^2_j \\ &\leqq C \|w^{2\sigma}\|_{L^{s/(s-2)}(B^c_R)} \|g_j\|^2_{L^s_\mu(\tilde{\Omega}_j)} + C \int_{\tilde{\Omega}_j \cap B_R} g^2_j \qquad (6.21) \\ &\leqq C R^{\frac{n(s-2)}{s}} e^{-2\sigma R} + C \int_{\tilde{\Omega}_j \cap B_R} g^2_j \, , \end{aligned}$$

where $R \geqq 1$ is an arbitrary number and C is independent of R. Since $g_j \to 0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ by Step 2, we obtain

$$\begin{aligned} \overline{\lim_{j \to \infty}} \|f'(w) g_j\|_{L^s_\mu(\tilde{\Omega}_j)} &\leqq C e^{-\sigma R} \, , \\ \overline{\lim_{j \to \infty}} \|f'(w) g_j\|_{L^2_\mu(\tilde{\Omega}_j)} &\leqq C R^{\frac{n(s-2)}{2s}} e^{-\sigma R} \, . \end{aligned}$$

Letting $R \to \infty$, we conclude from (6.17) that

$$\|g_j\|_{W_\mu^{2,s}(\tilde{\Omega}_j)} = o(1) .$$

COROLLARY 6.8. $\|\phi_\varepsilon\|_{W_\mu^{2,s}(\tilde{\Omega}_\varepsilon)} \leqq C(s)$ *for* $s > n$.

Proof: Noting that $\|\phi_\varepsilon\|_{L_\mu^s(\tilde{\Omega}_\varepsilon)} \leqq C(s)$, we simply observe that our conclusion follows from the arguments in Step 1 of the proof of Lemma 6.7.

By Lemma 6.4 (iii), we can extend ϕ_ε to a C^2 function on $\mathbb{R}^n$ (still denoted by ϕ_ε) such that $\|\phi_\varepsilon\|_{W_\mu^{2,s}(\mathbb{R}^n)} \leqq C(s)$ for $n < s < \infty$. Now we fix $s > n$. For any subsequence ε_j, we can take a further sequence, still denoted by ε_j, such that $\phi_{\varepsilon_j} \to \phi_0$ weakly in $W_\mu^{2,s}(\mathbb{R}^n)$ and $\phi_{\varepsilon_j} \to \phi_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. As before, we denote ϕ_{ε_j} as $\phi_j, \ldots$. We are now ready to prove Proposition 6.2.

Proof of Proposition 6.2: Part (i) is just Corollary 6.8. We next show that ϕ_0 is a solution of equation (6.3). To this end, we need to show that $F_j(\phi_j) \to -f'(w)\tilde{V}_0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. But this can be easily deduced from (6.15). Hence ϕ_0 is a weak (thus classical) solution of equation (6.3).

Now we derive a property of ϕ_0. We claim that

$$\|\phi_0\|_{W_\mu^{2,s}(\mathbb{R}^n)} < \infty \qquad \text{for all } s > 1 .$$

In fact by (6.3), ϕ_0 satisfies

$$\Delta\phi_0 - \phi_0 + f_0 = 0$$

where $f_0 = f'(w)\phi_0 - f'(w)\tilde{V}_0$ and we have

$$\begin{aligned} |f_0(y)| &\leqq Ce^{-\sigma|y|}e^{\mu|y|} + Ce^{-\sigma|y|}e^{(1+\sigma_2)|y|} \\ &\leqq Ce^{(1-\sigma+\sigma_2)|y|} . \end{aligned}$$

Hence

$$\|f_0\|_{L_\mu^s(\mathbb{R}^n)} < \infty \qquad \text{for all } s > 1 .$$

Therefore, by Lemma 6.4 (ii), our assertion above is established.

It remains to prove that $\phi_j \to \phi_0$ in the sense described in part (ii). Let $\chi(r) = 1$ for $r \leqq 1 - \delta_2$, $\chi(r) = 0$ for $r \geqq 1 - \delta_2/2$ where $\delta_2 = \frac{\sigma_2}{10}$. Setting $\chi_j(y) = \chi(|y|/\rho_j)$ and $\overline{\phi}_j = \chi_j(y)\phi_0$, we see that $\phi_j - \overline{\phi}_j$ satisfies the following equation

$$\begin{aligned} &\Delta(\phi_j - \overline{\phi}_j) - (\phi_j - \overline{\phi}_j) + f'(w)(\phi_j - \overline{\phi}_j) \\ (6.22) \quad &= -(F_j(\phi_j) + f'(w)\tilde{V}_0) + (1 - \chi_j)f'(w)\tilde{V}_0 - 2\nabla\chi_j\nabla\phi_0 - (\Delta\chi_j)\phi_0 \\ &= I_5 + I_6 + I_7 + I_8 \end{aligned}$$

where I_5, I_6, I_7, I_8 are defined by the last equality and can be estimated as follows. First, from Lemma 6.6 it follows that

$$\|I_5\|_{L^s_\mu(\tilde\Omega_j)} + \|I_5\|_{L^2_\mu(\tilde\Omega_j)} = o(1)\,. \tag{6.23}$$

Then

$$\begin{aligned}
&\|I_6\|_{L^s_\mu(\tilde\Omega_j)} + \|I_6\|_{L^2_\mu(\tilde\Omega_j)} \\
&\leqq C\|f'(w)\tilde V_0\|_{L^s_\mu\left(\tilde\Omega_j\cap B^c_{(1-\delta_2)\rho_j}\right)} + C\|f'(w)\tilde V_0\|_{L^2_\mu\left(\tilde\Omega_j\cap B^c_{(1-\delta_2)\rho_j}\right)} \\
&\leqq C\rho_j^{\frac{n}{2}} e^{((1-\sigma)+\sigma_2-\mu)\rho_j} = o(1)\,,
\end{aligned} \tag{6.24}$$

$$\begin{aligned}
&\|I_7\|_{L^s_\mu(\tilde\Omega_j)} + \|I_7\|_{L^2_\mu(\tilde\Omega_j)} + \|I_8\|_{L^s_\mu(\tilde\Omega_j)} + \|I_8\|_{L^2_\mu(\tilde\Omega_j)} \\
&\leqq (C/\rho_j)\left(\|\phi_0\|_{W^{2,s}_\mu(\mathbb{R}^n)} + \|\phi_0\|_{W^{2,2}_\mu(\mathbb{R}^n)}\right) = o(1)\,.
\end{aligned} \tag{6.25}$$

The same argument leading to (6.20) and (6.21) yields

$$\left\|f'(w)\left(\phi_j - \overline{\phi}_j\right)\right\|^s_{L^s_\mu(\tilde\Omega_j)} \leqq Ce^{-s\sigma R} + C\int_{\tilde\Omega_j\cap B_R} \left|\phi_j - \overline{\phi}_j\right|^s\,, \tag{6.26}$$

$$\left\|f'(w)\left(\phi_j - \overline{\phi}_j\right)\right\|^2_{L^2_\mu(\tilde\Omega_j)} \leqq CR^{\frac{n(s-2)}{s}} e^{-2\sigma R} + C\int_{\tilde\Omega_j\cap B_R} |\phi_j - \overline{\phi}_j|^2 \tag{6.27}$$

where $R \geqq 1$ is an arbitrary number and C is independent of R. By part (i) of Lemma 6.4

$$\begin{aligned}
\|\phi_j - \overline{\phi}_j\|_{W^{2,s}_\mu(\tilde\Omega_j)} \leqq\ & C\left\|f'(w)(\phi_j - \overline{\phi}_j)\right\|_{L^s_\mu(\tilde\Omega_j)} \\
&+ C\|f'(w)(\phi_j - \overline{\phi}_j)\|_{L^2_\mu(\tilde\Omega_j)} \\
&+ C\sum_{k=5}^{8}\left(\|I_k\|_{L^s_\mu(\tilde\Omega_j)} + \|I_k\|_{L^2_\mu(\tilde\Omega_j)}\right).
\end{aligned}$$

Putting (6.23)–(6.27) together, we have

$$\overline{\lim_{j\to\infty}}\left\|\phi_j - \overline{\phi}_j\right\|_{W^{2,s}_\mu(\tilde\Omega_j)} \leqq Ce^{-\sigma R} + CR^{\frac{n(s-2)}{2s}} e^{-\sigma R}$$

since we have already proved $\phi_j \to \phi_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Our assertion now follows easily by letting $R \to \infty$.

7. Appendices

7.1. Appendix A: Proof of Lemma 4.3

In this appendix, we prove Lemma 4.3; some of the proof will overlap that of Theorem 5.2 and Theorem 6.1 of [13].

We will give a self-contained proof, and it is divided into two steps:

Step 1. We prove that $\psi_0(x) = \inf_{P\in\partial\Omega}(|P-Q| + L(P,x))$ is the maximum element of $\mathcal{S}$ which is the set of all functions $v \in W^{1,\infty}(\Omega)$ satisfying $v(x) \leqq |x-Q|$ on $\partial\Omega$ and $|\nabla v| \leqq 1$ almost everywhere in Ω.

Step 2. We prove that for any sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k_l} \to 0$ such that $\psi^{\varepsilon_{k_l}} \to \psi_0$ uniformly in $\overline{\Omega}$ as $\varepsilon_{k_l} \to 0$. Then it follows that $\psi^{\varepsilon} \to \psi_0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$.

We first prove Step 1. To begin we show that $\psi_0 \in \mathcal{S}$. In fact, since $L(x,y)$ is the length of the shortest path in $\overline{\Omega}$ connecting x and y, we see that $|L(x,y)-L(\overline{x},y)| \leqq L(x,\overline{x})$. Therefore $|\psi_0(x) - \psi_0(\overline{x})| \leqq L(x,\overline{x})$. Moreover, when $x \in \Omega$ and $\overline{x} \in \Omega$ are close, it is easy to see that $L(x,\overline{x}) = |x-\overline{x}|$ and $|\psi_0(x) - \psi_0(\overline{x})| \leqq |x-\overline{x}|$. Hence $\psi_0 \in W^{1,\infty}(\Omega)$ and $|\nabla\psi_0| \leqq 1$ almost everywhere in Ω. It is easy to see that $\psi_0(x) = |x-Q|$ on $\partial\Omega$ since $|x-Q| - |y-Q| \leqq L(x,y)$ for $x, y \in \partial\Omega$. We next show that ψ_0 is the maximum element of $\mathcal{S}$. In fact, let $v \in \mathcal{S}$. Since Ω is smooth, we can extend v in the following way: for h_0 small enough, there exists $\tilde{v} \in W^{1,\infty}(\Omega^{h_0})$ such that $\tilde{v} = v$ in Ω and $|\nabla\tilde{v}| \leqq \tilde{k}$, almost everywhere in Ω^{h_0}, where $\Omega^{h_0} := \overline{\Omega}\cup\{x \in \mathbb{R}^n\backslash\overline{\Omega} \mid d(x,\partial\Omega) < h_0\}$, $\tilde{k} \in C(\overline{\Omega^{h_0}})$ and $\tilde{k} \equiv 1$ on $\overline{\Omega}$. Indeed, if h_0 is small enough, each point x in $\overline{\Omega^{h_0}}\backslash\overline{\Omega}$ is uniquely determined by the equation: $x = z + h\nu(z)$ where $z \in \partial\Omega, h > 0$ and $\nu(z)$ is the unit outer normal to $\partial\Omega$ at the point z. In addition, the map $x \to (z,h)$ is C^1 diffeomorphism on $\overline{\Omega^{h_0}}\backslash\overline{\Omega}$. Then, we set $\tilde{v}(x) = v(z)$ and $\tilde{k}(x) = 1 + Ch$ for some large constant $C > 0$ (independent of h). Next we regularize v in the following way: for α small enough, we may define $v_\alpha = \tilde{v} * \rho_\alpha$, where $\rho_\alpha = \alpha^{-n}\rho(\cdot/\alpha), \rho \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\rho \subset B_1, \int_{\mathbb{R}^n}\rho(y)\,dy = 1$. Then we have

$$|\nabla v_\alpha|^2 \leqq (|\nabla\tilde{v}|^2) * \rho_\alpha \leqq \tilde{k}^2 * \rho_\alpha \leqq 1 + C\alpha \tag{7.1}$$

on $\overline{\Omega}$ and $v_\alpha \to v$ in $C(\overline{\Omega})$ as $\alpha \to 0$. Let now $x, y \in \overline{\Omega}$ and for every $\eta > 0$, let ξ, T_0 be such that $\xi(0) = x$, $\xi(T_0) = y$, $|\frac{d\xi}{dt}| \leqq 1$ almost everywhere in $[0, T_0]$, $\xi(t) \in \overline{\Omega}$ for all $t \in [0, T_0]$ and $T_0 \leqq L(x,y)+\eta$. Since Ω is smooth, it is clear that there exist x_α, y_α, ξ_α, T_α such that $\xi_\alpha(0) = x_\alpha$, $\xi(T_\alpha) = y_\alpha$, $|\frac{d\xi_\alpha}{dt}| \leqq 1+C\alpha$ in $[0,T_\alpha]$, $\xi_\alpha \in C^1([0,T_\alpha], \overline{\Omega^{h_0}})$ and $\xi_\alpha \to \xi, T_\alpha \to T_0$ as $\alpha \to 0$. (For instance, we can take ξ_α to be the regularization of ξ.) Now we have

$$\begin{aligned} |v_\alpha(y_\alpha) - v_\alpha(x_\alpha)| &= \left|\int_0^{T_\alpha} \nabla v_\alpha(\xi_\alpha(t)) \cdot \frac{d\xi_\alpha}{dt}(t)\,dt\right| \\ &\leqq \int_0^{T_\alpha} (1 + C\alpha)^2\,dt\,. \end{aligned} \tag{7.2}$$

Letting $\alpha \to 0$ and then $\eta \to 0$, we obtain $|v(y) - v(x)| \leqq L(x,y)$. Hence $v(x) \leqq v(y) + L(x,y) \leqq |y-Q| + L(x,y)$ for all $y \in \partial\Omega$. So $v \leqq \psi_0$.

We next prove Step 2. By Lemma 4.2 and the Arzela-Ascoli theorem, for any sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k_l} \to 0$ such that $\psi^{\varepsilon_{k_l}} \to \psi^0$ uniformly in $\overline{\Omega}$ as $\varepsilon_{k_l} \to 0$. We next show that $\psi^0 = \psi_0$. First we observe that $\psi^0 \in \mathcal{S}$. In fact, by taking limits in the sense of distributions we obtain $|\nabla\psi^0|^2 \leqq 1$ in $\mathcal{D}'(\Omega)$. Thus $\psi^0 \in W^{1,\infty}(\Omega)$, $|\nabla\psi^0| \leqq 1$ almost everywhere in Ω and $\psi^0(x) = |x - Q|$ on $\partial\Omega$. Hence $\psi^0 \leqq \psi_0$. On the other hand, let $v \in \mathcal{S}$. Similarly to Step 1, we extend v to $\tilde{v}$ in Ω^{h_0} and regularize $\tilde{v}$ to v_α in such a way that we have $\|v - v_\alpha\|_{L^\infty(\overline{\Omega})} \leqq C\alpha$ and $|\nabla\tilde{v}| \leqq \tilde{k}$. Hence as before we have

$$|\nabla v_\alpha|^2 \leqq (|\nabla\tilde{v}|^2) * \rho_\alpha \leqq \tilde{k}^2 * \rho_\alpha \leqq 1 + C\alpha \tag{7.3}$$

and $v_\alpha \to v$ in $C(\overline{\Omega})$ as $\alpha \to 0$.

In addition we have

$$\begin{cases} \varepsilon\Delta v_\alpha - |\nabla v_\alpha|^2 + 1 + C\alpha + A_\alpha\varepsilon \geqq 0 & \text{in } \Omega\,, \\ v_\alpha(x) \leqq |x - Q| + C\alpha & \text{on } \partial\Omega\,, \end{cases} \tag{7.4}$$

where $A_\alpha \geqq 0$. Let now

$$\tilde{v}_\alpha := \frac{v_\alpha}{\sqrt{1 + C\alpha + A_\alpha\varepsilon}}\,.$$

Then by comparison we deduce that

$$\tilde{v}_\alpha \leqq \psi^{\varepsilon/(\sqrt{1+C\alpha+A_\alpha\varepsilon})} + C\alpha\,. \tag{7.5}$$

Choosing $\varepsilon = \varepsilon'_{k_l}$ in (7.5) such that

$$\varepsilon_{k_l} = \frac{\varepsilon'_{k_l}}{\sqrt{1 + C\alpha + A_\alpha\varepsilon'_{k_l}}}\,,$$

we see that

$$\frac{v_\alpha}{\sqrt{1 + C\alpha}} \leqq \psi^0 + C\alpha$$

as $\varepsilon'_{k_l} \to 0$. Then, letting $\alpha \to 0$, we obtain $v \leqq \psi^0$. In particular, we have $\psi_0 \leqq \psi^0$. Hence $\psi^0(x) = \psi_0(x)$.

7.2. Appendix B: Proof of Lemma 6.4

In this appendix, we prove Lemma 6.4.

In fact, part (ii) follows easily from part (i) by truncation. A proof of (iii) can be found in Lemma 4.2 (2) of [17]. So we just need to prove (i). Our proof will follow closely to that of Lemma 1.1 in [8]. *Therefore, for the convenience of the reader, we will try to use the same notation as that of* [8], *and, it should be noted that to a large extent our notation in this appendix is independent of that of the rest of this paper.* The point here is to find out the restrictions on μ. More precisely, we will show that $\mu_0 = 1$ in our case where μ_0 is defined in Lemma 1.1 of [8]. We will use slightly different norms, but equivalent to those of [8]. We now replace $\tilde{\Omega}_\varepsilon$ by Ω_ε and change $\overline{f}$ to f. We will also use $W^{2,p}(\Omega_\varepsilon)$ instead of $W^{2,s}(\Omega_\varepsilon)$.

First we claim that: there exist constants δ_0 and C^* (independent of $\varepsilon \leqq \varepsilon_0$), such that for each $y \in \partial\Omega_\varepsilon$, the set $\partial\Omega_\varepsilon \cap \{x : |x-y| < \delta_0\}$ can be represented in the form

$$x_i - y_i = \Phi(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

for some $i, 1 \leqq i \leqq n$ and

$$|\Phi| + \sum \left| \frac{\partial \Phi}{\partial x_j} \right| + \sum \left| \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right| \leqq C^* . \tag{7.6}$$

In fact, at each point $y \in \partial\Omega_\varepsilon$, we can just use the transformation given in Section 3. Then we introduce a mesh in $\mathbb{R}^n$ made up of cubes with sides parallel to the coordinate axes and having length $\eta = \delta_0/(3\sqrt{n})$. Denote by $\Gamma_1, \dots, \Gamma_{h_0}$ those cubes whose closure intersects $\partial\Omega_\varepsilon$. Denote the center of Γ_j by y_j. Let Γ'_j, Γ''_j be cubes with center y_j and with sides parallel to the coordinate axes, having length 2η and 3η repectively. Then $\Gamma'_1, \dots, \Gamma'_{h_0}$ form an open covering of $\partial\Omega_\varepsilon$. Further, for any $y \in \partial\Omega_\varepsilon$ there is a cube Γ'_j such that $y \in \Gamma_j$ and dist. $(y, \partial\Gamma'_j) \geqq \delta_0/2$.

Let Ψ be a C^∞ function such that

$$\begin{array}{rclll} \Psi(x) & = & 1 & \text{if } |x_i| < \eta & \text{for all } i = 1, 2, \dots, n , \\ \Psi(x) & = & 0 & \text{if } |x_i| > \frac{5}{4}\eta & \text{for some } i , \\ 0 & \leqq & \Psi & \leqq 1 & \text{elsewhere} , \end{array}$$

and set $\Psi_j(x) = \Psi(y_j + x)$. Then $\Psi_j = 1$ in Γ'_j and $\Psi_j = 0$ in a small neighborhood of $\partial\Gamma''_j$ and outside Γ''_j.

Denote by $\Omega_{\varepsilon,\eta}$ the set of all points in Ω_ε whose distance to $\partial\Omega_\varepsilon$ is $\geqq \eta/2$. We now introduce a mesh made up of cubes with sides parallel to the coordinate axes and having length $\eta_0 = \delta_0/(8\sqrt{n})$. Denote by $\Delta_1, \dots, \Delta_{h_1}$ those cubes whoses closure intersects Ω_ε. Let Δ'_j, Δ''_j be the cubes with the same center z_j as Δ_j and with sides parallel to the coordinate axes, having length $2\eta_0$ and $3\eta_0$ repectively. The cubes $\Delta'_1, \dots, \Delta'_{h_1}$ form an open covering of $\overline{\Omega}_{\varepsilon,\eta}$ and the cubes $\Delta''_1, \dots, \Delta''_{h_1}$ lie entirely in Ω_ε.

Let χ be the C^∞ function

$$\chi(x) = \Psi\left(\frac{\eta}{\eta_0} x\right).$$

and let $\chi_j(x) = \chi(z_j + x)$. Let

$$\begin{array}{rcll} \varphi_j & = & \frac{\Psi_j}{\sum \Psi_k + \sum \chi_k} & \text{if } 1 \leqq j \leqq h_0 , \\ \varphi_{j+h_0} & = & \frac{\chi_j}{\sum \Psi_k + \sum \chi_k} & \text{if } 1 \leqq j \leqq h_1 , \\ G_j & = & \Gamma''_j,\ G'_j = \Gamma'_j & \text{if } 1 \leqq j \leqq h_0 , \\ G_{j+h_0} & = & \Delta''_j,\ G'_{j+h_0} = \Delta'_j & \text{if } 1 \leqq j \leqq h_1 , \end{array}$$

and let $h = h_0 + h_1$. Then $G_1, \dots, G_h$ form an open covering of $\overline{\Omega}_\varepsilon$ and $\varphi_1, \dots, \varphi_h$ form a partition of unity subordinate to this covering, such that

(a) $G_1, \dots, G_{h_0}$ intersects $\partial\Omega_\varepsilon$, and $G_{h_0+1}, \dots, G_h$ lie entirely in Ω_ε;

(b) $\varphi_k \in C_0^\infty(G_k)$;

(c) each $x \in \overline{\Omega_\varepsilon}$ belongs to at most N_1 sets G_k, where N_1 is a positive integer independent of $\varepsilon \leqq \varepsilon_0$;

(d) $\varphi_k \geqq 1/N_1$ on the set G_k', the sets $G_1', \dots, G_h'$ form an open covering of $\overline{\Omega_\varepsilon}$;

(e) there is a constant N_2 independent of k, ε, such that

$$|D^\alpha \varphi_k| \leqq N_2 \quad \text{if} \quad |\alpha| \leqq 2\,, \quad x \in G_k\,, \quad 1 \leqq k \leqq h\,. \tag{7.7}$$

Let

$$G_{k,\varepsilon} = G_k \cap \Omega_\varepsilon$$

and

$$Au = -\Delta u + u\,.$$

Note that φ_k has compact support in G_k. By standard regularity theorem, we have

$$\|u\varphi_k\|_{W^{2,p}(G_{k,\varepsilon})} \leqq C\left(\|A(u\varphi_k)\|_{L^p(G_{k,\varepsilon})} + \|u\varphi_k\|_{L^p(G_{k,\varepsilon})}\right) \tag{7.8}$$

where C is a constant independent of $k, \varepsilon \leqq \varepsilon_0$. But we note that

$$A(u\varphi_k) = f\varphi_k - 2\nabla u \nabla\varphi_k - u\Delta\varphi_k\,.$$

Hence

$$|A(u\varphi_k)|^p \leqq C|f|^p + C|Du|^p + C|u|^p\,. \tag{7.9}$$

By (7.8), we have

$$\begin{aligned}\int_{G_{k,\varepsilon}} (|D^2 u|\varphi_k)^p\,dx \quad &\leqq \quad C\int_{G_{k,\varepsilon}} (|Du|\,|D\varphi_k|)^p\,dx + C\int_{G_{k,\varepsilon}} (|u|\,|D^2\varphi_k|)^p\,dx \\ &\quad + C\int_{G_{k,\varepsilon}} (A(u\varphi_k))^p\,dx + C\int_{G_{k,\varepsilon}} (|u\varphi_k|)^p\,dx\,.\end{aligned}$$

Multiplying both sides by $\exp(-p\mu|z_k|)$, where z_k is the center of the cube G_k, and noting that

$$Ce^{-p\mu|x|} \leqq e^{-p\mu|z_k|} \leqq Ce^{-p\mu|x|} \qquad \text{if } x \in G_k\,,$$

we obtain, by using (7.8) and (7.9),

$$\begin{aligned}\int_{G_{k,\varepsilon}'} (e^{-\mu|x|}|D^2 u|\varphi_k)^p\,dx \quad &\leqq \quad C\int_{G_{k,\varepsilon}} (e^{-\mu|x|}|f|)^p\,dx \\ &\quad + \int_{G_{k,\varepsilon}} (e^{-\mu|x|}|Du|)^p\,dx + \int_{G_{k,\varepsilon}} (e^{-\mu|x|}|u|)^p\,dx\,.\end{aligned}$$

Summing for $k = 1, \dots, h$, we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} (e^{-\mu|x|}|D^2u|)^p\,dx \;&\leqq\; C\int_{\Omega_\varepsilon} (e^{-\mu|x|}|f|)^p\,dx \\ &\quad + \int_{\Omega_\varepsilon}(e^{-\mu|x|}|Du|)^p\,dx + \int_{\Omega_\varepsilon}(e^{-\mu|x|}|u|)^p\,dx\,. \end{aligned} \tag{7.10}$$

Now we derive the so-called Garding's inequality in Ω_ε. The space $L^2_\mu(G)$ is a real Hilbert space with the scalar product

$$(u,v)_{\mu,G} = \int_G e^{-2\mu\langle x\rangle}u(x)v(x)\,dx\,, \qquad \text{where } \langle x\rangle = (1+|x|^2)^{\frac{1}{2}}\,.$$

When $G = \Omega_\varepsilon$, we write $(u,v)_{\mu,G} = (u,v)_{\mu,\varepsilon}$. Then,

$$\begin{aligned}(Au,u)_{\mu,\varepsilon} &= \int_{\Omega_\varepsilon} e^{-2\mu\langle x\rangle}(-\Delta u + u)u\,dx \\ &= \int_{\Omega_\varepsilon} e^{-2\mu\langle x\rangle}(-\Delta u)u\,dx + (u,u)_{\mu,\varepsilon} \\ &= -2\mu\int_{\Omega_\varepsilon} e^{-2\mu\langle x\rangle}u\nabla u\cdot\frac{x}{\langle x\rangle} + \sum_{i=1}^n\left(\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial x_i}\right)_{\mu,\varepsilon} + (u,u)_{\mu,\varepsilon} \\ &\geqq (1-\mu)\left[\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial x_i}\right)_{\mu,\varepsilon} + (u,u)_{\mu,\varepsilon}\right]\end{aligned}$$

when $0 < \mu < 1$.

We have thus proved that

$$\int_{\Omega_\varepsilon} e^{-2\mu\langle x\rangle}Au(x)u(x)\,dx \geqq C\|u\|^2_{W^{1,2}_\mu(\Omega_\varepsilon)}\,. \tag{7.11}$$

Therefore,

$$\|u\|_{W^{1,2}_\mu(\Omega_\varepsilon)} \leqq C\|f\|_{L^2_\mu(\Omega_\varepsilon)}\,, \tag{7.12}$$

and for $p = 2$, (i) follows from (7.10) and (7.12).

By using (7.12) and a variant of Sobolev's inequalities (see [8]), the rest of the proof is exactly the same as that of Lemma 1.1 in [8].

7.3. Appendix C: Proof of Lemma 6.5

In this appendix, we prove Lemma 6.5.

Let $\varphi \in \operatorname{Ker}(L) \cap W^{2,s}_\mu(\mathbb{R}^n)$, then $\|\varphi\|_{W^{2,s}_\mu} \leqq C$. Hence

$$\varphi(y) \leqq Ce^{\mu<y>}\,. \tag{7.13}$$

Moreover, by elliptic regularity theorems, $\varphi \in C^\infty(\mathbb{R}^n)$. By part (ii) of Lemma 6.4, we have

$$\varphi(y) = \int_{\mathbb{R}^n} G(y-z) f'(w)(z)\varphi(z)\,dz \tag{7.14}$$

where $G(y-z)$ is the Green's function for $-\Delta + 1$ and $0 < G(y-z) < \frac{C_n}{|y-z|^{n-2}}(1 + |y-z|)^{(n-3)/2} e^{-|y-z|}$. Setting $\mu_1 = \mu - \sigma$, we see that $\mu_1 < \mu < 1$. If $\mu_1 \leqq 0$, then we have

$$\varphi(y) \leqq C \int_{\mathbb{R}^n} \frac{(1 + |y-z|)^{(n-3)/2}}{|y-z|^{n-2}} e^{-|y-z|} e^{-\sigma|z|} e^{\mu|z|}\,dz \leqq C\,.$$

Substituting this into (7.14) we obtain

$$\begin{aligned}
&e^{\sigma'|y|}\varphi(y)\\
\leqq\ & C \int_{\mathbb{R}^n} \frac{(1 + |y-z|)^{(n-3)/2}}{|y-z|^{n-2}} e^{-|y-z|} e^{-\sigma|z|} e^{\sigma'|y|}\,dz\\
\leqq\ & C \int_{\mathbb{R}^n} \frac{(1 + |y-z|)^{(n-3)/2}}{|y-z|^{n-2}} e^{-(1-\sigma')|y-z|} e^{\sigma'(|y|-|z|-|y-z|)} e^{(\sigma'-\sigma)|z|}\,dz\\
\leqq\ & C
\end{aligned}$$

for $y \in \mathbb{R}^n$ and $0 < \sigma' < \sigma$, i.e., φ decays exponentially.

In case $\mu_1 \geqq 0$, we observe that

$$\begin{aligned}
&e^{-\mu_1|y|}\varphi(y)\\
\leqq\ & C \int_{\mathbb{R}^n} \frac{(1 + |y-z|)^{(n-3)/2}}{|y-z|^{n-2}} e^{-|y-z|} e^{-\sigma|z|} e^{\mu|z|} e^{-\mu_1|y|}\,dz\\
=\ & C \int_{\mathbb{R}^n} \frac{(1 + |y-z|)^{(n-3)/2}}{|y-z|^{n-2}} e^{-(1-\mu_1)|y-z|} e^{\mu_1(|z|-|y|-|y-z|)}\,dz\,.
\end{aligned}$$

Therefore

$$\varphi(y) \leqq Ce^{\mu_1|y|}\,. \tag{7.15}$$

Now let k be the largest integer such that $\mu_k := \mu - k\sigma > 0$. Then iterating the above argument leading to (7.15), we arrive at

$$\varphi(y) \leqq Ce^{\mu_k|y|} \tag{7.16}$$

for $y \in \mathbb{R}^n$. Substituting (7.16) into (7.14), we obtain $\varphi(y) \leqq C$ for all $y \in \mathbb{R}^n$, which implies as before that φ decays exponentially.

Once φ decays exponentially, standard elliptic regularity estimates guarantee that $\varphi(x) \in W^{2,s}(\mathbb{R}^n)$, for all $s > 1$. By Lemma 4.2 in [15], we finish the proof.

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Bibliography

[1] Adams, R., *Sobolev Spaces,* Academic Press, New York, 1975.
[2] Amann, H., and Crandall, M. G., *On some existence theorems for semilinear elliptic equations,* Indiana Univ. Math. J. 27, 1978, pp. 779–790.
[3] Ambrosetti, A., and Rabinowitz, P. H., *Dual variational methods in critical point theory and applications,* J. Funct. Anal. 14, 1973, pp. 349–381.
[4] Benci, V., and Cerami, G., *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems,* Arch. Rational Mech. Anal. 114, 1991, pp. 79–94.
[5] Caffarelli, L. A., and Littman, W., *Representation formulas for solutions to* $\Delta u - u = 0$ *in* $\mathbb{R}^n$, pp. 249–263 in: *Studies in Partial Differential Equations,* W. Littman, ed., Studies in Mathematics No. 23, Mathematical Association of America, 1982.
[6] Chen, C.-C., and Lin, C.-S., *Uniqueness of the ground state solution of* $\Delta u + f(u) = 0$, Comm. Partial Differential Equations 16, 1991, pp. 1549–1572.
[7] Esteban, M. J., and Lions, P. L., *Existence and non-existence results for semilinear elliptic problems in unbounded domains,* Proc. Roy. Soc. Edinburgh Sect. A 93, 1982, pp. 1–14.
[8] Friedman, A., *Regularity theorems for variational inequalities in unbounded domains and applications to stopping time problems,* Arch. Rational Mech. Anal. 52, 1973, pp. 134–160.
[9] Gidas, B., Ni, W.-M., and Nirenberg, L., *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, pp. 369–402 in: *Mathematical Analysis and Applications,* Part A, L. Nachbin, ed., Adv. Math. Suppl. Studies 7A, Academic Press, New York, 1981.
[10] Gilbarg, D., and Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order,* 2nd ed., Springer-Verlag, Berlin, 1983.
[11] Kwong, M. K., and Zhang, L., *Uniqueness of positive solutions of* $\Delta u + f(u) = 0$ *in an annulus,* Differential Integral Equations 4, 1991, pp. 583–599.
[12] Lin, C.-S., Ni, W.-M., and Takagi, I., *Large amplitude stationary solutions to a chemotaxis system,* J. Differential Equations 72, 1988, pp. 1–27.
[13] Lions, P. L., *Generalized Solutions of Hamilton-Jacobi Equations,* Pitman Research Notes Series No. 69, Pitman, London, 1982.
[14] Ni, W.-M., and Takagi, I., *On the shape of least energy solutions to a semilinear Neumann problem,* Comm. Pure Appl. Math. 44, 1991, pp. 819–851.
[15] Ni, W.-M., and Takagi, I., *Locating the peaks of least energy solutions to a semilinear Neumann problem,* Duke Math. J. 70, 1993, pp. 247–281.
[16] Wang, Z. Q., *On the existence of multiple single-peaked solutions for a semilinear Neumann problem,* Arch. Rational Mech. Anal. 120, 1992, pp. 375–399.
[17] Wei, J., *Locating the blow-up point of a semilinear Dirichlet problem involving critical Sobolev exponent,* preprint.

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