DESINGULARIZATION OF CLIFFORD TORUS AND NONRADIAL SOLUTIONS TO YAMABE PROBLEM WITH MAXIMAL RANK

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ABSTRACT. Through desingularization of Clifford torus, we prove the existence of a sequence of nondegenerate (in the sense of Duyckaerts-Kenig-Merle ([?])) nodal nonradial solutions to the critical Yamabe problem

$$-\Delta u = \frac{n(n-2)}{4}|u|^{\frac{4}{n-2}}u, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^n).$$

The case $n = 4$ is the first example in the literature of a solution with *maximal rank* $\mathcal{N} = 2n + 1 + \frac{n(n-1)}{2}$.

Introduction

Consider the problem

$$-\Delta u = \gamma |u|^{p-1}u \text{ in } \mathbb{R}^n, \qquad \gamma := \frac{n(n-2)}{4}, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \tag{0.0.1}$$

where $n \geqslant 4$, $p = \frac{n+2}{n-2}$ and $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with the norm $\|\nabla u\|_{L^2(\mathbb{R}^n)}$.

When $u > 0$, problem (**??**) arises in the classical Yamabe problem or extremal equation for Sobolev inequality. For positive or sign-changing u Problem (**??**) corresponds to the steady state of the energy-critical focusing nonlinear wave equation

$$\partial_t^2 u - \Delta u - |u|^{\frac{4}{n-2}}u = 0, \ (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{0.0.2}$$

These are classical problems that have attracted the attention of many researchers ([**?, ?, ?, ?, ?, ?**]). The study of (**??**) naturally relies on complete classification of the set of non-zero finite energy solutions to Problem (**??**) which is defined by

$$\Sigma := \left\{ Q \in \mathcal{D}^{1,2}(\mathbb{R}^n) \backslash \{0\} : \ -\Delta Q = \frac{n(n-2)}{4}|Q|^{\frac{4}{n-2}}Q \right\}. \tag{0.0.3}$$

By the classical work of Caffarelli-Gidas-Spruck [**?**] all positive solutions to (**??**) are given by

$$U(y) = \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}}, \tag{0.0.4}$$

and all its translations and dilations

$$U_{\alpha,\bar{y}} := \alpha^{-\frac{n-2}{2}} U\left(\frac{y - \bar{y}}{\alpha}\right), \qquad \alpha > 0, \ \bar{y} \in \mathbb{R}^n. \tag{0.0.5}$$

For sign-changing solutions much less is known. A direct application of Pohozaev's identity gives that all sign-changing solutions to Problem (**??**) are non-radial. The existence of elements of Σ that are nonradial sign-changing, and with arbitrary large energy was first proved by Ding [**?**] using Ljusternik-Schnirelman category theory. However no other qualitative properties are known for Ding's solutions. Recently more explicit constructions of sign changing solutions to Problem (**??**) have been obtained by del Pino-Musso-Pacard-Pistoia [**?, ?**]. In [**?**], the second and the third authors established the rigidity of the solutions constructed in [**?**] by showing that they are *nondegenerate*, (in the sense of Duyckaerts-Kenig-Merle ([**?**])). (See definitions below.)

The purpose of this work is to give a positive answer to an open question formulated in the work of M. Musso and J. Wei ([**?**]): whether there exists a solution that, apart from nondegenerate, is *maximal*. To properly explain this framework, let us denote by

$$\Sigma := \left\{ Q \in \mathcal{D}^{1,2}(\mathbb{R}^n) \setminus \{0\} : -\Delta Q = \gamma |Q|^{p-1} Q \right\}$$

the set of non trivial finite energy solutions of (**??**). It can be seen that the equation in (**??**) is invariant under four transformations: translation, dilation, orthogonal transformation and Kelvin transform. More precisely, if $Q \in \Sigma$, then:

   (i) $Q(y + a) \in \Sigma$ for every $a \in \mathbb{R}^n$;
   (ii) $\lambda^{\frac{n-2}{2}} Q(\lambda y) \in \Sigma$ for every $\lambda > 0$;
   (iii) $Q(Py) \in \Sigma$ for every $P \in \mathcal{O}_n$, where $\mathcal{O}_n$ denotes the classical orthogonal group;
   (iv) $|y|^{2-n} Q(|y|^{-2} y) \in \Sigma$.

Denote by $\mathcal{M}$ the group of isometries of $\mathcal{D}^{1,2}(\mathbb{R}^n)$ generated by these transformations. Then, $\mathcal{M}$ generates a family of transformations in a neighborhood of the identity (see [**?**, Lemma 3.8]) of dimension

$$\mathcal{N} := 2n + 1 + \frac{n(n-1)}{2}. \tag{0.0.6}$$

In particular, $\mathcal{M}$ generates the vector space

$$\tilde{\mathcal{J}}_Q = \text{span} \left\{ \begin{array}{l} (2 - n) y_\alpha Q + |y|^2 \partial_{y_\alpha} Q - 2 y_\alpha y \cdot \nabla Q, \ \partial_{y_\alpha} Q, \ 1 \leqslant \alpha \leqslant n, \\ (y_\alpha \partial_{y_\beta} - y_\beta \partial_{y_\alpha}) Q, \ 1 \leqslant \alpha < \beta \leqslant n, \ \frac{n-2}{2} Q + y \cdot Q \end{array} \right\}.$$

Consider the associated linearized operator around $Q \in \Sigma$, i.e.,

$$L_Q := -\Delta - \gamma p |Q|^{p-2} Q,$$

and its kernel

$$\mathcal{J}_Q := \{ f \in \mathcal{D}^{1,2}(\mathbb{R}^n) : L_Q f = 0 \}.$$

Clearly $\tilde{\mathcal{J}}_Q \subseteq \mathcal{J}_Q$. Indeed, following the work of T. Duyckaerts, C. Kenig and F. Merle ([**?**]), we can define the notion of *nondegeneracy*.

Definition 0.0.1. $Q \in \Sigma$ is said to be *nondegenerate* if $\mathcal{J}_Q = \tilde{\mathcal{J}}_Q$.

Let Q be nondegenerate. Its *rank* is defined as the dimension of $\tilde{\mathcal{J}}_Q$, which is at most $\mathcal{N}$. Actually, for positive solutions $Q = U$ can be proved to be nondegenerate as a consequence of the radial symmetry, and $\tilde{\mathcal{J}}_W$, which is

$$\tilde{\mathcal{J}}_W = \left\{ \frac{n-2}{2} W + y \cdot \nabla W, \ \partial_{y_\alpha} W, \ 1 \leqslant \alpha \leqslant n \right\},$$

has rank $n + 1$ ([**?**]). In this case, the rank is strictly less than $\mathcal{N}$.

In [?], the authors give the first example of nodal nonradial sign-changing solution satisfying the nondegeneracy condition. Indeed, they consider the solution u_k of (??) built in [?, Theorem 1] given by

$$u_k(y) = U(y) - \sum_{j=1}^{k} \mu_k^{-\frac{n-2}{2}} U(\mu_k^{-1}(y - \xi_j)) + o(1),$$

where

$$\mu_k := \frac{c_n}{k^2}, \qquad \xi_j := (e^{\frac{2j\pi i}{k}}, 0, \ldots), \qquad U(y) := \left(\frac{2}{1 + |y|^2}\right)^{\frac{n-2}{2}},$$

and they prove that $\tilde{\mathfrak{J}}_{u_k} = \mathfrak{J}_{u_k}$, where the dimension of these vector spaces is $3n - 1$, i.e. the rank is $3n - 1$. Also in this case, the rank is strictly less than $\mathcal{N}$.

The purpose of this work is to provide the first example in the literature of a nondegenerate solution u to (??) which has the *maximal rank* $\mathcal{N}$.

Definition 0.0.2. A nondegenerate solution $Q \in \Sigma$ is said to be *maximal* if

$$\dim(\tilde{\mathfrak{J}}_Q) = \dim(\mathfrak{J}_Q) = \mathcal{N},$$

where $\mathcal{N}$ was defined in (??).

Thus, our main result can be formulated as follows.

Theorem 0.0.3. *Let $n \geqslant 4$. Then, there exists a sequence of nodal solutions to (??), with arbitrarily large energy, which are nondegenerate according to Definition ??. If $n = 4$ these solutions are maximal in the sense of Definition ??.*

To prove this result, we will build a solution in the following way: let k and h be two large positive integers (not necessarily equal) , and

$$\mu := \frac{\delta^{\frac{2}{n-2}}}{k^2}, \qquad \lambda := \frac{\varepsilon^{\frac{2}{n-2}}}{h^2}, \tag{0.0.7}$$

where δ and ε are positive parameters so that

$$c_1 < \delta < c_1^{-1}, \qquad c_2 < \varepsilon < c_2^{-1},$$

for some constants $c_1, c_2 > 0$ which are independent of k and h as they tend to infinity. Consider now the points

$$\xi_j := \sqrt{1 - \mu^2}(e^{\frac{2\pi i(j-1)}{k}}, 0, \ldots, 0) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}, j = 1, \ldots, k,$$
$$\eta_l := \sqrt{1 - \lambda^2}(0, 0, e^{\frac{2\pi i(l-1)}{h}}, 0, \ldots, 0) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{n-4}, l = 1, \ldots, h, \tag{0.0.8}$$

which clearly satisfy

$$|\xi_j|^2 + \mu^2 = 1, \qquad |\eta_l|^2 + \lambda^2 = 1. \tag{0.0.9}$$

Consider

$$u(y) = U(y) - \sum_{j=1}^{k} U_{\mu,\xi_j}(y) - \sum_{l=1}^{h} U_{\lambda,\eta_l}(y) + \phi(y) \tag{0.0.10}$$

where U is defined in (??),

$$U_{\mu,\xi_j}(y) := \mu^{-\frac{n-2}{2}} U\left(\frac{y - \xi_j}{\mu}\right), \qquad U_{\lambda,\eta_l}(y) := \lambda^{-\frac{n-2}{2}} U\left(\frac{y - \eta_l}{\lambda}\right), \tag{0.0.11}$$

and ϕ is a small function when compared with the other terms (for the sake of simplicity we do not make explicit the dependence of u in k and h).

Notice that functions U, U_{μ,ξ_j} and U_{λ,η_l} are invariant under rotation of angle $\frac{2\pi}{k}$ in the (y_1, y_2) plane and of angle $\frac{2\pi}{h}$ in the (y_3, y_4) angle. Furthermore, they are even in the y_α-coordinates, for $\alpha = 2, 4, 5, \ldots, n$ and invariant under Kelvin's transform (due to (??)). Assume that ϕ also satisfies these properties (we will prove this in Part ??).

Consider the following set of functions:

$$z_0(y) := \frac{n-2}{2} u(y) + \nabla u(y) \cdot y, \tag{0.0.12}$$

$$z_\alpha(y) := \frac{\partial}{\partial y_\alpha} u(y), \qquad \alpha = 1, \ldots, n, \tag{0.0.13}$$

$$z_{n+1}(y) := -y_2 \frac{\partial}{\partial y_1} u(y) + y_1 \frac{\partial}{\partial y_2} u(y),$$
$$z_{n+2}(y) := -y_4 \frac{\partial}{\partial y_3} u(y) + y_3 \frac{\partial}{\partial y_4} u(y), \tag{0.0.14}$$

$$z_{n+2+\alpha}(y) := -2y_1 z_0(y) + |y|^2 z_1(y), \quad \alpha = 1, 2, 3, 4, \tag{0.0.15}$$

$$z_{n+\alpha+4}(y) := -y_\alpha z_1(y) + y_1 z_\alpha(y), \qquad \alpha = 3, \ldots, n,$$
$$z_{2n+\alpha+2}(y) := -y_\alpha z_2(y) + y_2 z_\alpha(y), \qquad \alpha = 3, \ldots, n, \tag{0.0.16}$$

and

$$z_{3n+\alpha-2}(y) := -y_\alpha z_3(y) + y_3 z_\alpha(y), \qquad \alpha = 5, \ldots, n,$$
$$z_{4n+\alpha-6}(y) := -y_\alpha z_4(y) + y_4 z_\alpha(y), \qquad \alpha = 5, \ldots, n. \tag{0.0.17}$$

Functions (??) and (??) are related to the invariance of problem (??) under dilations and translations respectively, and (??) to the rotation in the (y_1, y_2) and (y_3, y_4) planes. Likewise, (??) arise from the invariance under Kelvin transform, and (??), (??) from the rotation in the planes (y_1, y_α), (y_2, y_α), for $\alpha = 3, \ldots, n$, and (y_3, y_α), (y_4, y_α) for $\alpha = 5, \ldots, n$. If we denote by L the linearized operator around u associated to (??), i.e.,

$$L(\varphi) := \Delta\varphi + p\gamma|u|^{p-2}u\varphi, \tag{0.0.18}$$

(??)-(??) provide $N_0 := 5(n-1)$ elements of the kernel of L.

We will prove that these are indeed all the elements in the kernel, i.e., solution (??) is a second example of nodal nondegenerate solution of (??). But what is more remarkable here is that if $n = 4$, then $N_0 = \mathcal{N}$, that is, the solution is maximal in the sense of Definition ??, which is the first example of a nondegenerate maximal solution in the literature, and answers the open question formulated in [?].

Remark 0.0.4. When $\mu \neq \lambda, h \neq k$, our solution is different from the ones constructed in [?, ?]. In [?] the symmetric case $\mu = \lambda, h = k$ is considered, which corresponds to the *Clifford torus*. In this case the solution has an additional symmetry which reduces the problem to one dimensional. Because of this symmetry the rank of the solutions constructed in [?] can be shown to be strictly less than $\mathcal{N}$. Thus our solutions are *new*. Our construction can be considered as a sort of *desingularization* of Clifford torus. For geometric application of desingularization of Clifford torus, we refer to recent papers [?, ?] and the references therein.

Remark 0.0.5. The construction can be extended to higher even dimensions, that is, one can set bubbles in an analogous way in the (y_5, y_6), (y_7, y_8), ..., planes, in such a way that the solution is expected to be nondegenerate and the elements corresponding to the invariances generate a space of dimension exactly $\mathcal{N}$. Therefore, this type of construction presumably provides a sequence of nodal nondegenerate and maximal rank solutions of (??) for any even $n \geqslant 4$. For odd dimensions, the existence of a maximal solution is still an open question.

Remark 0.0.6. Nondegenerate solutions to (??) play an important role in the analysis of possible singularity formations in energy-critical wave equations. We refer to [?, ?, ?, ?, ?, ?] and the references therein.

Remark 0.0.7. The existence of sign-changing solutions for critical exponents in other contexts has been studied in [?, ?, ?].

Remark 0.0.8. There have been many papers on the uniqueness and nondegeneracy of *positive solutions* to semilinear equations, whether or not for classical nonlinear Schrodinger equations [?] or for nonlinear fractional equations [?, ?]. The rank of the positive solutions is at most $n+1$. For sign-changing solutions the nondegeneracy question is in general quite difficult without knowing the precise behavior of the solution. Our result is the first of the type for sign-changing solutions with maximal rank.

Along the work we will denote points $y \in \mathbb{R}^n$, $n \geqslant 4$, as

$$y = (\overline{y}, \hat{y}), \ \overline{y} := (y_1, y_2), \ \hat{y} := (y_3, y_4), \ \text{if } n = 4,$$

$$y = (\overline{y}, \hat{y}, y'), \ \overline{y} := (y_1, y_2), \ \hat{y} := (y_3, y_4), \ y' := (y_5, \ldots, y_n), \ \text{if } n \geqslant 5,$$

and we will work with the norms

$$\|h\|_{**} := \|(1 + |y|)^{n+2-\frac{2n}{q}} h\|_{L^q(\mathbb{R}^n)}, \qquad \|\phi\|_* := \|(1 + |y|^{n-2})\phi\|_{L^\infty(\mathbb{R}^n)}, \tag{0.0.19}$$

where $\frac{n}{2} < q < n$ is a fixed number.

Part ?? of the paper is devoted to prove that (??) solves (??), and Part ?? concerns the proof of its nondegeneracy.

Part 1. **Construction of the solution**

To prove that (??) is a solution of (??) we use a Lyapunov-Schmidt reduction method, following the ideas of [?]. We linearize the equation around a first approximation and take advantage of the invertibility tools available for this setting. Then, performing a careful analysis of the error of the approximation and of the non linear terms we solve the problem by a fixed point argument. Let us point out that the precise scaling of the parameters μ and λ plays a fundamental role here.

Recalling the definitions given in (??), (??) and (??), the main result of this part can be stated as follows.

Theorem 1.0.1. *Let $n \geqslant 4$, and let k, h be positive integers so that $k = O(h)$. Then, for sufficiently large k and h there is a finite energy solution of the form*

$$u_{k,h}(x) = U(x) - \sum_{j=1}^{k} U_{\mu, \xi_j}(x) - \sum_{l=1}^{h} U_{\lambda, \eta_l}(x) + o_k(1) + o_h(1),$$

where $o_k(1)$ and $o_h(1)$ denote quantities that tend to zero when k and h tend to infinity respectively.

1.1. Error of the approximation

Denote

$$U_*(y) := U(y) - \sum_{j=1}^{k} U_{\mu,\xi_j}(y) - \sum_{l=1}^{h} U_{\lambda,\eta_l}(y),$$

and suppose that the solution u we are looking for has the form

$$u = U_* + \phi,$$

where ϕ is a small function when compared with U_*. Then solving equation (**??**) is equivalent to find ϕ such that

$$\Delta\phi + p\gamma|U_*|^{p-1}\phi + E + \gamma N(\phi) = 0, \tag{1.1.1}$$

where

$$E := \Delta U_* + \gamma|U_*|^{p-1}U_*,$$
$$N(\phi) := |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\phi.$$

In this section we try to estimate the error term E. In particular,

$$\gamma^{-1}E = \left| U - \sum_{j=1}^{k} U_{\mu,\xi_j} - \sum_{l=1}^{h} U_{\lambda,\eta_l} \right|^{p-1} \left(U - \sum_{j=1}^{k} U_{\mu,\xi_j} - \sum_{l=1}^{h} U_{\lambda,\eta_l} \right)$$
$$- U^p - \sum_{j=1}^{k} U_{\mu,\xi_j}^p - \sum_{l=1}^{h} U_{\lambda,\eta_l}^p$$

We divide the study of the error in three diffentent regions. Roughly speaking, we will estimate first the $\|\cdot\|_{**}$ norm of the error far from the points ξ_j and η_l, then around ξ_j, and finally around η_l, for any $j = 1, \ldots, k$ and $l = 1, \ldots, h$. Indeed, let $\overline{\alpha}$ and $\hat{\alpha}$ be positive numbers independent of k and h.

Exterior region: $y \in \{\cap_{j=1}^{k}\{|y - \xi_j| > \frac{\overline{\alpha}}{k}\}\} \cap \{\cap_{l=1}^{h}\{|y - \eta_l| > \frac{\hat{\alpha}}{h}\}\}$.
For y in this region we can estimate

$$|E| \leqslant C \left[\frac{1}{(1+|y|^2)^2} + \left| \sum_{j=1}^{k} \frac{\mu^{\frac{n-2}{2}}}{|y - \xi_j|^{n-2}} \right|^{\frac{4}{n-2}} + \left| \sum_{l=1}^{h} \frac{\lambda^{\frac{n-2}{2}}}{|y - \eta_l|^{n-2}} \right|^{\frac{4}{n-2}} \right]$$
$$\cdot \left(\sum_{j=1}^{k} \frac{\mu^{\frac{n-2}{2}}}{|y - \xi_j|^{n-2}} + \sum_{l=1}^{h} \frac{\lambda^{\frac{n-2}{2}}}{|y - \eta_l|^{n-2}} \right) \tag{1.1.2}$$
$$\leqslant \frac{C}{(1+|y|^2)^2} \left[\sum_{j=1}^{k} \frac{\mu^{\frac{n-2}{2}}}{|y - \xi_j|^{n-2}} + \sum_{l=1}^{h} \frac{\lambda^{\frac{n-2}{2}}}{|y - \eta_l|^{n-2}} \right],$$

where in the last inequality we have used that here

$$\sum_{j=1}^{k} \frac{\mu^{\frac{n-2}{2}}}{|y - \xi_j|^{n-2}} \leqslant C\left(1 - \frac{k-1}{k^{n-2}}\right), \qquad \sum_{l=1}^{h} \frac{\lambda^{\frac{n-2}{2}}}{|y - \eta_l|^{n-2}} \leqslant C\left(1 - \frac{h-1}{h^{n-2}}\right).$$

Thus

$$\begin{aligned}
\|(1 + |y|)^{n+2-\frac{2n}{q}} E\|_{L^q(\{\cap_{j=1}^{k}\{|y-\xi_j|>\frac{\bar{\alpha}}{k}\}\} \cap \{\cap_{l=1}^{h}\{|y-\eta_l|>\frac{\hat{\alpha}}{h}\}\})} \\
\leqslant C\left(\frac{\mu^{\frac{n-2}{2}} k^{n-2}}{k^{\frac{n}{q}-1}} + \frac{\lambda^{\frac{n-2}{2}} h^{n-2}}{h^{\frac{n}{q}-1}}\right) \\
\leqslant C(k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}).
\end{aligned} \tag{1.1.3}$$

Interior regions around ξ_j: $y \in \{|y - \xi_j| < \frac{\bar{\alpha}}{k}\}$ for some $j = 1, \ldots, k$.
Let j be fixed. For some $s \in (0, 1)$ we have

$$\begin{aligned}
\gamma^{-1} E = {} & p(U_{\mu,\xi_j} + s(-\sum_{i \neq j} U_{\mu,\xi_i} + U - \sum_{l=1}^{h} U_{\lambda,\eta_l}))^{p-1}(-\sum_{i \neq j} U_{\mu,\xi_i} + U - \sum_{l=1}^{h} U_{\lambda,\eta_l}) \\
& - U^p - \sum_{i \neq j} U_{\mu,\xi_i}^p - \sum_{l=1}^{h} U_{\lambda,\eta_l}^p.
\end{aligned}$$

Let us define

$$\overline{E}_j(y) := \mu^{\frac{n+2}{2}} E(\xi_j + \mu y), \qquad |y| < \frac{\bar{\alpha}}{\mu k}.$$

Thus,

$$\begin{aligned}
\gamma^{-1} \overline{E}_j(y) = {} & p\left(-U(y) + s(-\sum_{i \neq j} U(y - \mu^{-1}(\xi_j - \xi_i)) + \mu^{\frac{n-2}{2}} U(\xi_j + \mu y)\right. \\
& \left. - \sum_{l=1}^{h} \mu^{\frac{n-2}{2}} \lambda^{-\frac{n-2}{2}} U(\lambda^{-1}(\xi_j + \mu y - \eta_l))\right)^{p-1}(-\sum_{i \neq j} U(y - \mu^{-1}(\xi_j - \xi_i)) \\
& + \mu^{\frac{n-2}{2}} U(\xi_j + \mu y) - \sum_{l=1}^{h} \mu^{\frac{n-2}{2}} \lambda^{-\frac{n-2}{2}} U(\lambda^{-1}(\xi_j + \mu y - \eta_l)) \\
& - \mu^{\frac{n+2}{2}} U^p(\xi_j + \mu y) - \sum_{i \neq j} U^p(y - \mu^{-1}(\xi_i - \xi_j)) \\
& - \sum_{l=1}^{h} \mu^{\frac{n+2}{2}} \lambda^{-\frac{n+2}{2}} U^p(\lambda^{-1}(\xi_j + \mu y - \eta_l)),
\end{aligned} \tag{1.1.4}$$

and consequently

$$
\begin{aligned}
|\overline{E}_j(y)| &\leqslant C \left[\frac{(k\mu)^{n-2} + (\mu\lambda)^{\frac{n-2}{2}} h}{1 + |y|^4} + \mu^{\frac{n+2}{2}} + (\mu\lambda)^{\frac{n+2}{2}} h^{\frac{n+2}{n-2}} \right] \\
&\leqslant C \left[\frac{\mu^{\frac{n-2}{2}}}{1 + |y|^4} + \mu^{\frac{n+2}{2}} \right].
\end{aligned}
\tag{1.1.5}
$$

Noticing that $h = O(k)$ we can compute

$$
\|(1 + |y|)^{n+2-\frac{2n}{q}} \overline{E}_j\|_{L^q(|y| < \frac{\overline{\alpha}}{\mu k})} \leqslant C\mu^{\frac{n}{2q}} \leqslant Ck^{-\frac{n}{q}}.
\tag{1.1.6}
$$

Interior regions around η_l: $y \in \{|y - \eta_l| < \frac{\hat{\alpha}}{h}\}$ for some $l = 1, \ldots, h$.
The estimates in this region follow analogously to the previous case, but interchanging the role of μ, k and λ, h. Thus, considering

$$
\hat{E}_l(y) := \lambda^{\frac{n+2}{2}} E(\eta_l + \mu y), \qquad |y| < \frac{\hat{\alpha}}{\lambda h},
$$

we get

$$
\begin{aligned}
|\hat{E}_l(y)| &\leqslant C \left[\frac{(h\lambda)^{n-2} + (\mu\lambda)^{\frac{n-2}{2}} k}{1 + |y|^4} + \lambda^{\frac{n+2}{2}} + (\mu\lambda)^{\frac{n+2}{2}} k^{\frac{n+2}{n-2}} \right] \\
&\leqslant C \left[\frac{\lambda^{\frac{n-2}{2}}}{1 + |y|^4} + \lambda^{\frac{n+2}{2}} \right]
\end{aligned}
\tag{1.1.7}
$$

and therefore

$$
\|(1 + |y|)^{n+2-\frac{2n}{q}} \hat{E}_l\|_{L^q(|y| < \frac{\hat{\alpha}}{\lambda h})} \leqslant C\lambda^{\frac{n}{2q}} \leqslant Ch^{-\frac{n}{q}}.
\tag{1.1.8}
$$

1.2. Building the solution

Recall from Section **??** that to find a solution to (**??**) we will prove the existence of a function ϕ that solves (**??**). We will try to build this function in a special form.

Let $\zeta(s)$ be a smooth function such that $\zeta(s) = 1$ for $s > 1$ and $\zeta(s) = 0$ for $s > 2$, and let $\overline{\alpha}, \hat{\alpha} > 0$ be fixed numbers independent of k and h. Define

$$
\overline{\zeta}_j(y) := \begin{cases} \zeta(k\overline{\alpha}^{-1}|y|^{-2}|y - \xi_j|y|^2|) \text{ if } |y| > 1, \\ \zeta(k\overline{\alpha}^{-1}|y - \xi_j|) \text{ if } |y| \leqslant 1, \end{cases}
$$

$$
\hat{\zeta}_l(y) := \begin{cases} \zeta(h\hat{\alpha}^{-1}|y|^{-2}|y - \eta_l|y|^2|) \text{ if } |y| > 1, \\ \zeta(h\hat{\alpha}^{-1}|y - \eta_l|) \text{ if } |y| \leqslant 1. \end{cases}
$$

A function of the form

$$
\phi = \sum_{j=1}^{k} \overline{\phi}_j + \sum_{l=1}^{h} \hat{\phi}_l + \psi
\tag{1.2.1}
$$

is a solution of (**??**) if we solve the system

$$
\Delta\overline{\phi}_j + p\gamma|U_*|^{p-1}\overline{\zeta}_j\overline{\phi}_j + \overline{\zeta}_j \left[p\gamma|U_*|^{p-1}\psi + E + \gamma N(\phi) \right] = 0, \; ; j = 1, \ldots, k,
\tag{1.2.2}
$$

$$
\Delta\hat{\phi}_l + p\gamma|U_*|^{p-1}\hat{\zeta}_l\hat{\phi}_l + \hat{\zeta}_l \left[p\gamma|U_*|^{p-1}\psi + E + \gamma N(\phi) \right] = 0, \; l = 1, \ldots, h,
\tag{1.2.3}
$$

$$\Delta\psi + p\gamma U^{p-1}\psi + [p\gamma(|U_*|^{p-1} - U^{p-1})(1 - \sum_{j=1}^{k}\overline{\zeta}_j - \sum_{l=1}^{h}\hat{\zeta}_l)$$

$$+ p\gamma U^{p-1}(\sum_{j=1}^{k}\overline{\zeta}_j + \sum_{l=1}^{h}\hat{\zeta}_l)]\psi + p\gamma|U_*|^{p-1}\sum_{j=1}^{k}(1 - \overline{\zeta}_j)\overline{\phi}_j \tag{1.2.4}$$

$$+ p\gamma|U_*|^{p-1}\sum_{l=1}^{h}(1 - \hat{\zeta}_l)\hat{\phi}_l + (1 - \sum_{j=1}^{k}\overline{\zeta}_j - \sum_{l=1}^{h}\hat{\zeta}_l)(E + \gamma N(\phi)) = 0.$$

We assume in addition the following symmetry properties on $\overline{\phi}_j$ and $\hat{\phi}_l$,

$$\overline{\phi}_j(\overline{y}, \hat{y}, y') = \overline{\phi}_1(e^{-\frac{2\pi(j-1)}{k}i}\overline{y}, \hat{y}, y'), \qquad j = 1, \ldots k, \tag{1.2.5}$$

where

$$\overline{\phi}_1(y_1, \ldots, y_j, \ldots, y_n) = \overline{\phi}_1(y_1, \ldots, -y_j, \ldots, y_n), \qquad j = 2, 4, 5, \ldots, n,$$
$$\overline{\phi}_1(y) = |y|^{2-n}\overline{\phi}_1(|y|^{-2}y), \tag{1.2.6}$$
$$\overline{\phi}_1(\overline{y}, \hat{y}, y') = \overline{\phi}_1(\overline{y}, e^{\frac{2\pi(l-1)}{h}i}\hat{y}, y'), \qquad l = 1, \ldots, h,$$

and

$$\hat{\phi}_j(\overline{y}, \hat{y}, y') = \hat{\phi}_1(\overline{y}, e^{-\frac{2\pi(l-1)}{h}i}\hat{y}, y'), \qquad l = 1, \ldots h, \tag{1.2.7}$$

where

$$\hat{\phi}_1(y_1, \ldots, y_j, \ldots, y_n) = \hat{\phi}_1(y_1, \ldots, -y_j, \ldots, y_n), \qquad j = 2, 4, 5, \ldots, n,$$
$$\hat{\phi}_1(y) = |y|^{2-n}\hat{\phi}_1(|y|^{-2}y), \tag{1.2.8}$$
$$\hat{\phi}_1(\overline{y}, \hat{y}, y') = \hat{\phi}_1(e^{\frac{2\pi(j-1)}{k}i}\overline{y}, \hat{y}, y'), \qquad j = 1, \ldots, k.$$

Assume in addition that

$$\|\overline{\overline{\phi}}_1\|_* \leqslant \rho, \qquad \overline{\overline{\phi}}_1(y) := \mu^{\frac{n-2}{2}}\overline{\phi}_1(\xi_1 + \mu y),$$
$$\|\hat{\hat{\phi}}_1\|_* \leqslant \rho, \qquad \hat{\hat{\phi}}_1(y) := \lambda^{\frac{n-2}{2}}\hat{\phi}_1(\eta_1 + \lambda y), \tag{1.2.9}$$

for $\rho > 0$ small.

Lemma 1.2.1. *There exist constants k_0, h_0, C, ρ_0 such that, for all $k \geqslant k_0$ and $h \geqslant h_0$, if $\overline{\phi}_j$, $j = 1, \ldots, k$, and $\hat{\phi}_l$, $l = 1, \ldots, h$ satisfy conditions (??)-(??) with $\rho < \rho_0$ then there exists a unique solution $\psi = \Psi(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)$ to equation (??), that satisfies the symmetries*

$$\psi(y_1, \ldots, y_\alpha, \ldots) = \psi(y_1, \ldots, -y_\alpha, \ldots), \quad \alpha = 5, \ldots, n, \tag{1.2.10}$$
$$\psi(\overline{y}, \hat{y}, y') = \psi(e^{\frac{2\pi(j-1)}{k}i}\overline{y}, \hat{y}, y'), \quad j = 1, \ldots, k, \tag{1.2.11}$$
$$\psi(\overline{y}, \hat{y}, y') = \psi(\overline{y}, e^{\frac{2\pi(l-1)}{h}i}\hat{y}, y'), \quad l = 1, \ldots, h, \tag{1.2.12}$$
$$\psi(y) = |y|^{2-n}\psi(|y|^{-2}y), \tag{1.2.13}$$

and such that

$$\|\psi\|_* \leqslant C\left[\|\overline{\overline{\phi}}_1\|_* + \|\hat{\hat{\phi}}_1\|_* + k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}\right]. \tag{1.2.14}$$

Moreover, the operator Ψ satisfies

$$\|\Psi(\overline{\overline{\phi}}_1^1, \hat{\hat{\phi}}_1^1) - \Psi(\overline{\overline{\phi}}_1^2, \hat{\hat{\phi}}_1^2)\|_* \leqslant C(\|\overline{\overline{\phi}}_1^1 - \overline{\overline{\phi}}_1^2\|_* + \|\hat{\hat{\phi}}_1^1 - \hat{\hat{\phi}}_1^2\|_*). \tag{1.2.15}$$

Proof. We write equation (**??**) as

$$\Delta\psi + p\gamma U^{p-1}\psi + V(y)\psi + p\gamma|U_*|^{p-1}\Big(\sum_{j=1}^{k}(1-\overline{\zeta}_j)\overline{\phi}_j + \sum_{l=1}^{h}(1-\hat{\zeta}_l)\hat{\phi}_l\Big) + M(\psi) = 0, \quad (1.2.16)$$

where

$$V(y) := p\gamma(|U_*|^{p-1} - U^{p-1})(1 - \sum_{j=1}^{k}\overline{\zeta}_j - \sum_{l=1}^{h}\hat{\zeta}_l) + p\gamma U^{p-1}(\sum_{j=1}^{k}\overline{\zeta}_j + \sum_{l=1}^{h}\hat{\zeta}_l),$$

$$=: V_1(y) + V_2(y),$$

and

$$M(\psi) := (1 - \sum_{j=1}^{k}\overline{\zeta}_j - \sum_{l=1}^{h}\hat{\zeta}_l)(E + \gamma N(\phi)) = 0.$$

Consider first the problem

$$\Delta\psi + p\gamma U^{p-1}\psi = h, \quad (1.2.17)$$

where h is a function satisfying (**??**)-(**??**), $\|h\|_{**} < +\infty$ and

$$h(y) = |y|^{-n-2}h(|y|^{-2}y). \quad (1.2.18)$$

Let

$$Z_\alpha := \partial_{y_\alpha}U, \quad \alpha = 1, \cdots, n \text{ and } Z_0 = y \cdot \nabla U + \frac{n-2}{2}U. \quad (1.2.19)$$

Due to the oddness of Z_α and assumption (**??**) on h it yields

$$\int_{\mathbb{R}^n} Z_\alpha h = 0 \text{ for all } \alpha = 5, \cdots, n.$$

The cases $\alpha = 0, 1, 2, 3, 4$ also vanish proceeding as in the proof of [**?**, Lemma 4.1] as a consequence of (**??**)-(**??**). Thus we can apply the linear existence result [**?**, Lemma 3.1] to ensure the existence of a unique solution ψ to (**??**) such that

$$\int_{\mathbb{R}^n} U^{p-1}Z_\alpha\psi = 0 \text{ for all } \alpha = 0, 1, \ldots, n,$$

and $\|\psi\|_* \leqslant C\|h\|_{**}$. Notice in addition that the functions

$$\psi_\alpha(y) := \psi(\overline{y}, \hat{y}, \ldots, -y_i, \ldots, y_n), \quad \alpha = 5, \cdots, n,$$

$$\psi_{12j}(y) := \psi(e^{\frac{2\pi(j-1)}{k}i}\overline{y}, \hat{y}, y'), \quad j = 1, \ldots, k,$$

$$\psi_{34l}(y) := \psi(\overline{y}, e^{\frac{2\pi(l-1)}{h}i}\hat{y}, y'), \quad l = 1, \ldots, h,$$

$$\psi_{n+1}(y) := |y|^{2-n}\psi(|y|^{-2}y),$$

also satisfy (**??**) and thus, by the uniqueness,

$$\psi = \psi_\alpha = \psi_{12j} = \psi_{34l} = \psi_{n+1}$$

for all $\alpha = 5, \ldots, n$, $j = 1, \ldots, k$, $l = 1, \ldots, h$, i.e., ψ satisfies (**??**)-(**??**). Therefore, (**??**) has a unique bounded solution $\psi = T(h)$ satisfying symmetries (**??**)-(**??**) and

$$\|\psi\|_* \leqslant C\|h\|_{**}$$

for a constant depending only on q and n.

We will solve (**??**) by means of a fixed point argument, writting

$$\psi = -T(V\psi + p\gamma|U_*|^{p-1}(\sum_{j=1}^{k}(1 - \overline{\zeta}_j)\overline{\phi}_j + \sum_{l=1}^{h}(1 - \hat{\zeta}_l)\hat{\phi}_l) + M(\psi)) =: \mathcal{M}(\psi),$$

$\psi \in X$, where X is the space of continuous functions ψ with $\|\psi\|_* < +\infty$ satisfying (**??**)-(**??**). Thanks to the special form of U_* and to the symmetry assumptions on $\overline{\phi}_j$ and $\hat{\phi}_l$,

$$V\psi + p\gamma|U_*|^{p-1}(\sum_{j=1}^{k}(1 - \overline{\zeta}_j)\overline{\phi}_j + \sum_{l=1}^{h}(1 - \hat{\zeta}_l)\hat{\phi}_l) + M(\psi)$$

satisfies (**??**)-(**??**) and (**??**) if $\psi \in X$, and $\mathcal{M}$ is well defined. Actually, we claim that $\mathcal{M}$ is a contraction mapping in the $\|\|_*$ norm in a small ball around the origin in X. Indeed,

$$|V_1(y)| \leqslant \gamma p(p-1)\left|U - s(\sum_{j=1}^{k}U_{\mu,\xi_j} + \sum_{l=1}^{h}U_{\lambda,\eta_l})\right|^{p-2}\left(\sum_{j=1}^{k}U_{\mu,\xi_j} + \sum_{l=1}^{h}U_{\lambda,\eta_l}\right)$$

$$\leqslant CU^{p-2}(y)\left(\sum_{j=1}^{k}\frac{\mu^{\frac{n-2}{2}}}{|y - \xi_j|^{n-2}} + \sum_{l=1}^{h}\frac{\lambda^{\frac{n-2}{2}}}{|y - \eta_l|^{n-2}}\right),$$

and thus

$$|V_1\psi(y)| \leqslant C\|\psi\|_*U^{p-1}(y)\left(\sum_{j=1}^{k}\frac{\mu^{\frac{n-2}{2}}}{|y - \xi_j|^{n-2}} + \sum_{l=1}^{h}\frac{\lambda^{\frac{n-2}{2}}}{|y - \eta_l|^{n-2}}\right).$$

Proceeding as in (**??**) we get

$$\|V_1\psi\|_{**} \leqslant C\|\psi\|_*(k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}). \tag{1.2.20}$$

On the other hand,

$$\|V_2\psi\|_{**} \leqslant \sum_{j=1}^{k}\|pU^{p-1}\psi\overline{\zeta}_j\|_{**} + \sum_{l=1}^{h}\|pU^{p-1}\psi\hat{\zeta}_l\|_{**}$$

$$\leqslant C\|\psi\|_*(k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}), \tag{1.2.21}$$

and putting together (**??**) and (**??**) we conclude

$$\|V\psi\|_{**} \leqslant C\|\psi\|_*(k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}). \tag{1.2.22}$$

Assume $|y - \xi_j| > \frac{\overline{\alpha}}{2k}$ and $|y - \eta_l| > \frac{\hat{\alpha}}{2h}$ for all j and l. We knew that in this region

$$\|E\|_{**} \leqslant C(k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}).$$

Moreover,

$$\left|N\left(\sum_{j=1}^{k}\overline{\phi}_j + \sum_{l=1}^{h}\hat{\phi}_l + \psi\right)\right| \leqslant CU^{p-2}\left(\left|\sum_{j=1}^{k}\overline{\phi}_j\right|^2 + \left|\sum_{l=1}^{h}\hat{\phi}_l\right|^2 + |\psi|^2\right).$$

From (**??**), we get

$$|\overline{\phi}_j(y)| \leqslant C\|\overline{\overline{\phi}}_1\|_*\frac{\mu^{\frac{n-2}{2}}}{\mu^{n-2} + |y - \xi_j|^{n-2}}, \quad |\hat{\phi}_l(y)| \leqslant C\|\hat{\hat{\phi}}_1\|_*\frac{\lambda^{\frac{n-2}{2}}}{\lambda^{n-2} + |y - \eta_l|^{n-2}}.$$

Moreover,

$$U^{p-2}|\psi|^2 \leqslant U^p \|\psi\|_*^2.$$

Hence, proceeding again as in (??), we obtain

$$\|M(\psi)\|_{**} \leqslant Ck^{1-\frac{n}{q}}(1+\|\overline{\overline{\phi}}_1\|_*^2) + Ch^{1-\frac{n}{q}}(1+\|\hat{\hat{\phi}}_1\|_*^2) + C\|\psi\|_*^2. \tag{1.2.23}$$

Likewise, if $|y - \xi_j| > \frac{\overline{\alpha}}{2k}$ and $|y - \eta_l| > \frac{\hat{\alpha}}{2h}$,

$$\||U_*|^{p-1}(\sum_{j=1}^{k}\overline{\phi}_j + \sum_{l=1}^{h}\hat{\phi}_l)\|_{**} \leqslant Ck^{1-\frac{n}{q}}\|\overline{\overline{\phi}}_1\|_* + Ch^{1-\frac{n}{q}}\|\hat{\hat{\phi}}_1\|_*. \tag{1.2.24}$$

Moreover, for ψ_1, ψ_2 satisfying $\|\psi_1\| < \rho, \|\psi_2\| < \rho$ it follows

$$\|M(\psi_1) - M(\psi_2)\|_{**} \leqslant C\rho\|\psi_1 - \psi_2\|_*.$$

Joining (??), (??) and (??), we see that for ρ small enough the operator $\mathcal{M}$ defines a contraction map in the set of functions $\psi \in X$ with

$$\|\psi\|_* \leqslant C[\|\overline{\overline{\phi}}_1\|_* + k^{1-\frac{n}{q}} + \|\hat{\hat{\phi}}_1\|_* + h^{1-\frac{n}{q}}], \quad \|\overline{\overline{\phi}}_1\|_* < \rho, \quad \|\hat{\hat{\phi}}_1\|_* < \rho,$$

for ρ small. Therefore, there exists a solution of (??) satisfying conditions (??)-(??). The Lipschitz condition (??) easily follows. $\qquad\square$

Consider the operator $\Psi(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)$. Equations (??) and (??) reduce to solve one of each, for example for $\overline{\phi}_1$ and $\hat{\phi}_1$.

We try to solve first

$$\Delta\overline{\phi}_1 + p\gamma|U_*|^{p-1}\overline{\zeta}_1\overline{\phi}_1 + \overline{\zeta}_1\left[p\gamma|U_*|^{p-1}\Psi(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) + E + \gamma N(\phi)\right] = 0 \text{ in } \mathbb{R}^n,$$

or equivalently,

$$\Delta\overline{\phi}_1 + p\gamma|U_{\mu,\xi_1}|^{p-1}\overline{\phi}_1 + \overline{\zeta}_1 E + \gamma\overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) = 0, \tag{1.2.25}$$

where

$$\overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) := p(|U_*|^{p-1}\overline{\zeta}_1 - |U_{\mu,\xi_1}|^{p-1})\overline{\phi}_1 + \overline{\zeta}_1\left[p|U_*|^{p-1}\Psi(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) + N(\phi)\right].$$

Consider first a general function $\overline{h}$ and the problem

$$\Delta\overline{\phi} + p\gamma U_{\mu,\xi_1}^{p-1}\overline{\phi} + \overline{h} = \overline{c}_0 U_{\mu,\xi_1}^{p-1}\overline{Z}_0 \text{ in } \mathbb{R}^n, \tag{1.2.26}$$

where

$$\overline{Z}_0(y) := \mu^{-\frac{n-2}{2}}Z_0\left(\frac{y-\xi_1}{\mu}\right), \qquad \overline{c}_0 := \frac{\int_{\mathbb{R}^n}\overline{h}\,\overline{Z}_0}{\int_{\mathbb{R}^n}U_{\mu,\xi_1}^{p-1}\overline{Z}_0^2},$$

with Z_0 defined in (??).

Lemma 1.2.2. *Suppose that $\overline{h}$ is even with respect to each of the variables $y_2, y_4, y_5, \ldots, y_n$ and such that*

$$\overline{h}(y) = |y|^{-n-2}\overline{h}(|y|^{-2}y), \qquad \overline{h}(y) = \overline{h}(\overline{y}, e^{\frac{2\pi(l-1)}{h}}\hat{y}, y'), \ \ l = 1, \ldots, h. \tag{1.2.27}$$

Assume that

$$h(y) := \mu^{\frac{n+2}{2}}\overline{h}(\xi_1 + \mu y)$$

satisfies $\|h\|_{**} < +\infty$. Then problem (??) has a unique solution $\overline{\phi} := \overline{T}(\overline{h})$ that is even with respect to the variables $y_2, y_4, y_5, \ldots, y_n$, invariant under Kelvin's tranform, i.e.,

$$\overline{\phi}(y) = |y|^{2-n}\overline{\phi}(|y|^{-2}y),$$

and with $\overline{\overline{\phi}}(y) := \mu^{\frac{n-2}{2}}\overline{\phi}(\xi_1 + \mu y)$ satisfying

$$\int_{\mathbb{R}^n} \overline{\overline{\phi}} U^{p-1} Z_0 = 0, \qquad \|\overline{\overline{\phi}}\|_* \leqslant C\|h\|_{**}.$$

Proof. We assume with no loss of generality that

$$\int_{\mathbb{R}^n} \overline{h}\, \overline{Z}_0 = 0, \qquad \text{i.e.,} \qquad \overline{c}_0 = 0.$$

Thus, equation (??) is equivalent to

$$\Delta\overline{\overline{\phi}} + p\gamma|U|^{p-1}\overline{\overline{\phi}} = -h \qquad \text{in } \mathbb{R}^n.$$

Due to the evenness of h we know that

$$\int_{\mathbb{R}^n} hZ_\alpha = 0, \qquad \alpha = 2, 4, 5, \cdots, n. \tag{1.2.28}$$

The proof of

$$\int_{\mathbb{R}^n} hZ_1 = 0$$

follows exactly as in the proof of [?, Lemma 4.2], so we focus on the case $\alpha = 3$.

Indeed, denote by $w_\mu(y) := \mu^{-\frac{n-2}{2}}U(\mu^{-1}y)$, and $J(t) := \int_{\mathbb{R}^n} w_\mu(y - \xi_1 + te_3)\overline{h}(y)\, dy$. Notice first that

$$\frac{d}{dt}J(t)\Big|_{t=0} = \int_{\mathbb{R}^n} \partial_{y_3} w_\mu(y - \xi_1)\overline{h}(y)\, dy = \int_{\mathbb{R}^n} hZ_3. \tag{1.2.29}$$

On the other hand, defining $\tilde{y} := (\overline{y}, e^{\frac{2\pi(l-1)}{h}}\hat{y}, y')$ for some $l = 2, 3, \ldots, h$, it can be checked that

$$|\tilde{y} - \xi_1 + te_3|^2 = |y - \xi_1 + t\tilde{e}|^2, \qquad t \in \mathbb{R}^n,$$

where $\tilde{e} := (0, 0, \cos(\frac{2\pi(l-1)}{h}), -\sin(\frac{2\pi(l-1)}{h}), 0, \ldots)$. Thus, after a change of variables, by (??),

$$J(t) = \int_{\mathbb{R}^n} w_\mu(\tilde{y} - \xi_1 + te_3)\overline{h}(\tilde{y})\, d\tilde{y} = \int_{\mathbb{R}^n} w_\mu(y - \xi_1 + t\tilde{e})\overline{h}(y)\, dy.$$

Differentiating here,

$$\frac{d}{dt}J(t)\Big|_{t=0} = \cos\left(\frac{2\pi(l-1)}{h}\right)\int_{\mathbb{R}^n} \partial_{y_3} w_\mu(y - \xi_1)\overline{h}(y)\, dy$$

$$- \sin\left(\frac{2\pi(l-1)}{h}\right)\int_{\mathbb{R}^n} \partial_{y_4} w_\mu(y - \xi_1)\overline{h}(y)\, dy$$

$$= \cos\left(\frac{2\pi(l-1)}{h}\right)\int_{\mathbb{R}^n} hZ_3\, dy - \sin\left(\frac{2\pi(l-1)}{h}\right)\int_{\mathbb{R}^n} hZ_4\, dy.$$

Applying (??) and (??) we conclude that necessarily $\int_{\mathbb{R}^n} hZ_3 = 0$. Thus, by [?, Lemma 3.1] there exists a unique solution $\overline{\overline{\phi}}$ satisfying

$$\|\overline{\overline{\phi}}\|_* \leqslant C\|h\|_{**}, \qquad \int_{\mathbb{R}^n} \overline{\overline{\phi}} U^{p-1} Z_{n+1} = 0.$$

The invariance under Kelvin transform and the symmetries are obtained as a consequence of the uniqueness. □

Likewise, we rewrite

$$\Delta\hat{\phi}_1 + p\gamma|U_*|^{p-1}\hat{\zeta}_1\hat{\phi}_1 + \hat{\zeta}_1\left[p\gamma|U_*|^{p-1}\Psi(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) + E + \gamma N(\phi)\right] = 0 \text{ in } \mathbb{R}^n,$$

as

$$\Delta\hat{\phi}_1 + p\gamma|U_{\lambda,\eta_1}|^{p-1}\hat{\phi}_1 + \hat{\zeta}_1 E + \gamma\hat{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) = 0, \tag{1.2.30}$$

with

$$\hat{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) := p(|U_*|^{p-1}\hat{\zeta}_1 - |U_{\lambda,\eta_1}|^{p-1})\hat{\phi}_1 + \hat{\zeta}_1\left[p|U_*|^{p-1}\Psi(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) + N(\phi)\right],$$

and we consider the problem

$$\Delta\hat{\phi} + p\gamma U_{\lambda,\eta_1}^{p-1}\hat{\phi} + \hat{h} = \hat{c}_0 U_{\lambda,\eta_1}^{p-1}\hat{Z}_0 \text{ in } \mathbb{R}^n, \tag{1.2.31}$$

where $\hat{h}$ is a general function and

$$\hat{Z}_0(y) := \lambda^{-\frac{n-2}{2}} Z_0\left(\frac{y-\eta_1}{\lambda}\right), \qquad \hat{c}_0 := \frac{\int_{\mathbb{R}^n}\hat{h}\hat{Z}_0}{\int_{\mathbb{R}^n}U_{\lambda,\eta_1}^{p-1}\hat{Z}_0^2}.$$

Lemma 1.2.3. *Suppose that $\hat{h}$ is even with respect to each of the variables $y_2, y_4, y_5, \ldots, y_n$ and such that*

$$\hat{h}(y) = |y|^{-n-2}\hat{h}(|y|^{-2}y), \qquad \hat{h}(y) = \hat{h}(e^{\frac{2\pi(j-1)}{k}}\overline{y}, \hat{y}, y'), \;\; j = 1, \ldots, k.$$

Assume that

$$h(y) := \lambda^{\frac{n+2}{2}}\hat{h}(\eta_1 + \lambda y)$$

satisfies $\|h\|_{**} < +\infty$. Then problem (??) has a unique solution $\hat{\phi} := \hat{T}(\hat{h})$ that is even with respect to the variables $y_2, y_4, y_5, \ldots, y_n$, invariant under Kelvin's tranform, i.e.,

$$\hat{\phi}(y) = |y|^{2-n}\hat{\phi}(|y|^{-2}y),$$

and with $\hat{\hat{\phi}}(y) := \lambda^{\frac{n-2}{2}}\hat{\phi}(\eta_1 + \lambda y)$ satisfying

$$\int_{\mathbb{R}^n}\hat{\hat{\phi}}U^{p-1}Z_0 = 0, \qquad \|\hat{\hat{\phi}}\|_* \leqslant C\|h\|_{**}.$$

The proof of this result is analogous to the one for Lemma **??**, interchanging the roles of μ and ξ_1 with λ and η_1, so we skip it.

We use these lemmas to solve the projected versions of (**??**) and (**??**), that is,

$$\begin{aligned}\Delta\overline{\phi}_1 + p\gamma|U_{\mu,\xi_1}|^{p-1}\overline{\phi}_1 + \overline{\zeta}_1 E + \gamma\overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) &= \overline{c}_0 U_{\mu,\xi_1}^{p-1}\overline{Z}_0, \\ \Delta\hat{\phi}_1 + p\gamma|U_{\lambda,\eta_1}|^{p-1}\hat{\phi}_1 + \hat{\zeta}_1 E + \gamma\hat{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) &= \hat{c}_0 U_{\lambda,\eta_1}^{p-1}\hat{Z}_0,\end{aligned} \tag{1.2.32}$$

in $\mathbb{R}^n$, with

$$\overline{c}_0 := \frac{\int_{\mathbb{R}^n}(\overline{\zeta}_1 E + \gamma\overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1))\overline{Z}_0}{\int_{\mathbb{R}^n}U_{\mu,\xi_1}^{p-1}\hat{Z}_0^2}, \qquad \hat{c}_0 := \frac{\int_{\mathbb{R}^n}(\hat{\zeta}_1 E + \gamma\hat{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1))\hat{Z}_0}{\int_{\mathbb{R}^n}U_{\lambda,\eta_1}^{p-1}\hat{Z}_0^2}.$$

Proposition 1.2.4. *There exists a unique solution $\phi_1 = (\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) = (\overline{\overline{\phi}}_1(\delta), \hat{\hat{\phi}}_1(\varepsilon))$ to* **(??)** *such that*

$$\|\phi_1\|_* := \|\overline{\overline{\phi}}_1\|_* + \|\hat{\hat{\phi}}_1\|_* \leqslant C(k^{-\frac{n}{q}} + h^{-\frac{n}{q}}),$$

and

$$\|\overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)\|_{**} \leqslant Ck^{-\frac{2n}{q}}, \qquad \|\hat{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)\|_{**} \leqslant Ch^{-\frac{2n}{q}}. \tag{1.2.33}$$

Proof. Denote by $\overline{T}$ and $\hat{T}$ the linear operators predicted by Lemma **??** and Lemma **??** respectively. Thus, solving **(??)** is equivalent to solve the fixed point problem

$$\phi_1 = \begin{pmatrix} \overline{\phi}_1 \\ \hat{\phi}_1 \end{pmatrix} = \begin{pmatrix} \overline{T}(\overline{\zeta}_1 E + \gamma \overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)) \\ \hat{T}(\hat{\zeta}_1 E + \gamma \hat{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)) \end{pmatrix} = \begin{pmatrix} \overline{\mathcal{M}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) \\ \hat{\mathcal{M}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) \end{pmatrix} =: \mathcal{M}(\phi_1).$$

Let us focus first on $\overline{\mathcal{M}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) := \overline{T}(\overline{\zeta}_1 E + \gamma \overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1))$. Recall that

$$\overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) := p(|U_*|^{p-1}\overline{\zeta}_1 - |U_{\mu,\xi_1}|^{p-1})\overline{\phi}_1 + \overline{\zeta}_1 \left[p|U_*|^{p-1}\Psi(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1) + N(\phi) \right].$$

Denote in general

$$\tilde{f}(y) = \mu^{\frac{n+2}{2}} f(\xi_1 + \mu y).$$

Consider

$$f_1(y) := p\overline{\zeta}_1(|U_*|^{p-1} - |U_{\mu,\xi_1}|^{p-1})\overline{\phi}_1.$$

For $|y| < \frac{\overline{\alpha}}{\mu k}$,

$$|\tilde{f}_1(y)| = \left| p((U(y) + \sum_{j=2}^{k} U(y + \mu^{-1}(\xi_1 - \xi_j)) \right.$$

$$+ \sum_{l=1}^{h} \mu^{\frac{n-2}{2}} \lambda^{-\frac{n-2}{2}} U(\lambda^{-1}(\xi_1 + \mu y - \eta_l))$$

$$\left. - \mu^{\frac{n-2}{2}} U(\xi_1 + \mu y))^{p-1} - U^{p-1}(y))\overline{\overline{\phi}}_1(y) \right|.$$

Noticing that

$$\sum_{j=2}^{k} U(y + \mu^{-1}(\xi_1 - \xi_j)) \leqslant C\mu^{n-2}k^{n-2} \sum_{j=1}^{k} \frac{1}{j^{n-2}} \leqslant C\mu^{\frac{n-2}{2}}, \tag{1.2.34}$$

and

$$\sum_{l=1}^{h} \mu^{\frac{n-2}{2}} \lambda^{-\frac{n-2}{2}} U(\lambda^{-1}(\xi_1 + \mu y - \eta_l))$$

$$\leqslant C \sum_{l=1}^{h} \mu^{\frac{n-2}{2}} \lambda^{-\frac{n-2}{2}} \leqslant C\mu^{\frac{n-2}{2}} \lambda^{-\frac{n-2}{2}} h \leqslant C\mu^{\frac{n-2}{2}}, \tag{1.2.35}$$

doing a Taylor expansion we get

$$|\tilde{f}_1(y)| \leqslant C\mu^{\frac{n-2}{2}} U^{p-2}(y)|\overline{\overline{\phi}}_1(y)| \leqslant C\mu^{\frac{n-2}{2}} U^{p-1}(y)\|\overline{\overline{\phi}}_1\|_* \tag{1.2.36}$$

and, proceeding as in the computations for the interior error,

$$\|\tilde{f}_1\|_{**} \leqslant C\mu^{\frac{n}{2q}}\|\overline{\overline{\phi}}_1\|_*.$$

For the term
$$f_2 := (\overline{\zeta}_1 - 1)U_{\mu,\xi_1}^{p-1}\overline{\phi}_1,$$
we have
$$|\tilde{f}_2(y)| \leqslant U^p(y)\|\overline{\overline{\phi}}_1\|_*, \qquad |y| > \frac{\overline{\alpha}}{\mu k}, \qquad \|\tilde{f}_2\|_{**} \leqslant C\mu^{\frac{n}{2q}}\|\overline{\overline{\phi}}_1\|_*. \qquad (1.2.37)$$

Consider now
$$f_3 := \overline{\zeta}_1 p |U_*|^{p-1}\Psi(\overline{\phi}_1, \hat{\phi}_1).$$
Using (**??**), (**??**) and (**??**) we get that, for $|y| \leqslant \frac{\overline{\alpha}}{\mu k}$,
$$\begin{aligned}
|\tilde{f}_3(y)| &\leqslant CU^{p-1}\mu^{\frac{n-2}{2}}\|\Psi(\overline{\phi}_1, \hat{\phi}_1)\|_{L^\infty(\mathbb{R}^n)} \\
&\leqslant CU^{p-1}\mu^{\frac{n-2}{2}}(\|\overline{\overline{\phi}}_1\|_* + \|\hat{\hat{\phi}}_1\|_* + k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}),
\end{aligned} \qquad (1.2.38)$$
$$\|\tilde{f}_3\|_{**} \leqslant C\mu^{\frac{n}{2q}}(\|\overline{\overline{\phi}}_1\|_* + \|\hat{\hat{\phi}}\|_* + k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}).$$

Denote
$$f_4 := \overline{\zeta}_1 N(\phi), \qquad f_5 := \overline{\zeta}_1 E.$$
Notice that
$$\tilde{N}(\phi) = |V_* + \tilde{\phi}_1|^{p-1}(V_* + \tilde{\phi}_1) - |V_*|^{p-1}V_* - p|V_*|^{p-1}\tilde{\phi}_1,$$
where $\tilde{\phi}_1(y) := \mu^{\frac{n-2}{2}}\phi(\xi_1 + \mu y)$, and

$$\begin{aligned}
V_*(y) := &- U(y) - \sum_{j=2}^{k} U(y + \mu^{-1}(\xi_1 - \xi_j)) \\
&- \sum_{l=1}^{h} \mu^{\frac{n-2}{2}}\lambda^{-\frac{n-2}{2}}U(\lambda^{-1}(\xi_1 + \mu y - \eta_l)) \\
&+ \mu^{\frac{n-2}{2}}U(\xi_1 + \mu y).
\end{aligned}$$

Hence, for
$$\phi = \overline{\phi}_1 + \sum_{j=2}^{k}\overline{\phi}_j + \sum_{l=1}^{h}\hat{\phi}_l + \Psi(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1),$$
one has
$$|\tilde{\phi}_1| \leqslant C\mu^{\frac{n-2}{2}}(\|\overline{\overline{\phi}}_1\|_* + \|\hat{\hat{\phi}}_1\|_*) + \mu^{\frac{n-2}{2}}\|\Psi(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)\|_{L^\infty(\mathbb{R}^n)}.$$

Furthermore, in the region $|y| < \frac{\overline{\alpha}}{\mu k}$ it holds $U(y) \sim \mu^{\frac{n-2}{2}}$ and thus, after a second order Taylor expansion one has
$$\begin{aligned}
|\tilde{f}_4(y)| &\leqslant C|V_*|^{p-2}|\tilde{\phi}_1|^2 \leqslant CU^{p-2}\mu^{\frac{n-2}{2}}|\tilde{\phi}_1| \\
&\leqslant CU^{p-1}\mu^{\frac{n-2}{2}}(\|\overline{\overline{\phi}}_1\|_* + \|\hat{\hat{\phi}}_1\|_* + \|\Psi\|_*),
\end{aligned} \qquad (1.2.39)$$
and
$$\|\tilde{f}_4\|_{**} \leqslant C\mu^{\frac{n}{2q}}(\|\overline{\overline{\phi}}_1\|_* + \|\hat{\hat{\phi}}_1\|_* + k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}).$$
Finally, by (**??**), we know
$$\|\tilde{f}_5\|_{**} \leqslant C\mu^{\frac{n}{2q}}. \qquad (1.2.40)$$

Likewise, one can obtain analogous estimates for

$$\hat{T}(\hat{\zeta}_1 E + \gamma \hat{\mathcal{N}}(\overline{\hat{\phi}}_1, \hat{\hat{\phi}}_1))$$

to conclude that $\mathcal{M}$ maps functions ϕ_1 with $\|\phi_1\|_* \leqslant C(\mu^{\frac{n}{2q}} + \lambda^{\frac{n}{2q}})$ into the same class of functions. Besides, one can prove that the map is indeed a contraction, and thus we conclude the existence of a unique solution to the system (**??**). □

Remark 1.2.5. The symmetry conditions (**??**) and (**??**) follow straightforward as consequence of the uniqueness.

1.3. PROOF OF THEOREM **??**

Thanks to Proposition **??** we have $\overline{\phi}_1$ and $\hat{\phi}_1$ solutions to (**??**). Thus, if we find δ and ε in (**??**) so that $\overline{c}_0(\delta, \varepsilon) = \hat{c}_0(\delta, \varepsilon) = 0$ they actually solve (**??**) and (**??**). Repeating this argument for every $j = 1, \ldots, k-1$ and $l = 1, \ldots, h-1$ we conclude that

$$u = U_* + \phi$$

with ϕ defined in (**??**) is the solution to problem (**??**) we were looking for. Thus, we want to prove the existence of δ and ε so that (we keep the names in an abuse of notation)

$$\overline{c}_0(\delta, \varepsilon) = \int_{\mathbb{R}^n} (\overline{\zeta}_1 E + \gamma \overline{\mathcal{N}}(\overline{\hat{\phi}}_1, \hat{\hat{\phi}}_1)) \overline{Z}_0 = 0,$$

$$\hat{c}_0(\delta, \varepsilon) = \int_{\mathbb{R}^n} (\hat{\zeta}_1 E + \gamma \hat{\mathcal{N}}(\overline{\hat{\phi}}_1, \hat{\hat{\phi}}_1)) \hat{Z}_0 = 0.$$

Indeed, we will prove that

$$\overline{c}_0(\delta, \varepsilon) = -A_n \frac{\delta}{k^{n-2}} [\delta a_{n,k}^1 - a_{n,k}^2] + \frac{1}{k^{n-1}} \Theta_{k,h}(\delta, \varepsilon),$$
$$\hat{c}_0(\delta, \varepsilon) = -A_n \frac{\varepsilon}{h^{n-2}} [\varepsilon b_{n,h}^1 - b_{n,h}^2] + \frac{1}{h^{n-1}} \Theta_{k,h}(\delta, \varepsilon).$$
$$(1.3.1)$$

Here A_n is a fixed positive constant that depends on n, while for $i = 1, 2$, $a_{n,k}^i$, $b_{n,h}^i$ are positive constants, of the form $a_{n,k}^i = a_n^i + O(\frac{1}{k})$, $b_{n,h}^i = b_n^i + O(\frac{1}{h})$, as $k, h \to \infty$, with a_n^i and b_n^i positive constants. Furthermore, $\Theta_{k,h}(\delta, \varepsilon)$ denotes a generic function, which is smooth in its variables, and it is uniformly bounded, together with its first derivatives, in δ and ε satisfying the bounds (**??**), when $k \to \infty$ and $h \to \infty$. By a fixed point argument one can prove the existence of a solution (δ, ε) to the system

$$\overline{c}_0(\delta, \varepsilon) = \hat{c}_0(\delta, \varepsilon) = 0. \qquad (1.3.2)$$

Thus, if we prove that (**??**) holds, we conclude the proof of Theorem **??**.

Both estimates in (**??**) follow in the same way, so let us prove the first one. We write

$$\overline{c}_0(\delta, \varepsilon) = \int_{\mathbb{R}^n} E\overline{Z}_0 + \int_{\mathbb{R}^n} (\overline{\zeta}_1 - 1) E\overline{Z}_0 + \gamma \int_{\mathbb{R}^n} \overline{\mathcal{N}}(\overline{\hat{\phi}}_1, \hat{\hat{\phi}}_1) \overline{Z}_0. \qquad (1.3.3)$$

and we analyze every term independently. For δ and ε satisfying (**??**), we have that

Claim 1:

$$\int_{\mathbb{R}^n} E\overline{Z}_0 = -A_n \frac{\delta}{k^{n-2}} [\delta a_{n,k}^1 - a_{n,k}^2] + \frac{1}{k^{n-1}} \Theta_{k,h}(\delta, \varepsilon). \qquad (1.3.4)$$

Claim 2:

$$\int_{\mathbb{R}^n} (\overline{\zeta}_1 - 1) E\overline{Z}_0 = \frac{1}{k^{n-1}} \Theta_{k,h}(\delta, \varepsilon). \tag{1.3.5}$$

Claim 3:

$$\int_{\mathbb{R}^n} \overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\phi}_1) \overline{Z}_0 = \frac{1}{k^{n-1}} \Theta_{k,h}(\delta, \varepsilon), \tag{1.3.6}$$

as k and $h \to \infty$. It is clear that these claims imply the validity of the first equation in (**??**).

Proof of Claim 1. Let us denote

$$Ext := \{\cap_{j=1}^k \{|y - \xi_j| > \frac{\overline{\alpha}}{k}\}\} \cap \{\cap_{l=1}^h \{|y - \eta_l| > \frac{\hat{\alpha}}{h}\}\}.$$

For $\overline{\alpha} > 0$ independent of k we can write the first term as

$$\int_{\mathbb{R}^n} E\overline{Z}_0 = \int_{B(\xi_1, \frac{\overline{\alpha}}{k})} E\overline{Z}_0 + \int_{Ext} E\overline{Z}_0 + \sum_{j \neq 1} \int_{B(\xi_j, \frac{\overline{\alpha}}{k})} E\overline{Z}_0 + \sum_{l=1}^h \int_{B(\eta_l, \frac{\hat{\alpha}}{h})} E\overline{Z}_0. \tag{1.3.7}$$

Considering $\overline{E}_1(y) = \mu^{\frac{n+2}{2}} E(\xi_1 + \mu y)$ and using (**??**) we obtain

$$\int_{B(\xi_1, \frac{\overline{\alpha}}{k})} E\overline{Z}_0 = \int_{B(0, \frac{\overline{\alpha}}{\mu k})} \overline{E}_1(y) Z_0(y) \, dy$$

$$= -\gamma p \sum_{j \neq 1} \int_{B(0, \frac{\overline{\alpha}}{\mu k})} U^{p-1} U(y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \, dy$$

$$+ \gamma p \mu^{\frac{n-2}{2}} \int_{B(0, \frac{\overline{\alpha}}{\mu k})} U^{p-1} U(\xi_1 + \mu y) Z_{n+1} \, dy$$

$$- \gamma p \sum_{l=1}^h \mu^{\frac{n-2}{2}} \lambda^{-\frac{n-2}{2}} \int_{B(0, \frac{\overline{\alpha}}{\mu k})} U^{p-1} U(\lambda^{-1}(\xi_1 + \mu y - \eta_l)) Z_{n+1} \, dy$$

$$+ \gamma p \int_{B(0, \frac{\overline{\alpha}}{\mu k})} [(U(y) + sV)^{p-1} - U^{p-1}] V(y) Z_{n+1} \, dy$$

$$- \mu^{\frac{n+2}{2}} \int_{B(0, \frac{\overline{\alpha}}{\mu k})} U^p(\xi_1 + \mu y) Z_{n+1} \, dy$$

$$- \sum_{j \neq 1} \int_{B(0, \frac{\overline{\alpha}}{\mu k})} U^p(y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \, dy$$

$$- \sum_{l=1}^h \mu^{\frac{n+2}{2}} \lambda^{-\frac{n+2}{2}} \int_{B(0, \frac{\overline{\alpha}}{\mu k})} U^p(\lambda^{-1}(\xi_1 + \mu y - \eta_l)) Z_{n+1} \, dy,$$

where

$$V(y) := -\sum_{j \neq 1} U(y - \mu^{-1}(\xi_j - \xi_1)) + \mu^{\frac{n-2}{2}} U(\xi_1 + \mu y)$$

$$- \sum_{l=1}^h \mu^{\frac{n-2}{2}} \lambda^{-\frac{n-2}{2}} U(\lambda^{-1}(\xi_1 + \mu y - \eta_l)).$$

Doing a Taylor expansion, for $j \neq 1$ there holds

$$\int_{B(0,\frac{\overline{\alpha}}{\mu k})} U^{p-1} U(y - \mu^{-1}(\xi_j - \xi_1)) Z_0 \, dy = \frac{c_1 \mu^{n-2}}{|\xi_j - \xi_1|^{n-2}} \left(1 + \frac{\mu^2}{|\xi_j - \xi_1|^2} \Theta_{k,h}(\delta, \varepsilon)\right),$$

and

$$\int_{B(0,\frac{\overline{\alpha}}{\mu k})} U^{p-1} U(\lambda^{-1}(\xi_1 + \mu y - \eta_l)) Z_0 \, dy = \frac{c_1 \lambda^{n-2}}{|\xi_1 - \eta_l|^{n-2}} \left(1 + \frac{\lambda^2}{|\xi_j - \xi_1|^2} \Theta_{k,h}(\delta, \varepsilon)\right),$$

where c_1 is some positive constant, and, as before, $\Theta_{k,h}(\delta, \varepsilon)$ denotes a generic function, which is smooth in its variables, and it is uniformly bounded, together with its first derivatives, in δ and ε satisfying the bounds (??), when $k \to \infty$ and $h \to \infty$. Proceeding in a similar way,

$$\mu^{\frac{n-2}{2}} \int_{B(0,\frac{\overline{\alpha}}{\mu k})} U^{p-1} U(\xi_1 + \mu y) Z_0 \, dy = c_2 \mu^{\frac{n-2}{2}} (1 + (\mu k)^2 \Theta_{k,h}(\delta, \varepsilon)),$$

for some positive constant c_2. On the other hand,

$$\left| \mu^{\frac{n+2}{2}} \int_{B(0,\frac{\overline{\alpha}}{\mu k})} U^p(\xi_1 + \mu y) Z_0 \, dy \right| \leqslant C \mu^{\frac{n+2}{2}} \int_{B(0,\frac{\overline{\alpha}}{\mu k})} \frac{1}{(1 + |y|)^{n-2}} \leqslant C \mu^{\frac{n-2}{2}} k^{-2}, \qquad (1.3.8)$$

$$\left| \sum_{j \neq 1} \int_{B(0,\frac{\overline{\alpha}}{\mu k})} U^p(y - \mu^{-1}(\xi_j - \xi_1)) Z_0 \, dy \right|$$

$$\leqslant \sum_{j \neq 1} \frac{\mu^{n+2}}{|\xi_j - \xi_1|^{n+2}} \int_{B(0,\frac{\overline{\alpha}}{\mu k})} \frac{1}{(1 + |y|)^{n-2}} \qquad (1.3.9)$$

$$\leqslant C(\mu k)^{-2} \sum_{j \neq 1} \frac{\mu^{n+2}}{|\xi_j - \xi_1|^{n+2}},$$

and

$$\left| \sum_{l=1}^{h} \mu^{\frac{n-2}{2}} \lambda^{-\frac{n+2}{2}} \int_{B(0,\frac{\overline{\alpha}}{\mu k})} U^p(\lambda^{-1}(\xi_1 + \mu y - \eta_l)) Z_0 \right|$$

$$\leqslant C \sum_{l=1}^{h} \mu^{\frac{n-2}{2}} \lambda^{\frac{n+2}{2}} \int_{B(0,\frac{\overline{\alpha}}{\mu k})} \frac{1}{(1 + |y|)^{n-2}} \qquad (1.3.10)$$

$$\leqslant C \mu^{\frac{n+2}{2}} \lambda^{\frac{n+2}{2}} h k^2.$$

Finally, putting together (??), (??) and (??),

$$\left| \int_{B(0,\frac{\overline{\alpha}}{\mu k})} [(U(y) + sV)^{p-1} - U^{p-1}] V(y) Z_0 \, dy \right|$$

$$\leqslant C \left(\mu^{\frac{n-2}{2}} k^{-2} + (\mu k)^{-2} \sum_{j \neq 1} \frac{\mu^{n+2}}{|\xi_j - \xi_1|^{n+2}} + \mu^{\frac{n+2}{2}} \lambda^{\frac{n+2}{2}} h k^2 \right). \qquad (1.3.11)$$

To estimate the second term in (??) we apply Hölder inequality to get

$$\left| \int_{Ext} E \overline{Z}_0 \right| \leqslant C \|(1 + |y|)^{n+2 - \frac{2n}{q}} E\|_{L^q(Ext)} \|(1 + |y|)^{-n-2 + \frac{2n}{q}} \overline{Z}_0\|_{L^{\frac{q}{q-1}}(Ext)}. \qquad (1.3.12)$$

Proceeding as in [**?**] we have

$$\|(1+|y|)^{-n-2+\frac{2n}{q}}\overline{Z}_0\|_{L^{\frac{q}{q-1}}(Ext)} \leqslant C\mu^{\frac{n-2}{2}}k^{n-2}k^{\frac{n}{q}-n},$$

and using the estimates obtained in Section **??** we see that

$$\|(1+|y|)^{n+2-\frac{2n}{q}}E\|_{L^q(Ext)} \leqslant C(\mu^{\frac{n-2}{2}}k^{n-2}k^{1-\frac{n}{q}}+\lambda^{\frac{n-2}{2}}h^{n-2}h^{1-\frac{n}{q}}),$$

and thus, substituting in (**??**),

$$\left|\int_{Ext}E\overline{Z}_0\right| \leqslant C\left(\frac{\mu^{n-2}k^{2(n-2)}}{k^{n-1}}+\frac{\mu^{\frac{n-2}{2}}\lambda^{\frac{n-2}{2}}k^{n-2}h^{n-2}}{k^{n-\frac{n}{q}}h^{\frac{n}{q}-1}}\right). \tag{1.3.13}$$

Arguing as in [**?**] the third term in (**??**) can be estimated as

$$\left|\sum_{j\neq 1}\int_{B(\xi_j,\frac{\overline{\alpha}}{k})}E\overline{Z}_0\right| \leqslant \frac{\mu^{\frac{n-2}{2}}}{(\mu k)^{n-4}}\left[\mu^{n-2}\sum_{j\neq 1}\frac{1}{|\xi_j-\xi_1|^{n-2}}\right]. \tag{1.3.14}$$

Likewise,

$$\left|\int_{B(\eta_l,\frac{\hat{\alpha}}{h})}E\overline{Z}_0\right| = \left|\lambda^{\frac{n-2}{2}}\int_{B(0,\frac{\hat{\alpha}}{\lambda h})}\hat{E}_l(y)\overline{Z}_0(\lambda y+\eta_l)\right|$$

$$\leqslant \lambda^{\frac{n-2}{2}}\|(1+|y|)^{n+2-\frac{2n}{q}}\hat{E}_l\|_{L^q(|y|<\frac{\hat{\alpha}}{\lambda h})}$$

$$\cdot \|(1+|y|)^{-n-2+\frac{2n}{q}}\mu^{-\frac{n-2}{2}}Z_0(\mu^{-1}(\lambda y+\eta_l-\xi_1))\|_{L^{\frac{q}{q-1}}(|y|<\frac{\hat{\alpha}}{\lambda h})}.$$

Noticing that

$$\|(1+|y|)^{-n-2+\frac{2n}{q}}\mu^{-\frac{n-2}{2}}Z_0(\mu^{-1}(\lambda y+\eta_l-\xi_1))\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n)}$$

$$\leqslant C\mu^{\frac{n-2}{2}}\left(\int_1^{\frac{\hat{\alpha}}{\lambda h}}\frac{t^{n-1}\,dt}{t^{(n+2-\frac{2n}{q})\frac{q}{q-1}}}\right)^{\frac{q-1}{q}} \leqslant C\mu^{\frac{n-2}{2}}(\lambda h)^{2-\frac{n}{q}},$$

by (**??**) we conclude that

$$\left|\sum_{l=1}^h\int_{B(\eta_l,\frac{\hat{\alpha}}{h})}E\overline{Z}_0\right| \leqslant C\lambda^{\frac{n-2}{2}}h^{-\frac{n}{q}}\mu^{\frac{n-2}{2}}(\lambda h)^{2-\frac{n}{q}}h \tag{1.3.15}$$

Claim 1 follows from these estimates applying the fact that $h = O(k)$.

Proof of Claim 2. Let us estimate the second term of (**??**). Notice first that

$$\left|\int_{\mathbb{R}^n}(\overline{\zeta}_1-1)E\overline{Z}_0\right| \leqslant C\left|\int_{\{|y-\xi_1|>\frac{\overline{\alpha}}{k}\}}E\overline{Z}_0\right|,$$

and we separate this integral as

$$\int_{\{|y-\xi_1|>\frac{\overline{\alpha}}{k}\}}E\overline{Z}_0 = \int_{Ext}E\overline{Z}_0 + \sum_{j=2}^k\int_{\{|y-\xi_j|\leqslant\frac{\overline{\alpha}}{k}\}}E\overline{Z}_0 + \sum_{l=1}^h\int_{\{|y-\eta_l|\leqslant\frac{\hat{\alpha}}{h}\}}E\overline{Z}_0.$$

Thus, Claim 2 follows from (**??**), (**??**) and (**??**).

Proof of Claim 3. Notice that

$$\int_{\mathbb{R}^n} \overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)\overline{Z}_0 = \mu^{\frac{n+2}{2}} \int_{\mathbb{R}^n} \overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)(\xi_1 + \mu y)Z_0(y),$$

and thus, from estimates $(\textbf{??}) - (\textbf{??})$ and the fact $h = O(k)$ we conclude

$$\int_{\mathbb{R}^n} \overline{\mathcal{N}}(\overline{\overline{\phi}}_1, \hat{\hat{\phi}}_1)\overline{Z}_0 \leqslant Ck^{3-n-\frac{n}{q}} \int_{\mathbb{R}^n} U^{p-1}|Z_0|,$$

and the claim follows.

Part 2. Nondegeneracy

As stated before, the goal of this part is to prove the nondegeneracy (see Definition **??**) of the solution u provided by Theorem **??** (we drop the dependence on k and h by simplicity). Recalling the functions z_α defined in (**??**)-(**??**) we can formulate the result in Theorem **??** as follows.

Theorem 2.0.1. *There exists a sequence of solutions u to Problem* (**??**) *among the ones constructed in Theorem* **??** *for which all bounded solutions to the equation*

$$-\Delta\varphi - \gamma p|u|^{p-2}u\varphi = 0 \tag{2.0.1}$$

are linear combination of the functions z_α for $\alpha = 0, \ldots, N_0 - 1$. (Recall that $N_0 := 5(n-1)$.)

For later simplification, we introduce the following functions

$$\mathbf{z}_\beta := z_\beta, \quad \text{if} \quad \beta \neq n+3, n+4, n+5, n+6,$$

$$\mathbf{z}_{n+2+\alpha} := \frac{z_\alpha - z_{n+2+\alpha}}{2}, \quad \text{if} \quad \alpha = 1, 2, 3, 4. \tag{2.0.2}$$

Since $\mathbf{z}_\beta$ are linear combinations of the original functions z_β, the statement of Theorem **??** is equivalent to say that there exists a sequence of solutions among the ones constructed in Theorem **??** for which all bounded solutions to (**??**) are linear combinations of $\mathbf{z}_\beta$, for $\beta = 0, \ldots, N_0 - 1$.

Thus, let φ be a bounded solution of (**??**), namely $L(\varphi) = 0$, with L defined in (**??**). We decompose φ as

$$\varphi(y) = \sum_{\beta=0}^{N_0-1} a_\beta \mathbf{z}_\beta(y) + \tilde{\varphi}(y), \tag{2.0.3}$$

where a_β are chosen so that

$$\int_{\mathbb{R}^n} u^{p-1}\mathbf{z}_\beta\tilde{\varphi} = 0 \tag{2.0.4}$$

holds. Notice that, since $\mathbf{z}_\beta \in \ker\{L\}$, one has $L(\tilde{\varphi}) = 0$ and thus our goal will be to prove that actually $\tilde{\varphi} \equiv 0$.

Recall that our solution u has the form

$$u(y) = U(y) - \sum_{j=1}^{k} U_{\mu,\xi_j}(y) - \sum_{l=1}^{h} U_{\lambda,\eta_l}(y) + \phi(y), \tag{2.0.5}$$

where ϕ is defined in (**??**) as $\phi = \sum_{j=1}^{k} \overline{\phi}_j + \sum_{l=1}^{h} \hat{\phi}_l + \psi$.

We introduce the following functions

$$Z_{00}(y) := \frac{n-2}{2}\left[U + \psi\right](y) + \nabla[U + \psi](y) \cdot y,$$

$$Z_{\alpha 0}(y) := \frac{\partial}{\partial y_\alpha}U(y) + \frac{\partial}{\partial y_\alpha}\psi(y) \qquad \alpha = 1, \dots, n. \tag{2.0.6}$$

For j fixed in $\{1, \dots, k\}$, we define

$$\overline{Z}_{0j}(y) := \frac{n-2}{2}[U_{\mu,\xi_j} + \bar\phi_j](y) + \nabla[U_{\mu,\xi_j} + \bar\phi_j](y) \cdot (y - \xi_j)$$

$$\overline{Z}_{1j}(y) := \xi_j \cdot \nabla_y[U_{\mu,\xi_j} + \bar\phi_j](y), \qquad \overline{Z}_{2j}(y) := \xi_j^\perp \cdot \nabla_y[U_{\mu,\xi_j} + \bar\phi_j](y),$$

$$\overline{Z}_{\alpha j}(y) := \frac{\partial}{\partial y_\alpha}[U_{\mu,\xi_j}(y) + \bar\phi_j], \qquad \alpha = 3, \dots, n.$$

For l fixed in $\{1, \dots, h\}$, we define

$$\hat{Z}_{0j}(y) := \frac{n-2}{2}[U_{\lambda,\eta_l} + \hat\phi_l](y) + \nabla[U_{\lambda,\eta_l} + \hat\phi_l](y) \cdot (y - \eta_l)$$

$$\hat{Z}_{3l}(y) := \eta_l \cdot \nabla_y[U_{\lambda,\eta_l} + \hat\phi_l](y), \qquad \hat{Z}_{4l}(y) := \eta_l^\perp \cdot \nabla_y[U_{\lambda,\eta_l} + \hat\phi_l](y),$$

$$\hat{Z}_{\alpha l}(y) := \frac{\partial}{\partial y_\alpha}[U_{\lambda,\eta_l}(y) + \hat\phi_l], \qquad \alpha = 1, 2, 5, 6, \dots, n.$$

In Appendix **??** we provide the expressions of the functions $\mathbf{z}_\beta$, $\beta = 0, \dots, N_0 - 1$, in terms of the functions $Z_{\alpha 0}$, $\overline{Z}_{\alpha j}$, $j = 1, \dots, k$, and $\hat{Z}_{\alpha,l}$, $l = 1, \dots, h$, for any $\alpha = 0, \dots, n$. These relations will be useful in other parts of our argument.

We rearrange the functions above in $(n+1)$ vector fields as

$$\Pi_\alpha := \left[Z_{\alpha 0}, \overline{Z}_{\alpha 1}, \dots, \overline{Z}_{\alpha k}, \hat{Z}_{\alpha 1}, \dots, \hat{Z}_{\alpha h}\right]^T, \quad \alpha = 0, 1, \dots, n, \tag{2.0.7}$$

and, for any given vector $d = \left[d_0, \bar{d}_1, \dots, \bar{d}_k, \hat{d}_1, \dots, \hat{d}_h\right]^T \in \mathbb{R}^{1+k+h}$ we use the notation

$$d \cdot \Pi_\alpha := d_0 Z_{\alpha 0} + \sum_{j=1}^k \bar{d}_j \overline{Z}_{\alpha j} + \sum_{l=1}^h \hat{d}_l \hat{Z}_{\alpha l}.$$

With this in mind, we write the function $\tilde\varphi$ in (**??**) as

$$\tilde\varphi(y) = \sum_{\alpha=0}^n c_\alpha \cdot \Pi_\alpha(y) + \varphi^\perp(y), \tag{2.0.8}$$

where

$$c_\alpha := [c_{\alpha 0}, \bar{c}_{\alpha 1}, \dots, \bar{c}_{\alpha k}, \hat{c}_{\alpha 1}, \dots, \hat{c}_{\alpha h}]^T, \qquad \alpha = 0, \dots, n,$$

are $(n+1)$ vectors in $\mathbb{R}^{k+h+1}$ chosen so that

$$\int_{\mathbb{R}^n} U^{p-1} Z_{\alpha 0}\varphi^\perp = 0, \qquad \alpha = 0, 1, \dots, n,$$

$$\int_{\mathbb{R}^n} U_{\mu,\xi_j}^{p-1} \overline{Z}_{\alpha j}\varphi^\perp = 0, \qquad j = 1, \dots, k, \ \ \alpha = 0, 1, \dots, n,$$

$$\int_{\mathbb{R}^n} U_{\lambda,\eta_l}^{p-1} \hat{Z}_{\alpha l}\varphi^\perp = 0, \qquad l = 1, \dots, h, \ \ \alpha = 0, 1, \dots, n.$$

Hence, to prove that $\tilde{\varphi} \equiv 0$ we have to see that $c_\alpha = 0$ for every α and $\varphi^\perp \equiv 0$. This will be consequence of the following three facts.

Fact 1: Since $L(\tilde{\varphi}) = 0$, one has that

$$\sum_{\alpha=0}^{n} c_\alpha \cdot L(\Pi_\alpha) = -L(\varphi^\perp), \tag{2.0.9}$$

with L defined in (**??**). We write $\varphi^\perp = \varphi_0^\perp + \sum_{j=1}^{k} \overline{\varphi}_j^\perp + \sum_{l=1}^{h} \hat{\varphi}_l^\perp$, where

$$-L(\varphi_0^\perp) = \sum_{\alpha=0}^{n} c_{\alpha 0} L(Z_{\alpha 0}),$$

$$-L(\overline{\varphi}_j^\perp) = \sum_{\alpha=0}^{n} \overline{c}_{\alpha j} L(\overline{Z}_{\alpha j}), \qquad j = 1, \ldots, k,$$

$$-L(\hat{\varphi}_l^\perp) = \sum_{\alpha=0}^{n} \hat{c}_{\alpha l} L(\hat{Z}_{\alpha l}), \qquad l = 1, \ldots, h.$$

Furthermore, let us define

$$\overline{\overline{\varphi}}_j^\perp(y) := \mu^{\frac{n-2}{2}} \overline{\varphi}_j^\perp(\mu y + \xi_j), \qquad \hat{\hat{\varphi}}_l^\perp(y) := \lambda^{\frac{n-2}{2}} \hat{\varphi}_l^\perp(\lambda y + \eta_l),$$

and

$$\|\varphi^\perp\| := \|\varphi_0^\perp\|_* + \sum_{j=1}^{k} \|\overline{\overline{\varphi}}_j^\perp\|_* + \sum_{l=1}^{h} \|\hat{\hat{\varphi}}_l^\perp\|_*.$$

Thus, as we will prove in Section **??**, there exists a positive constant C such that

$$\|\varphi^\perp\| \leqslant Ck^{-2+\frac{2}{n}} \sum_{\alpha=0}^{n} \|c_\alpha\|. \tag{2.0.10}$$

Fact 2: Condition (**??**) is equivalent to

$$\sum_{\alpha=0}^{n} c_\alpha \cdot \int_{\mathbb{R}^n} \Pi_\alpha u^{p-1} \mathbf{z}_\beta = \sum_{\alpha=0}^{n} \left[c_{\alpha 0} \int_{\mathbb{R}^n} Z_{\alpha 0} u^{p-1} \mathbf{z}_\beta + \sum_{j=1}^{k} \overline{c}_{\alpha j} \int_{\mathbb{R}^n} \overline{Z}_{\alpha j} u^{p-1} \mathbf{z}_\beta \right.$$

$$\left. + \sum_{l=1}^{h} \hat{c}_{\alpha l} \int_{\mathbb{R}^n} \hat{Z}_{\alpha l} u^{p-1} \mathbf{z}_\beta \right] \tag{2.0.11}$$

$$= -\int_{\mathbb{R}^n} \varphi^\perp u^{p-1} \mathbf{z}_\beta, \qquad \beta = 0, \ldots, N_0 - 1.$$

Let us denote

$$
\overline{\cos} := \begin{bmatrix} 1 \\ \cos\overline{\theta}_2 \\ \dots \\ \cos\overline{\theta}_{k-1} \end{bmatrix}, \ \overline{\sin} := \begin{bmatrix} 0 \\ \sin\overline{\theta}_2 \\ \dots \\ \sin\overline{\theta}_{k-1} \end{bmatrix}, \qquad \overline{\theta}_j := \frac{2\pi}{k}(j-1),
$$

$$
\hat{\cos} := \begin{bmatrix} 1 \\ \cos\hat{\theta}_2 \\ \dots \\ \cos\hat{\theta}_{h-1} \end{bmatrix}, \ \hat{\sin} := \begin{bmatrix} 0 \\ \sin\hat{\theta}_2 \\ \dots \\ \sin\hat{\theta}_{h-1} \end{bmatrix}, \qquad \hat{\theta}_l := \frac{2\pi}{h}(l-1),
$$

(2.0.12)

and

$$
\overline{1} := \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}, \qquad \hat{1} := \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}, \qquad \overline{0} := \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \qquad \hat{0} := \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \qquad (2.0.13)
$$

where $\overline{1}$ and $\overline{0}$ are k-dimensional vectors, and $\hat{1}$ and $\hat{0}$ are vectors of dimension h. Likewise, define

$$
\tilde{c}_0 := \begin{bmatrix} c_{00} \\ \dots \\ c_{n0} \end{bmatrix} \in \mathbb{R}^{n+1}, \qquad \overline{c}_\alpha := \begin{bmatrix} \overline{c}_{\alpha 1} \\ \dots \\ \overline{c}_{\alpha k} \end{bmatrix} \in \mathbb{R}^k, \qquad \hat{c}_\alpha := \begin{bmatrix} \hat{c}_{\alpha 1} \\ \dots \\ \hat{c}_{\alpha h} \end{bmatrix} \in \mathbb{R}^h,
$$

and

$$
\overline{c} := \begin{bmatrix} \overline{c}_0 \\ \overline{c}_1 \\ \dots \\ \overline{c}_{n+1} \end{bmatrix} \in \mathbb{R}^{(n+1)k}, \qquad \hat{c} := \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \dots \\ \hat{c}_n \end{bmatrix} \in \mathbb{R}^{(n+1)h}.
$$

Thus,

Proposition 2.0.2. *Solving system* (**??**) *is equivalent to solve*

$$
c_0 \cdot \begin{bmatrix} 1 \\ -\overline{1} \\ \hat{1} \end{bmatrix} + c_1 \cdot \begin{bmatrix} 0 \\ -\overline{1} \\ \hat{0} \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ -\hat{1} \end{bmatrix} = t_0
$$
$$
+ R_{h,k}[c_0, c_1, \dots c_n],
$$

(2.0.14)

$$
c_1 \cdot \begin{bmatrix} 1 \\ -\overline{\cos} \\ -\hat{1} \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ \overline{\sin} \\ \hat{0} \end{bmatrix} = t_1 + R_{h,k}[c_0, c_1, \dots c_n], \qquad (2.0.15)
$$

$$
c_1 \cdot \begin{bmatrix} 0 \\ -\overline{\sin} \\ \hat{0} \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ -\overline{\cos} \\ -\hat{1} \end{bmatrix} = t_2 + R_{h,k}[c_0, c_1, \dots c_n], \qquad (2.0.16)
$$

$$
c_3 \cdot \begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{\cos} \end{bmatrix} + c_4 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\sin} \end{bmatrix} = t_3 + R_{h,k}[c_0, c_1, \dots c_n], \qquad (2.0.17)
$$

$$c_3 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ -\hat{\sin} \end{bmatrix} + c_4 \cdot \begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{\cos} \end{bmatrix} = t_4 + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.18}$$

$$c_\alpha \cdot \begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{1} \end{bmatrix} = t_\alpha + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.19}$$

for $\alpha = 5, \ldots, n,$

$$c_2 \cdot \begin{bmatrix} 0 \\ \overline{1} \\ \hat{0} \end{bmatrix} = t_{n+1} + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.20}$$

$$c_4 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{1} \end{bmatrix} = t_{n+2} + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.21}$$

$$c_0 \cdot \begin{bmatrix} 0 \\ \overline{\cos} \\ \hat{0} \end{bmatrix} - c_1 \cdot \begin{bmatrix} 0 \\ \overline{\cos} \\ \hat{0} \end{bmatrix} = t_{n+3} + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.22}$$

$$c_0 \cdot \begin{bmatrix} 0 \\ \overline{\sin} \\ \hat{0} \end{bmatrix} - c_1 \cdot \begin{bmatrix} 0 \\ \overline{\sin} \\ \hat{0} \end{bmatrix} = t_{n+4} + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.23}$$

$$c_0 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\cos} \end{bmatrix} - c_3 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\cos} \end{bmatrix} = t_{n+5} + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.24}$$

$$c_0 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\sin} \end{bmatrix} - c_3 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\sin} \end{bmatrix} = t_{n+6} + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.25}$$

$$c_1 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\cos} \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ -\overline{\cos} \\ \hat{0} \end{bmatrix} = t_{n+7} + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.26}$$

$$c_1 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\sin} \end{bmatrix} + c_4 \cdot \begin{bmatrix} 0 \\ -\overline{\cos} \\ \hat{0} \end{bmatrix} = t_{n+8} + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.27}$$

$$c_\alpha \cdot \begin{bmatrix} 0 \\ -\overline{\cos} \\ \hat{0} \end{bmatrix} = t_{n+\alpha+4} + R_{h,k}[c_0, c_1, \ldots c_n], \tag{2.0.28}$$

for $\alpha = 5, \ldots, n$,

$$c_2 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\cos} \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ -\overline{\sin} \\ \hat{0} \end{bmatrix} = t_{2n+5} + R_{h,k}[c_0, c_1, \ldots c_n], \qquad (2.0.29)$$

$$c_2 \cdot \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\sin} \end{bmatrix} + c_4 \cdot \begin{bmatrix} 0 \\ -\overline{\sin} \\ \hat{0} \end{bmatrix} = t_{2n+6} + R_{h,k}[c_0, c_1, \ldots c_n], \qquad (2.0.30)$$

and, for $\alpha = 5, \ldots, n$,

$$c_\alpha \cdot \begin{bmatrix} 0 \\ -\overline{\sin} \\ \hat{0} \end{bmatrix} = t_{2n+\alpha+2} + R_{h,k}[c_0, c_1, \ldots c_n], \qquad (2.0.31)$$

$$c_\alpha \cdot \begin{bmatrix} 0 \\ \overline{0} \\ -\hat{\sin} \end{bmatrix} = t_{3n+\alpha-2} + R_{h,k}[c_0, c_1, \ldots c_n], \qquad (2.0.32)$$

$$c_\alpha \cdot \begin{bmatrix} 0 \\ \overline{0} \\ -\hat{\cos} \end{bmatrix} = t_{4n+\alpha-6} + R_{h,k}[c_0, c_1, \ldots c_n]. \qquad (2.0.33)$$

Here t_i, $i = 0, \ldots, 5n - 6$, are fixed numbers such that

$$\|t_i\| \leqslant C\|\varphi^\perp\|.$$

Moreover, $R_{h,k}[c_0, c_1, \ldots c_n]$ stands for a function, whose specific definition changes from line to line, which can be described as follows:

$$R_{h,k}[c_0, c_1, \ldots c_n] = \Theta_{k,h}\mathcal{L}\left(c_{00}, \ldots, c_{n0}\right) + \overline{\Theta}_{k,h}\overline{\mathcal{L}}\left(\overline{c}_1, \ldots, \overline{c}_k\right)$$
$$+ \hat{\Theta}_{k,h}\hat{\mathcal{L}}\left(\hat{c}_1, \ldots, \hat{c}_h\right)$$

and $\mathcal{L} : \mathbb{R}^{n+1} \to \mathbb{R}$, $\overline{\mathcal{L}} : \mathbb{R}^{k(n+1)} \to \mathbb{R}$, $\hat{\mathcal{L}} : \mathbb{R}^{h(n+1)} \to \mathbb{R}$ are linear functions uniformly bounded when $k, h \to \infty$, and

$$\Theta_{k,h} = O(k^{1-\frac{n}{q}}), \qquad \overline{\Theta}_{k,h} = O(k^{-\frac{n}{q}}), \qquad \hat{\Theta}_{k,h} = O(k^{-\frac{n}{q}}),$$

*where $O(1)$ denotes a quantity uniformly bounded when $k, h \to \infty$, and $\frac{n}{2} < q < n$ is the number fixed in (**??**).*

We will prove this result in Section **??**.

Fact 3: Multiplying (**??**) for every $Z_{\alpha 0}$, $\overline{Z}_{\alpha j}$, $\hat{Z}_{\alpha l}$, $\alpha = 0, 1, \ldots, n$, $j = 1, \ldots, k$ and $l = 1, \ldots, h$ and integrating in $\mathbb{R}^n$ we get a system of the form

$$
M \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = - \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_n \end{bmatrix} \qquad \text{with} \qquad r_\alpha := \begin{bmatrix} \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{\alpha 0} \\ \int_{\mathbb{R}^n} L(\varphi^\perp) \overline{Z}_{\alpha 1} \\ \vdots \\ \int_{\mathbb{R}^n} L(\varphi^\perp) \overline{Z}_{\alpha k} \\ \int_{\mathbb{R}^n} L(\varphi^\perp) \hat{Z}_{\alpha 1} \\ \vdots \\ \int_{\mathbb{R}^n} L(\varphi^\perp) \hat{Z}_{\alpha h} \end{bmatrix}. \tag{2.0.34}
$$

Due to the symmetries, the matrix M has the form

$$
M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},
$$

where M_1 and M_2 are square matrices of dimensions $(5 \times (k+h+1))^2$ and $((n-4) \times (k+h+1))^2$ of the form

$$
M_1 = \begin{bmatrix} \tilde{A} & \tilde{B} & \tilde{C} & \tilde{D} & \tilde{E} \\ \tilde{B}^T & \tilde{F} & \tilde{G} & \tilde{H} & \tilde{I} \\ \tilde{C}^T & \tilde{G}^T & \tilde{J} & \tilde{K} & \tilde{L} \\ \tilde{D}^T & \tilde{H}^T & \tilde{K}^T & \tilde{M} & \tilde{N} \\ \tilde{E}^T & \tilde{I}^T & \tilde{L}^T & \tilde{N}^T & \tilde{P} \end{bmatrix}, \qquad M_2 = \begin{bmatrix} \tilde{H}_5 & 0 & 0 & 0 \\ 0 & \tilde{H}_6 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \tilde{H}_n \end{bmatrix},
$$

with

$$
\tilde{H}_\alpha = \begin{bmatrix} \int L(Z_{\alpha 0}) Z_{\alpha 0} & \left(\int L(Z_{\alpha 0}) \overline{Z}_{\alpha j} \right)_j & \left(\int L(Z_{\alpha 0}) \hat{Z}_{\alpha l} \right)_l \\ \left(\int L(\overline{Z}_{\alpha i}) Z_{\alpha 0} \right)_i & \left(\int L(\overline{Z}_{\alpha i}) \overline{Z}_{\alpha j} \right)_{i,j} & \left(\int L(\overline{Z}_{\alpha i}) \hat{Z}_{\alpha l} \right)_{i,l} \\ \left(\int L(\hat{Z}_{\alpha m}) Z_{\alpha 0} \right)_m & \left(\int L(\hat{Z}_{\alpha m}) \overline{Z}_{\alpha j} \right)_{m,j} & \left(\int L(\hat{Z}_{\alpha m}) \hat{Z}_{\alpha l} \right)_{m,l} \end{bmatrix}, \tag{2.0.35}
$$

for $i, j = 1, \ldots, k$ and $m, l = 1, \ldots, h$. Thus, solving (**??**) is equivalent to find a solution of

$$
M_1 \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}, \qquad \tilde{H}_\alpha c_\alpha = r_\alpha \text{ for } \alpha = 5, \ldots, n, \tag{2.0.36}
$$

with r_α defined in (**??**).

Proposition 2.0.3. *There exists k_0, h_0 such that, for all $k > k_0$, $h > h_0$, system (**??**) is solvable. Moreover, the solution has the form*

$$
c_0 = v_0 + t_0 \begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{1} \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}
$$
$$
+ \overline{t}_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \overline{t}_1 \begin{bmatrix} 0 \\ \overline{\cos} \\ \hat{0} \end{bmatrix} + \overline{t}_2 \begin{bmatrix} 0 \\ \overline{\sin} \\ \hat{0} \end{bmatrix} + \hat{t}_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\cos} \end{bmatrix} + \hat{t}_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\sin} \end{bmatrix},
$$

$$c_1 = v_1 + t_0 \begin{bmatrix} 0 \\ -\overline{1} \\ \hat{0} \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\overline{\cos} \\ -\hat{1} \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{1-\mu^2}}\overline{\sin} \\ \hat{0} \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}$$

$$+ \bar{t}_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \bar{t}_1 \begin{bmatrix} 0 \\ -\overline{\cos} \\ \hat{0} \end{bmatrix} + \bar{t}_2 \begin{bmatrix} 0 \\ -\overline{\sin} \\ \hat{0} \end{bmatrix} + \hat{t}_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix},$$

$$c_2 = v_2 + t_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ \frac{1}{\sqrt{1-\mu^2}}\overline{\sin} \\ \hat{0} \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\overline{\cos} \\ -\hat{1} \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}$$

$$+ \bar{t}_0 \begin{bmatrix} 0 \\ \overline{1} \\ \hat{0} \end{bmatrix} + \bar{t}_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \bar{t}_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix},$$

$$c_3 = v_3 + t_0 \begin{bmatrix} 0 \\ \overline{0} \\ -\hat{1} \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ -\overline{1} \\ -\frac{1}{\sqrt{1-\lambda^2}}\hat{\cos} \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ \overline{0} \\ -\frac{1}{\sqrt{1-\lambda^2}}\hat{\sin} \end{bmatrix}$$

$$+ \bar{t}_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \bar{t}_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \bar{t}_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_1 \begin{bmatrix} 0 \\ \overline{0} \\ -\hat{\cos} \end{bmatrix} + \hat{t}_2 \begin{bmatrix} 0 \\ \overline{0} \\ -\hat{\sin} \end{bmatrix},$$

$$c_4 = v_4 + t_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ \overline{0} \\ \frac{1}{\sqrt{1-\lambda^2}}\hat{\sin} \end{bmatrix} + t_4 \begin{bmatrix} 1 \\ -\overline{1} \\ -\frac{1}{\sqrt{1-\lambda^2}}\hat{\cos} \end{bmatrix}$$

$$+ \bar{t}_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \bar{t}_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \bar{t}_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_0 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{1} \end{bmatrix} + \hat{t}_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix} + \hat{t}_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \end{bmatrix},$$

and, for $\alpha = 5, \ldots, n$,

$$c_\alpha = v_\alpha + t_\alpha \begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{1} \end{bmatrix} + \overline{\nu}_{\alpha 1} \begin{bmatrix} 0 \\ \overline{\cos} \\ \hat{0} \end{bmatrix} + \overline{\nu}_{\alpha 2} \begin{bmatrix} 0 \\ \overline{\sin} \\ \hat{0} \end{bmatrix} + \hat{\nu}_{\alpha 1} \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\cos} \end{bmatrix} + \hat{\nu}_{\alpha 2} \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\sin} \end{bmatrix},$$

for any $t_0, t_1, t_2, t_3, t_4, \bar{t}_0, \bar{t}_1, \bar{t}_2, \hat{t}_0, \hat{t}_1, \hat{t}_2,$ and $t_\alpha, \overline{\nu}_{\alpha 1}, \overline{\nu}_{\alpha 2}, \hat{\nu}_{\alpha 1}, \hat{\nu}_{\alpha 2}$ real parameters. The vectors $v_\alpha \in \mathbb{R}^{k+h+1}$ are fixed and satisfy

$$\|v_\alpha\| \leqslant C\|\varphi^\perp\|, \ \alpha = 0, 1, \ldots, n.$$

Proof. Proceeding as in [?, Proposition 6.1] it can be checked that, for any $\alpha = 0, \ldots, n$,

$$\|\overline{r}_\alpha\| \leqslant C\mu^{\frac{n-2}{2}}\|\varphi^\perp\|, \qquad \|\hat{r}_\alpha\| \leqslant C\lambda^{\frac{n-2}{2}}\|\varphi^\perp\|,$$

and combining this estimate with Lemma **??** and Lemma **??** we obtain the result. $\qquad\square$

We shall use the following notations: for any $\alpha = 0, 1, \ldots, n$,

$$\tilde{c}_\alpha := \begin{bmatrix} \overline{c}_\alpha \\ \hat{c}_\alpha \end{bmatrix} \in \mathbb{R}^{k+h}, \ \ \overline{c}_\alpha := \begin{bmatrix} \overline{c}_{\alpha 1} \\ \ldots \\ \overline{c}_{\alpha k} \end{bmatrix} \in \mathbb{R}^k, \ \ \hat{c}_\alpha := \begin{bmatrix} \hat{c}_{\alpha 1} \\ \ldots \\ \hat{c}_{\alpha h} \end{bmatrix} \in \mathbb{R}^h, \qquad (2.0.37)$$

$$\tilde{r}_\alpha := \begin{bmatrix} \overline{r}_\alpha \\ \hat{r}_\alpha \end{bmatrix} \in \mathbb{R}^{k+h},$$

where

$$\overline{r}_\alpha := \begin{bmatrix} \int_{\mathbb{R}^n} L(\varphi^\perp)\overline{Z}_{\alpha 1} \\ \vdots \\ \int_{\mathbb{R}^n} L(\varphi^\perp)\overline{Z}_{\alpha k} \end{bmatrix} \in \mathbb{R}^k, \quad \hat{r}_\alpha := \begin{bmatrix} \int_{\mathbb{R}^n} L(\varphi^\perp)\hat{Z}_{\alpha 1} \\ \vdots \\ \int_{\mathbb{R}^n} L(\varphi^\perp)\hat{Z}_{\alpha h} \end{bmatrix} \in \mathbb{R}^h.$$

2.1. Solving the second system in (??)

Let α be fixed in $\{5, \ldots, n\}$. This section is devoted to solve

$$\tilde{H}_\alpha c_\alpha = r_\alpha, \tag{2.1.1}$$

where r_α is the vector defined in (??).

Using (??) and (??) and the fact that $L(\mathbf{z}_\alpha) = 0$ it follows that

$$\operatorname{row}_1(\tilde{H}_\alpha) = \sum_{l=2}^{k+h+1} \operatorname{row}_l(\tilde{H}_\alpha).$$

As a consequence, $\begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{1} \end{bmatrix} \in \ker(\tilde{H}_\alpha)$, and hence $\tilde{H}_\alpha c_\alpha = r_\alpha$ has a solution only if $r_\alpha \cdot \begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{1} \end{bmatrix} = 0$. This last orthogonality condition is indeed fulfilled since one has

$$\operatorname{row}_1(r_\alpha) = \sum_{j=2}^{k+1} \operatorname{row}_j(r_\alpha) + \sum_{l=k+2}^{h+k+1} \operatorname{row}_l(r_\alpha), \tag{2.1.2}$$

again as consequence of the fact that $L(\mathbf{z}_\alpha) = 0$. Thus, the general solution to (??) has the form

$$c_\alpha = \begin{bmatrix} 0 \\ \tilde{c}_\alpha \end{bmatrix} + t \begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{1} \end{bmatrix}, \quad t \in \mathbb{R},$$

where $\tilde{c}_\alpha$ solves

$$H_\alpha \tilde{c}_\alpha = \tilde{r}_\alpha, \tag{2.1.3}$$

with

$$H_\alpha = \begin{bmatrix} \overline{H}_\alpha & \gamma_\alpha \mathbb{1}_{k \times h} \\ \gamma_\alpha \mathbb{1}_{h \times k} & \hat{H}_\alpha \end{bmatrix}, \quad \alpha = 5, \ldots, n,$$

being $\overline{H}_\alpha$ and $\hat{H}_\alpha$ square matrices of dimensions $k \times k$ and $h \times h$ respectively, defined by

$$\overline{H}_\alpha := \left(\int L(\overline{Z}_{\alpha,i})\overline{Z}_{\alpha,j} \, dy \right)_{i,j=1,\ldots k}, \quad \hat{H}_\alpha := \left(\int L(\hat{Z}_{\alpha,l})\hat{Z}_{\alpha,m} \, dy \right)_{l,m=1,\ldots h}$$

and

$$\gamma_\alpha := \int_{\mathbb{R}^n} L(\overline{Z}_{\alpha 1})\hat{Z}_{\alpha 1}.$$

By $\mathbb{1}_{s \times t}$ we mean a $s \times t$-dimensional matrix whose entries are all 1. Observe that

$$|\gamma_\alpha| \leqslant Ck^{4-2n}, \tag{2.1.4}$$

for some fixed constant C. Arguing as in [**?**], one can show that $\bar{H}_\alpha$ and $\hat{H}_\alpha$ are circulant matrices of dimensions $(k \times k)$ and $(h \times h)$ respectively (see [**?**] for properties). Moreover, [**?**, Proposition 5.1] ensures that

$$\overline{H}_\alpha[\overline{c}_\alpha] = \overline{s}_\alpha, \quad \hat{H}_\alpha[\hat{c}_\alpha] = \hat{s}_\alpha, \tag{2.1.5}$$

has a solution if

$$\overline{s}_\alpha \cdot \overline{\cos} = \overline{s}_\alpha \cdot \overline{\sin} = 0 \quad \text{and} \quad \hat{s}_\alpha \cdot \hat{\cos} = \hat{s}_\alpha \cdot \hat{\sin} = 0.$$

Actually, if a solution to (**??**) exists, it has the form

$$\overline{c}_\alpha = \overline{w}_\alpha + \overline{\nu}_1\overline{\cos} + \overline{\nu}_2\overline{\sin}, \quad \hat{c}_\alpha = \hat{w}_\alpha + \hat{\nu}_1\hat{\cos} + \hat{\nu}_2\hat{\sin},$$

for all $\overline{\nu}_1, \overline{\nu}_2, \hat{\nu}_1, \hat{\nu}_2 \in \mathbb{R}$, where $\overline{w}_\alpha$, $\hat{w}_\alpha$ are the unique solutions to

$$\overline{H}_\alpha\overline{w}_\alpha = \overline{s}_\alpha, \quad \overline{w}_\alpha \cdot \overline{\cos} = \overline{w}_\alpha \cdot \overline{\sin} = 0$$

and

$$\hat{H}_\alpha\hat{w}_\alpha = \hat{s}_\alpha, \quad \hat{w}_\alpha \cdot \hat{\cos} = \hat{w}_\alpha \cdot \hat{\sin} = 0.$$

Furthermore, in [**?**, Proposition 5.1] it is proved that there exists a constant C independent of k so that, for all k large

$$\|\overline{w}_\alpha\| \leqslant Ck^{n-4}\|\overline{s}_\alpha\| \quad \text{and} \quad \|\hat{w}_\alpha\| \leqslant Ck^{n-4}\|\hat{s}_\alpha\|.$$

We start with the observation that system (**??**) is solvable. Indeed, since $L(\mathbf{z}_{n+\alpha+4}) = L(\mathbf{z}_{2n+\alpha+2}) = 0$, one has that $\overline{r}_\alpha \cdot \overline{\cos} = \overline{r}_\alpha \cdot \overline{\sin} = 0$. Similarly, one gets that $\hat{r}_\alpha \cdot \hat{\cos} = \hat{r}_\alpha \cdot \hat{\sin} = 0$, as consequence of the fact that $L(\mathbf{z}_{3n+\alpha-2}) = L(\mathbf{z}_{4n+\alpha-6}) = 0$. Moreover, the vector $\mathbb{1}_{k \times h}\hat{c}_\alpha$ is a multiple of $\overline{1}$, and $\mathbb{1}_{h \times k}\overline{c}_\alpha$ is a multiple of $\hat{1}$. Thus $\mathbb{1}_{k \times h}\hat{c}_\alpha \cdot \cos = \mathbb{1}_{k \times h}\hat{c}_\alpha \cdot \sin = 0$, and $\mathbb{1}_{h \times k}\overline{c}_\alpha \cdot \hat{\cos} = \mathbb{1}_{h \times k}\overline{c}_\alpha \cdot \hat{\sin} = 0$. Now we observe that the solution of system (**??**) has the form

$$\begin{aligned} \overline{c}_\alpha &= \overline{w}_\alpha + \overline{\nu}_1\overline{\cos} + \overline{\nu}_2\overline{\sin} \\ \hat{c}_\alpha &= \hat{w}_\alpha + \hat{\nu}_1\overline{\cos} + \hat{\nu}_2\overline{\sin}, \end{aligned} \tag{2.1.6}$$

for any value for $\overline{\nu}_1, \overline{\nu}_2, \hat{\nu}_1, \hat{\nu}_2 \in \mathbb{R}$, where $\overline{w}_\alpha$ and $\hat{w}_\alpha$ are the unique solutions to

$$\begin{aligned} \overline{H}_\alpha\overline{w}_\alpha &= \overline{r}_\alpha - \gamma_\alpha\mathbb{1}_{k \times h}\hat{w}_\alpha, \quad \overline{w}_\alpha \cdot \overline{\cos} = \overline{w}_\alpha \cdot \overline{\sin} = 0, \\ \hat{H}_\alpha\hat{w}_\alpha &= \hat{r}_\alpha - \gamma_\alpha\mathbb{1}_{h \times k}\overline{w}_\alpha, \quad \hat{w}_\alpha \cdot \hat{\cos} = \hat{w}_\alpha \cdot \hat{\sin} = 0. \end{aligned} \tag{2.1.7}$$

Moreover, there exists a constant C so that

$$\|\overline{w}_\alpha\| \leqslant Ck^{n-4}\|\overline{r}_\alpha\| \quad \text{and} \quad \|\hat{w}_\alpha\| \leqslant Ck^{n-4}\|\hat{r}_\alpha\|. \tag{2.1.8}$$

Existence and uniqueness of solutions to (**??**) satisfying (**??**) follows from a contraction map argument. Indeed, $\begin{bmatrix} \overline{w}_\alpha \\ \hat{w}_\alpha \end{bmatrix}$ is a solution if and only if it is a fixed point to $A_\alpha \begin{bmatrix} \overline{w}_\alpha \\ \hat{w}_\alpha \end{bmatrix} :=$ $T_\alpha^{-1}\left(\begin{bmatrix} \overline{r}_\alpha - \gamma_\alpha\mathbb{1}_{k \times h}\hat{w}_\alpha \\ \hat{r}_\alpha - \gamma_\alpha\mathbb{1}_{h \times k}\overline{w}_\alpha \end{bmatrix}\right)$, where we denote by T_α the linear map $T_\alpha\left(\begin{bmatrix} \overline{w}_\alpha \\ \hat{w}_\alpha \end{bmatrix}\right) = \left(\begin{bmatrix} \overline{H}_\alpha\overline{w}_\alpha \\ \hat{H}_\alpha\hat{w}_\alpha \end{bmatrix}\right)$, which is invertible for vectors that are orthogonal to $\overline{\cos}$, $\overline{\sin}$, in their first components, and to $\hat{\cos}$, $\hat{\sin}$ in their second components. Let

$$B_r := \{\begin{bmatrix} \overline{w}_\alpha \\ \hat{w}_\alpha \end{bmatrix} \in K_\alpha \; : \; \|\overline{w}_\alpha\| \leqslant rk^{n-4}\|\overline{r}_\alpha\| \quad \text{and} \quad \|\hat{w}_\alpha\| \leqslant rk^{n-4}\|\hat{r}_\alpha\|\},$$

where $K_\alpha := \{ \begin{bmatrix} \overline{w}_\alpha \\ \hat{w}_\alpha \end{bmatrix} \in \mathbb{R}^{k+h} \ : \ \overline{w}_\alpha \cdot \overline{\cos} = \overline{w}_\alpha \cdot \overline{\sin} = 0, \hat{w}_\alpha \cdot \hat{\cos} = \hat{w}_\alpha \cdot \hat{\sin} = 0 \}$. Then, choosing r large but fixed, and thanks to (**??**), one has that A_α is a contraction in B_r. This gives the existence of solutions to (**??**), satisfying (**??**).

Summarizing the above arguments, we have

Lemma 2.1.1. *Let* $\alpha \in \{5, \ldots, n\}$ *be fixed. Then system* (**??**) *is solvable, and the solution has the form*

$$c_\alpha = \begin{bmatrix} 0 \\ \overline{w}_\alpha \\ \hat{w}_\alpha \end{bmatrix} + t \begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{1} \end{bmatrix} + \overline{\nu}_1 \begin{bmatrix} 0 \\ \overline{\cos} \\ \hat{0} \end{bmatrix} + \overline{\nu}_2 \begin{bmatrix} 0 \\ \overline{\sin} \\ \hat{0} \end{bmatrix} + \hat{\nu}_1 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\cos} \end{bmatrix} + \hat{\nu}_2 \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\sin} \end{bmatrix}, \qquad (2.1.9)$$

for any values of $t, \overline{\nu}_1, \overline{\nu}_2, \hat{\nu}_1, \hat{\nu}_2 \in \mathbb{R}$. *In the above formula* $\begin{bmatrix} \overline{w}_\alpha \\ \hat{w}_\alpha \end{bmatrix}$ *is the unique solution to* (**??**), *and satisfies* (**??**).

2.2. SOLVING THE FIRST SYSTEM IN (**??**)

This section is devoted to solve the first system in (**??**), namely

$$M_1 \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}. \qquad (2.2.1)$$

Using (**??**) and (**??**) and the fact that $L(\mathbf{z}_\alpha) = 0$ for every $\alpha = 0, \ldots, 4$, together with the result in Section **??**, we observe that

$$\mathrm{row}_1(M_1) = \sum_{i=1}^{k} \mathrm{row}_{1+i}(M_1) + \mathrm{row}_{k+h+2+i}(M_1) + \sum_{i=1}^{h} \mathrm{row}_{k+1+i}(M_1) + \mathrm{row}_{4k+3h+4+i}(M_1),$$

$$\mathrm{row}_{k+h+2}(M_1) = \frac{1}{\sqrt{1-\mu^2}} \left[\sum_{i=1}^{k} \cos\overline{\theta}_i \mathrm{row}_{k+h+2+i}(M_1) - \sin\overline{\theta}_i \mathrm{row}_{2k+2h+3+i}(M_1) \right]$$
$$+ \sum_{i=1}^{h} \mathrm{row}_{2k+h+2+i}(M_1),$$

$$\mathrm{row}_{2k+2h+3}(M_1) = \frac{1}{\sqrt{1-\mu^2}} \left[\sum_{i=1}^{k} \sin\overline{\theta}_i \mathrm{row}_{k+h+2+i}(M_1) + \cos\overline{\theta}_i \mathrm{row}_{2k+2h+3+i}(M_1) \right]$$
$$+ \sum_{i=1}^{h} \mathrm{row}_{3k+2h+3+i}(M_1),$$

$$\text{row}_{3k+3h+4}(M_1) = \sum_{i=1}^{k} \text{row}_{3k+3h+4+i}(M_1)$$

$$+ \frac{1}{\sqrt{1-\lambda^2}} \left[\sum_{i=1}^{h} \cos\hat{\theta}_i \text{row}_{4k+3h+4+i}(M_1) - \sin\hat{\theta}_i \text{row}_{5k+4h+5+i}(M_1) \right],$$

$$\text{row}_{4k+4h+5}(M_1) = \sum_{i=1}^{k} \text{row}_{4k+4h+5+i}(M_1)$$

$$+ \frac{1}{\sqrt{1-\lambda^2}} \left[\sum_{i=1}^{h} \sin\hat{\theta}_i \text{row}_{4k+3h+4+i}(M_1) + \cos\hat{\theta}_i \text{row}_{5k+4h+5+i}(M_1) \right].$$

From these facts we deduce that system (**??**) is solvable only if

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \cdot w_j = 0, \qquad j = 0, 1, \ldots, 4, \tag{2.2.2}$$

where (recall definitions (**??**) and (**??**))

$$w_0 := \begin{bmatrix} 1 \\ -\overline{1} \\ -\hat{1} \\ 0 \\ -\overline{1} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ -\hat{1} \\ 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}, \quad w_1 := \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\overline{\cos} \\ -\hat{1} \\ 0 \\ \frac{1}{\sqrt{1-\mu^2}}\overline{\sin} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}, \quad w_2 := \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ -\frac{1}{\sqrt{1-\mu^2}}\overline{\sin} \\ \hat{0} \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\overline{\cos} \\ -\hat{1} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}, \tag{2.2.3}$$

$$
w_3 := \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 1 \\ -\overline{1} \\ -\frac{1}{\sqrt{1-\lambda^2}}\hat{\cos} \\ 0 \\ \overline{0} \\ \frac{1}{\sqrt{1-\lambda^2}}\hat{\sin} \end{bmatrix}, \quad w_4 := \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ -\frac{1}{\sqrt{1-\lambda^2}}\hat{\sin} \\ 1 \\ -\overline{1} \\ -\frac{1}{\sqrt{1-\lambda^2}}\hat{\cos} \end{bmatrix}, \tag{2.2.4}
$$

which belong all to $\ker(M_1)$. On the other hand, using again that $L(\mathbf{z}_\alpha) = 0$ for every $\alpha = 0, \ldots, 4$ one sees that the vectors r_α satisfy the following relations

$$
\mathrm{row}_1(r_0) = \sum_{j=2}^{k+1} [\mathrm{row}_j(r_0) + \mathrm{row}_j(r_1)] + \sum_{l=k+2}^{h+k+1} [\mathrm{row}_l(r_0) + \mathrm{row}_l(r_3)]
$$

$$
\mathrm{row}_1(r_1) = \frac{1}{\sqrt{1-\mu^2}} \sum_{j=2}^{k+1} \left[\cos\overline{\theta}_{j-1}\mathrm{row}_j(r_1) - \sin\overline{\theta}_{j-1}\mathrm{row}_j(r_2)\right] + \sum_{l=k+2}^{h+k+1} \mathrm{row}_l(r_1),
$$

$$
\mathrm{row}_1(r_2) = \frac{1}{\sqrt{1-\mu^2}} \sum_{j=2}^{k+1} \left[\sin\overline{\theta}_{j-1}\mathrm{row}_j(r_1) + \cos\overline{\theta}_{j-1}\mathrm{row}_j(r_2)\right] + \sum_{l=k+2}^{h+k+1} \mathrm{row}_l(r_2),
$$

$$
\mathrm{row}_1(r_3) = \sum_{j=2}^{k+1} \mathrm{row}_j(r_3) + \frac{1}{\sqrt{1-\lambda^2}} \sum_{l=k+2}^{h+k+1} \left[\cos\hat{\theta}_{l-1}\mathrm{row}_l(r_3) - \sin\hat{\theta}_{l-1}\mathrm{row}_l(r_4)\right],
$$

$$
\mathrm{row}_1(r_4) = \sum_{j=2}^{k+1} \mathrm{row}_j(r_4) + \frac{1}{\sqrt{1-\lambda^2}} \sum_{l=k+2}^{h+k+1} \left[\sin\hat{\theta}_{l-1}\mathrm{row}_l(r_3) + \cos\hat{\theta}_{l-1}\mathrm{row}_l(r_4)\right].
$$

These facts imply that the orthogonality conditions (**??**) are satisfied, and thus (**??**) is solvable. The solution to (**??**) has the form

$$
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{c}_0 \\ 0 \\ \tilde{c}_1 \\ 0 \\ \tilde{c}_2 \\ 0 \\ \tilde{c}_3 \\ 0 \\ \tilde{c}_4 \end{bmatrix} + tw_0 + sw_1 + rw_2 + uw_3 + vw_4,
$$

for any values of $t, s, r, u, v \in \mathbb{R}$, where $\tilde{c}_\alpha := \begin{bmatrix} \bar{c}_\alpha \\ \hat{c}_\alpha \end{bmatrix}$ are solutions of

$$Q \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \\ \tilde{c}_4 \end{bmatrix} = \begin{bmatrix} \tilde{r}_0 \\ \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \\ \tilde{r}_4 \end{bmatrix}, \quad \tilde{r}_\alpha := \begin{bmatrix} \bar{r}_\alpha \\ \hat{r}_\alpha \end{bmatrix} \tag{2.2.5}$$

Here Q is the square matrix of dimension $[5(k+h) \times (k+h)]^2$ defined as

$$Q := \begin{bmatrix} A & B & C & D & E \\ B^T & F & G & H & I \\ C^T & G^T & J & K & L \\ D^T & H^T & K^T & M & N \\ E^T & I^T & L^T & N^T & P \end{bmatrix},$$

where every submatrix of Q has dimension $(k+h) \times (k+h)$ and entries of the form

$$\int_{\mathbb{R}^n} L(V)W,$$

where

   (i) In A: $V, W \in \{(\overline{Z}_{0j})_{j=1,\ldots,k}, (\hat{Z}_{0l})_{l=1,\ldots,h}\}$.
  (ii) In B: $V \in \{(\overline{Z}_{0j})_{j=1,\ldots,k}, (\hat{Z}_{0l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{1j})_{j=1,\ldots,k}, (\hat{Z}_{1l})_{l=1,\ldots,h}\}$.
 (iii) In C: $V \in \{(\overline{Z}_{0j})_{j=1,\ldots,k}, (\hat{Z}_{0l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{2j})_{j=1,\ldots,k}, (\hat{Z}_{2l})_{l=1,\ldots,h}\}$.
 (iv) In D: $V \in \{(\overline{Z}_{0j})_{j=1,\ldots,k}, (\hat{Z}_{0l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{3j})_{j=1,\ldots,k}, (\hat{Z}_{3l})_{l=1,\ldots,h}\}$.
  (v) In E: $V \in \{(\overline{Z}_{0j})_{j=1,\ldots,k}, (\hat{Z}_{0l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{4j})_{j=1,\ldots,k}, (\hat{Z}_{4l})_{l=1,\ldots,h}\}$.
 (vi) In F: $V, W \in \{(\overline{Z}_{1j})_{j=1,\ldots,k}, (\hat{Z}_{1l})_{l=1,\ldots,h}\}$.
(vii) In G: $V \in \{(\overline{Z}_{1j})_{j=1,\ldots,k}, (\hat{Z}_{1l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{2j})_{j=1,\ldots,k}, (\hat{Z}_{2l})_{l=1,\ldots,h}\}$.
(viii) In H: $V \in \{(\overline{Z}_{1j})_{j=1,\ldots,k}, (\hat{Z}_{1l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{3j})_{j=1,\ldots,k}, (\hat{Z}_{3l})_{l=1,\ldots,h}\}$.
 (ix) In I: $V \in \{(\overline{Z}_{1j})_{j=1,\ldots,k}, (\hat{Z}_{1l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{4j})_{j=1,\ldots,k}, (\hat{Z}_{4l})_{l=1,\ldots,h}\}$.
  (x) In J: $V, W \in \{(\overline{Z}_{2j})_{j=1,\ldots,k}, (\hat{Z}_{2l})_{l=1,\ldots,h}\}$.
 (xi) In K: $V \in \{(\overline{Z}_{2j})_{j=1,\ldots,k}, (\hat{Z}_{2l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{3j})_{j=1,\ldots,k}, (\hat{Z}_{3l})_{l=1,\ldots,h}\}$.
 (xii) In L: $V \in \{(\overline{Z}_{2j})_{j=1,\ldots,k}, (\hat{Z}_{2l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{4j})_{j=1,\ldots,k}, (\hat{Z}_{4l})_{l=1,\ldots,h}\}$.
(xiii) In M: $V, W \in \{(\overline{Z}_{3j})_{j=1,\ldots,k}, (\hat{Z}_{3l})_{l=1,\ldots,h}\}$.
(xiv) In N: $V \in \{(\overline{Z}_{3j})_{j=1,\ldots,k}, (\hat{Z}_{3l})_{l=1,\ldots,h}\}$, $W \in \{(\overline{Z}_{4j})_{j=1,\ldots,k}, (\hat{Z}_{4l})_{l=1,\ldots,h}\}$.
 (xv) In P: $V, W \in \{(\overline{Z}_{4j})_{j=1,\ldots,k}, (\hat{Z}_{4l})_{l=1,\ldots,h}\}$.

Let us analize the structure of every matrix, where we will make use of the notation

$$\beta_{\alpha_1,\alpha_2} := \int L(\overline{Z}_{\alpha_1 1})\hat{Z}_{\alpha_2 1}, \tag{2.2.6}$$

$$\overline{\theta}_j := \frac{2\pi}{k}(j-1) \qquad \text{and} \qquad \hat{\theta}_l := \frac{2\pi}{h}(l-1).$$

Matrix A. Due to the invariance properties one can check that for $i, j = 1, \ldots, k$ and $l, m = 1, \ldots, h$,

$$\int_{\mathbb{R}^n} L(\overline{Z}_{0i})\overline{Z}_{0j} = \int_{\mathbb{R}^n} L(\overline{Z}_{01})\overline{Z}_{0,|i-j|+1},$$

$$\int_{\mathbb{R}^n} L(\hat{Z}_{0l})\hat{Z}_{0m} = \int_{\mathbb{R}^n} L(\hat{Z}_{01})\hat{Z}_{0,|l-m|+1},$$

$$\int_{\mathbb{R}^n} L(\overline{Z}_{0j})\hat{Z}_{0l} = \int_{\mathbb{R}^n} L(\hat{Z}_{0l})\overline{Z}_{0j} = \int_{\mathbb{R}^n} L(\overline{Z}_{01})\hat{Z}_{01} = \beta_{00}.$$

Thus we can write

$$A = \begin{bmatrix} \overline{A} & A_1 := \beta_{00}\mathbb{1}_{k \times h} \\ A_2 := \beta_{00}\mathbb{1}_{h \times k} & \hat{A} \end{bmatrix},$$

where $\overline{A}$ and $\hat{A}$ are circulant matrices of dimensions $(k \times k)$ and $(h \times h)$ (see [?]).

Matrix B. Applying the invariance properties like in matrix A we obtain

$$B = \begin{bmatrix} \overline{B} & B_1 := \left(\int L(\overline{Z}_{0j})\hat{Z}_{1l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ B_2 := \left(\int L(\hat{Z}_{0l})\overline{Z}_{1j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & 0 \end{bmatrix},$$

where $\overline{B}$ is a $(k \times k)$ circulant matrix. Rotating in the (y_1, y_2) and (y_3, y_4) one obtains

$$\int L(\overline{Z}_{0j})\hat{Z}_{1l} = \cos\overline{\theta}_j\beta_{01} - \sin\overline{\theta}_j\beta_{02} = \cos\overline{\theta}_j\beta_{01},$$

$$\int L(\hat{Z}_{0l})\overline{Z}_{1j} = \cos\overline{\theta}_j\beta_{10} - \sin\overline{\theta}_j\beta_{20} = \cos\overline{\theta}_j\beta_{10},$$

for $j = 1, \ldots, k$, $l = 1, \ldots, h$, since $\beta_{02} = \beta_{20} = 0$ due to the symmetry properties. Notice that both expressions are independent of l.

Matrix C. Likewise,

$$C = \begin{bmatrix} \overline{C} & C_1 := \left(\int L(\overline{Z}_{0j})\hat{Z}_{2l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ C_2 := \left(\int L(\hat{Z}_{0l})\overline{Z}_{2j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & 0 \end{bmatrix},$$

whith $\overline{C}$ being a $(k \times k)$ circulant matrix and

$$\int L(\overline{Z}_{0j})\hat{Z}_{2l} = \sin\overline{\theta}_j\beta_{01}, \qquad j = 1, \ldots, k,\, l = 1, \ldots, h,$$

$$\int L(\hat{Z}_{0l})\overline{Z}_{2j} = \sin\overline{\theta}_j\beta_{10}, \qquad j = 1, \ldots, k,\, l = 1, \ldots, h.$$

Matrix D.

$$D = \begin{bmatrix} 0 & D_1 := \left(\int L(\overline{Z}_{0j})\hat{Z}_{3l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ D_2 := \left(\int L(\hat{Z}_{0l})\overline{Z}_{3j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & \hat{D} \end{bmatrix},$$

whith $\hat{D}$ being a $(h \times h)$ circulant matrix and

$$\int L(\overline{Z}_{0j})\hat{Z}_{3l} = \cos\hat{\theta}_l\beta_{03}, \qquad j = 1, \ldots, k,\, l = 1, \ldots, h,$$

$$\int L(\hat{Z}_{0l})\overline{Z}_{3j} = \cos\hat{\theta}_l\beta_{30}, \qquad j = 1, \ldots, k,\, l = 1, \ldots, h,$$

where we have used that $\beta_{04} = \beta_{40} = 0$.

Matrix E.

$$E = \begin{bmatrix} 0 & E_1 := \left(\int L(\overline{Z}_{0j})\hat{Z}_{4l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ E_2 := \left(\int L(\hat{Z}_{0l})\overline{Z}_{4j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & \hat{E} \end{bmatrix},$$

whith $\hat{E}$ being a $(h \times h)$ circulant matrix and

$$\int L(\overline{Z}_{0j})\hat{Z}_{4l} = \sin\hat{\theta}_l\beta_{03}, \qquad j = 1, \ldots, k,\, l = 1, \ldots, h,$$

$$\int L(\hat{Z}_{0l})\overline{Z}_{4j} = \sin\hat{\theta}_l\beta_{30}, \qquad j = 1, \ldots, k,\, l = 1, \ldots, h.$$

Matrix F.

$$F = \begin{bmatrix} \overline{F} & F_1 := \left(\int L(\overline{Z}_{1j})\hat{Z}_{1l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ F_2 := \left(\int L(\hat{Z}_{1l})\overline{Z}_{1j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & \hat{F} \end{bmatrix},$$

where $\overline{F}$ and $\hat{F}$ are $(k \times k)$ and $(h \times h)$ circulant matrices respectively and

$$\int L(\overline{Z}_{1j})\hat{Z}_{1l} = \int L(\hat{Z}_{1l})\overline{Z}_{1j} = \cos^2\overline{\theta}_j\beta_{11} + \sin^2\overline{\theta}_j\beta_{22},$$

for $j = 1, \ldots, k,\, l = 1, \ldots, h$, using that $\beta_{12} = \beta_{21} = 0$.

Matrix G.

$$G = \begin{bmatrix} \overline{G} & G_1 := \left(\int L(\overline{Z}_{1j})\hat{Z}_{2l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ G_2 := \left(\int L(\hat{Z}_{1l})\overline{Z}_{2j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & 0 \end{bmatrix},$$

where $\overline{G}$ is a $(k \times k)$ circulant matrix and

$$\int L(\overline{Z}_{1j})\hat{Z}_{2l} = \int L(\hat{Z}_{1l})\overline{Z}_{2j} = \cos\overline{\theta}_j\sin\overline{\theta}_j\beta_{11} - \sin\overline{\theta}_j\cos\overline{\theta}_j\beta_{22},$$

for $j = 1, \ldots, k,\, l = 1, \ldots, h$.

Matrix H.

$$H = \begin{bmatrix} 0 & H_1 := \left(\int L(\overline{Z}_{1j})\hat{Z}_{3l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ H_2 := \left(\int L(\hat{Z}_{1l})\overline{Z}_{3j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & 0 \end{bmatrix},$$

where, since $\beta_{14} = \beta_{41} = \beta_{23} = \beta_{32} = \beta_{24} = \beta_{42} = 0$,

$$\int L(\overline{Z}_{1j})\hat{Z}_{3l} = \cos\overline{\theta}_j \cos\hat{\theta}_l \beta_{13},$$

$$\int L(\hat{Z}_{1l})\overline{Z}_{3j} = \cos\overline{\theta}_j \cos\hat{\theta}_l \beta_{31},$$

for $j = 1, \ldots, k$, $l = 1, \ldots, h$.

Matrix I.

$$I = \begin{bmatrix} 0 & I_1 := \left(\int L(\overline{Z}_{1j})\hat{Z}_{4l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ I_2 := \left(\int L(\hat{Z}_{1l})\overline{Z}_{4j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & 0 \end{bmatrix},$$

where

$$\int L(\overline{Z}_{1j})\hat{Z}_{4l} = \cos\overline{\theta}_j \sin\hat{\theta}_l \beta_{13},$$

$$\int L(\hat{Z}_{1l})\overline{Z}_{4j} = \cos\overline{\theta}_j \sin\hat{\theta}_l \beta_{31},$$

for $j = 1, \ldots, k$, $l = 1, \ldots, h$.

Matrix J.

$$J = \begin{bmatrix} \overline{J} & J_1 := \left(\int L(\overline{Z}_{2j})\hat{Z}_{2l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ J_2 := \left(\int L(\hat{Z}_{2l})\overline{Z}_{2j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & \hat{J} \end{bmatrix},$$

where $\overline{J}$ and $\hat{J}$ are $(k \times k)$ and $(h \times h)$ circulant matrices respectively and

$$\int L(\overline{Z}_{2j})\hat{Z}_{2l} = \int L(\hat{Z}_{2l})\overline{Z}_{2j} = \sin^2\overline{\theta}_j\beta_{11} + \cos^2\overline{\theta}_j\beta_{22},$$

for $j = 1, \ldots, k$, $l = 1, \ldots, h$.

Matrix K.

$$K = \begin{bmatrix} 0 & K_1 := \left(\int L(\overline{Z}_{2j})\hat{Z}_{3l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ K_2 := \left(\int L(\hat{Z}_{2l})\overline{Z}_{3j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & 0 \end{bmatrix},$$

where

$$\int L(\overline{Z}_{2j})\hat{Z}_{3l} = \sin\overline{\theta}_j \cos\hat{\theta}_l \beta_{13},$$

$$\int L(\hat{Z}_{2l})\overline{Z}_{3j} = \sin\overline{\theta}_j \cos\hat{\theta}_l \beta_{31},$$

for $j = 1, \ldots, k$, $l = 1, \ldots, h$.

Matrix L.

$$L = \begin{bmatrix} 0 & L_1 := \left(\int L(\overline{Z}_{2j})\hat{Z}_{4l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ L_2 := \left(\int L(\hat{Z}_{2l})\overline{Z}_{4j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & 0 \end{bmatrix},$$

where

$$\int L(\overline{Z}_{2j})\hat{Z}_{4l} = \sin\overline{\theta}_j \sin\hat{\theta}_l \beta_{13},$$

$$\int L(\hat{Z}_{2l})\overline{Z}_{4j} = \sin\overline{\theta}_j \sin\hat{\theta}_l \beta_{31},$$

for $j = 1, \ldots, k$, $l = 1, \ldots, h$.

Matrix M.

$$M = \begin{bmatrix} \overline{M} & M_1 := \left(\int L(\overline{Z}_{3j})\hat{Z}_{3l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ M_2 := \left(\int L(\hat{Z}_{3l})\overline{Z}_{3j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & \hat{M} \end{bmatrix},$$

where $\overline{M}$ and $\hat{M}$ are $(k \times k)$ and $(h \times h)$ circulant matrices respectively and, since $\beta_{34} = \beta_{43} = 0$,

$$\int L(\overline{Z}_{3j})\hat{Z}_{3l} = \int L(\hat{Z}_{3l})\overline{Z}_{3j} = \cos^2\hat{\theta}_l \beta_{33} + \sin^2\hat{\theta}_l \beta_{44},$$

for $j = 1, \ldots, k$, $l = 1, \ldots, h$.

Matrix N.

$$N = \begin{bmatrix} 0 & N_1 := \left(\int L(\overline{Z}_{3j})\hat{Z}_{4l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ N_2 := \left(\int L(\hat{Z}_{3l})\overline{Z}_{4j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & \hat{N} \end{bmatrix},$$

where $\hat{N}$ is a $(h \times h)$ circulant matrix and

$$\int L(\overline{Z}_{3j})\hat{Z}_{4l} = \int L(\hat{Z}_{3l})\overline{Z}_{4j} = \cos\hat{\theta}_l \sin\hat{\theta}_l \beta_{33} - \sin\hat{\theta}_l \cos\hat{\theta}_l \beta_{44},$$

for $j = 1, \ldots, k$, $l = 1, \ldots, h$.

Matrix P.

$$P = \begin{bmatrix} \overline{P} & P_1 := \left(\int L(\overline{Z}_{4j})\hat{Z}_{4l}\right)_{j=1,\ldots,k,\, l=1,\ldots,h} \\ P_2 := \left(\int L(\hat{Z}_{4l})\overline{Z}_{4j}\right)_{l=1,\ldots,h,\, j=1,\ldots,k} & \hat{P} \end{bmatrix},$$

where $\overline{P}$ and $\hat{P}$ are $(k \times k)$ and $(h \times h)$ circulant matrices respectively and

$$\int L(\overline{Z}_{4j})\hat{Z}_{4l} = \int L(\hat{Z}_{4l})\overline{Z}_{4j} = \sin^2\hat{\theta}_l \beta_{33} + \cos^2\hat{\theta}_l \beta_{44},$$

for $j = 1, \ldots, k$, $l = 1, \ldots, h$.

Notice that

$$\hat{F} = \hat{J} \qquad \text{and} \qquad \overline{M} = \overline{P}.$$

Henceforth, system (**??**) can be decomposed in two different systems in the following way,

$$\begin{bmatrix} \overline{A} & \overline{B} & \overline{C} & 0 & 0 \\ \overline{B}^T & \overline{F} & \overline{G} & 0 & 0 \\ \overline{C}^T & \overline{G}^T & \overline{J} & 0 & 0 \\ 0 & 0 & 0 & \overline{M} & 0 \\ 0 & 0 & 0 & 0 & \overline{P} \end{bmatrix} \begin{bmatrix} \overline{c}_0 \\ \overline{c}_1 \\ \overline{c}_2 \\ \overline{c}_3 \\ \overline{c}_4 \end{bmatrix} = \begin{bmatrix} \overline{r}_0 \\ \overline{r}_1 \\ \overline{r}_2 \\ \overline{r}_3 \\ \overline{r}_4 \end{bmatrix} - \begin{bmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ B_2^T & F_1 & G_1 & H_1 & I_1 \\ C_2^T & G_2^T & J_1 & K_1 & L_1 \\ D_2^T & H_2^T & K_2^T & M_1 & N_1 \\ E_2^T & I_2^T & L_2^T & N_2^T & P_1 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{bmatrix}, \quad (2.2.7)$$

$$
\begin{bmatrix}
\hat{A} & 0 & 0 & \hat{D} & \hat{E} \\
0 & \hat{F} & 0 & 0 & 0 \\
0 & 0 & \hat{J} & 0 & 0 \\
\hat{D}^T & 0 & 0 & \hat{M} & \hat{N} \\
\hat{E}^T & 0 & 0 & \hat{N}^T & \hat{P}
\end{bmatrix}
\begin{bmatrix}
\hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4
\end{bmatrix}
=
\begin{bmatrix}
\hat{r}_0 \\ \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4
\end{bmatrix}
-
\begin{bmatrix}
A_2 & B_2 & C_2 & D_2 & E_2 \\
B_1^T & F_2 & G_2 & H_2 & I_2 \\
C_1^T & G_1^T & J_2 & K_2 & L_2 \\
D_1^T & H_1^T & K_1^T & M_2 & N_2 \\
E_1^T & I_1^T & L_1^T & N_1^T & P_2
\end{bmatrix}
\begin{bmatrix}
\bar{c}_0 \\ \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \bar{c}_4
\end{bmatrix}. \quad (2.2.8)
$$

By [**?**, Proposition 5.1] we know that the systems

$$
\begin{bmatrix}
\overline{A} & \overline{B} & \overline{C} & 0 & 0 \\
\overline{B}^T & \overline{F} & \overline{G} & 0 & 0 \\
\overline{C}^T & \overline{G}^T & \overline{J} & 0 & 0 \\
0 & 0 & 0 & \overline{M} & 0 \\
0 & 0 & 0 & 0 & \overline{P}
\end{bmatrix}
\begin{bmatrix}
\bar{c}_0 \\ \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \bar{c}_4
\end{bmatrix}
=
\begin{bmatrix}
\bar{s}_0 \\ \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \\ \bar{s}_4
\end{bmatrix},
$$

$$
\begin{bmatrix}
\hat{A} & 0 & 0 & \hat{D} & \hat{E} \\
0 & \hat{F} & 0 & 0 & 0 \\
0 & 0 & \hat{J} & 0 & 0 \\
\hat{D}^T & 0 & 0 & \hat{M} & \hat{N} \\
\hat{E}^T & 0 & 0 & \hat{N}^T & \hat{P}
\end{bmatrix}
\begin{bmatrix}
\hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4
\end{bmatrix}
=
\begin{bmatrix}
\hat{s}_0 \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4
\end{bmatrix}
$$

are solvable if the orthogonality conditions

$$
\begin{aligned}
&\bar{s}_2 \cdot \overline{1} = (\bar{s}_0 + \bar{s}_1) \cdot \overline{\cos} = (\bar{s}_0 + \bar{s}_1) \cdot \overline{\sin} = 0, \\
&\bar{s}_3 \cdot \overline{\cos} = \bar{s}_3 \cdot \overline{\sin} = 0, \\
&\bar{s}_4 \cdot \overline{\cos} = \bar{s}_4 \cdot \overline{\sin} = 0, \\
&\hat{s}_4 \cdot \hat{1} = (\hat{s}_0 + \hat{s}_3) \cdot \hat{\cos} = (\hat{s}_0 + \hat{s}_3) \cdot \hat{\sin} = 0, \\
&\hat{s}_1 \cdot \hat{\cos} = \hat{s}_1 \cdot \hat{\sin} = 0, \\
&\hat{s}_2 \cdot \hat{\cos} = \hat{s}_2 \cdot \hat{\sin} = 0,
\end{aligned}
\quad (2.2.9)
$$

hold. Moreover,

$$
\begin{aligned}
&\begin{bmatrix} \bar{c}_0 \\ \bar{c}_1 \\ \bar{c}_2 \end{bmatrix}
= \begin{bmatrix} \overline{w}_0 \\ \overline{w}_1 \\ \overline{w}_2 \end{bmatrix}
+ \bar{t}_1 \begin{bmatrix} 0 \\ 0 \\ \overline{1} \end{bmatrix}
+ \bar{t}_2 \begin{bmatrix} \overline{\cos} \\ -\overline{\cos} \\ 0 \end{bmatrix}
+ \bar{t}_3 \begin{bmatrix} \overline{\sin} \\ -\overline{\sin} \\ 0 \end{bmatrix}, \quad \forall \bar{t}_1, \bar{t}_2, \bar{t}_3 \in \mathbb{R}, \\
&\bar{c}_3 = \overline{w}_3 + \bar{t}_4 \overline{\cos} + \bar{t}_5 \overline{\sin} \quad \forall \bar{t}_4, \bar{t}_5 \in \mathbb{R}, \\
&\bar{c}_4 = \overline{w}_4 + \bar{t}_6 \overline{\cos} + \bar{t}_7 \overline{\sin}, \quad \forall \bar{t}_6, \bar{t}_7 \in \mathbb{R}, \\
&\hat{c}_1 = \hat{w}_1 + \hat{t}_1 \hat{\cos} + \hat{t}_2 \hat{\sin} \quad \forall \hat{t}_1, \hat{t}_2 \in \mathbb{R}, \\
&\hat{c}_2 = \hat{w}_2 + \hat{t}_3 \hat{\cos} + \hat{t}_4 \hat{\sin}, \quad \forall \hat{t}_3, \hat{t}_4 \in \mathbb{R}, \\
&\begin{bmatrix} \hat{c}_0 \\ \hat{c}_3 \\ \hat{c}_4 \end{bmatrix}
= \begin{bmatrix} \hat{w}_0 \\ \hat{w}_3 \\ \hat{w}_4 \end{bmatrix}
+ \hat{t}_5 \begin{bmatrix} 0 \\ 0 \\ \hat{1} \end{bmatrix}
+ \hat{t}_6 \begin{bmatrix} \hat{\cos} \\ -\hat{\cos} \\ 0 \end{bmatrix}
+ \hat{t}_7 \begin{bmatrix} \hat{\sin} \\ -\hat{\sin} \\ 0 \end{bmatrix}, \quad \forall \hat{t}_5, \hat{t}_6, \hat{t}_7 \in \mathbb{R},
\end{aligned}
\quad (2.2.10)
$$

with $\begin{bmatrix} \overline{w}_0 \\ \cdots \\ \overline{w}_4 \end{bmatrix}$, $\begin{bmatrix} \hat{w}_0 \\ \cdots \\ \hat{w}_4 \end{bmatrix}$ fixed vectors such that

$$\left\| \begin{bmatrix} \overline{w}_0 \\ \cdots \\ \overline{w}_4 \end{bmatrix} \right\| \leqslant \frac{C}{k^n \mu^{n-2}} \left\| \begin{bmatrix} \overline{s}_0 \\ \cdots \\ \overline{s}_4 \end{bmatrix} \right\|, \quad \left\| \begin{bmatrix} \hat{w}_0 \\ \cdots \\ \hat{w}_4 \end{bmatrix} \right\| \leqslant \frac{C}{h^n \lambda^{n-2}} \left\| \begin{bmatrix} \hat{s}_0 \\ \cdots \\ \hat{s}_4 \end{bmatrix} \right\|. \tag{2.2.11}$$

We will prove that **(??)** and **(??)** have a solution $\begin{bmatrix} \overline{c}_0 \\ \hat{c}_0 \\ \cdots \\ \overline{c}_4 \\ \hat{c}_4 \end{bmatrix}$ in the space

$$X := \left\{ \begin{bmatrix} \overline{c}_0 \\ \hat{c}_0 \\ \cdots \\ \overline{c}_4 \\ \hat{c}_4 \end{bmatrix} : \begin{array}{ll} \overline{c}_3 \cdot \overline{\cos} = \overline{c}_3 \cdot \overline{\sin} = 0, & \overline{c}_4 \cdot \overline{\cos} = \overline{c}_4 \cdot \overline{\sin} = 0, \\ \hat{c}_1 \cdot \hat{\cos} = \hat{c}_1 \cdot \hat{\sin} = 0, & \hat{c}_2 \cdot \hat{\cos} = \hat{c}_2 \cdot \hat{\sin} = 0. \end{array} \right\}.$$

We need the following auxiliar result, whose proof follows straightforward using the same argument as in Lemma **??** so we skip it.

Lemma 2.2.1. *Let h, g be functions in $\mathbb{R}^n$ such that $h(y) = h(e^{\frac{2\pi}{k}(j-1)}\overline{y}, \hat{y}, y')$ for all $j = 1, \dots, k$ and $g(y) = g(\overline{y}, e^{\frac{2\pi}{h}(l-1)}\hat{y}, y')$ for all $l = 1, \dots, h$. Then,*

$$\int_{\mathbb{R}^n} \hat{Z}_{1l}(y) h(y) \, dy = \int_{\mathbb{R}^n} \hat{Z}_{2l}(y) h(y) \, dy = 0, \ \forall \, l = 1, \dots, h,$$

$$\int_{\mathbb{R}^n} \overline{Z}_{3j}(y) g(y) \, dy = \int_{\mathbb{R}^n} \overline{Z}_{4j}(y) g(y) \, dy = 0, \ \forall \, j = 1, \dots, k.$$

Let us focus on **(??)**. By **(??)**,

$$\overline{r}_2 \cdot \overline{1} = \sum_{j=1}^k \int L(\varphi^\perp) \overline{Z}_{2j}$$

$$= \sum_{j=1}^k \left[\int L(\sum_{\alpha=0}^n \sum_{i=1}^k \overline{Z}_{\alpha i}) \overline{Z}_{2j} + \int L(\sum_{\alpha=0}^n \sum_{l=1}^h \hat{Z}_{\alpha l}) \overline{Z}_{2j} \right].$$

Notice that the second term vanishes due to Lemma **??** since, by the symmetry properties of the functions,

$$\sum_{j=1}^k \int L(\sum_{\alpha=0}^n \sum_{l=1}^h \hat{Z}_{\alpha l}) \overline{Z}_{2j} = \sum_{j=1}^k \int L(\sum_{l=1}^h \hat{Z}_{2l}) \overline{Z}_{2j} = \sum_{l=1}^h \int \hat{Z}_{2l} L(\sum_{j=1}^k \overline{Z}_{2j}),$$

and $L(\sum_{j=1}^k \overline{Z}_{2j})$ is invariant under rotation of angle $e^{\frac{2\pi}{k}(j-1)}$ in the (y_1, y_2) plane. Therefore,

$$\overline{r}_2 \cdot \overline{1} = \left(\int L(\sum_{\alpha=0}^n \sum_{i=1}^k \overline{Z}_{\alpha i}) \overline{Z}_{21} \right) \sum_{j=1}^k \sin \overline{\theta}_j + \left(\int L(\sum_{\alpha=0}^n \sum_{i=1}^k \overline{Z}_{\alpha i}) \overline{Z}_{22} \right) \sum_{j=1}^k \cos \overline{\theta}_j = 0. \tag{2.2.12}$$

On the other hand, as a consequence of [**?**, Lemma 6.1],

$$(\overline{r}_0 + \overline{r}_1) \cdot \overline{\cos} = (\overline{r}_0 + \overline{r}_1) \cdot \overline{\sin} = 0. \tag{2.2.13}$$

Using the invariances under rotation in the planes (y_1, y_2) and (y_3, y_4) and Lemma **??** we get

$$\int L(\varphi^{\perp})\overline{Z}_{3j} = \int L(\sum_{\alpha=1}^{n}\sum_{i=1}^{k}\overline{Z}_{\alpha i})\overline{Z}_{3j} + \int L(\sum_{\alpha=1}^{n}\sum_{l=1}^{h}\hat{Z}_{\alpha l})\overline{Z}_{3j}$$

$$= \int L(\sum_{\alpha=1}^{n}\sum_{i=1}^{k}\overline{Z}_{\alpha i})\overline{Z}_{31},$$

and thus

$$\overline{r}_3 \cdot \overline{\cos} = \int L(\sum_{\alpha=1}^{n}\sum_{i=1}^{k}\overline{Z}_{\alpha i})\overline{Z}_{31}\left(\sum_{j=1}^{k}\cos\theta_j\right) = 0. \tag{2.2.14}$$

Analogously,

$$\overline{r}_3 \cdot \overline{\sin} = \overline{r}_4 \cdot \overline{\cos} = \overline{r}_4 \cdot \overline{\sin} = 0. \tag{2.2.15}$$

Let us now check the last term in (**??**). We expect

$$(C_2^T[\hat{c}_0] + G_2^T[\hat{c}_1] + J_1[\hat{c}_2] + K_1[\hat{c}_3] + L_1[\hat{c}_4]) \cdot \overline{1} = 0, \tag{2.2.16}$$

$$(A_1[\hat{c}_0] + B_1[\hat{c}_1] + C_1[\hat{c}_2] + D_1[\hat{c}_3] + E_1[\hat{c}_4]$$
$$+ B_2^T[\hat{c}_0] + F_1[\hat{c}_1] + G_1[\hat{c}_2] + H_1[\hat{c}_3] + I_1[\hat{c}_4]) \cdot \overline{\cos} = 0, \tag{2.2.17}$$

$$(A_1[\hat{c}_0] + B_1[\hat{c}_1] + C_1[\hat{c}_2] + D_1[\hat{c}_3] + E_1[\hat{c}_4]$$
$$+ B_2^T[\hat{c}_0] + F_1[\hat{c}_1] + G_1[\hat{c}_2] + H_1[\hat{c}_3] + I_1[\hat{c}_4]) \cdot \overline{\sin} = 0, \tag{2.2.18}$$

$$(D_2^T[\hat{c}_0] + H_2^T[\hat{c}_1] + K_2^T[\hat{c}_2] + M_1[\hat{c}_3] + N_1[\hat{c}_4]) \cdot \overline{\cos} = 0, \tag{2.2.19}$$

$$(D_2^T[\hat{c}_0] + H_2^T[\hat{c}_1] + K_2^T[\hat{c}_2] + M_1[\hat{c}_3] + N_1[\hat{c}_4]) \cdot \overline{\sin} = 0, \tag{2.2.20}$$

$$(E_2^T[\hat{c}_0] + I_2^T[\hat{c}_1] + L_2^T[\hat{c}_2] + N_2^T[\hat{c}_3] + P_1[\hat{c}_4]) \cdot \overline{\cos} = 0, \tag{2.2.21}$$

$$(E_2^T[\hat{c}_0] + I_2^T[\hat{c}_1] + L_2^T[\hat{c}_2] + N_2^T[\hat{c}_3] + P_1[\hat{c}_4]) \cdot \overline{\sin} = 0, \tag{2.2.22}$$

where $\hat{c}_0, \ldots, \hat{c}_4$ satisfy

$$\hat{c}_1 \cdot \hat{\cos} = \hat{c}_1 \cdot \hat{\sin} = 0,$$
$$\hat{c}_2 \cdot \hat{\cos} = \hat{c}_2 \cdot \hat{\sin} = 0. \tag{2.2.23}$$

Notice that, since C_2 and G_2 have all their rows identical,

$$C_2^T[\hat{c}_0] \cdot \overline{1} = \beta_{10}\left(\sum_{l=1}^{h}\hat{c}_{0l}\right)\left(\sum_{j=1}^{k}\sin\overline{\theta}_j\right) = 0,$$

$$G_2^T[\hat{c}_1] \cdot \overline{1} = (\beta_{11} - \beta_{22})\left(\sum_{l=1}^{h}\hat{c}_{1l}\right)\left(\sum_{j=1}^{k}\sin\overline{\theta}_j\cos\overline{\theta}_j\right) = 0.$$

Likewise, using the definition of J_1 and Lemma **??**,

$$J_1[\hat{c}_2] \cdot \overline{1} = \left(\sum_{l=1}^{h}\hat{c}_{2l}\right)\int L\left(\sum_{j=1}^{k}\overline{Z}_{2j}\right)\hat{Z}_{11} = 0,$$

since $L\left(\sum_{j=1}^{k} \overline{Z}_{2j}\right)$ is invariant under rotation of angle θ_j, and

$$K_1[\hat{c}_3] \cdot \overline{1} = \beta_{13}(\hat{c}_3 \cdot \hat{\cos})\left(\sum_{j=1}^{k} \sin \overline{\theta}_j\right) = 0,$$

$$L_1[\hat{c}_4] \cdot \overline{1} = \beta_{13}(\hat{c}_4 \cdot \hat{\sin})\left(\sum_{j=1}^{k} \sin \overline{\theta}_j\right) = 0.$$

Thus, (??) follows. Furthermore, again by [?, Lemma 6.1],

$$(A_1 + B_2^T)_{jl} = \int L(\hat{Z}_{0l})(\overline{Z}_{0j} + \overline{Z}_{1j}) = 0,$$

$$(B_1 + F_1)_{jl} = \int L(\hat{Z}_{1l})(\overline{Z}_{0j} + \overline{Z}_{1j}) = 0,$$

$$(C_1 + G_1)_{jl} = \int L(\hat{Z}_{2l})(\overline{Z}_{0j} + \overline{Z}_{1j}) = 0,$$

and thus $(A_1 + B_2^T)[\hat{c}_0] = (B_1 + F_1)[\hat{c}_1] = (C_1 + G_1)[\hat{c}_2] = 0$. Likewise,

$$(D_1 + H_1)_{jl} = \int L(\hat{Z}_{3l})(\overline{Z}_{0j} + \overline{Z}_{1j}) = 0,$$

$$(E_1 + I_1)_{jl} = \int L(\hat{Z}_{4l})(\overline{Z}_{0j} + \overline{Z}_{1j}) = 0,$$

and therefore $(D_1 + H_1)[\hat{c}_3] = (E_1 + I_1)[\hat{c}_4] = 0$. (??) can be analogously proved. Furthermore,

$$D_2^T[\hat{c}_0] \cdot \overline{\cos} = \beta_{31}(\hat{c}_0 \cdot \hat{\cos})\left(\sum_{j=1}^{k} \cos \overline{\theta}_j\right) = 0,$$

$$M_1[\hat{c}_3] \cdot \overline{\cos} = \left[\beta_{33}\left(\sum_{l=1}^{h} \hat{c}_{3l} \cos^2 \hat{\theta}_l\right) + \beta_{44}\left(\sum_{l=1}^{h} \hat{c}_{3l} \sin^2 \hat{\theta}_l\right)\right]\left(\sum_{j=1}^{k} \cos \overline{\theta}_j\right) = 0,$$

$$N_1[\hat{c}_4] \cdot \overline{\cos} = (\beta_{33} - \beta_{44})\left(\sum_{l=1}^{h} \hat{c}_{4l} \sin \hat{\theta}_l \cos \hat{\theta}_l\right)\left(\sum_{j=1}^{k} \cos \overline{\theta}_j\right) = 0,$$

and, due to (??),

$$H_2^T[\hat{c}_1] \cdot \overline{\cos} = \beta_{31}(\hat{c}_1 \cdot \hat{\cos})\left(\sum_{j=1}^{k} \cos^2 \overline{\theta}_j\right) = 0,$$

$$K_2^T[\hat{c}_2] \cdot \overline{\cos} = \beta_{31}(\hat{c}_2 \cdot \hat{\cos})\left(\sum_{j=1}^{k} \sin \overline{\theta}_j \cos \overline{\theta}_j\right) = 0,$$

so (??) holds. Identities (??)-(??) can be obtained in a similar way, and thus (??) is solvable. An analogous reasoning proves the solvability of (??). Thus, the systems (??) and (??) have a

solution in X with the form (**??**), where $[\overline{w}_0, \ldots, \overline{w}_4, \hat{w}_0, \ldots, \hat{w}_4]$ satisfies (**??**). It can be checked that

$$|\beta_{\alpha\alpha}| \leqslant Ck^{-2n+4}, \ \alpha = 0, 2, 4, \qquad |\beta_{\alpha\alpha}| \leqslant Ck^{-2n+6}, \ \alpha = 1, 3,$$
$$|\beta_{\alpha_1\alpha_2}| \leqslant Ck^{-2n+6}, \ \alpha_1, \alpha_2 = 0, 1, 3, \ \alpha_1 \neq \alpha_2. \tag{2.2.24}$$

where $\beta_{\alpha_1,\alpha_2}$ was defined in (**??**), and henceforth, by (**??**) and recalling that $h = O(k)$,

$$\|\overline{w}_\alpha\| \leqslant Ck^{n-4}\|\overline{r}_\alpha\|, \qquad \|\hat{w}_\alpha\| \leqslant Ck^{n-4}\|\hat{r}_\alpha\|, \qquad \alpha = 0, 1, \ldots, 4.$$

As it was done in the case $\alpha \geqslant 5$, we will solve the systems by means of a fixed point argument. If we denote

$$\overline{M}_1 := \begin{bmatrix} \overline{A} & \overline{B} & \overline{C} & 0 & 0 \\ \overline{B}^T & \overline{F} & \overline{G} & 0 & 0 \\ \overline{C}^T & \overline{G}^T & \overline{J} & 0 & 0 \\ 0 & 0 & 0 & \overline{M} & 0 \\ 0 & 0 & 0 & 0 & \overline{P} \end{bmatrix}, \ \overline{\overline{M}}_1 := \begin{bmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ B_2^T & F_1 & G_1 & H_1 & I_1 \\ C_2^T & G_2^T & J_1 & K_1 & L_1 \\ D_2^T & H_2^T & K_2^T & M_1 & N_1 \\ E_2^T & I_2^T & L_2^T & N_2^T & P_1 \end{bmatrix},$$

$$\hat{M}_1 := \begin{bmatrix} \hat{A} & 0 & 0 & \hat{D} & \hat{E} \\ 0 & \hat{F} & 0 & 0 & 0 \\ 0 & 0 & \hat{J} & 0 & 0 \\ \hat{D}^T & 0 & 0 & \hat{M} & \hat{N} \\ \hat{E}^T & 0 & 0 & \hat{N}^T & \hat{P} \end{bmatrix}, \ \hat{\hat{M}}_1 := \begin{bmatrix} A_2 & B_2 & C_2 & D_2 & E_2 \\ B_1^T & F_2 & G_2 & H_2 & I_2 \\ C_1^T & G_1^T & J_2 & K_2 & L_2 \\ D_1^T & H_1^T & K_1^T & M_2 & N_2 \\ E_1^T & I_1^T & L_1^T & N_1^T & P_2 \end{bmatrix},$$

then $[\overline{w}_0, \ldots, \overline{w}_4, \hat{w}_0, \ldots, \hat{w}_4]$ is a solution of (**??**)-(**??**) if and only if it is a fixed point of

$$F\begin{bmatrix} \overline{w}_0 \\ \ldots \\ \overline{w}_4 \\ \hat{w}_0 \\ \ldots \\ \hat{w}_4 \end{bmatrix} := S^{-1}\left(\begin{bmatrix} \overline{r}_0 \\ \overline{r}_1 \\ \overline{r}_2 \\ \overline{r}_3 \\ \overline{r}_4 \end{bmatrix} - \overline{\overline{M}}_1 \begin{bmatrix} \hat{w}_0 \\ \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \\ \hat{w}_4 \end{bmatrix}, \begin{bmatrix} \hat{r}_0 \\ \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \end{bmatrix} - \hat{\hat{M}}_1 \begin{bmatrix} \overline{w}_0 \\ \overline{w}_1 \\ \overline{w}_2 \\ \overline{w}_3 \\ \overline{w}_4 \end{bmatrix}\right),$$

where

$$S\left(\begin{bmatrix} \overline{w}_0 \\ \overline{w}_1 \\ \overline{w}_2 \\ \overline{w}_3 \\ \overline{w}_4 \end{bmatrix}, \begin{bmatrix} \hat{w}_0 \\ \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \\ \hat{w}_4 \end{bmatrix}\right) := \left(\overline{M}_1 \begin{bmatrix} \overline{w}_0 \\ \overline{w}_1 \\ \overline{w}_2 \\ \overline{w}_3 \\ \overline{w}_4 \end{bmatrix}, \hat{M}_1 \begin{bmatrix} \hat{w}_0 \\ \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \\ \hat{w}_4 \end{bmatrix}\right).$$

Notice that S is a linear map which is invertible for vectors satisfying the orthogonality conditions (**??**). Let

$$B_r := \{ \begin{bmatrix} \overline{w}_0 \\ \ldots \\ \overline{w}_4 \end{bmatrix}, \begin{bmatrix} \hat{w}_0 \\ \ldots \\ \hat{w}_4 \end{bmatrix} \in K : \|\begin{bmatrix} \overline{w}_0 \\ \ldots \\ \overline{w}_4 \end{bmatrix}\| \leqslant rk^{n-4}\|\begin{bmatrix} \overline{r}_0 \\ \ldots \\ \overline{r}_4 \end{bmatrix}\|,$$

$$\|\begin{bmatrix} \hat{w}_0 \\ \ldots \\ \hat{w}_4 \end{bmatrix}\| \leqslant rk^{n-4}\|\begin{bmatrix} \hat{r}_0 \\ \ldots \\ \hat{r}_4 \end{bmatrix}\|\},$$

for some fixed r large, where

$$K := \{ \begin{bmatrix} \overline{w}_0 \\ \dots \\ \overline{w}_4 \end{bmatrix} \in \mathbb{R}^{5 \times k}, \begin{bmatrix} \hat{w}_0 \\ \dots \\ \hat{w}_4 \end{bmatrix} \in \mathbb{R}^{5 \times h} \text{ satisfying } (\textbf{??}).\}.$$

Thanks to the particular form of the matrices $\overline{\overline{M}}_1$, $\hat{M}$ (all their submatrices are combinations of sinus and cosinus multiplied by a term $\beta_{\alpha_1,\alpha_2}$) and (**??**) it can be checked that F is a contraction mapping that sends B_r into B_r. This finishes the proof of the existence of a solution to (**??**)-(**??**) satisfying (**??**).

Define the vectors

$$\overline{u}_0 := \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{1} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}, \overline{u}_1 := \begin{bmatrix} 0 \\ \overline{\cos} \\ \hat{0} \\ 0 \\ -\overline{\cos} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}, \overline{u}_2 := \begin{bmatrix} 0 \\ \overline{\sin} \\ \hat{0} \\ 0 \\ -\overline{\sin} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}, \tag{2.2.25}$$

$$\hat{u}_0 := \begin{bmatrix} 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{1} \end{bmatrix}, \hat{u}_1 := \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\cos} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ -\hat{\cos} \\ 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}, \hat{u}_2 := \begin{bmatrix} 0 \\ \overline{0} \\ \hat{\sin} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ \hat{0} \\ 0 \\ \overline{0} \\ -\hat{\sin} \\ 0 \\ \overline{0} \\ \hat{0} \end{bmatrix}. \tag{2.2.26}$$

We can summarize this section in the following lemma.

Lemma 2.2.2. *System* (**??**) *is solvable, and the solution has the form*

$$\begin{bmatrix} \tilde{c}_0 \\ \dots \\ \tilde{c}_4 \end{bmatrix} = w + t_0 w_0 + t_1 w_1 + t_2 w_2 + t_3 w_3 + t_4 w_4$$

$$+ \bar{t}_0 \bar{u}_0 + \bar{t}_1 \bar{u}_1 + \bar{t}_2 \bar{u}_2 + \hat{t}_0 \hat{u}_0 + \hat{t}_1 \hat{u}_1 + \hat{t}_2 \hat{u}_2,$$

where

$$w := \begin{bmatrix} 0 & \overline{w}_0 & \hat{w}_0 & 0 & \overline{w}_1 & \hat{w}_1 & 0 & \overline{w}_2 & \hat{w}_2 & 0 & \overline{w}_3 & \hat{w}_3 & 0 & \overline{w}_4 & \hat{w}_4 \end{bmatrix}^T$$

$(\overline{w}_0, \dots, \overline{w}_4, \hat{w}_0, \dots, \hat{w}_4$ *being the unique solution to the system* (**??**)-(**??**) *satisfying* (**??**)*and* (**??**)), $t_0, \dots, t_4, \bar{t}_0, \bar{t}_1, \bar{t}_2, \hat{t}_0, \hat{t}_1, \hat{t}_2 \in \mathbb{R}$ *and* $\tilde{c}_0, \dots, \tilde{c}_4, w_0, \dots, w_4, \bar{u}_0, \bar{u}_1, \bar{u}_2, \hat{u}_0, \hat{u}_1, \hat{u}_2$ *are defined in* (**??**), (**??**), (**??**), (**??**) *and* (**??**).

2.3. PROOF OF PROPOSITION **??**

With this in mind, we observe that Proposition **??** is a consequence of the following estimates: if $k, h \to \infty$

$$\int |u|^{p-1} Z_\alpha Z_\beta = \int U^{p-1} Z_0^2 + O(\mu^{\frac{n-2}{2}} + \lambda^{\frac{n-2}{2}}) \qquad \text{if } \alpha = \beta = 0,$$

$$= \int U^{p-1} Z_1^2 + O(\mu^{\frac{n-2}{2}} + \lambda^{\frac{n-2}{2}}) \qquad \text{if } \alpha = \beta \neq 0, \qquad (2.3.1)$$

$$= O(\mu^{\frac{n-2}{2}} + \lambda^{\frac{n-2}{2}}) \qquad \text{otherwise,}$$

$$\int |u|^{p-1} \overline{Z}_{\alpha i} \overline{Z}_{\beta j} = \int U^{p-1} Z_0^2 + O(\mu^{\frac{n-2}{2}}) \qquad \text{if } \alpha = \beta = 0, i = j,$$

$$= \int U^{p-1} Z_1^2 + O(\mu^{\frac{n-2}{2}}) \qquad \text{if } \alpha = \beta \neq 0, i = j \qquad (2.3.2)$$

$$= O(\mu^{\frac{n-2}{2}}) \qquad \text{otherwise,}$$

$$\int |u|^{p-1} \hat{Z}_{\alpha l} \hat{Z}_{\beta m} = \int U^{p-1} Z_0^2 + O(\lambda^{\frac{n-2}{2}}) \qquad \text{if } \alpha = \beta = 0, l = m,$$

$$= \int U^{p-1} Z_1^2 + O(\lambda^{\frac{n-2}{2}}) \qquad \text{if } \alpha = \beta \neq 0, l = m, \qquad (2.3.3)$$

$$= O(\lambda^{\frac{n-2}{2}}) \qquad \text{otherwise,}$$

$$\int |u|^{p-1} \overline{Z}_{\alpha i} Z_\beta = O(\mu^{\frac{n-2}{2}}), \quad \int |u|^{p-1} \hat{Z}_{\alpha l} Z_\beta = O(\lambda^{\frac{n-2}{2}}), \qquad (2.3.4)$$

$$\int |u|^{p-1} \overline{Z}_{\alpha i} \hat{Z}_{\beta l} = O(\mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}}). \qquad (2.3.5)$$

In the formulas above, $i, j = 1, \dots, k$, $l, m = 1, \dots, h$, $\alpha, \beta = 0, \dots, n$.

The proof of (**??**) and (**??**) follows like (8.3) in [**?**], and (**??**), (**??**) are obtained analogously. Let us prove (**??**).

The key point here is to notice that if $y \in B(\xi_i, \frac{\overline{\alpha}}{k})$ then $|y - \eta_l| \geqslant C$, with C independent of i, l and k, and $|y - \xi_i| \geqslant C$ whenever $y \in B(\eta_l, \frac{\hat{\alpha}}{h})$, where C is independent of i, l and h. Consider the case $\alpha = \beta = 0$. We split the integral into four parts.

$$\int |u|^{p-1} \overline{Z}_{0i} \hat{Z}_{0l} = \int_{B(\xi_i, \frac{\overline{\alpha}}{k})} |u|^{p-1} \overline{Z}_{0i} \hat{Z}_{0j} + \int_{B(\eta_l, \frac{\hat{\alpha}}{h})} |u|^{p-1} \overline{Z}_{0i} \hat{Z}_{0j}$$

$$+ \int_{\mathbb{R}^n \backslash B(0,2)} |u|^{p-1} \overline{Z}_{0i} \hat{Z}_{0j}$$

$$+ \int_{B(0,2) \backslash (B(\xi_i, \frac{\overline{\alpha}}{k}) \cup B(\eta_l, \frac{\hat{\alpha}}{h}))} |u|^{p-1} \overline{Z}_{0i} \hat{Z}_{0j}$$

$$=: i_1 + i_2 + i_3 + i_4,$$

Firstly, using the definition of $\overline{Z}_{0i}$ and $\hat{Z}_{0l}$,

$$i_1 \leqslant C \int_{B(\xi_i, \frac{\overline{\alpha}}{k})} |u|^{p-1} \overline{Z}_{0i} \frac{\lambda^{\frac{n-2}{2}}}{|y - \eta_l|^{n-2}} \leqslant C \mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}} \int_{B(0, \frac{\overline{\alpha}}{\mu k})} U^{p-1} Z_0$$

$$\leqslant C \mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}},$$

where the second inequality follows by the change of variable $x = \xi_i + \mu y$. i_2 follows in the same way only by translating to $x = \eta_l + \lambda y$. On the other hand, we have that

$$i_3 \leqslant C \mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}} \int_{\mathbb{R}^n \backslash B(0,2)} \frac{1}{|y|^4 |y|^{n-2} |y|^{n-2}} \, dy \leqslant C \mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}}.$$

To estimate i_4 we take into account that, since ξ_i and η_l are separated, $|y - \xi_i|^{-(n-2)}$ and $|y - \eta_l|^{-(n-2)}$ cannot be singular at the same time, so the behavior of the integral comes determined by the singularity of only one of them. That is,

$$i_4 \leqslant C \mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}} \int_{B(0,2) \backslash (B(\xi_i, \frac{\overline{\alpha}}{k}) \cup B(\eta_l, \frac{\hat{\alpha}}{h}))} |u|^{p-1} \frac{1}{|y - \xi_i|^{n-2}} \frac{1}{|y - \eta_l|^{n-2}} \, dy$$

$$\leqslant C \mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}} \int_{B(0,2) \backslash (B(\xi_i, \frac{\overline{\alpha}}{k}) \cup B(\eta_l, \frac{\hat{\alpha}}{h}))} \left(\frac{1}{|y - \xi_i|^{n-2}} + \frac{1}{|y - \eta_l|^{n-2}} \right) dy$$

$$\leqslant C \mu^{\frac{n-2}{2}} \lambda^{\frac{n-2}{2}}.$$

The case $\alpha, \beta \neq 0$ follows analogously just by noticing that

$$\overline{Z}_{\alpha i} \sim \frac{\mu^{\frac{n-2}{2}}}{|y - \xi_i|^{n-1}}, \qquad \hat{Z}_{\beta l} \sim \frac{\lambda^{\frac{n-2}{2}}}{|y - \eta_l|^{n-1}},$$

and hence (**??**) is proved.

We now need the following result.

Proposition 2.3.1. *The functions π_α can be decomposed as*

$$\pi_\alpha(y) = \sum_{j=1}^{k} \overline{\pi}_{\alpha j}(y) + \sum_{l=1}^{h} \hat{\pi}_{\alpha l} + \tilde{\pi}_\alpha(y),$$

where

$$\overline{\pi}_{\alpha j}(y) = \overline{\pi}_{\alpha 1}(e^{-\frac{2\pi(j-1)}{k}} \overline{y}, \hat{y}, y'), \qquad \hat{\pi}_{\alpha l}(y) = \hat{\pi}_{\alpha 1}(\overline{y}, e^{-\frac{2\pi(l-1)}{h}} \hat{y}, y').$$

Furthermore, there exists a positive constant C such that

$$\|\tilde{\pi}_\alpha\|_* \leqslant C(k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}), \qquad \alpha = 0, 1, \ldots, n,$$

$$\|\overline{\overline{\pi}}_{\alpha 1}\|_* \leqslant Ck^{-\frac{n}{q}}, \ \alpha = 0, \ldots, n, \quad where \ \ \overline{\overline{\pi}}_{\alpha 1} := \mu^{\frac{n-2}{2}} \overline{\pi}_{\alpha 1}(\xi_1 + \mu y),$$

$$\|\hat{\hat{\pi}}_{\alpha 1}\|_* \leqslant Ch^{-\frac{n}{q}}, \ \alpha = 0, \ldots, n, \quad where \ \ \hat{\hat{\pi}}_{\alpha 1} := \lambda^{\frac{n-2}{2}} \hat{\pi}_{\alpha 1}(\eta_1 + \lambda y).$$

We omit the proof of this result.

Thanks to Proposition **??**, we get

$$\left| \int |u|^{p-1} Z_\alpha \pi_\beta \right| \leqslant C\|\tilde{\pi}_\beta\|_*, \tag{2.3.6}$$

$$\left| \int |u|^{p-1} \overline{Z}_{\alpha i} \pi_\beta \right| \leqslant C\|\overline{\pi}_{\beta 1}\|_*, \qquad \left| \int |u|^{p-1} \hat{Z}_{\alpha l} \pi_\beta \right| \leqslant C\|\hat{\pi}_{\beta 1}\|_*. \tag{2.3.7}$$

Notice next (see (8.5) in [**?**] for a proof) that

$$\int U^{p-1} Z_0^2 = \int U^{p-1} Z_1^2 > 0,$$

and thus we can define

$$t_\beta := -\frac{1}{\int U^{p-1} Z_0^2} \int \varphi^\perp u^{p-1} \mathbf{z}_\beta,$$

which satisfies $|t_\beta| \leqslant C\|\varphi^\perp\|_*$, with C independent of k and h.

Consider (**??**) for $\beta = 0$. Thus, by using the definition of z_0, (**??**)-(**??**), (**??**), (**??**) and Proposition **??** we get

$$\sum_{\alpha=0}^n \left[c_{\alpha 0} \int_{\mathbb{R}^n} Z_{\alpha 0} u^{p-1} z_0 + \sum_{j=1}^k \overline{c}_{\alpha j} \int_{\mathbb{R}^n} \overline{Z}_{\alpha j} u^{p-1} z_0 + \sum_{l=1}^h \hat{c}_{\alpha l} \int_{\mathbb{R}^n} \hat{Z}_{\alpha l} u^{p-1} z_0 \right]$$

$$= c_{00} \int U^{p-1} Z_0^2 - \sum_{j=1}^k \overline{c}_{0j} \int U^{p-1} Z_0^2 - \sum_{l=1}^h \hat{c}_{0l} \int U^{p-1} Z_0^2$$

$$- \sum_{j=1}^k \overline{c}_{1j} \int U^{p-1} Z_0^2 - \sum_{l=1}^h \hat{c}_{3l} \int U^{p-1} Z_0^2 + O(k^{1-\frac{n}{q}} + h^{1-\frac{n}{q}}) \mathcal{L} \begin{bmatrix} c_{00} \\ \ldots \\ c_{n0} \end{bmatrix}$$

$$+ O(k^{-\frac{n}{q}}) \overline{\mathcal{L}} \begin{bmatrix} \overline{c}_0 \\ \ldots \\ \overline{c}_n \end{bmatrix} + O(h^{-\frac{n}{q}}) \hat{\mathcal{L}} \begin{bmatrix} \hat{c}_0 \\ \ldots \\ \hat{c}_n \end{bmatrix},$$

where $\mathcal{L}$, $\overline{\mathcal{L}}$ and $\hat{\mathcal{L}}$ are linear functions with coefficients uniformly bounded in k and h. Identity (**??**) follows straightforward from here. (**??**)-(**??**) are obtained in the same way.

2.4. Proof of (??)

We proceed as in [?, Section 9]. Indeed, we decompose $\varphi_0^\perp$ as

$$\varphi_0^\perp = \sum_{\alpha=0}^n c_{\alpha 0}\varphi_{\alpha 0}^\perp, \qquad \text{with} \qquad L(\varphi_{\alpha 0}^\perp) = -L(Z_{\alpha 0}),$$

which is equivalent to

$$\Delta(\varphi_{\alpha 0}^\perp) + p\gamma U^{p-1}(\varphi_{\alpha 0}^\perp) + a_0(y)\varphi_{\alpha 0}^\perp = -L(Z_{\alpha 0}), \tag{2.4.1}$$

where $a_0(y) := p\gamma(|u|^{p-1} - U^{p-1})$. Adapting the arguments of [?] it can be seen that $a \in L^{\frac{n}{2}}(\mathbb{R}^n)$,

$$|y|^{-n-2}L(Z_{00})(|y|^{-2}y) = -L(Z_{00})(y), \tag{2.4.2}$$

$$|y|^{-n-2}L(Z_{\alpha 0})(|y|^{-2}y) = L(Z_{\alpha 0})(y), \ \alpha = 1, \dots, n, \tag{2.4.3}$$

and

$$\|L(Z_{\alpha 0})\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leqslant C(\mu^{\frac{n-1}{n}} + \lambda^{\frac{n-1}{n}}). \tag{2.4.4}$$

We will solve (??) as a fixed point problem. Let us consider the problem

$$L_0(\varphi) = h - a_0(y)\phi,$$

where $L_0(\varphi) := \Delta\varphi + p\gamma U^{p-1}\varphi$ and $h \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$ satisfies

$$h(y) = |y|^{-n-2}h(|y|^{-2}y).$$

Let T be the operator that associates to every ϕ the solution φ to this problem, that is,

$$\varphi = T(h - a_0(y)\phi).$$

Naming $A(\phi) := T(h - a_0(y)\phi)$ we are going to see that this operator is a contraction and that maps the ball

$$B := \{\phi \in \mathcal{D}^{1,2}(\mathbb{R}^n) : \|\phi\|_* \leqslant C(\mu^{\frac{n-1}{n}} + \lambda^{\frac{n-1}{n}}), \phi(y) = |y|^{-n+2}\phi(|y|^{-2}y)\},$$

into herself. Indeed, assume $\phi \in B$. Thus,

$$a_0(y)\phi(y) = |y|^{-n-2}a_0(|y|^{-2}y)\phi(|y|^{-2}y),$$

and, by [?, Proposition 9.1], we know that

$$\|\varphi\|_* \leqslant C\|h - a_0(y)\phi\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leqslant C\left(\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} + \|a_0(y)\phi\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}\right),$$

and $\varphi(y) = |y|^{2-n}\varphi(|y|^{-2}y)$.

We study the last term in two different regions. First, in

$$\mathbb{R}^n \setminus (\{\cup_{j=1}^k B(\xi_j, \tfrac{\overline{\alpha}}{k})\} \cup \{\cup_{l=1}^h B(\eta_l, \tfrac{\hat{\alpha}}{h})\})$$

we can estimate a_0 as

$$|a_0(y)| \leqslant CU^{p-2}\left[\sum_{j=1}^k \frac{\mu^{\frac{n-2}{2}}}{|y - \xi_j|^{n-2}} + \sum_{l=1}^h \frac{\lambda^{\frac{n-2}{2}}}{|y - \eta_l|^{n-2}}\right],$$

and consequently,

$$\int_{\mathbb{R}^n \setminus (\{\cup_{j=1}^k B(\xi_j, \frac{\overline{\alpha}}{k})\} \cup \{\cup_{l=1}^h B(\eta_l, \frac{\hat{\alpha}}{h})\})} |a_0(y)|^{\frac{2n}{n+2}} \, dy \leqslant C\left(k^{-(n-1)} + h^{-(n-1)}\right). \tag{2.4.5}$$

Consider now $j \in \{1, \ldots, k\}$ and the ball $B(\xi_j, \frac{\overline{\alpha}}{k})$. Here

$$|a_0(y)| \leqslant C|U_{\mu,\xi_j}(y)|^{p-1},$$

and thus

$$\sum_{j=1}^{k} \int_{B(\xi_j, \frac{\overline{\alpha}}{k})} |a_0(y)|^{\frac{2n}{n+2}} \, dy \leqslant C \sum_{j=1}^{k} \int_{B(\xi_j, \frac{\overline{\alpha}}{k})} \left[\frac{\mu^{-\frac{n-2}{2}}}{1 + |y - \xi_j|^{n-2}} \right]^{(p-1)\frac{2n}{n+2}}$$

$$\leqslant C k \mu^{-2\frac{2n}{n+2}} \mu^n \int_{B(0, \frac{1}{\mu k})} \left[\frac{1}{1 + |y|^{n-2}} \right]^{(p-1)\frac{2n}{n+2}} \, dy \qquad (2.4.6)$$

$$\leqslant C k^{-(n-1)}.$$

Likewise,

$$\sum_{l=1}^{k} \int_{B(\eta_l, \frac{\hat{\alpha}}{h})} |a_0(y)|^{\frac{2n}{n+2}} \, dy \leqslant C h^{-(n-1)}. \qquad (2.4.7)$$

Putting (**??**), (**??**) and (**??**) together we conclude that if $\phi \in B$, then

$$\|a_0(y)\phi\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leqslant \|\phi\|_{L^{\infty}(\mathbb{R}^n)} \|a_0(y)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leqslant C(\mu^{\frac{n-1}{n}} + \lambda^{\frac{n-1}{n}}).$$

Furthermore

$$\|A(\phi_1) - A(\phi_2)\|_* \leqslant C\|a_0(y)(\phi_1 - \phi_2)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$$

$$\leqslant C\|a_0(y)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \|\phi_1 - \phi_2\|_* \qquad (2.4.8)$$

$$= o(1)\|\phi_1 - \phi_2\|_*,$$

where $o(1)$ denotes a quantity which goes to zero when k, h tend to infinity. Thus, A defines a contraction mapping whenever

$$\|h\|_{L^{\frac{2n}{n+2}}} \leqslant C(\mu^{\frac{n-1}{n}} + \lambda^{\frac{n-1}{n}}).$$

Hence, considering $h = L(Z_{\alpha 0})$, by (**??**) we conclude the existence of a solution to (**??**) satisfying

$$\|\varphi_{\alpha 0}^{\perp}\|_* \leqslant C(\mu^{\frac{n-1}{n}} + \lambda^{\frac{n-1}{n}}).$$

Consider now $j \in \{1, \ldots, k\}$, $l \in \{1, \ldots, h\}$, and let us write,

$$\overline{\varphi}_j^{\perp} = \sum_{\alpha=0}^{n} c_{\alpha j} \overline{\varphi}_{\alpha j}^{\perp}, \qquad \text{with} \qquad L(\overline{\varphi}_{\alpha j}^{\perp}) = -L(\overline{Z}_{\alpha j}),$$

$$\hat{\varphi}_l^{\perp} = \sum_{\alpha=0}^{n} c_{\alpha l} \hat{\varphi}_{\alpha l}^{\perp}, \qquad \text{with} \qquad L(\hat{\varphi}_{\alpha l}^{\perp}) = -L(\hat{Z}_{\alpha l}).$$

Performing the change of variables

$$\overline{\overline{\varphi}}_j^{\perp}(y) := \mu^{\frac{n-2}{2}} \overline{\varphi}_{\alpha j}^{\perp}(\mu y + \xi_j), \qquad \hat{\hat{\varphi}}_l^{\perp}(y) := \lambda^{\frac{n-2}{2}} \hat{\varphi}_{\alpha l}^{\perp}(\lambda y + \eta_l),$$

the previous equations turn into

$$\Delta(\overline{\overline{\varphi}}_j^{\perp}) + p\gamma U^{p-1}(\overline{\overline{\varphi}}_j^{\perp}) + p\gamma \overline{a}_j(y)\overline{\overline{\varphi}}_j^{\perp} = \overline{h}_j(y),$$

$$\Delta(\hat{\hat{\varphi}}_l^{\perp}) + p\gamma U^{p-1}(\hat{\hat{\varphi}}_l^{\perp}) + p\gamma \hat{a}_l(y)\hat{\hat{\varphi}}_l^{\perp} = \hat{h}_l(y),$$

where

$$\overline{a}_j(y) := p\gamma[(\mu^{-\frac{n-2}{2}}|u|(\mu y + \xi_j))^{p-1} - U^{p-1}], \quad \overline{h}_j(y) := -\mu^{\frac{n+2}{2}}L(\overline{Z}_{\alpha j})(\mu y + \xi_j),$$

$$\hat{a}_l(y) := p\gamma[(\lambda^{-\frac{n-2}{2}}|u|(\lambda y + \eta_l))^{p-1} - U^{p-1}], \quad \hat{h}_l(y) := -\lambda^{\frac{n+2}{2}}L(\hat{Z}_{\alpha l})(\lambda y + \eta_l).$$

Performing an analogous fixed point argument we conclude (**??**).

2.5. Final argument

Let $\begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix}$ be the solution to (**??**) provided by Proposition **??**, and let $t_0, t_1, t_2, t_3, t_4, \overline{t}_0, \overline{t}_1, \overline{t}_2,$

$\hat{t}_0, \hat{t}_1, \hat{t}_2,$ and $t_\alpha, \overline{\nu}_{\alpha 1}, \overline{\nu}_{\alpha 2}, \hat{\nu}_{\alpha 1}, \hat{\nu}_{\alpha 2}, \alpha = 5, \dots, n,$ be the associated parameters. Thus, it follows straightforward the existence of a unique vector of parameters

$$(t_0^*, \dots, t_4^*, \overline{t}_0^*, \overline{t}_1^*, \overline{t}_2^*, \hat{t}_0^*, \hat{t}_1^*, \hat{t}_2^*, t_5^*, \overline{\nu}_{51}^*, \overline{\nu}_{52}^*, \hat{\nu}_{51}^*, \hat{\nu}_{52}^*, \dots, t_n^*, \overline{\nu}_{n1}^*, \overline{\nu}_{n2}^*, \hat{\nu}_{n1}^*, \hat{\nu}_{n2}^*)$$

such that $\begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix}$ solves the system in Proposition **??** and, equivalently, (**??**). Moreover,

$$\|(t_0^*, t_1^*, t_2^*, t_3^*, t_4^*, \overline{t}_0^*, \overline{t}_1^*, \overline{t}_2^*, \hat{t}_0^*, \hat{t}_1^*, \hat{t}_2^*, t_5^*, \overline{\nu}_{51}^*, \overline{\nu}_{52}^*, \hat{\nu}_{51}^*, \hat{\nu}_{52}^*, \dots, t_n^*, \overline{\nu}_{n1}^*, \overline{\nu}_{n2}^*, \hat{\nu}_{n1}^*, \hat{\nu}_{n2}^*)\| \leqslant C\|\varphi^\perp\|,$$

and therefore

$$\left\| \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix} \right\| \leqslant C\|\varphi^\perp\|.$$

This estimate, together with (**??**), allows us to conclude

$$c_\alpha = 0 \qquad \forall\, \alpha = 0, \dots, n,$$

and thus $\varphi^\perp \equiv 0$. Replacing this in (**??**) the proof of Theorem **??** is complete.

2.6. Appendix

According to their definitions, see (**??**)–(**??**) and (**??**), it is convenient to rewrite the functions z_α as

$$z_0(y) = Z_0(y) - \sum_{j=1}^k \left[\overline{Z}_{0j}(y) + \overline{Z}_{1j}(y)\right] - \sum_{l=1}^h \left[\hat{Z}_{0l}(y) + \hat{Z}_{3l}(y)\right],$$

$$z_1(y) = Z_1(y) - \sum_{j=1}^{k} \frac{\cos \overline{\theta}_j \overline{Z}_{1j}(y) - \sin \overline{\theta}_j \overline{Z}_{2j}(y)}{\sqrt{1-\mu^2}} - \sum_{l=1}^{h} \hat{Z}_{1l}(y),$$

$$z_2(y) = Z_2(y) - \sum_{j=1}^{k} \frac{\sin \overline{\theta}_j \overline{Z}_{1j}(y) + \cos \overline{\theta}_j \overline{Z}_{2j}(y)}{\sqrt{1-\mu^2}} - \sum_{l=1}^{h} \hat{Z}_{2l}(y),$$

$$z_3(y) = Z_3(y) - \sum_{j=1}^{k} \overline{Z}_{3j}(y) - \sum_{l=1}^{h} \frac{\cos \hat{\theta}_l \hat{Z}_{3l}(y) - \sin \hat{\theta}_l \hat{Z}_{4l}(y)}{\sqrt{1-\lambda^2}},$$

$$z_4(y) = Z_4(y) - \sum_{j=1}^{k} \overline{Z}_{4j}(y) - \sum_{l=1}^{h} \frac{\sin \hat{\theta}_l \hat{Z}_{3l}(y) + \cos \hat{\theta}_l \hat{Z}_{4l}(y)}{\sqrt{1-\lambda^2}},$$

$$z_\alpha(y) = Z_\alpha(y) - \sum_{j=1}^{k} \overline{Z}_{\alpha j}(y) - \sum_{l=1}^{h} \hat{Z}_{\alpha l}(y), \quad \alpha = 5, \ldots, n$$

$$z_{n+1}(y) = -\sum_{j=1}^{k} \overline{Z}_{2j}(y), \quad z_{n+2}(y) = -\sum_{l=1}^{h} \hat{Z}_{4l}(y),$$

$$z_{n+7}(y) = -\sqrt{1-\mu^2} \sum_{j=1}^{k} \cos \overline{\theta}_j \overline{Z}_{3j}(y) + \sqrt{1-\lambda^2} \sum_{l=1}^{h} \cos \hat{\theta}_l \hat{Z}_{1l}(y),$$

$$z_{n+8}(y) = -\sqrt{1-\mu^2} \sum_{j=1}^{k} \cos \overline{\theta}_j \overline{Z}_{4j}(y) + \sqrt{1-\lambda^2} \sum_{l=1}^{h} \sin \hat{\theta}_l \hat{Z}_{1l}(y),$$

$$z_{n+\alpha+4}(y) = -\sqrt{1-\mu^2} \sum_{j=1}^{k} \cos \overline{\theta}_j \overline{Z}_{\alpha j}(y), \quad \alpha = 5, \ldots, n,$$

$$z_{2n+5}(y) = -\sqrt{1-\mu^2} \sum_{j=1}^{k} \sin \overline{\theta}_j \overline{Z}_{3j} + \sqrt{1-\lambda^2} \sum_{l=1}^{h} \cos \hat{\theta}_l \hat{Z}_{2l}$$

$$z_{2n+6}(y) = -\sqrt{1-\mu^2} \sum_{j=1}^{k} \sin \overline{\theta}_j \overline{Z}_{4j} + \sqrt{1-\lambda^2} \sum_{l=1}^{h} \sin \hat{\theta}_l \hat{Z}_{2l}$$

and, for $\alpha = 5, \ldots, n$,

$$z_{2n+\alpha+2}(y) = -\sqrt{1-\mu^2} \sum_{j=1}^{k} \sin \overline{\theta}_j \overline{Z}_{\alpha j}, \quad z_{3n+\alpha-2}(y) = -\sqrt{1-\lambda^2} \sum_{l=1}^{h} \sin \hat{\theta}_l \hat{Z}_{3l},$$

$$z_{4n+\alpha-6}(y) = -\sqrt{1-\lambda^2} \sum_{l=1}^{h} \cos \hat{\theta}_l \hat{Z}_{4l}.$$

The proof of the above identities follows from straightforward computations, and the symmetry properties for $U(y) + \psi(y)$, for $U_{\mu,\xi_j}(y) + \overline{\phi}_j(y)$ and $U_{\lambda,\eta_l}(y) + \hat{\phi}_l(y)$ respectively.

A less straightforward computation gives that, $\alpha = n+3, n+4, n+5, n+6$, we have

$$\mathbf{z}_{n+3}(y) = z_1 - 2\sqrt{1-\mu^2} \sum_{j=1}^{k} \cos\overline{\theta}_j \left[\overline{Z}_{0j}(y) + \overline{Z}_{1j} \right],$$

$$\mathbf{z}_{n+4}(y) = z_2 - 2\sqrt{1-\mu^2} \sum_{j=1}^{k} \sin\overline{\theta}_j \left[\overline{Z}_{0j}(y) + \overline{Z}_{1j} \right],$$

$$\mathbf{z}_{n+5}(y) = z_1 - 2\sqrt{1-\lambda^2} \sum_{l=1}^{h} \cos\hat{\theta}_l \left[\hat{Z}_{0l}(y) + \hat{Z}_{3l} \right],$$

$$\mathbf{z}_{n+6}(y) = z_1 - 2\sqrt{1-\lambda^2} \sum_{j=1}^{h} \sin\hat{\theta}_l \left[\hat{Z}_{0l}(y) + \hat{Z}_{4l} \right].$$

$$(2.6.1)$$

We shall prove the validity of the first identity in (**??**). The proofs of the validity of the the other expressions in (**??**) are similar. We write

$$\mathbf{z}_{n+3} = z_1 + T(u), \qquad T(u) := (|y|^2 - 1)\frac{\partial u}{\partial y_1} - 2y_1(\frac{n-2}{2}u(y) + \nabla u(y) \cdot y).$$

Thus (**??**) follows from (**??**) and from

$$T(u) = -2\sum_{j=1}^{k} \xi_{j1} \left[\overline{Z}_{0j} + \nabla(U_{\mu,\xi_j} + \overline{\phi}_j)(y) \cdot \xi_j \right].$$

$$(2.6.2)$$

From the explicit expression of u in (**??**), we get

$$T(u) = T(U + \psi) - \sum_{j=1}^{k} T(U_{\mu,\xi_j} + \overline{\phi}_j) - \sum_{l=1}^{h} T(U_{\lambda,\eta_l} + \hat{\phi}_l).$$

We shall first show that $T(U + \psi)(y) \equiv 0$, and $T(U_{\lambda,\eta_l} + \hat{\phi}_l)(y) \equiv 0$ for any $l = 1, \ldots, h$.

Observe that, if v is any smooth function and if we define $h(z) := \frac{\partial}{\partial z_1}\left(|z|^{2-n}v(\frac{z}{|z|^2})\right)$, then we have

$$h(z) = -\frac{2z_1}{|z|^n}\left[\frac{n-2}{2}v\left(\frac{z}{|z|^2}\right) + \nabla v\left(\frac{z}{|z|^2}\right)\cdot\left(\frac{z}{|z|^2}\right)\right]$$
$$+ \frac{1}{|z|^n}\frac{\partial v}{\partial z_1}\left(\frac{z}{|z|^2}\right),$$

and $g(y) := \frac{1}{|y|^{n-2}}h(\frac{y}{|y|^2})$ takes the form

$$g(y) = -2y_1\left[\frac{n-2}{2}v(y) + \nabla v(y)\cdot y\right] + |y|^2\frac{\partial v}{\partial y_1}(y).$$

With this is mind, one gets that if v is Kelvin invariant $v(y) = |y|^{n-2}v\left(\frac{y}{|y|^2}\right)$, then

$$\frac{1}{|y|^{n-2}}\frac{\partial v}{\partial y_1}\left(\frac{y}{|y|^2}\right) = -2y_1\left[\frac{n-2}{2}v(y) + \nabla v(y)\cdot y\right] + |y|^2\frac{\partial v}{\partial y_1}(y).$$

$$(2.6.3)$$

On the other hand, if v is Kelvin invariant (with respect to the origin) and even in y_1, then also the function $\frac{\partial v}{\partial y_1}$ is Kelvin invariant, that is $\frac{\partial v}{\partial y_1}(y) = |y|^{n-2}\frac{\partial v}{\partial y_1}\left(\frac{y}{|y|^2}\right)$. By (**??**), we get that any function v which is invariant under Kelvin transform (with respect to the origin) and even in the y_1 direction, one that $T(v)(y) \equiv 0$. Since the functions $(U + \psi)(y)$, and $(U_{\lambda,\eta_l} + \hat{\phi}_l)(y)$ for any $l = 1, \ldots, h$ are invariant under Kelvin transform and even in y_1, we get the proof of our claim.

Let us fix $j \in \{1, \ldots, k\}$. We write, for $v(y) = (U_{\mu,\xi_j} + \overline{\phi}_j)(y)$,

$$T(U_{\mu,\xi_j} + \overline{\phi}_j)(y) = \underbrace{(|y|^2 - 1)\frac{\partial v}{\partial y_1}(y) - 2(y - \xi_j)_1[\frac{n-2}{2}v(y) + \nabla v(y) \cdot y]}_{=:T_j(v)}$$
$$- 2(\xi_j)_1\left[\overline{Z}_{0j}(y) + \nabla v(y) \cdot \xi_j\right].$$

We claim that $T_j(U_{\mu,\xi_j} + \overline{\phi}_j)(y) \equiv 0$. To prove this fact, we recall that

$$v(y) := (U_{\mu,\xi_j} + \overline{\phi}_j)(y) = \mu^{-\frac{n-2}{2}}(U + \overline{\overline{\phi}}_1)\left(\frac{y - \xi_j}{\mu}\right),$$

see Section **??**. Also, μ and $|\xi|$ are related so that $U_{\mu,\xi_j} + \overline{\phi}_j$ is invariant under Kelvin transform. Thus, from (**??**), we get

$$T_j(v)(y) = \frac{1}{|y|^{n-2}}\frac{\partial v}{\partial y_1}\left(\frac{y}{|y|^2}\right) - \frac{\partial v}{\partial y_1}(y) + 2(\xi_j)_1\left[\frac{n-2}{2}v(y) + \nabla v(y) \cdot y\right].$$

We note that, in this case, $U_{\mu,\xi_j} + \overline{\phi}_j$ is not even in the y_1 variable, so that one gets

$$\frac{1}{|y|^{n-2}}\frac{\partial v}{\partial y_1}\left(\frac{y}{|y|^2}\right) = \frac{\partial v}{\partial y_1}(y) - (\xi_j)_1\left[(n-2)v(y) + 2\nabla v(y) \cdot y\right].$$

This concludes the proof of (**??**).

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