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# On the Parabolic Gluing Method and Singularity Formation

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Abstract. Singularity formation for evolution equations has attracted much attention in recent years. In this survey article, we will introduce some recent progress on the *parabolic gluing method* and its applications in investigating the mechanism of singularity formation for parabolic flows. Two model problems will be revisited to illustrate the ideas, and recent developments and techniques will be presented.

Résumé. La formation de singularités pour les équations d'évolution a attiré beaucoup d'attention ces dernières années. Dans cet article d'enquête, nous présenterons quelques progrès récents sur la *méthode de collage parabolique* et ses applications dans l'étude du mécanisme de formation de singularités pour les écoulements paraboliques. Deux problèmes modèles seront revisités pour illustrer les idées, et les développements et techniques récents seront présentés.

1. **Introduction** Singularity formation for evolution equations has attracted much attention in recent years, probably because of the connection to the possible singularity or global regularity for the incompressible Navier-Stokes equation in $\mathbb{R}^3$, a Clay Millennium Problem, as well as the motivations from geometric flows (Ricci flow and mean curvature flow). As a matter of fact, the resolution of Poincare's conjecture (another Clay Millennium problem) by G. Perelman [47, 48] is a manifesto of the importance of the analysis of singularity formulation in evolution equations. Many equations, such as Fujita equations and harmonic map heat flows, which certainly have their own interest and significance, might be regarded as testing fields for the analysis of singularity formation in evolution equations. In this survey, we shall report some recent development on the *parabolic gluing method* and its applications in constructing finite- and infinite-time blow-up solutions to various evolution equations.

In the elliptic context, the *inner–outer gluing method* was developed by del Pino, Kowalczyk, and Wei [13, 15] to investigate concentration on higher dimensional sets such as curves and surfaces. The climax of this method is the resolution of De Giorgi's Conjecture in dimensions greater than 8 [15]. Since then, many new phenomena and features have been found in the Allen-Cahn equations, critical or supercritical elliptic problems, and other settings. Its parabolic

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analogue, motivated by a recent surge of interests in the singularity formation in evolution settings, was developed by Dávila, del Pino, Musso, and Wei to investigate finite- and infinite-time blow-up solutions for energy critical heat equations and heat flow of harmonic maps [7,12]. Later, the gluing method was generalized and applied to a wider class of evolution equations. Without being exhaustive, these include Euler equations and related fluid equations, geometric flows, parabolic equations, and systems arising from mathematical biology and physics such as Keller-Segel systems, nematic liquid crystal flow, the LLG equation, and others. We refer the reader to [9–11,17,18,40,53,57] and the references therein.

Let us first explain some of the ideas and recent progress in the development of the gluing method in an abstract fashion and then give two specific examples to better illustrate the ideas. Roughly speaking, the parabolic gluing method is a refined type of perturbative argument. Our aim is to construct solutions exhibiting singular asymptotic behavior near some concentration points as $t \to T$ or $t \to +\infty$. The construction starts with a well-chosen blow-up profile, usually driven by energy concentration. Then one looks for a perturbation that consists of inner and outer parts, where the inner part captures the heart of the singularity formation and the outer part handles all the external noises. This leads to a coupled inner–outer gluing system involving the inner and outer solutions and the (typically scaling and translation) parameter functions. The full system is then solved by a fixed point argument provided that one can obtain suitable linear theories for inner and outer problems as well as the reduced problems that determine the dynamics of parameters. The linear theories are designed such that the full system is decoupled or less coupled, namely the gluing procedure can be implemented, and it usually involves careful and rather precise choices of weighted topologies in a pointwise sense for solution spaces. On the other hand, the linearization for the inner problem is surely not invertible in the presence of an infinitesimal generator of rigid motions, and thus for an inner solution with sufficient decay to exist, orthogonality conditions are required ensuring the development of the linear theory. These orthogonalities in turn determine the dynamics of parameters, yielding the desired blow-up speed and location.

A typical first approximate solution is the steady state invariant under rescaling and translation. These invariances naturally imply kernels in the linearized operator. We call the case where all the kernels decay sufficiently fast the $L^2$ case, while the case with slowly decaying kernels ($\notin L^2(\mathbb{R}^n)$) is called the non-$L^2$ case. Usually the non-$L^2$ case happens in lower dimensions, and under such circumstances, well chosen nonlocal corrections are needed in order to improve the spatial decay. The new error terms introduced by nonlocal corrections enter the orthogonality condition (at the corresponding mode) as leading order, leading to a certain integro-differential operator in the reduced equation. This global feature has been observed, for instance, in [8,12,17,58]. Techniques such as the Laplace transform and Riemann-Liouville type can be applied for certain cases, but for the other threshold cases, one has to take advantage of the Hölder regularity inherited from the outer problem to control the nonlocal operator.

In general, the development of the linear theory for the outer problem is more straightforward compared to the one for the inner problem. In parabolic settings, the maximum principle, or the direct and careful use of Duhamel's formula, can be employed. However, the design of the weighted space can be more delicate in the absence of the maximum principle. The gluing method in such set-ups has been developed and applied equally well. See [9–11, 57] for recent progress on the incompressible Euler equations and LLG equation.

For the inner problem, solutions with sufficient decay in space-time can only be expected with orthogonality conditions imposed, and careful choice of initial data might be needed if instability is present for the corresponding linearized operator, resulting in codimension stability. There are various techniques and tools available, and the spectral information plays a key role. A refined version, that gets less deteriorated in the innermost region, can be achieved by the re-gluing process, namely another inner–outer gluing procedure. The distorted Fourier transform also turns out to be a powerful tool in the gluing method and is present in [57]. This is motivated by [28] on the spectral analysis of the Schrödinger operator and [36–39] on the singularity formation for wave equations and wave maps.

In the rest of this survey, we plan to revisit two model problems, the Fujita equation with critical exponent in $\mathbb{R}^5$ ($L^2$ case) and in $\mathbb{R}^4$ (non-$L^2$ case) to illustrate the ideas and techniques in the parabolic gluing method. In the last section, recent application of the distorted Fourier transform in the LLG equation will be presented. In the appendix we include the proof of a priori estimates of linear theory by the blow-up arguments.

## 2. Fujita Equation: A Brief Introduction

Let us start with a brief introduction to singularity formation for the Fujita equation,

$$u_t = \Delta u + |u|^{p-1}u \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$

where $\Omega$ is the entire space $\mathbb{R}^n$ or a smooth domain in $\mathbb{R}^n$ and $0 < T \le +\infty$. This semilinear heat equation with $p > 1$ has been widely studied since Fujita's celebrated work [26]. The Fujita equation might be the one of the most simple-looking semilinear parabolic equations. However, rich and sophisticated phenomena arise, and those are intimately related to the power nonlinearity in a rather precise manner. Much literature has been devoted to studying this problem concerning the singularity formation. For a comprehensive survey in the literature, we refer the readers to the book of Quittner and Souplet [49].

For the finite time blow-up, the solution $u$ is said to be *type I* if

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty < +\infty,$$

and *type II* if

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty = +\infty.$$

Type I blow-up is more generic and similar to that of the ODE $u_t = u^p$, while type II blow-up, where the Laplacian dominates, is much more difficult to detect. In particular, two different types of blow-up phenomena in problem (1) depend sensitively on the power nonlinearity. For instance, it is known after a series of works, including [29,30], that type I is the only way possible if $p < p_s$ in the case that $\Omega$ is $\mathbb{R}^n$ or a convex domain, where $p_s$ is the critical Sobolev exponent

$$p_s := \begin{cases} \frac{n+2}{n-2} & \text{if } n \geq 3, \\ +\infty & \text{if } n = 1, 2. \end{cases}$$

The critical exponent $p_s$ is special in various ways. For the energy critical case $p = p_s$, in the positive radial and monotonically decreasing class, Filippas, Herrero and Velázquez [24] excluded the possibility of type II blow-up for $n \geq 3$, and Matano and Merle [42, Theorem 1.7] removed the monotone assumption and obtained the same result. Wang and Wei [56] generalized the result to the non-radial positive class in higher dimensions $n \geq 7$. For $p < p_s$, finite time type I blow-up solution was found and its stability was studied in [43]. For the critical case $p = p_s$ in $\mathbb{R}^n$ with $n \geq 7$, classification results were proved near the ground state of the energy critical heat equation in [5]. In the aspect of type II blow-ups, the first example was discovered by Herrero-Velázquez [33, 34], for $p > p_{JL}$ where $p_{JL}$ is the Joseph-Lundgren exponent [35]

$$p_{JL} = \begin{cases} 1 + \frac{4}{n-4-2\sqrt{n-1}} & \text{if } n \geq 11, \\ +\infty, & \text{if } n \leq 10. \end{cases}$$

See, for instance, [4, 6, 19, 44, 52] and references therein for more results on existence and construction of type II blow-ups. For the critical case $p = p_s$ in dimensions $n = 3, 4, 5, 6$, sign-changing type II blow-up solutions were conjectured to exist, via formal matched asymptotic analysis, by Filippas, Herrero and Velazquez [24] and have been rigorously constructed recently in [16, 20, 22, 31, 32, 41, 51].

In view of the results mentioned above, regarding finite-time blow-up for positive solutions to the Fujita equation (1), we mention three interesting open questions/Conjectures.

**Conjecture 1.** For $3 \leq n \leq 6$ and $p = \frac{n+2}{n-2}$, all positive finite time blow-ups to (1) are Type I.

**Conjecture 2.** For $n \geq 7$ and $p = \frac{n+2}{n-2}$, all (sign-changing) finite time blow-ups to (1) are Type I.

**Conjecture 3.** For $\frac{n+2}{n-2} < p < p_{JL}(n)$ and $p \neq \frac{n-m+2}{n-m-2}$, all finite time blow-ups to (1) are Type I.

On the other hand, infinite time blow-ups for $p = p_s$ have also received some attention recently. In dimensions $n \geq 3$, Galaktionov and King [27] investigated positive, radially symmetric, infinite time blow-up solutions for problem (1) in the case of the unit ball with Dirichlet boundary condition. See also [55, Theorem 1.4] for the case that the domain is convex and symmetric. In the non-radial setting, the positive infinite time blow-up solutions for problem (1) with zero Dirichlet boundary condition and $n \geq 5$ was constructed in [7], where the role of the Green's function in the bubbling phenomenon was studied, in parallel to the seminal works [2] and [3] in elliptic settings. See also [21] for the construction based on non-degenerate sign-changing profile and [18, 54] for the bubble towers in higher dimensions at forward and backward time infinity. Infinite time blow-ups for the lower dimensions $n = 3, 4$ have been constructed in [17,58] confirming a conjecture by Fila and King [23].

**3. $L^2$ Case: Critical Fujita Equation in $\mathbb{R}^5$** In this section, we introduce the first (simpler) parabolic gluing method when the kernels are in $L^2$.

The first example is the type II singularity for Fujita equation with critical exponent in $\mathbb{R}^5$. The first step is to find a suitable blow-up profile whose natural choice is the steady state. We recall that all positive entire solutions of the equation

$$\Delta u + |u|^{\frac{4}{n-2}} u = 0 \quad \text{in } \mathbb{R}^n$$

are given by the family of *Aubin-Talenti bubbles*

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\mu}\right) \tag{2}$$

where

$$U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{n-2}{4}}.$$

The solutions we construct do change sign, and look at main order near the blow-up points like one of the bubbles (2) with time dependent parameters and $\mu(t) \to 0$ as $t \to T$. Thus we consider the equation

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u \quad \text{in } \Omega \times (0,T), \\ u(\cdot, 0) = u_0 \quad \text{in } \Omega \end{cases} \tag{3}$$

in the case $n = 5$, $p = 7/3$. Let us fix arbitrary points $q_1, q_2, \dots q_k \in \Omega$. We consider a smooth function $Z_0^* \in L^\infty(\Omega)$ with the property that

$$Z_0^*(q_j) < 0 \ \text{ for all } \ j = 1, \dots, k.$$

The sign condition is required to ensure the existence of the desired blow-up dynamics.

THEOREM 1 ([16]). *Let $n = 5$. For each $T > 0$ sufficiently small there exists an initial condition $u_0$ such that the solution of problem (3) blows up at time $T$ exactly at the $k$ points $q_1, \ldots, q_k$. It looks at main order like*

$$u(x,t) = \sum_{j=1}^{k} U_{\mu_j(t),\xi_j(t)}(x) + Z_0^*(x) + \theta(x,t)$$

*where*

$$\mu_j(t) \to 0, \quad \xi_j(t) \to q_j \quad as \quad t \to T,$$

*and $\|\theta\|_{L^\infty} \le T^a$ for some $a > 0$. More precisely, for numbers $\beta_j > 0$ we have*

$$\mu_j(t) \;=\; \beta_j (T-t)^2\,(1 + o(1)).$$

We observe that, in particular, the solution constructed in Theorem 1 is type II since

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^5)} \sim (T-t)^{-3} \gg (T-t)^{-3/4}.$$

For notational simplicity we shall only sketch the proof in the single-bubble case $k = 1$. The general case requires relatively minor changes.

• **Ansatzes and error estimates.**

We fix a point $q \in \Omega$. Let us consider a function $Z_0^*$ smooth in $\bar{\Omega}$ with $Z_0^* = 0$ on $\partial\Omega$. We assume in addition that

$$Z_0^*(q) < 0. \tag{4}$$

We let $Z^*(x,t)$ be the unique solution of the initial-boundary value problem

$$\begin{cases} \partial_t Z^* = \Delta Z^* \quad \text{in } \ \Omega \times (0,\infty), \\ \quad Z^* = 0 \quad \text{on } \partial\Omega \times (0,\infty), \quad Z^*(\cdot,0) = Z_0^* \quad \text{in } \Omega. \end{cases} \tag{5}$$

We consider functions $\xi(t) \to q$, and parameters $\mu(t) \to 0$ as $t \to T$. We look for a solution of the form

$$u(x,t) = U_{\mu(t),\xi(t)}(x) + Z^*(x,t) + \varphi(x,t) \tag{6}$$

with a remainder $\varphi$ consisting of inner and outer parts

$$\varphi(x,t) = \mu^{-\frac{n-2}{2}} \phi\,(y,t)\,\eta_R(y) + \psi(x,t), \quad y = \frac{x-\xi(t)}{\mu(t)} \tag{7}$$

where

$$\eta_R(y) = \eta_0\left(\frac{|y|}{R}\right)$$

and $\eta_0(s)$ is a smooth cut-off function with $\eta_0(s) = 1$ for $s < 1$ and $= 0$ for $s > 1$.

Let us define the error of $u$ as

$$S(u) = -u_t + \Delta u + u^p.$$

Then

$$\begin{aligned} S(U_{\lambda,\xi} + Z^* + \varphi) \; &= \; -\varphi_t + \Delta\varphi + pU_{\mu,\xi}^{p-1}(\varphi + Z^*) + \mu^{-\frac{n+2}{2}} E + N(Z^* + \varphi) \\ &= \; \eta_R \mu^{-\frac{n+2}{2}} \big[ -\mu^2 \phi_t + \Delta_y \phi + pU(y)^{p-1}[\phi + \mu^{\frac{n-2}{2}}(Z^* + \psi)] + E \big] \\ &\quad - \psi_t + \Delta_x \psi + p\mu^{-2}(1-\eta_R) U(y)^{p-1}(Z^* + \psi) + A[\phi] \\ &\quad + B[\phi] + \mu^{-\frac{n+2}{2}} E(1-\eta_R) + N(Z^* + \varphi) \end{aligned}$$

where

$$E(y,t) \; := \mu\dot\mu[y \cdot \nabla U(y) + \frac{n-2}{2} U(y)] + \mu\dot\xi \cdot \nabla U(y), \tag{8}$$

$$\begin{aligned} N_{\mu,\xi}(Z) &:= |U_{\mu,\xi} + Z|^{p-1}(U_{\mu,\xi} + Z) - U_{\mu,\xi}^p - pU_{\mu,\xi}^{p-1} Z, \\ A[\phi] &:= \mu^{-\frac{n+2}{2}} \left\{ \Delta_y \eta_R \phi + 2\nabla_y \eta_R \nabla_y \phi \right\}, \\ B[\phi] &:= \mu^{-\frac{n}{2}} \left\{ \dot\mu\big[y \cdot \nabla_y \phi + \frac{n-2}{2}\phi\big]\eta_R + \dot\xi \cdot \nabla_y \phi \, \eta_R + \big[\dot\mu y \cdot \nabla_y \eta_R + \dot\xi \cdot \nabla_y \eta_R\big] \phi \right\} \end{aligned}$$

and we have used $U_{\mu,\xi}^{p-1}\varphi = \mu^{-2} U(y)^{p-1}\varphi$. Thus, we will have a solution if the pair $(\phi(y,t), \psi(x,t))$ solves the following inner–outer gluing system

$$\mu^2 \phi_t \; = \; \Delta_y \phi + pU(y)^{p-1}\phi + H(\psi, \mu, \xi) \quad \text{in } B_{2R}(0) \times (0,T) \tag{9}$$

$$\begin{cases} \psi_t &= \Delta_x \psi + G(\phi, \psi, \mu, \xi) \quad \text{in } \Omega \times (0,T) \\ \psi &= -U_{\mu,\xi} \qquad\qquad\qquad \text{on } \partial\Omega \times (0,T), \\ \psi(\cdot, 0) &= 0 \qquad\qquad\qquad\qquad \text{in } \Omega \end{cases} \tag{10}$$

where

(11)

$$\begin{aligned} H(\psi,\mu,\xi)(y,t) \; &:= \mu^{\frac{n-2}{2}} pU(y)^{p-1}(Z^*(\xi + \mu y, t) + \psi(\xi + \mu y, t)) + E(y,t), \\ G(\phi,\psi,\mu,\xi)(x,t) \; &:= p\mu^{-2}(1-\eta_R)U(y)^{p-1}(Z^* + \psi) + A[\phi] \; + B[\phi] \\ &\quad + \mu^{-\frac{n+2}{2}} E(1-\eta_R) + N(Z^* + \varphi), \quad y = \frac{x-\xi}{\mu}. \end{aligned}$$

• **Formal derivation of $\mu$ and $\xi$.**

Next we do a formal consideration that allows us to identify the parameters $\mu(t)$ and $\xi(t)$ at main order. Leaving aside smaller order terms, the inner problem (9) is approximately an equation of the form

$$\begin{aligned} \mu^2\phi_t = \Delta_y\phi + pU(y)^{p-1}\phi + h(y,t) \quad \text{in } \mathbb{R}^n \times (0,T) \\ \phi(y,t) \to 0 \quad \text{as} \quad |y| \to \infty \end{aligned} \tag{12}$$

with

$$\begin{aligned} h(y,t) \ &= \mu\dot{\mu}\,(U(y) + y\cdot\nabla U(y)) + p\mu^{\frac{n-2}{2}}U(y)^{p-1}Z_0^*(q) \\ &+ \ \mu\dot{\xi}\cdot\nabla U(y) + p\mu^{\frac{n}{2}}U(y)^{p-1}\nabla Z_0^*(q)\cdot y. \end{aligned} \tag{13}$$

The condition of spatial decay in $y$ for the inner problem solution $\phi$ mitigates the effect of $\phi$ in the outer problem (10), making at main order (9) and (10) decoupled.

Roughly speaking, for $n \geq 5$, the elliptic equation

$$\begin{aligned} L[\phi] := \Delta_y\phi + pU(y)^{p-1}\phi = g(y) \quad \text{in } \mathbb{R}^n \\ \phi(y) \to 0 \quad \text{as} \quad |y| \to \infty, \end{aligned}$$

with $g(y) = O((1+|y|)^{-2-a})$ and $0 < a < 1$, is solved by $\phi = O((1+|y|)^{-a})$ provided that

$$\int_{\mathbb{R}^n} g(y)Z_i(y)\,dy \ = \ 0 \ \text{ for all } \ i = 1,\ldots,n+1,$$

where

$$Z_i(y) = \partial_i U(y), \quad i = 1,\ldots,n, \quad Z_{n+1}(y) = \frac{n-2}{2}U(y) + y\cdot\nabla U(y).$$

These are in fact all bounded solutions of the linearized equation $L[Z] = 0$.

It seems reasonable to get an approximation to a solution of equation (12) (valid up to large $|y|$) by solving the elliptic equation

$$\begin{aligned} \Delta_y\phi + pU(y)^{p-1}\phi + h(y,t) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\ \phi(y,t) \to 0 \quad \text{as} \quad |y| \to \infty, \end{aligned}$$

which we can indeed do under the orthogonality conditions

$$\int_{\mathbb{R}^n} h(y,t)Z_i(y)\,dy \ = \ 0 \ \text{ for all } \ i = 1,\ldots,n+1, \quad t \in [0,T). \tag{14}$$

These orthogonalities imply the dynamics of the parameters. Indeed, integrating against $Z_{n+1}(y)$ we get

$$\int_{\mathbb{R}^n} h(y,t)Z_{n+1}(y)\,dy \;=\; \mu\dot{\mu}(t)\int_{\mathbb{R}^n} Z_{n+1}^2 dy - \frac{n-2}{2}\mu(t)^{\frac{n-2}{2}} Z_0^*(q)\int_{\mathbb{R}^n} U^p dy.$$

Clearly, one sees from above that integrability issues arise for lower dimensional cases, yielding nonlocal/global features, which we shall discuss in next Section for the case $n = 4$. This quantity is zero if and only if for a certain explicit constant $\beta_n > 0$

$$\dot{\mu}(t) = -\beta_n |Z_0^*(q)|\mu(t)^{\frac{n-4}{2}}, \quad \mu(T) = 0,$$

and thus for $n = 5$

$$\mu_*(t) = \alpha(T-t)^2, \quad \alpha = \frac{1}{4}\beta_n^2 |Z_0^*(q)|^2. \tag{15}$$

In a similar way, the remaining $n$ relations in (14) lead us to $\dot{\xi}(t) = \mu(t)^{\frac{n-2}{2}} b$ for a certain vector $b$. Hence $\dot{\xi}(t) = O(T-t)^3$ and

$$\xi(t) = q + O(T-t)^2.$$

To solve the actual inner problem (12), even assuming orthogonalities (14) is not sufficient, and further constraints are needed. Indeed, let us recall that the operator $L$ has a positive radially symmetric bounded eigenfunction $Z_0$ associated to the only positive eigenvalue $\lambda_0$ to the problem

$$L[\phi] = \lambda_0 \phi, \quad \phi \in L^\infty(\mathbb{R}^n).$$

It is known that $\lambda_0$ is a simple eigenvalue and

$$Z_0(y) \sim |y|^{-\frac{n-1}{2}} e^{-\sqrt{\lambda_0}\,|y|} \quad \text{as} \quad |y| \to \infty.$$

Let us write

$$p(t) = \int_{\mathbb{R}^n} \phi(y,t) Z_0(y)\,dy, \quad q(t) = \int_{\mathbb{R}^n} h(y,t) Z_0(y)\,dy.$$

One expects instability produced by $Z_0$ along the flow without restriction on the initial data. Then we compute

$$\mu(t)^2 \dot{p}(t) - \lambda_0 p(t) = q(t).$$

Since $\mu(t) \sim (T-t)^{-2}$, then $p(t)$ will have exponential growth in time $p(t) \sim e^{\frac{c}{T-t}}$ unless

$$p(t) = e^{\int_0^t \frac{d\tau}{\mu^2(\tau)}} \int_t^T e^{-\int_0^s \frac{d\tau}{\mu^2(\tau)}}\, \mu(s)^{-2}\, q(s)\,ds$$

This relation imposes a linear constraint on the initial data $\phi(y,0)$ to the desired solution $\phi(y,t)$ to (9)

$$\int_{\mathbb{R}^n} \phi(y,0) Z_0(y)\,dy = \int_0^T e^{-\int_0^s \frac{d\tau}{\mu^2(\tau)}} \mu(s)^{-2} \int_{\mathbb{R}^n} h(y,s) Z_0(y)\,dy\,ds. \tag{16}$$

For this reason, we impose an initial data for the inner problem along $Z_0$-direction to get rid of such instability.

• **The linear theories.**

The outer problem (10) in linear version is actually simpler than its counterpart (19), corresponding just to the standard heat equation with nearly singular right hand sides and zero initial and boundary conditions. Thus we consider the problem

$$\begin{cases} \psi_t &= \Delta_x \psi + g(x,t) \quad \text{in } \Omega \times (0,T) \\ \psi &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ \psi(\cdot,0) &= 0 \quad \text{in } \Omega. \end{cases} \tag{17}$$

The class of right hand sides $g$ that we want to take are naturally controlled by the following norms. Let $0 < a < 1$, $q \in \Omega$ and $\mu_0(t) = (T-t)^2$. We define the norms $\|g\|_{o*}$ and $\|\psi\|_o$ to be respectively the least numbers $K_1$ and $K_2$ such that for all $(x,t) \in \Omega \times [0,T)$,

$$\begin{aligned} |g(x,t)| &\le K_1 \left[ \frac{1}{\mu_0(t)^2} \frac{1}{1+|y|^{2+a}} + 1 \right] \\ |\psi(x,t)| &\le K_2 \left[ \frac{1}{1+|y|^a} + T^{\frac{3}{2}a} \right] \end{aligned}, \qquad y = \frac{x-q}{\mu_0(t)}.$$

Then the following estimate holds.

LEMMA 3.1. ([16, Lemma 4.2]) *There exists a constant $C$ such that for all sufficiently small $T > 0$ and any $g$ with $\|g\|_o < +\infty$, the unique solution $\psi = \mathcal{T}^{out}[g]$ of problem (17) satisfies the estimate*

$$\|\psi\|_{o*} \le C \|g\|_o. \tag{18}$$

The proof can be carried out either by barriers or Duhamel's representation.

The inner problem (19) in linear version is actually harder and more delicate than its counterpart (10). In order to deal with the inner problem (9), we need to solve a linear problem like (12) restricted to a large ball $B_{2R}$ where orthogonality conditions like (14) are assumed and the initial condition of the solution depends on a scalar parameter which is part of the unknown, connected with constraint

(16). We construct a solution $(\phi, \ell)$ which defines a linear operator of functions $h(y,t)$ defined on

$$\mathcal{D}_{2R} = B_{2R} \times (0,T)$$

to the initial value problem

$$\begin{aligned} \mu^2 \phi_t &= \Delta_y \phi + pU(y)^{p-1}\phi + h(y,t) \quad \text{in } \mathcal{D}_{2R} \\ \phi(y,0) &= \ell Z_0(y) \quad \text{in } B_{2R}, \end{aligned} \tag{19}$$

for some constant $\ell$, under the orthogonality conditions

$$\int_{B_{2R}} h(y,t)\, Z_i(y)\, dy = 0 \;\; \text{for all} \;\; i = 1, \ldots, Z_{n+1},\;\; t \in [0,T). \tag{20}$$

We impose on the parameter function $\mu$ the following constraints, which are motivated on the discussion earlier: let us write

$$\mu_0(t) = (T-t)^2.$$

For some positive constants $\alpha$ and $\beta$ (to be fixed later), we impose

$$\alpha\mu_0(t) \;\leq\; \mu(t) \;\leq\; \beta\mu_0(t) \;\; \text{for all} \;\; t \in [0,T].$$

Let us fix numbers $0 < a < 1$ and $\nu > 0$. We will consider functions $h$ satisfying

$$|h(y,t)| \lesssim \frac{\mu_0(t)^\nu}{1+|y|^{2+a}} \quad \text{in } \mathcal{D}_{2R}.$$

The formal analysis of the previous section would make us hope to find a solution to (19) such that

$$|\phi(y,t)| \lesssim \frac{\mu_0(t)^\nu}{1+|y|^{a}} \quad \text{in } \mathcal{D}_{2R}.$$

We will find a solution so that a somewhat worse bound for $\phi(y,t)$ in space variable is found but coinciding with the expected behavior in the gluing regime $|y| \sim R$. Let us define the following norms. We let $\|h\|_{2+a,\nu}$ be the least number $K$ such that

$$|h(y,t)| \;\leq\; K\frac{\mu_0(t)^\nu}{1+|y|^{2+a}} \quad \text{in } \mathcal{D}_{2R} \tag{21}$$

and let $\|\phi\|_{*a,\nu}$ be the least number $K$ with

$$|\phi(y,t)| \;\leq\; K\mu_0(t)^\nu \frac{R^{n+1-a}}{1+|y|^{n+1}} \quad \text{in } \mathcal{D}_{2R}. \tag{22}$$

We observe that $\|\phi\|_{*a,\nu} \leq \|\phi\|_{a,\nu}$.

The following is the key linear result associated to the inner problem.

LEMMA 3.2. ([7, Proposition 7.1],[16, Lemma 4.1]) *There is a $C > 0$ such that for all sufficiently large $R > 0$ and any $h$ with $\|h\|_{2+a,\nu} < +\infty$ that satisfies relations* (20) *there exist linear operators*

$$\phi = \mathcal{T}_{\mu}^{in}[h], \quad \ell = \ell[h]$$

*which solve Problem* (19) *and define linear operators of $h$ with*

$$|\ell[h]| + \|(1+|y|)\nabla_y \phi\|_{*a,\nu} \ + \ \|\phi\|_{*a,\nu} \ \leq \ C\,\|h\|_{\nu,2+a}.$$

The proof is by using the self-similar variables $(y,\tau)$ with

$$\tau = \tau_0 + \int_0^t \mu(s)^{-2} ds.$$

Expressing $\phi = \phi(y,\tau)$, problem (19) becomes

$$\begin{aligned} \phi_\tau &= \Delta_y \phi + pU(y)^{p-1}\phi + h(y,\tau) \quad \text{in } B_{2R}\times(\tau_0,\infty), \\ \phi(y,0) &= \ell Z_0(y) \quad \text{in } B_{2R}. \end{aligned}$$

Then finding solution with sufficient decay in space-time consisting of three steps:

Step 1: solving an elliptic equation by orthogonality;
Step 2: solving a parabolic equation with slower decay by a spectrum gap estimate (see Lemma 4.1) and energy estimates, then improving the pointwise estimate;
Step 3: acting the linearized operator on both sides of the parabolic equation in Step 2 yields a desired solution.

In fact, for the higher dimensional case $n \geq 5$, blow-up argument can be employed to show a more refined version of the linear theory for the inner problem. The result obtained via blow-up argument turns out to be exactly what we have discussed formally before, and there is no loss of $R$'s in the innermost region. We give a detailed proof in Appendix 5. Since we do not use maximum principle in the blow-up argument, this argument is rather general and flexible and can be applied to a larger class of equations. One good application of this blow-up argument is the construction of infinite time blow-ups for the fractional Fujita equation

$$u_t + (-\Delta)^s u = |u|^{\frac{4s}{n-2s}} u$$

in which the localization argument in the above does not work, since the operator $(-\Delta)^s$ has to be defined globally. Another applictaion is the infinite-time blow-up for sign-changing blow-ups for the Fujita equation. See [21, 45].

**• Proof of Theorem 1: fixed point argument.**

With the above preliminaries we are now ready to carry out the proof of Theorem 1 for the case $k = 1$. We want to find a tuple $\vec{p} = (\phi, \psi, \mu, \xi)$ solving the inner–outer gluing system (9)-(10) so that a desired blow-up solution $u$ is constructed. This is achieved by formulating the problem as a fixed point problem for $\vec{p}$ in a small region of a suitable Banach space.

We first set up inner problem. For a function $h(y,t)$ defined in $\mathcal{D}_{2R}$, we write

$$c_j[h](t) = \frac{\int_{B_{2R}} h(y,t)\, Z_j(y)\, dy}{\int_{B_{2R}} |Z_j(y)|^2 dy}$$

so that the function

$$\bar{h}(y,t) = h(y,t) - \sum_{j=1}^{n+1} c_j[h](t)\, Z_j(y)$$

satisfies

$$\int_{B_{2R}} \bar{h}(y,t)\, Z_j(y)\, dy \ = \ 0 \ \text{ for all } \ j = 1, \ldots, n+1, \quad t \in [0,T)$$

which makes the result of Lemma 3.2 applicable to the equation

$$\begin{cases} \mu^2 \phi_t \ = \ \Delta_y \phi + pU(y)^{p-1}\phi + \bar{H}(\psi,\mu,\xi) & \text{in } \mathcal{D}_{2R} \\ \phi(\cdot,0) = \ell\, Z_0 \quad \text{in } B_{2R} \end{cases} \tag{23}$$

where

$$\bar{H}(\psi,\mu,\xi) = H(\psi,\mu,\xi) - \sum_{j=1}^{n+1} c_j[H(\psi,\mu,\xi)]\, Z_j$$

and $H(\psi,\mu,\xi)$ is defined in (11). Using Lemma 3.2, we find a solution to (23) if the following equation is satisfied

$$\phi = \mathcal{T}_{\mu}^{in}[\, \bar{H}(\psi,\mu,\xi)\,] =: \mathcal{F}_1(\phi,\psi,\mu,\xi). \tag{24}$$

Then the inner equation (9) is satisfied if in addition we have

$$c_j[H(\psi,\mu,\xi)] = 0 \ \text{ for all } \ j = 1, \ldots, n+1. \tag{25}$$

In addition, the outer equation (10) is satisfied provided

$$\psi = \mathcal{T}^{out}[\, G(\phi,\psi,\mu,\xi)\,] =: \mathcal{F}_2(\phi,\psi,\mu,\xi). \tag{26}$$

where the operator $G(\phi,\psi,\mu,\xi)$ is defined in (11). We will solve system (23)-(25)-(26) using a degree-theoretical argument.

For $\lambda \in [0,1]$, we define the homotopy

$$\begin{aligned} H_\lambda(\psi,\mu,\xi)(y,t) \ &= \mu^{\frac{n-2}{2}} pU(y)^{p-1} Z_0^*(q) + \mu\dot{\mu} Z_{n+1}(y) + \mu \sum_{j=1}^{n} \dot{\xi}_j Z_j(y) \\ &\quad + \ \lambda\, \mu^{\frac{n-2}{2}} pU(y)^{p-1} (Z^*(\xi+\mu y,t) - Z_0^*(q) + \psi(\xi+\mu y,t)\,), \end{aligned}$$

and consider the system of equations

$$\begin{cases} \phi = \mathcal{T}_{\mu}^{in}[\,H_\lambda(\psi,\mu,\xi) - \displaystyle\sum_{j=1}^{n+1} c_j[H_\lambda(\psi,\mu,\xi)]\,Z_j\,] \\ c_j[H_\lambda(\psi,\mu,\xi)] \;=\; 0 \;\text{ for all }\; j=1,\dots,n+1, \\ \psi = \mathcal{T}^{out}[\,\lambda G(\phi,\psi,\mu,\xi)\,]. \end{cases} \tag{27}$$

We observe that for $\lambda = 1$ this problem precisely corresponds to the system (23)-(25)-(26) that we want to solve.

It is convenient to write

$$\mu(t) = \mu_*(t) + \mu^{(1)}(t), \quad \xi(t) = q + \xi^{(1)}(t), \quad t \in [0,T]$$

where $\mu_*(t)$ is defined in (15), and $\mu^{(1)}(T) = 0$, $\xi^{(1)}(t) = 0$.

We assume that we have a solution $(\phi,\psi,\mu^{(1)},\xi^{(1)})$ to system (27) with

$$\begin{cases} |\dot\mu^{(1)}(t)| + |\dot\xi^{(1)}(t)| \;\le \delta_0 \\ \|\phi\|_{*a,\nu} + \|\psi\|_\infty \;\le \delta_1 \end{cases} \tag{28}$$

where $\delta_0,\delta_1$ are small positive constants to be adjusted later. We will also assume that $Z^*$ is sufficiently small but fixed independently of $T$, i.e., $\|Z^*\|_\infty \ll 1$.

The function $\mu_*(t)$ solves the equation

$$\dot\mu_*(t)\int_{\mathbb{R}^N} Z_{n+1}^2\,dy + \mu_*(t)^{\frac{n-4}{2}} Z_0^*(q)\int_{\mathbb{R}^n} pU^{p-1}Z_{n+1}\,dy = 0. \tag{29}$$

The equation

$$c_{n+1}(H_\lambda(\psi,\mu_*+\mu_1,\xi))(t) = 0, \quad t\in[0,T) \tag{30}$$

which corresponds to

$$\begin{aligned} 0 \;=\;& \dot\mu(t)\Big(\int_{B_{2R}} Z_{n+1}^2\,dy\Big) \;+\; \mu(t)^{\frac{n-4}{2}} Z_0^*(q)\int_{B_{2R}} pU^{p-1}Z_{n+1}\,dy \\ &+\; \lambda\,\mu(t)^{\frac{n-4}{2}}\int_{\mathbb{R}^n} pU(y)^{p-1}(Z^*(\xi(t)+\mu(t)y,t) - Z_0^*(q) + \psi(\xi(t)+\mu(t)y,t)\,)Z_{n+1}(y)\,dy \end{aligned}$$

can be written as

$$\dot\mu(t) + \beta\mu(t)^{\frac{n-4}{2}} = \mu(t)^{\frac{n-4}{2}}(\delta_R + \lambda\theta(\psi,\xi,\mu_1))$$

for a suitable number $\beta > 0$, $\delta_R = O(R^{-2})$ and the operator $\theta$ satisfies

$$|\theta(\psi,\xi,\mu_1)| \;\le\; C\;(T + \|\psi\|_\infty)$$

for some constant $C$. From (29), the equation for $\mu_1$ can then be written, in the "linearized" form, as

$$\dot{\mu}_1 + \frac{\gamma}{T-t}\mu_1 = (T-t)g_0(\psi, \mu, \xi)$$

for a suitable $\gamma > 0$, where

$$|g_0(\psi, \xi, \mu^{(1)}, \lambda)(t)| \ \leq \ C\,(\|\psi\|_\infty + T + R^{-2}).$$

The linear problem

$$\dot{\mu} + \frac{\gamma}{T-t}\mu = (T-t)g(t), \quad \mu_1(T) = 0$$

can be uniquely solved by the following operator in $g$

$$\mu(t) = \mathcal{T}^0[g](t) := -(T-t)^{-\gamma}\int_t^T (T-s)^{\gamma+1} g_0(s)\,ds.$$

It defines a linear operator on $g$ with estimates

$$\|(T-t)^{-1}\dot{\mu}\|_\infty + \|(T-t)^{-2}\mu\|_\infty \leq C\|g_0\|_\infty.$$

Equation (30) then becomes

$$\mu^{(1)}(t) = \mathcal{T}^{(0)}[\,g_0(\psi, \xi, \mu^{(1)}, \lambda)\,](t) \ \text{ for all } \ t \in [0, T)$$

and we get

$$\|(T-t)^{-1}\dot{\mu}^{(1)}\|_\infty + \|(T-t)^{-2}\mu^{(1)}\|_\infty \ \leq \ C\,(\|\psi\|_\infty + T + R^{-2}). \tag{31}$$

Similarly, equations

$$c_j[H_\lambda(\psi, \mu, \xi)] \ = \ 0 \ \text{ for all } \ j = 1, \ldots, n,$$

can be written in vector form as

$$\xi^{(1)}(t) = \mathcal{T}^{(1)}[g_1(\psi, \mu_1, \xi_1)](t) \ \text{ for all } \ t \in [0, T), \tag{32}$$

where

$$\mathcal{T}^{(1)}[g] := \int_t^T (T-s)g(s)\,ds$$

and

$$|g_1(\psi, \xi, \mu^{(1)}, \lambda)(t)| \leq C\,(\|\psi\|_\infty + T).$$

From equation (32), we thus find

$$\|(T-t)^{-1}\dot\xi^{(1)}\|_\infty + \|(T-t)^{-2}\xi^{(1)}\|_\infty \ \le\ C\,(\|\psi\|_\infty + T). \tag{33}$$

On the other hand, we have

$$|H(\psi,\mu,\xi)(y,t)| \ \le\ C\frac{\mu(t)^{\frac{n-2}{2}}}{1+|y|^4}(\|\psi\|_\infty + \|Z^*\|_\infty) + \frac{\mu\dot\mu}{1+|y|^{n-2}} + \frac{\mu|\dot\xi|}{1+|y|^{n-1}}$$

and thus

$$|H(\psi,\mu,\xi)(y,t)| \ \le\ C\frac{\mu_0(t)^{\frac{n-2}{2}}}{1+|y|^{2+a}}\big(\|\psi\|_\infty + \|Z^*\|_\infty\big)$$

for $0<a<1$. From the first equation in (27) and Lemma 3.2, we obtain

$$\|\phi\|_{*a,\nu} \ \le\ C\big(\|\psi\|_\infty + \|Z^*\|_\infty\big), \quad \nu = \frac{n-2}{2}. \tag{34}$$

with the $\|\cdot\|_{*a,\nu}$-norm defined in (22). Next we consider the last equation in (27). We recall that

$$\begin{aligned}
G(\phi,\psi,\mu,\xi)(x,t) \ &= p\mu^{-2}(1-\eta_R)U(y)^{p-1}(Z^*+\psi) + A[\phi] \ + B[\phi]\\
&\quad + \mu^{-\frac{n+2}{2}}E(1-\eta_R) + N(Z^* + \mu^{-\frac{n-2}{2}}\eta_R\phi + Z^* + \psi),\\
E(y,t) \ &= \ \mu\dot\mu[y\cdot\nabla U(y) + \frac{n-2}{2}U(y)] + \mu\dot\xi\cdot\nabla U(y),\\
A[\phi] &= \mu^{-\frac{n+2}{2}}\left\{\Delta_y\eta_R\phi + 2\nabla_y\eta_R\nabla_y\phi\right\},\\
B[\phi] &= \mu^{-\frac{n}{2}}\left\{\dot\mu\big[y\cdot\nabla_y\phi + \frac{n-2}{2}\phi\big]\eta_R + \dot\xi\cdot\nabla_y\phi\,\eta_R + \big[\dot\mu y\cdot\nabla_y\eta_R + \dot\xi\cdot\nabla_y\eta_R\big]\,\phi\right\}.
\end{aligned}$$

Let us consider for example the error terms

$$g_1(x,t) = \mu^{-2}(1-\eta_R)U^{p-1}(Z^*+\psi), \quad g_2(x,t) = \mu^{-\frac{n+2}{2}}E(1-\eta_R).$$

We see that

$$|g_1(x,t)| \ \le \frac{1}{R^{2-\sigma}}\mu^{-2}\frac{C}{1+|y|^{2+\sigma}}(\|Z^*\|_\infty + \|\psi\|_\infty)$$

and

$$|g_2(x,t)| \ \le\ \frac{1}{\mu^2}\Big[\frac{1}{|y|^{n-2}}\mu^{-\frac{n-2}{2}}(|\mu\dot\mu| + |\mu\dot\xi|) \ \le\ \frac{1}{R^{3-\sigma}}\mu^{-2}\frac{C}{1+|y|^{2+\sigma}}.$$

Let us now estimate the term $A[\phi]$. Let us choose $\sigma = \frac{a}{2}$, where $a$ is the number in the definition of $\|\phi\|_{*a,\nu}$. We have

$$\begin{aligned}\big|A[\phi](x,t)\big| &\leq \mu^{-2}\frac{1}{R^2}\frac{1}{1+R^{-2-\sigma}|y|^{2+\sigma}}\,\mu^{-\frac{n-2}{2}}\sup_{R<|y|<2R}(|\phi|+|y||\nabla\phi|)\\ &\leq \mu^{-2}\frac{R^{-\frac{a}{2}}}{1+|y|^{2+\sigma}}\|\phi\|_{*a,\frac{n-2}{2}}\end{aligned}$$

and similarly,

$$\big|B[\phi](x,t)\big| \leq C\mu^{-2}[\mu\dot\mu+\mu|\dot\xi|]\frac{R^{n+1-a}}{1+|y|^{n+1}}\|\phi\|_{*a,\frac{n-2}{2}} \leq C\mu^{-2}\frac{\mu^{\frac{3}{2}}R^{n+1-a}}{1+|y|^{2+\sigma}}\|\phi\|_{*a,\frac{n-2}{2}}.$$

Now for some $\sigma>0$ we have

$$\begin{aligned}\big|N(Z^*+\mu^{-\frac{n-2}{2}}\eta_R\phi+Z^*+\psi)\big| &\leq C\mu^{-2}\frac{\mu^{\sigma}}{1+|y|^{2+\sigma}}(\|\phi\|_{*a,\frac{n-2}{2}}R^{n+1-a}\\ &\qquad+\|Z_*\|_\infty+\|\psi\|_\infty)^2+C(\|Z_*\|_\infty+\|\psi\|_\infty)^p.\end{aligned}$$

According to the above estimates, it follows by Lemma 3.1 that

$$\|\psi\|_\infty \;\leq\; CT^{\sigma'}\|Z_*\|_\infty+R^{-\sigma'}\|\phi\|_{*a,\frac{n-2}{2}}. \tag{35}$$

Combining (34) and (35) and then using (31)-(33), we finally get

$$\begin{cases}\|\psi\|_\infty &\leq CT^{\sigma'}\|Z_*\|_\infty\\ \|\phi\|_{*a,\frac{n-2}{2}} &\leq C\|Z_*\|_\infty\\ \|(T-t)^{-1}\dot\xi^{(1)}\|_\infty+\|(T-t)^{-2}\xi^{(1)}\|_\infty &\leq C\,(T^{\sigma'}(\|Z_*\|_\infty+1)+R^{-2})\\ \|(T-t)^{-1}\dot\mu^{(1)}\|_\infty+\|(T-t)^{-2}\mu^{(1)}\|_\infty &\leq C\,T^{\sigma'}(\|Z_*\|_\infty+1).\end{cases} \tag{36}$$

We write System (27) in the form

$$\begin{cases}\phi &= \mathcal{T}^{in}_{\mu}[\,\bar H_\lambda(\mathcal{T}^{out}[\,\lambda G(\phi,\psi,\mu,\xi),\mu,\xi)]\\ \psi &= \mathcal{T}^{out}[\,\lambda G(\phi,\psi,\mu,\xi)\,]\\ \mu^{(1)} &= \mathcal{T}^{(0)}[\,\tilde g_0(\psi,\xi^{(1)},\mu^{(1)},\lambda)\,]\\ \xi^{(1)} &= \mathcal{T}^{(1)}[\tilde g_1(\psi,\mu^{(1)},\xi^{(1)},\lambda)].\end{cases} \tag{37}$$

Here, we can write

$$\begin{aligned}\tilde g_0(\psi,\xi^{(1)},\mu^{(1)},\lambda) =&c_R^1\int_{B_{2R}}H_\lambda(\mathcal{T}^{out}[\,\lambda G(\phi,\psi,\mu,\xi)],\mu,\xi)Z_{n+1}(y)dy\\ \tilde g_1(\psi,\xi^{(1)},\mu^{(1)},\lambda) =&c_R^2\int_{B_{2R}}H_\lambda(\mathcal{T}^{out}[\,\lambda G(\phi,\psi,\mu,\xi)],\mu,\xi)\nabla U(y)dy\end{aligned}$$

for suitable positive constants $c_R^\ell$, $\ell = 0, 1$. We fix an arbitrarily small $\varepsilon > 0$ and consider the problem defined only up to time $t = T - \varepsilon$.

$$(38)\qquad \begin{cases} \phi &= \mathcal{T}_\mu^{in}[\,\bar{H}_\lambda(\mathcal{T}^{out}[\,\lambda G(\phi,\psi,\mu,\xi),\mu,\xi)], \quad (y,t) \in \bar{B}_{2R} \times [0, T-\varepsilon] \\ \psi &= \mathcal{T}^{out}[\,\lambda G(\phi,\psi,\mu,\xi)\,], \quad (x,t) \in \bar{\Omega} \times [0, T-\varepsilon] \\ \mu^{(1)} &= \mathcal{T}_\varepsilon^{(0)}[\,\tilde{g}_0(\psi,\xi^{(1)},\mu^{(1)},\lambda)\,], \quad t \in [0, T-\varepsilon] \\ \xi^{(1)} &= \mathcal{T}_\varepsilon^{(1)}[\tilde{g}_1(\psi,\mu^{(1)},\xi^{(1)},\lambda)], \quad t \in [0, T-\varepsilon] \end{cases}$$

where

$$\mathcal{T}_\varepsilon^0[g](t) := -(T-t)^{-\gamma} \int_t^{T-\varepsilon} (T-s)^{\gamma+1} g_0(s)\, ds, \quad \mathcal{T}_\varepsilon^{(1)}[g] := \int_t^{T-\varepsilon} (T-s) g(s)\, ds.$$

The key is that the operators in the right hand side of (38) are compact when we regard them as defined in the space of functions

$$(\phi, \psi, \mu^{(1)}, \xi^{(1)}) \in X_1 \times X_2 \times X_3 \times X_4$$

with their respective norms defined as

$$X^1 = \{\phi \;/\; \phi \in C(B_{2R} \times [0, T-\varepsilon]),\ \nabla_y \phi \in C(B_{2R} \times [0, T-\varepsilon])\}, \quad \|\phi\|_{X_1} = \|\phi\|_\infty + \|\nabla_y \phi\|_\infty$$

$$X^2 = \{\psi \;/\; \phi \in C(\bar{\Omega} \times [0, T-ve])\}, \quad \|\psi\|_{X_2} = \|\psi\|_\infty$$

$$X^3 = \{\mu^{(1)} \;/\; \mu^{(1)} \in C^1[0, T-\varepsilon]\}, \quad \|\mu^{(1)}\|_{X_3} = \|\mu^{(1)}\|_\infty + \|\dot{\mu}^{(1)}\|_\infty$$

$$X^4 = \{\xi^{(1)} \;/\; \xi^{(1)} \in C^1[0, T-\varepsilon]\}, \quad \|\xi^{(1)}\|_{X_4} = \|\xi^{(1)}\|_\infty + \|\dot{\xi}^{(1)}\|_\infty.$$

Compactness on bounded sets of all the operators involved in the above expression is a direct consequence of the Hölder estimate for the operator $\mathcal{T}^{out}$ and Arzela-Ascoli's theorem. On the other hand, the a priori estimate we obtained for $\varepsilon = 0$ holds equally well, uniformly on arbitrary small $\varepsilon > 0$.

Leray Schauder degree applies in a suitable ball $\mathcal{B}$ that contains the origin in this space: essentially one slightly bigger than that defined by relations (36), which amounts to a choice of the parameters $\delta_0$ and $\delta_1$ in (28). In fact, the homotopy connects with the identity at $\lambda = 0$, and hence the total degree in the region defined by relations (36) is equal to 1. The existence of a solution to the approximate problem satisfying bounds (36) then follows. Finally, a standard diagonal argument yields a solution to the original problem with the desired asymptotics. The proof of Theorem 1 for the case $k = 1$ is concluded.

The general case of $k$ distinct points $q_1, \ldots, q_k$ is actually identical: in that case we have $k$ inner problems and one outer problem with analogous properties. We look for a solution of the form

(39)

$$u(x,t) = \sum_{j=1}^k U_{\mu_j,\xi_j}(x) + Z^*(x,t) + \mu_j^{-\frac{n-2}{2}} \phi(y_j,t)\eta_R(y_j) + \psi(x,t), \quad y_j = \frac{x - \xi_j}{\mu_j},$$

where $Z^*$ solves heat equation with initial condition $Z_0^*$ which is chosen so that (4) holds at all concentration points, namely $Z_0^*(q_j) < 0$, and $\xi_j(T) = q_j,\ \mu_j(T) = 0$.

A string of fixed point problems (with essentially decoupled equations associated at each point) then appears and can be solved in the same way. We omit the details. □

**4. Non-$L^2$ Case: Critical Fujita Equation in $\mathbb{R}^4$** In this section, we introduce the second parabolic gluing method when the kernels are not necessarily in $L^2$.

As a prototype, we consider the finite time blow-up for case $n = 4$ and $p = 3$.

In what follows we let $\Omega$ be a smooth bounded domain in $\mathbb{R}^4$ or $\Omega = \mathbb{R}^4$ and consider the equation

$$\begin{cases} u_t = \Delta u + u^3 \quad \text{in } \ \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{40}$$

Let us fix arbitrary points $q_1, q_2, \dots q_k \in \Omega$. We consider a smooth function $Z_0^* \in L^\infty(\Omega)$ with the property that

$$Z_0^*(q_j) < 0 \ \text{ for all } \ j = 1, \dots, k.$$

Theorem 2 ([22]). *For each $T > 0$ sufficiently small, there exists an initial condition $u_0$ such that the solution to problem (40) blows up at time $T$ exactly at the $k$ points $q_1, \dots, q_k$. The solution is of the sharply scaled form*

$$u(x,t) = \sum_{j=1}^{k} U_{\lambda_j(t),\xi_j(t)}(x) + Z_0^*(x) + \theta(x,t)$$

*where*

$$\lambda_j(t) \to 0, \quad \xi_j(t) \to q_j \quad \text{as} \quad t \to T,$$

*and $\|\theta\|_{L^\infty} \le T^a$ for some $a > 0$. More precisely,*

$$\lambda_j(t) \sim \frac{T-t}{|\log(T-t)|^2} \quad \text{as} \quad t \to T.$$

We observe that the solution constructed in Theorem 2 is type II. The result in Theorem 2 is the exact analog of Theorem 1 in dimension 5. We follow the same general approach of the *parabolic gluing method*. However, substantial differences and difficulties arise, due to the fact that the equation that determines $\lambda(t)$ involves a delicate nonlocal integro-differential operator. In dimension 5, the dynamics of $\lambda(t)$ is found in a much more direct way by just solving an ODE.

This nonlocal effect is related to the slower decay of the linear generator of dilations of the Aubin-Talenti bubbles in lower dimensions, i.e., $Z_5 \notin L^2(\mathbb{R}^4)$ (see (43)). A very similar difficulty was already encountered in the work [12] on blow-up in the harmonic map flow, where such nonlocal operator appeared in the reduced complex system at mode 0. The similarity between these problems in the presence of symmetries had already been noticed in [50, 51].

• **Approximation and correction.**

We first choose a proper approximate solution to (40) and compute its error. We consider the case $k = 1$ for simplicity and mention the minor changes for the general multi-bubble case when needed. We define the error operator

$$\mathcal{S}(u) := -u_t + \Delta u + u^3.$$

Recall that the Aubin-Talenti bubble

$$U(y) = \frac{\alpha_0}{1 + |y|^2} \tag{41}$$

solves the Yamabe problem

$$\Delta_y U + U^3 = 0 \quad \text{in} \quad \mathbb{R}^4,$$

where $\alpha_0 = 2\sqrt{2}$. It is well-known that the linearized operator around the bubble

$$L_0(\phi) := \Delta\phi + 3U^2\phi \tag{42}$$

is non-degenerate in the sense that all bounded solutions to $L_0(\phi) = 0$ are the linear combination of

$$Z_i(y) := \partial_{y_i} U(y), \quad i = 1, 2, 3, 4, \ \ Z_5(y) := U(y) + \nabla U(y) \cdot y. \tag{43}$$

Our first approximation is chosen as

$$U_{\lambda(t),\xi(t)} = \lambda^{-1}(t) U\left(\frac{x - \xi(t)}{\lambda(t)}\right),$$

where $\lambda(t)$ and $\xi(t)$ are scaling and translation parameter functions to be adjusted later. Direct computations yield

$$\begin{aligned} S(U_{\lambda(t),\xi(t)}) = -\partial_t U_{\lambda(t),\xi(t)} = \lambda^{-2}(t)\dot{\lambda}(t) &\left(-\frac{\alpha_0}{1 + |y|^2} + \frac{2\alpha_0}{(1 + |y|^2)^2}\right) \\ &+ \lambda^{-2}(t) \nabla_y U(y) \cdot \dot{\xi}(t), \end{aligned} \tag{44}$$

where $y = \frac{x - \xi(t)}{\lambda(t)}$. Observe that the slow decaying error in (44) is

$$\mathcal{E}_0 = -\frac{\alpha_0 \dot{\lambda}(t)}{\lambda^2(t) + \rho^2} \approx -\frac{\alpha_0 \dot{\lambda}(t)}{\rho^2} \notin L^2(\mathbb{R}^4),$$

where $\rho := |x - \xi(t)|$. In order to improve the approximation, we consider

$$\partial_t u_1 = \Delta u_1 + \mathcal{E}_0 \quad \text{in} \quad \mathbb{R}^4 \times (0, T). \tag{45}$$

By similar computations as in [12][1], a solution to (45) is given explicitly by

$$u_1 = -\alpha_0 \int_{-T}^{t} \dot\lambda(s) k(\rho, t-s) ds,$$

where

$$k(\rho, t) := \frac{1 - e^{-\frac{\rho^2}{4t}}}{\rho^2}. \tag{46}$$

We regularize the above $u_1$ and choose a correction $\Psi_0$ to be

$$\Psi_0(x, t) = -\alpha_0 \int_{-T}^{t} \dot\lambda(s) k(\zeta(\rho, t), t-s) ds, \tag{47}$$

where

$$\zeta(\rho, t) = \sqrt{\rho^2 + \lambda^2(t)}.$$

Then the new error produced by $\Psi_0$ is given by

$$\begin{aligned}
&\partial_t \Psi_0 - \Delta \Psi_0 - \mathcal{E}_0 \\
&= \alpha_0 \left[ \frac{y \cdot \dot\xi - \dot\lambda(t)}{(1 + |y|^2)^{1/2}} \right] \int_{-T}^{t} \dot\lambda(s) k_\zeta(\zeta, t-s) ds \\
&\quad + \frac{\alpha_0}{\lambda(t)(1 + |y|^2)^{3/2}} \int_{-T}^{t} \dot\lambda(s) \left[ -\zeta k_{\zeta\zeta}(\zeta, t-s) + k_\zeta(\zeta, t-s) \right] ds \\
&:= \mathcal{R}[\lambda].
\end{aligned} \tag{48}$$

It is thus reasonable to choose the corrected approximation as

$$u^* = U_{\lambda(t), \xi(t)} + \Psi_0$$

and its error is

$$\begin{aligned}
\mathcal{S}(u^*) &= \mathcal{S}(U_{\lambda(t),\xi(t)}) - \mathcal{E}_0 + (U_{\lambda(t),\xi(t)} + \Psi_0)^3 - U^3_{\lambda(t),\xi(t)} \\
&= \mathcal{K}[\lambda, \xi] + (U_{\lambda(t),\xi(t)} + \Psi_0)^3 - U^3_{\lambda(t),\xi(t)},
\end{aligned}$$

[1]See Section 17 in the full version available at https://personal.math.ubc.ca/~jcwei/hmf-2018-08-16.pdf

where $\mathcal{K}[\lambda,\xi]$ is defined as

$$\mathcal{K}[\lambda,\xi] := \frac{2\alpha_0\lambda^{-2}(t)\dot{\lambda}(t)}{(1+|y|^2)^2} + \lambda^{-2}(t)\nabla U(y)\cdot\dot{\xi}(t) - \mathcal{R}[\lambda] \tag{49}$$

with $\mathcal{R}[\lambda]$ given in (48).

• **Formulating the inner–outer gluing system.**

We look for a solution of the following form

$$u = u^* + \mathtt{w},$$

where $\mathtt{w}$ is a small perturbation consisting of inner and outer parts

$$\mathtt{w} = \varphi_{\rm in} + \varphi_{\rm out}, \quad \varphi_{\rm in} = \lambda^{-1}(t)\eta_R\phi(y,t), \quad \varphi_{\rm out} = \psi(x,t) + Z^*(x,t).$$

Here the cut-off function is defined by

$$\eta_R = \eta_{R(t)}(x,t) = \eta\left(\frac{|x-\xi(t)|}{\lambda(t)R(t)}\right)$$

where the smooth cut-off function $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ for $s > 2$, and $Z^*$ satisfies

$$\begin{cases} Z^*_t = \Delta_x Z^* & \text{in } \Omega\times(0,T), \\ Z^*(\cdot,t) = 0 & \text{on } \partial\Omega\times(0,T), \\ Z^*(\cdot,0) = Z^*_0 & \text{in } \Omega. \end{cases}$$

Denote

$$B_{2R} = \{x\in\Omega : |x-\xi(t)| \le 2\lambda R\}, \ \mathcal{D}_{2R} = B_{2R}\times(0,T),$$

and $\Psi^* = \psi + Z^*$. Then $u$ is a solution to the original problem (40) if

- $\phi$ solves the **inner problem**

$$\lambda^2\phi_t = \Delta_y\phi + 3U^2(y)\phi + \mathcal{H}(\phi,\psi,\lambda,\xi) \quad \text{in } \mathcal{D}_{2R} \tag{50}$$

where

$$\begin{aligned} \mathcal{H}(\phi,\psi,\lambda,\xi)(y,t) := {} & 3\lambda U^2(y)[\Psi_0 + \psi + Z^*](\lambda y + \xi, t) \\ & + \lambda\left[\dot{\lambda}(\nabla_y\phi\cdot y + \phi) + \nabla_y\phi\cdot\dot{\xi}\right] \\ & + \lambda^3\mathcal{N}(\mathtt{w}) + \lambda^3\mathcal{K}[\lambda,\xi] \end{aligned} \tag{51}$$

with $\mathcal{K}[\lambda,\xi]$ defined in (49), and

$$\mathcal{N}(\mathtt{w}) := (U_{\lambda,\xi} + \Psi_0 + \mathtt{w})^3 - U^3_{\lambda,\xi} - 3U^2_{\lambda,\xi}(\Psi_0 + \mathtt{w}). \tag{52}$$

- $\psi$ solves the **outer problem**

$$\psi_t = \Delta\psi + \mathcal{G}(\phi, \psi, \lambda, \xi) \quad \text{in } \Omega \times (0, T) \tag{53}$$

with

$$\begin{aligned} \mathcal{G}(\phi, \psi, \lambda, \xi) := {} & 3\lambda^{-2}(1 - \eta_R)U^2(y)(\Psi_0 + \psi + Z^*) \\ & + \lambda^{-3}\left[(\Delta_y \eta_R)\phi + 2\nabla_y \eta_R \cdot \nabla_y \phi - \lambda^2 \phi \partial_t \eta_R\right] \\ & + (1 - \eta_R)\mathcal{K}[\lambda, \xi] + (1 - \eta_R)\mathcal{N}(\mathtt{w}). \end{aligned} \tag{54}$$

- **The choices of parameters.**

We choose the leading orders $\lambda_*(t)$, $\xi_*(t)$ of the parameter functions $\lambda(t)$ and $\xi(t)$. As mentioned earlier, a good inner solution can be found provided approximately the following orthogonality conditions

$$\int_{\mathbb{R}^4} \mathcal{H}(\phi, \psi, \lambda, \xi) Z_j(y) dy = 0 \quad \text{for all } j = 1, \cdots, 5, \ t \in (0, T) \tag{55}$$

are satisfied. Here $Z_j$ are the kernel functions (c.f. (43)) of the linearized operator $L_0$ defined in (42). Basically, the scaling and translation parameters $\lambda(t)$ and $\xi(t)$ at main order will be derived from the orthogonality conditions (55).

Singling out the leading term $\mathcal{H}_*$ of $\mathcal{H}$ and computing

$$\int_{\mathbb{R}^4} \mathcal{H}_*[\lambda, \xi, \Psi^*] Z_\ell(y) dy = 0 \ \text{ for } \ \ell = 1, \cdots, 4$$

with

$$\begin{aligned} \mathcal{H}_*[\lambda, \xi, \Psi^*] &:= 3\lambda U^2(y)[\Psi_0 + \Psi^*](\lambda y + \xi, t) + \lambda^3 \mathcal{K}[\lambda, \xi] \\ &= 3\lambda U^2(y)[\Psi_0 + \Psi^*](\lambda y + \xi, t) + \frac{2\alpha_0 \lambda(t)\dot{\lambda}(t)}{(1 + |y|^2)^2} + \lambda(t)\nabla U(y) \cdot \dot{\xi}(t) \\ &\quad - \frac{\alpha_0 \lambda^2(t)}{(1 + |y|^2)^{3/2}} \int_{-T}^t \dot{\lambda}(s)\left[-\zeta k_{\zeta\zeta}(\zeta, t - s) + k_\zeta(\zeta, t - s)\right] ds \\ &\quad - \alpha_0 \lambda^3(t) \left[\frac{y \cdot \dot{\xi} - \dot{\lambda}(t)}{(1 + |y|^2)^{1/2}}\right] \int_{-T}^t \dot{\lambda}(s) k_\zeta(\zeta, t - s) ds \end{aligned}$$

imply that

$$\dot{\xi}_\ell = o(1),$$

where $\Psi^* = \psi + Z^*$ and $o(1) \to 0$ as $t \nearrow T$. So the choice of $\xi(t)$ at main order is

$$\xi(t) = q,$$

where $q$ is a prescribed point in $\Omega$.

The dynamics for $\lambda(t)$ from

$$\int_{\mathbb{R}^4} \mathcal{H}_*[\lambda, \xi, \Psi^*] Z_5(y) dy = 0$$

turns out to be more involved due to the non-local/global correction, and the reduced problem involves the following integro-differential operator

$$c_* \int_{-T}^{t-\lambda^2(t)} \frac{\dot\lambda(s)}{t-s} ds = -3c_0[Z_0^*(q) + \psi(q,0)] + o(1), \tag{56}$$

where

$$c_0 := \int_{\mathbb{R}^4} U^2(y) Z_5(y) dy < 0.$$

Here careful calculations are needed for nonlocal terms, and detailed derivation can be found in [22, Section 4]. Since $\lambda(t)$ decreases to 0 as $t \nearrow T$, we impose

$$a_* := Z_0^*(q) + \psi(q,0) < 0.$$

Now we claim that a good choice of $\lambda(t)$ at main order is

$$\dot\lambda(t) = -\frac{c}{|\log(T-t)|^2}, \tag{57}$$

where $c > 0$ is a constant to be determined later. Indeed, we get by substituting

$$\begin{aligned}\int_{-T}^{t-\lambda^2(t)} \frac{\dot\lambda(s)}{t-s} ds &= \int_{-T}^{t-(T-t)} \frac{\dot\lambda(s)}{t-s} ds + \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(t)}{t-s} ds - \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(t) - \dot\lambda(s)}{t-s} ds \\ &= \int_{-T}^{t-(T-t)} \frac{\dot\lambda(s)}{t-s} ds + \dot\lambda(t)(\log(T-t) - 2\log\lambda(t)) \\ &\quad - \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(t) - \dot\lambda(s)}{t-s} ds \\ &\approx \int_{-T}^{t} \frac{\dot\lambda(s)}{T-s} ds - \dot\lambda(t)\log(T-t) := \beta(t).\end{aligned}$$

By (57), we then get

$$\log(T-t)\frac{d\beta}{dt}(t) = \frac{d}{dt}\left(-\log^2(T-t)\dot\lambda(t)\right) = 0,$$

which means $\beta(t)$ is a constant. Thus, equation (56) can be approximately solved for

$$\dot\lambda(t) = -\frac{c}{|\log(T-t)|^2}$$

with the constant $c$ chosen as

$$-c\int_{-T}^{T}\frac{ds}{(T-s)|\log(T-s)|^2}=\kappa_*,$$

where $\kappa_*:=-\frac{3c_0a_*}{c_*}$. At main order, we obtain

$$\dot\lambda(t)=\kappa_*\dot\lambda_*(t)$$

with

$$\dot\lambda_*(t)=-\frac{|\log T|}{|\log(T-t)|^2}.$$

By imposing $\lambda_*(T)=0$, we obtain

$$\lambda_*(t)=\frac{|\log T|(T-t)}{|\log(T-t)|^2}(1+o(1))\quad\text{as}\quad t\nearrow T.$$

Choosing $Z_0^*(q_j)\sim-|\log T|^{-1}$, we can make $\lambda_j(t)\sim\frac{T-t}{|\log(T-t)|^2}$.

• **Linear theories.**

We start from the the outer problem (53) and consider

$$\begin{cases}\psi_t=\Delta\psi+f, & \text{in } \Omega\times(0,T),\\ \psi=0, & \text{on } \partial\Omega\times(0,T),\\ \psi(x,0)=0, & \text{in } \Omega,\end{cases}\tag{58}$$

where the non-homogeneous term $f$ in (58) is assumed to be bounded with respect to the weights appearing in the outer problem (53). Define the weights

$$\begin{cases}\varrho_1:=\lambda_*^{\nu-3}(t)R^{-2-\alpha}(t)\chi_{\{|x-\xi(t)|\le2\lambda_*R\}}\\ \varrho_2:=\frac{\lambda_*^{\nu2}}{|x-\xi(t)|^2}\chi_{\{|x-\xi(t)|\ge\lambda_*R\}}\\ \varrho_3:=1\end{cases}\tag{59}$$

where we choose $R(t)=\lambda_*^{-\beta}(t)$ for $\beta\in(0,1/2)$. We define the norms

$$\|f\|_{**}:=\sup_{(x,t)\in\Omega\times(0,T)}\left(\sum_{i=1}^3\varrho_i(x,t)\right)^{-1}|f(x,t)|,\tag{60}$$

(61)

$$\begin{aligned}\|\psi\|_*:=&\frac{\lambda_*^{1-\nu}(0)R^\alpha(0)}{|\log T|}\|\psi\|_{L^\infty(\Omega\times(0,T))}+\frac{\lambda_*^{2-\nu}(0)R^{1+\alpha}(0)}{|\log T|}\|\nabla\psi\|_{L^\infty(\Omega\times(0,T))}\\&+\sup_{(x,t)\in\Omega\times(0,T)}\left[\frac{\lambda_*^{1-\nu}(t)R^\alpha(t)}{|\log(T-t)|}|\psi(x,t)-\psi(x,T)|\right]\\&+\sup_{(x,t)\in\Omega\times(0,T)}\left[\frac{\lambda_*^{2-\nu}(t)R^{1+\alpha}(t)|}{|\log(T-t)|}|\nabla\psi(x,t)-\nabla\psi(x,T)|\right]\\&+\sup_{\Omega\times I_T}\frac{\lambda_*^{2\gamma+1-\nu}(t_2)R^{2\gamma+\alpha}(t_2)}{(t_2-t_1)^\gamma}|\psi(x,t_2)-\psi(x,t_1)|,\end{aligned}$$

where $\nu, \alpha, \gamma \in (0,1)$, and the last supremum is taken over

$$\Omega \times I_T = \left\{ (x, t_1, t_2) : \ x \in \Omega, \ 0 \le t_1 \le t_2 \le T, \ t_2 - t_1 \le \frac{1}{10}(T - t_2) \right\}.$$

For problem (58), we have the following estimates.

PROPOSITION 4.1. ([12, Appendix A],[22, Proposition 1]) *Let $\psi$ be the solution to problem* (58) *with $\|f\|_{**} < +\infty$. Then it holds that*

$$\|\psi\|_* \lesssim \|f\|_{**}.$$

Proposition 4.1 is established by estimating carefully Duhamel's formula with different right hand sides.

To solve the inner problem (50), we consider the associated linear problem

$$\lambda^2 \phi_t = \Delta_y \phi + 3U^2(y)\phi + h(y,t) \ \text{ in } \ \mathcal{D}_{2R}. \tag{62}$$

Recall that the linearized operator $L_0 = \Delta + 3U^2$ has only one positive eigenvalue $\mu_0$ such that

$$L_0(Z_0) = \mu_0 Z_0, \ Z_0 \in L^\infty(\mathbb{R}^4),$$

where the corresponding eigenfunction $Z_0$ is radially symmetric with the asymptotic behavior

$$Z_0(y) \sim |y|^{-3/2} e^{-\sqrt{\mu_0}|y|} \ \text{ as } \ |y| \to +\infty.$$

Similar to the discussion in previous section, such instability is reflected in the need for a careful choice of the initial data to ensure a well-behaved solution. Therefore, we consider the associated linear Cauchy problem of the inner problem (50)

$$\begin{cases} \lambda^2 \phi_t = \Delta_y \phi + 3U^2(y)\phi + h(y,t), & \text{in } \mathcal{D}_{2R}, \\ \phi(y,0) = e_0 Z_0(y), & \text{in } B_{2R(0)}, \end{cases} \tag{63}$$

where $R = R(t) = \lambda_*^{-\beta}(t)$ for $\beta \in (0, 1/2)$. On the other hand, the parabolic operator $-\lambda^2 \partial_t + L_0$ is certainly not invertible since all the time independent elements in the 5 dimensional kernel of $L_0$ (see (43)) also belong to the kernel of $-\lambda^2 \partial_t + L_0$. In order to construct solution to (63) with suitable space-time decay, we expect some orthogonality conditions to hold. We shall construct a solution $(\phi, e_0)$ to problem (63) under the orthogonality conditions

$$\int_{B_{2R}} h(y,t) Z_\ell(y) dy = 0 \ \text{ for } \ \ell = 1, \cdots, 5, \ t \in (0,T). \tag{64}$$

Define

$$\|h\|_{\nu, 2+a} := \sup_{(y,t) \in \mathcal{D}_{2R}} \lambda_*^{-\nu}(t)(1 + |y|^{2+a}) \left[ |h(y,t)| + (1 + |y|)|\nabla h(y,t)| \right]. \tag{65}$$

The construction of such solution is achieved by decomposing the equation into different spherical harmonic modes. Consider an orthonormal basis $\{\Theta_i\}_{i=0}^{\infty}$ made up of spherical harmonics in $L^2(\mathbb{S}^3)$, i.e.

$$\Delta_{\mathbb{S}^3}\Theta_i + \lambda_i\Theta_i = 0 \text{ in } \mathbb{S}^3$$

with $0 = \lambda_0 < \lambda_1 = \cdots = \lambda_4 = 3 < \lambda_5 \leq \cdots$. More precisely, $\Theta_0(y) = a_0,\ \Theta_i(y) = a_1 y_i,\ i = 1, \cdots, 4$ for two constants $a_0$, $a_1$ and

$$\lambda_i = i(2+i) \text{ with multiplicity } \frac{(3+i)!}{6i!} \text{ for } i \geq 0.$$

For $h \in L^2(\mathcal{D}_{2R})$, we decompose

$$h(y,t) = \sum_{j=0}^{\infty} h_j(r,t)\Theta_j(y/r),\ r = |y|,\ h_j(r,t) = \int_{\mathbb{S}^3} h(r\theta,t)\Theta_j(\theta)d\theta$$

and write $h = h^0 + h^1 + h^\perp$ with

$$h^0 = h_0(r,t),\ h^1 = \sum_{j=1}^{4} h_j(r,t)\Theta_j,\ h^\perp = \sum_{j=5}^{\infty} h_j(r,t)\Theta_j.$$

Also, we decompose $\phi = \phi^0 + \phi^1 + \phi^\perp$ in a similar form. Then looking for a solution to problem (63) is equivalent to finding the pairs $(\phi^0, h^0)$, $(\phi^1, h^1)$, $(\phi^\perp, h^\perp)$ in each mode.

The key linear result for the inner problem is stated as follows.

Proposition 4.2. *Let the constants* $a, \nu, \nu_1 \in (0,1)$, $a_1 \in (1,2)$ *be given. For* $T > 0$ *sufficiently small and any* $h(y,t)$ *satisfying* $\|h\|_{\nu,2+a} < +\infty$, $\|h^1\|_{\nu_1,2+a_1} < +\infty$*, and the orthogonality conditions* (64)*, there exists a pair* $(\phi, e_0)$ *solving* (63)*, and* $(\phi, e_0) = (\phi[h], e_0[h])$ *defines a linear operator of* $h(y,t)$ *that satisfies the estimates*

$$\begin{aligned} &|\phi(y,t)| + (1+|y|)|\nabla\phi(y,t)| \\ \lesssim\ & \frac{\lambda_*^{\nu}(t)R^{\delta}}{1+|y|^a}\|h^0\|_{\nu,2+a} + \frac{\lambda_*^{\nu_1}(t)}{1+|y|^{a_1}}\|h^1\|_{\nu_1,2+a_1} + \frac{\lambda_*^{\nu}(t)}{1+|y|^a}\|h^\perp\|_{\nu,2+a} \end{aligned}$$

*and*

$$|e_0[h]| \lesssim \|h\|_{\nu,2+a}$$

*where* $0 \leq \delta < 1$ *is small.*

In the proof of Proposition 4.2, mode 0 and higher modes can be carried out in a similar manner as in [58, Proposition 7.1] via a careful re-gluing process. The rougher version can be found in [7, Proposition 7.1]. Mode 1 is obtained by blow-up argument. The restriction $a_1 \in (1,2)$ is required to guarantee the integrability in the blow-up argument at translation mode 1.

If we define the norm

(66)
$$\|\phi^0\|_{*,\nu,a,\delta} := \sup_{(y,t)\in\mathcal{D}_{2R}} \lambda_*^{-\nu}(t) R^{-\delta}(1+|y|^a)\left[|\phi^0(y,t)| + (1+|y|)|\nabla\phi^0(y,t)|\right],$$

then Proposition 4.2 implies that

$$\|\phi^0\|_{*,\nu,a,\delta} \lesssim \|h^0\|_{\nu,2+a}.$$

We shall use the norm (66) when we solve the inner–outer gluing system.

The following spectrum gap plays a crucial role in the proof of the above Proposition, In fact, we have

LEMMA 4.1. *For all sufficiently large $R$ and all radially symmetric $\varphi \in H_0^1(B_R)$ with $\int_{B_{2R}} \varphi Z_0 = 0$, there exists a positive constant $\gamma$ independent of $R$ such that*

$$\int_{B_{2R}} \left(|\nabla\varphi|^2 - pU^{p-1}\varphi^2\right) \geq \gamma \begin{cases} \dfrac{1}{R^2}\displaystyle\int_{B_{2R}} \varphi^2, & \text{for } n=3, \\ \dfrac{1}{R^2\log R}\displaystyle\int_{B_{2R}} \varphi^2, & \text{for } n=4, \\ \dfrac{1}{R^{n-2}}\displaystyle\int_{B_{2R}} \varphi^2, & \text{for } n\geq 5. \end{cases}$$

Similar estimates for the linearization of harmonic map equation around degree 1 bubble are derived in [57, Lemma 9.2].

• **Solving the inner–outer gluing system.**

Our aim now is to find a solution $(\phi,\psi,\lambda,\xi)$ to the inner–outer gluing system such that the desired blow-up solution is constructed. We shall solve the inner–outer gluing system in the function space $\mathcal{X}$ defined in (97). We first make some assumptions about the parameter functions. Write

$$\lambda_*(t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2}$$

and assume that for some numbers $c_1, c_2 > 0$,

$$c_1|\dot\lambda_*(t)| \leq |\dot\lambda(t)| \leq c_2|\dot\lambda_*(t)| \quad \text{for all} \quad t\in(0,T).$$

For given $\|\phi^0\|_{*,\nu,a,\delta}$, $\|\phi^1\|_{\nu_1,a_1}$, $\|\phi^\perp\|_{\nu,a}$, $\|\psi\|_*$, $\|Z^*\|_\infty$, $\|\lambda\|_F$, $\|\xi\|_G$ bounded, we first estimate right hand sides $\mathcal{G}(\phi,\psi,\lambda,\xi)$ and $\mathcal{H}(\phi,\psi,\lambda,\xi)$ in the inner and outer problems. Here the above norms are defined in (66), (65), (61), (95) and (96).

**The outer problem: estimates of $\mathcal{G}$.**

Consider the outer problem

$$\psi_t = \Delta\psi + \mathcal{G}(\phi, \psi, \lambda, \xi) \quad \text{in } \Omega \times (0, T)$$

where

$$\begin{aligned}\mathcal{G}(\phi, \psi, \lambda, \xi) := {} & 3\lambda^{-2}(1-\eta_R)U^2(y)(\Psi_0 + \psi + Z^*) \\ & + \lambda^{-3}\left[(\Delta_y \eta_R)\phi + 2\nabla_y \eta_R \cdot \nabla_y \phi - \lambda^2 \phi \partial_t \eta_R\right] \\ & + (1-\eta_R)\mathcal{K}[\lambda, \xi] + (1-\eta_R)\mathcal{N}(\mathtt{w})\end{aligned}$$

with $\mathcal{K}[\lambda, \xi]$ and $\mathcal{N}(\mathtt{w})$ defined in (49) and (52) respectively.

In order to apply the linear theory Proposition 4.1, we estimate all the terms in $\mathcal{G}(\phi, \psi, \lambda, \xi)$ in the $\|\cdot\|_{**}$-norm, defined in (60). Direct computations imply that for a fixed number $\epsilon_0 > 0$

$$\begin{aligned}\|\mathcal{G}\|_{**} \lesssim T^{\epsilon_0}\big(&\|\psi\|_* + \|Z^*\|_\infty + \|\phi^0\|_{*,\nu,a,\delta} + \|\phi^1\|_{\nu_1,a_1} + \|\phi^\perp\|_{\nu,a} \\ & + \|\lambda\|_\infty + \|\xi\|_G + 1\big)\end{aligned} \tag{67}$$

if the parameters are chosen in the following range

$$\begin{aligned}& \nu - 1 + \beta(2+\alpha) - \nu_2 > 0, \quad 2\beta - \nu_2 > 0, \quad 0 < \alpha + \delta < a < 1, \\ & \beta + \upsilon - \nu_2 > 0, \quad 2\nu_1 - \nu + \beta(2a_1 - \alpha) > 0, \quad \nu_2 < 1, \\ & 2\nu - \nu_2 - 1 + 2\alpha\beta > 0, \quad \nu - \beta(\alpha + 2\delta - 2a) > 0.\end{aligned} \tag{68}$$

**The inner problem: Estimate of $\mathcal{H}$.**

Consider the inner problem

$$\lambda^2 \phi_t = \Delta_y \phi + 3U^2(y)\phi + \mathcal{H}(\phi, \psi, \lambda, \xi) \quad \text{in } \mathcal{D}_{2R}$$

where

$$\begin{aligned}\mathcal{H}(\phi, \psi, \lambda, \xi)(y, t) := {} & 3\lambda U^2(y)[\Psi_0 + \psi + Z^*](\lambda y + \xi, t) \\ & + \lambda\left[\dot\lambda(\nabla_y \phi \cdot y + \phi) + \nabla_y \phi \cdot \dot\xi\right] \\ & + \lambda^3 \mathcal{N}(\mathtt{w}) + \lambda^3 \mathcal{K}[\lambda, \xi]\end{aligned}$$

with $\mathcal{N}(\mathtt{w})$ and $\mathcal{K}[\lambda, \xi]$ defined in (52) and (49).

From the linear theory, we know that for $\mathcal{H} = \mathcal{H}^0 + \mathcal{H}^1 + \mathcal{H}^\perp$ satisfying

$$\|\mathcal{H}^0\|_{\nu, 2+a}, \ \|\mathcal{H}^1\|_{\nu_1, 2+a_1}, \ \|\mathcal{H}^\perp\|_{\nu, 2+a} < +\infty,$$

there exists a solution $(\phi^0, \phi^1, \phi^\perp, c^0, c^\ell)$ $(\ell = 1, \cdots, 4)$ solving the projected inner problems

$$\begin{cases} \lambda^2 \phi^0_t = \Delta_y \phi^0 + 3U^2(y)\phi^0 + \mathcal{H}^0(\phi, \psi, \lambda, \xi) + c^0 Z_5 & \text{in } \mathcal{D}_{2R}, \\ \phi^0(\cdot, 0) = 0 & \text{in } B_{2R}, \end{cases} \tag{69}$$

$$\begin{cases} \lambda^2 \phi^1_t = \Delta_y \phi^1 + 3U^2(y)\phi^1 + \mathcal{H}^1(\phi, \psi, \lambda, \xi) + \sum_{\ell=1}^{4} c^\ell Z_\ell & \text{in } \mathcal{D}_{2R}, \\ \phi^1(\cdot, 0) = 0 & \text{in } B_{2R}, \end{cases} \tag{70}$$

$$\begin{cases} \lambda^2 \phi^\perp_t = \Delta_y \phi^\perp + 3U^2(y)\phi^\perp + \mathcal{H}^\perp(\phi, \psi, \lambda, \xi) & \text{in } \mathcal{D}_{2R}, \\ \phi^\perp(\cdot, 0) = 0 & \text{in } B_{2R}, \end{cases} \tag{71}$$

and the inner solution $\phi[\mathcal{H}] = \phi^0[\mathcal{H}^0] + \phi^1[\mathcal{H}^1] + \phi^\perp[\mathcal{H}^\perp]$ with proper space-time decay can be obtained for the inner–outer gluing to be carried out. We have the following estimate for some fixed $\epsilon_0 > 0$

$$\begin{aligned} \|\mathcal{H}\|_{\nu, 2+a} \lesssim T^{\epsilon_0} \Big( & \|\phi^0\|_{*,\nu,a,\delta} + \|\phi^1\|_{\nu_1, a_1} + \|\phi^\perp\|_{\nu, a} + \|\psi\|_* + \|Z^*\|_\infty \\ & + \|\lambda\|_\infty + \|\xi\|_G + 1 \Big) \end{aligned} \tag{72}$$

provided

$$\begin{aligned} & 0 < \nu < 1, \quad 1 - \beta(2 + \frac{a}{2}) > 0, \quad 1 + \nu_1 - \nu - \beta(2 + a - a_1) > 0, \\ & 1 - 2\beta > 0, \quad \nu - \beta(4 - a) > 0, \quad 2\nu_1 - \nu > 0, \\ & 2 - \nu - a\beta > 0, \quad \nu - \beta(a - 2\alpha) > 0, \quad 2 - \nu - \beta(1 + a) > 0, \\ & 1 - \beta(\delta + 2) > 0, \quad \nu - 2\delta\beta > 0. \end{aligned} \tag{73}$$

Similar computations give that for some fixed $\epsilon_0 > 0$

$$\|\mathcal{H}^1\|_{\nu_1, 2+a_1} \lesssim T^{\epsilon_0} \left( \|\phi^1\|_{\nu_1, a_1} + \|\psi\|_* + \|Z^*\|_\infty + \|\lambda\|_\infty + \|\xi\|_G + 1 \right) \tag{74}$$

provided

$$\begin{aligned} & 0 < \nu_1 < 1, \quad \nu - \nu_1 + \alpha\beta > 0, \quad 2 - \nu_1 - a_1\beta > 0, \\ & 2\nu - \nu_1 + 2\alpha\beta - a_1\beta > 0, \quad 1 - \nu_1 - \beta(a_1 - 1) > 0. \end{aligned}$$

• **The parameter problems.**

From (69)–(71), it remains to adjust the parameter functions $\lambda(t)$, $\xi(t)$ such that

$$c^0[\lambda,\xi,\Psi^*]=0,\quad c^\ell[\lambda,\xi,\Psi^*]=0,\quad \ell=1,\cdots,4,\ \forall\, t\in(0,T),$$

where

$$c^0[\lambda,\xi,\Psi^*]=-\frac{\int_{B_{2R}}\mathcal{H}Z_5dy}{\int_{B_{2R}}|Z_5|^2dy},\tag{75}$$

$$c^\ell[\lambda,\xi,\Psi^*]=-\frac{\int_{B_{2R}}\mathcal{H}Z_\ell dy}{\int_{B_{2R}}|Z_\ell|^2dy}\quad\text{for}\quad \ell=1,\cdots,4.\tag{76}$$

It turns out that we can easily achieve at the translation mode (76), but the scaling mode (75) is more delicate.

We first consider the reduced equation for $\xi(t)$. Observe that (76) is equivalent to

$$\int_{B_{2R}}\mathcal{H}(\phi,\psi,\lambda,\xi)(y,t)Z_\ell(y)dy=0,\quad\text{for all } t\in(0,T),\ \ell=1,\cdots,4.$$

Write $\Psi^*=\psi+Z^*$ and $\xi(t)=(\xi_1(t),\cdots,\xi_4(t))$. Then for $\ell=1,\cdots,4$,

$$\int_{B_{2R}}\mathcal{H}(\phi,\psi,\lambda,\xi)(y,t)Z_\ell(y)dy=0$$

give that

$$\dot\xi_\ell=b_\ell[\lambda,\xi,\phi,\Psi^*],\tag{77}$$

where

$$b_\ell[\lambda,\xi,\phi,\Psi^*]=\int_{B_{2R}}\Big(\mathcal{H}[\lambda,\xi,\phi,\Psi^*](y,t)-\lambda U_{y_\ell}(y)\dot\xi_\ell\Big)\,Z_\ell(y)dy.$$

Furthermore, the size of $b_\ell[\lambda,\xi,\phi,\Psi^*]$ can be controlled by similarly estimating $\mathcal{H}$. Next, we analyze the reduced problem (77), which defines operators $\Xi_\ell$ ($\ell=1,\cdots,4$) that return the solutions $\xi_\ell$ ($\ell=1,\cdots,4$) respectively. Here we write

$$\Xi=(\Xi_1,\Xi_2,\Xi_3,\Xi_4)\tag{78}$$

and $\xi(t)=q+\xi^1(t)$ where $q=(q_1,\cdots,q_4)$ is a prescribed point in $\Omega$. We shall solve $\xi^1(t)$ under the norm

$$\|\xi\|_G=\|\xi\|_{L^\infty(0,T)}+\sup_{t\in(0,T)}\lambda_*^{-\upsilon}(t)|\dot\xi(t)|$$

for some fixed $\upsilon \in (0,1)$. From (77), we have

$$|\xi_\ell(t)| \le |q_\ell| + \|b_\ell[\lambda,\xi,\phi,\Psi^*]\|_{L^\infty(0,T)} (T-t).$$

Therefore, we obtain

$$\|\Xi_\ell\|_G \le |q_\ell| + (T-t)^{-\upsilon} \|b_\ell[\lambda,\xi,\phi,\Psi^*]\|_{L^\infty(0,T)} . \tag{79}$$

Since the reduced problem of $\lambda(t)$ is essentially the same as the real part of the reduced problem at mode 0 in [12], we shall follow the strategy and logic in [12, Section 8].

Direct computations show that (75) gives a non-local integro-differential equation

$$\int_{-T}^{t} \frac{\dot\lambda(s)}{t-s}\Gamma\left(\frac{\lambda^2(t)}{t-s}\right) ds + \mathbf{c_0}\dot\lambda = a[\lambda,\xi,\Psi^*](t) + \mathtt{a}_r[\lambda,\xi,\phi,\Psi^*](t), \tag{80}$$

where $\mathbf{c_0} = 2\alpha_0 \int_{\mathbb{R}^4} \frac{Z_5(y)}{(1+|y|^2)^2} dy$,

$$a[\lambda,\xi,\Psi^*] = -\int_{B_{2R}} 3U^2(y)\left(\Psi_0+\Psi^*\right) Z_5(y) dy, \tag{81}$$

and the remainder term $\mathtt{a}_r[\lambda,\xi,\phi,\Psi^*](t)$ turns out to be of smaller order and is controlled by

$$\begin{aligned}
&|\mathtt{a}_r[\lambda,\xi,\phi,\Psi^*](t)|\\
\lesssim\ & \lambda_*^\nu R^\delta \left(|\dot\lambda_*||\log(T-t)| + |\dot\xi|\right)\|\phi^0\|_{*,\nu,a,\delta}\\
&+ \lambda_*^\nu \left(|\dot\lambda_*|R^{2-a_1} + |\dot\xi|\right)\|\phi^1\|_{\nu_1,a_1} + \lambda_*^\nu\left(|\dot\lambda_*|R^{2-a} + |\dot\xi|R^{1-a}\right)\|\phi^\perp\|_{\nu,a}\\
&+ \lambda_*^{2\nu-1}R^{2\delta}\|\phi^0\|^2_{*,\nu,a,\delta} + \lambda_*^{2\nu_1-1}\|\phi^1\|^2_{\nu_1,a_1}\\
&+ \lambda_*^{2\nu-1}R^{2-2a}\|\phi^\perp\|^2_{\nu,a} + \lambda_*|\dot\lambda_*|^2|\log(T-t)|^3 + \lambda_*|\log(T-t)|\|Z^*\|^2_\infty\\
&+ \lambda_*(t)|\log(T-t)|\lambda_*^{2\nu-2}(0)R^{-2\alpha}(0)|\log T|^2\|\psi\|^2_*.
\end{aligned}$$

To solve $\lambda(t)$, we introduce the following norms

- $\|\cdot\|_{\Theta,l}$-norm
$$\|f\|_{\Theta,l} := \sup_{t\in[0,T]} \frac{|\log(T-t)|^l}{(T-t)^\Theta}|f(t)|,$$
where $f\in C([-T,T];\mathbb{R})$ with $f(T)=0$, and $\Theta\in(0,1)$, $l\in\mathbb{R}$.

- $[\ \cdot\ ]_{\gamma,m,l}$-seminorm

$$[g]_{\gamma,m,l} := \sup_{I_T} \frac{|\log(T-t)|^l}{(T-t)^m (t-s)^\gamma} |g(t)-g(s)|,$$

where $I_T = \left\{-T \le s \le t \le T : t-s \le \frac{1}{10}(T-t)\right\}$, $g \in C([-T,T];\mathbb{R})$ with $g(T)=0$ and $0<\gamma<1$, $m>0$, $l \in \mathbb{R}$.

Also, we define

$$\mathcal{B}_0[\lambda](t) := \int_{-T}^{t} \frac{\dot\lambda(s)}{t-s} \Gamma\left(\frac{\lambda^2(t)}{t-s}\right) ds + \mathbf{c_0}\dot\lambda \tag{82}$$

and write

$$c^0[\mathcal{H}] = \frac{\mathcal{B}_0[\lambda] - (a[\lambda,\xi,\Psi^*] + \mathsf{a}_r[\lambda,\xi,\phi,\Psi^*])}{\int_{B_{2R}} |Z_5(y)|^2 dy}. \tag{83}$$

We invoke a key proposition proved in [12, Proposition 6.5] concerning the solvability of $\lambda(t)$.

Proposition 4.3. *Let $\omega, \Theta \in (0,\frac{1}{2})$, $\gamma \in (0,1)$, $m \le \Theta - \gamma$ and $l \in \mathbb{R}$. If $a(t)$ satisfies $a(T)<0$ with $1/C \le a(T) \le C$ for some constant $C>1$, and*

$$T^\Theta |\log T|^{1+c-l} \|a(\cdot)-a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \le C_1 \tag{84}$$

*for some $c>0$, then there exist two operators $\mathcal{P}$ and $\mathcal{R}_0$ such that $\lambda = \mathcal{P}[a] : [-T,T] \to \mathbb{R}$ satisfies*

$$\mathcal{B}_0[\lambda](t) = a(t) + \mathcal{R}_0[a](t) \tag{85}$$

*with*

$$|\mathcal{R}_0[a](t)| \lesssim \left(T^{\frac{1}{2}+c} + T^\Theta \frac{\log|\log T|}{|\log T|} \|a(\cdot)-a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}\right) \frac{(T-t)^{m+(1+\omega)\gamma}}{|\log(T-t)|^l}.$$

When applying Proposition 4.3, the Hölder property is essentially inherited from regularity of the outer solution, and this is one of the reasons that we work in the weighted space (61).

- **The fixed point formulation.**

We first transform the inner–outer problems (50), (53) into the problems of finding solutions $(\psi, \phi^0, \phi^1, \phi^\perp, \lambda, \xi)$ solving the following *inner–outer gluing system*

$$\begin{cases} \psi_t = \Delta\psi + \mathcal{G}(\phi^0+\phi^1+\phi^\perp, \psi+Z^*, \lambda, \xi), & \text{in } \Omega\times(0,T),\\ \psi = 0, & \text{on } \partial\Omega\times(0,T),\\ \psi(x,0)=0, & \text{in } \Omega, \end{cases} \tag{86}$$

$$\begin{cases} \lambda^2 \phi^0_t = \Delta_y \phi^0 + 3U^2(y)\phi^0 + \mathcal{H}^0(\phi,\psi,\lambda,\xi) + \tilde{c}^0[\mathcal{H}^0] Z_5 & \text{in } \mathcal{D}_{2R}, \\ \phi^0(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{87}$$

$$\begin{cases} \lambda^2 \phi^1_t = \Delta_y \phi^1 + 3U^2(y)\phi^1 + \mathcal{H}^1(\phi,\psi,\lambda,\xi) + \sum_{\ell=1}^{4} c^\ell[\mathcal{H}^1] Z_\ell & \text{in } \mathcal{D}_{2R}, \\ \phi^1(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{88}$$

$$\begin{cases} \lambda^2 \phi^\perp_t = \Delta_y \phi^\perp + 3U^2(y)\phi^\perp + \mathcal{H}^\perp(\phi,\psi,\lambda,\xi) + c^0_*[\lambda,\xi,\Psi^*] Z_5 & \text{in } \mathcal{D}_{2R}, \\ \phi^\perp(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{89}$$

$$c^0[\mathcal{H}](t) - \tilde{c}^0[\lambda,\xi,\Psi^*](t) = 0 \quad \text{for all} \quad t \in (0,T), \tag{90}$$

$$c^1[\mathcal{H}](t) = 0 \quad \text{for all} \quad t \in (0,T), \tag{91}$$

where $\mathcal{G}$ is defined in (54), $\mathcal{H}^0$, $\mathcal{H}^1$, $\mathcal{H}^\perp$ are the projections of $\mathcal{H}$ (see (51)) on different modes. It is direct to see that if $(\psi,\phi^0,\phi^1,\phi^\perp,\lambda,\xi)$ satisfies the system (86)–(91), then

$$\Psi^* = \psi + Z^*, \quad \phi = \phi^0 + \phi^1 + \phi^\perp$$

solve the inner–outer problems (50), (53) and thus the desired blow-up solution is obtained.

The inner–outer gluing system (86)–(91) can be then formulated as a fixed point problem for operators we will describe below.

We first define the following function spaces

$$\begin{aligned} X_{\phi^0} &:= \left\{ \phi^0 \in L^\infty(\mathcal{D}_{2R}) : \ \nabla_y \phi^0 \in L^\infty(\mathcal{D}_{2R}), \ \|\phi^0\|_{*,\nu,a,\delta} < +\infty \right\}, \\ X_{\phi^1} &:= \left\{ \phi^1 \in L^\infty(\mathcal{D}_{2R}) : \ \nabla_y \phi^1 \in L^\infty(\mathcal{D}_{2R}), \ \|\phi^1\|_{\nu_1,a_1} < +\infty \right\}, \\ X_{\phi^\perp} &:= \left\{ \phi^\perp \in L^\infty(\mathcal{D}_{2R}) : \ \nabla_y \phi^\perp \in L^\infty(\mathcal{D}_{2R}), \ \|\phi^\perp\|_{\nu,a} < +\infty \right\}, \\ X_\psi &:= \left\{ \psi \in L^\infty(\Omega \times (0,T)) : \ \|\psi\|_* < +\infty \right\}. \end{aligned} \tag{92}$$

In order to introduce the space for the parameter function $\lambda(t)$, we recall from (82) that the integral operator $\mathcal{B}_0$ takes the following approximate form

$$\mathcal{B}_0[\lambda] = \int_{-T}^{t-\lambda_*^2(t)} \frac{\dot\lambda(s)}{t-s} ds + O(\|\dot\lambda\|_\infty).$$

Proposition 4.3 defines an approximate inverse operator $\mathcal{P}$ of the integral operator $\mathcal{B}_0$ such that for $a$ satisfying (84), $\lambda := \mathcal{P}[a]$ satisfies

$$\mathcal{B}_0[\lambda] = a + \mathcal{R}_0[a] \quad \text{in} \quad [-T,T],$$

where $\mathcal{R}_0[a]$ is a small remainder. Also, the proof as in [12, Proposition 6.6] implies a refined decomposition

$$\mathcal{P}[a] = \lambda_{0,\kappa} + \mathcal{P}_1[a] \tag{93}$$

with

$$\lambda_{0,\kappa} := \kappa|\log T| \int_t^T \frac{1}{|\log(T-s)|^2} ds, \ t \le T,$$

$\kappa = \kappa[a] \in \mathbb{R}$, and the function $\lambda_1 = \mathcal{P}_1[a]$ satisfies

$$\|\lambda_1\|_{*,3-\iota} \lesssim |\log T|^{1-\iota} \log^2(|\log T|) \tag{94}$$

for $0 < \iota < 1$, where the $\|\cdot\|_{*,3-\iota}$-norm is defined by

$$\|f\|_{*,k} := \sup_{t\in[-T,T]} |\log(T-t)|^k |\dot f(t)|.$$

Therefore, we define

$$X_\lambda := \{\lambda_1 \in C^1([-T,T]) : \lambda_1(T) = 0, \|\lambda_1\|_{*,3-\iota} < \infty\}.$$

Here by $(\kappa, \lambda_1)$ we represent $\lambda$ in the form

$$\lambda = \lambda_{0,\kappa} + \lambda_1,$$

and from [12, Proposition 6.6], one can write the norm

$$\|\lambda\|_F = |\kappa| + \|\lambda_1\|_{*,3-\iota}. \tag{95}$$

For the translation parameter function $\xi(t)$, we write $\xi(t) = q + \xi^1(t)$ and define the following space for $\xi^1(t)$

$$X_\xi = \Big\{\xi \in C^1((0,T);\mathbb{R}^4), \ \dot\xi(T) = 0, \ \|\xi\|_G < +\infty\Big\}$$

with

$$\|\xi\|_G = \|\xi\|_{L^\infty(0,T)} + \sup_{t\in(0,T)} \lambda_*^{-\upsilon}(t)|\dot\xi(t)| \tag{96}$$

for some fixed $\upsilon \in (0,1)$.

Define

$$\mathcal{X} = X_{\phi^0} \times X_{\phi^1} \times X_{\phi^\perp} \times X_\psi \times \mathbb{R} \times X_\lambda \times X_\xi. \tag{97}$$

We will solve the inner–outer gluing system in a closed ball $\mathcal{B}$ in which

$$(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1) \in \mathcal{X}$$

satisfies

$$
\begin{cases}
\|\phi^0\|_{*,\nu,a,\delta} + \|\phi^1\|_{\nu_1,a_1} + \|\phi^\perp\|_{\nu,a} \le 1 \\
\|\psi\|_* \le 1 \\
|\kappa - \kappa_0| \le |\log T|^{-1/2} \\
\|\lambda_1\|_{*,3-\iota} \le C|\log T|^{1-\iota}\log^2(|\log T|) \\
\|\xi\|_G \le 1
\end{cases}
\tag{98}
$$

for some large and fixed constant $C$, where $\kappa_0 = Z_0^*(0)$. The inner–outer gluing system (86)–(91) can be formulated as the following fixed point problem. We define an operator $\mathcal{F}$ which returns the solution from $\mathcal{B}$ to $\mathcal{X}$

$$
\begin{aligned}
\mathcal{F} : \mathcal{B} \subset \mathcal{X} &\to \mathcal{X} \\
v &\mapsto \mathcal{F}(v) = (\mathcal{F}_{\phi^0}(v), \mathcal{F}_{\phi^1}(v), \mathcal{F}_{\phi^\perp}(v), \mathcal{F}_\psi(v), \mathcal{F}_\kappa(v), \mathcal{F}_{\lambda_1}(v), \mathcal{F}_\xi(v))
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{F}_{\phi^0}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_0(\mathcal{H}^0[\lambda,\xi,\Psi^*]) \\
\mathcal{F}_{\phi^1}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_1(\mathcal{H}^1[\lambda,\xi,\Psi^*]) \\
\mathcal{F}_{\phi^\perp}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_\perp\left(\mathcal{H}^\perp[\lambda,\xi,\Psi^*] + c_*^0[\lambda,\xi,\Psi^*]Z_5\right) \\
\mathcal{F}_\psi(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_\psi\left(\mathcal{G}(\phi^0+\phi^1+\phi^\perp,\Psi^*,\lambda,\xi)\right) \\
\mathcal{F}_\kappa(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \kappa\left[a^0[\lambda,\xi,\Psi^*]\right] \\
\mathcal{F}_{\lambda_1}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{P}_1\left[a^0[\lambda,\xi,\Psi^*]\right] \\
\mathcal{F}_\xi(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \Xi(\phi^0,\phi^1,\phi^\perp,\psi,\lambda,\xi).
\end{aligned}
\tag{99}
$$

Here $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_\perp$ are the operators given in Proposition 4.2 which solve different modes of the inner problems (87)–(89). The operator $\mathcal{T}_\psi$ defined by Proposition 4.1 deals with the outer problem (86). Operators $\kappa[a]$, $\mathcal{P}_1$ and $\Xi$ handle the equations for $\lambda$ and $\xi$ which are defined in Proposition 4.3, (93) and (78), respectively.

• **Choices of constants.**

We list all the constraints of the constants $\beta$, $\alpha$, $a$, $a_1$, $\nu$, $\nu_1$, $\nu_2$, $\delta$ which are sufficient for the inner–outer gluing scheme to work.

First, we indicate all the parameters used in different norms.

- $R(t) = \lambda_*^{-\beta}(t)$ with $\beta \in (0, 1/2)$.
- The norm for $\phi^0$ solving mode 0 of the inner problem (87) is $\|\cdot\|_{*,\nu,a,\delta}$ which is defined in (66), where we require that $\nu, a \in (0,1)$ and $\delta \ge 0$ small enough.
- The norm for $\phi^1$ solving modes 1 to 4 of the inner problem (88) is $\|\cdot\|_{\nu_1,a_1}$ which is defined in (65), where we require that $\nu_1 \in (0,1)$ and $a_1 \in (1,2)$.

- The norm for $\phi^\perp$ solving higher modes ($j \geq 5$) of the inner problem (89) is $\|\cdot\|_{\nu,a}$ which is defined in (65), where $\nu, a \in (0,1)$.
- The norm for $\psi$ solving the outer problem (86) is $\|\cdot\|_*$ which is defined in (61), while the $\|\cdot\|_{**}$-norm for the right hand side of the outer problem (86) is defined in (60). Here we require that $\nu, \alpha, \nu_2, \gamma \in (0,1)$ .
- In Proposition 4.3, we have the parameters $\omega, \Theta, m, l, \gamma$. Here $\omega$ is the parameter used to describe the remainder $\mathcal{R}_\omega$ and $\omega \in (0, 1/2)$. To apply Proposition 4.3 in our setting, we let

$$\Theta = \nu - 1 + \alpha\beta, \quad m = \nu - 2 - \gamma + \beta(2+\alpha), \quad l < 1 + 2m,$$

  and require that $\beta > \frac{1-\omega}{2}$ such that $m + (1+\omega)\gamma > \Theta$ is guaranteed.

  In order to get the desired estimates for the outer problem (86), we need

$$\nu - 1 + \beta(2+\alpha) - \nu_2 > 0, \quad 2\beta - \nu_2 > 0, \quad 0 < \alpha < a < 1,$$
$$\beta + \upsilon - \nu_2 > 0, \quad 2\nu_1 - \nu + \beta(2a_1 - \alpha) > 0, \quad \nu_2 < 1,$$
$$2\nu - \nu_2 - 1 + 2\alpha\beta > 0, \quad \nu - \beta(\alpha + 2\delta - 2a) > 0.$$

In order to get the desired estimates for the inner problems at different modes (87)–(89), we require

$$0 < \nu < 1, \quad 1 - \beta(2 + \frac{a}{2}) > 0, \quad 1 + \nu_1 - \nu - \beta(2 + a - a_1) > 0,$$
$$1 - 2\beta > 0, \quad \nu - \beta(4-a) > 0, \quad 2\nu_1 - \nu > 0,$$
$$2 - \nu - a\beta > 0, \quad \nu - \beta(a - 2\alpha) > 0, \quad 2 - \nu - \beta(1+a) > 0,$$
$$1 - \beta(\delta + 2) > 0, \quad \nu - 2\delta\beta > 0,$$
$$0 < \nu_1 < 1, \quad \nu - \nu_1 + \alpha\beta > 0, \quad 2 - \nu_1 - a_1\beta > 0,$$
$$2\nu - \nu_1 + 2\alpha\beta - a_1\beta > 0, \quad 1 - \nu_1 - \beta(a_1 - 1) > 0.$$

It turns out that suitable choices of the parameters satisfying all the restrictions in this section can be found. Here we give a specific example:

$$\beta \approx \frac{1}{4} \ (\beta > \frac{1}{4}), \ \alpha \approx a \approx a_1 \approx 1, \ \nu \approx \nu_1 \approx 1, \ \nu_2 \approx 0, \ \delta \approx 0.$$

• **Proof of Theorem 2.**

Consider the operator

$$\mathcal{F} = (\mathcal{F}_{\phi^0}, \mathcal{F}_{\phi^1}, \mathcal{F}_{\phi^\perp}, \mathcal{F}_\psi, \mathcal{F}_\kappa, \mathcal{F}_{\lambda_1}, \mathcal{F}_\xi) \tag{100}$$

given in (99). To prove Theorem 2, our strategy is to show the existence of a fixed point for the operator $\mathcal{F}$ in $\mathcal{B}$ by the Schauder fixed point theorem, where the closed ball $\mathcal{B}$ is defined in (98). By collecting the estimates (67), (72), (74), (79), (94), and using Proposition 4.1, Proposition 4.2, Proposition 4.3, we conclude that for $(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1) \in \mathcal{B}$

$$\begin{cases} \|\mathcal{F}_{\phi^0}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{*,\nu,a,\delta} \le CT^\epsilon \\ \|\mathcal{F}_{\phi^1}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{\nu_1,a_1} \le CT^\epsilon \\ \|\mathcal{F}_{\phi^\perp}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{\nu,a} \le CT^\epsilon \\ \|\mathcal{F}_{\psi}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{*} \le CT^\epsilon \\ \left|\mathcal{F}_{\kappa}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1) - \kappa_0\right| \le C|\log T|^{-1} \\ \|\mathcal{F}_{\lambda_1}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{*,3-\iota} \le C|\log T|^{1-\iota}\log^2(|\log T|) \\ \|\mathcal{F}_{\xi}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{G} \le CT^\epsilon \end{cases} \tag{101}$$

where $C > 0$ is a constant independent of $T$, and $\epsilon > 0$ is a small fixed number. On the other hand, compactness of the operator $\mathcal{F}$ defined in (100) can be proved by proper variants of (101). Therefore, the existence of the desired blow-up solution for $k = 1$ is concluded from the Schauder fixed point theorem.

The proof of multi-bubble case follows similarly by taking the ansatz (39) with nonlocal corrections supported around each concentration zone added. □

5. **Distorted Fourier Transform in Gluing Method** In this section, we introduce the third parabolic gluing method when the kernels are not in $L^2$ and when the maximum principle is absent. To this end, we consider the Landau-Lifshitz-Gilbert equation in critical dimension

$$\begin{cases} u_t = a(\Delta u + |\nabla u|^2 u) - bu \wedge \Delta u & \text{in } \mathbb{R}^2 \times (0,T), \\ u(\cdot, 0) = u_0 \in S^2 & \text{in } \mathbb{R}^2, \end{cases}$$

where $a^2 + b^2 = 1$, $a > 0$, $b \in \mathbb{R}$. The inner linearization around degree 1 harmonic map, in self-similar variables, looks like

$$(a+ib)\partial_\tau - \left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho - \frac{(n+1)^2\rho^4 + (2n^2-6)\rho^2 + (n-1)^2}{(\rho^2+1)^2}\frac{1}{\rho^2}\right).$$

We utilize distorted Fourier transform at mode $n = -1$ in [57, Section 9.6]. The use of aforementioned techniques in Section 4 does imply a solution. However, such solution is not sufficient for the gluing to work as it loses too many $R$'s and makes the nonlinear terms non-controllable. So we use distorted Fourier transform (DFT) instead to get desired estimates. The reason behind this is that formally the worst mode is $-1$ as it corresponds to 2-dimensional heat operator, and usually estimates in 2D come with a logarithmic loss because of lack of maximum principle.

The derivation of desired estimates for mode $-1$ is done by first deducing the Duhamel's representation via DFT and then estimating pointwisely. For $\ell \in \mathbb{R}$ and $v(\tau) > 0$ and vectorial complex-valued function $f$, the weighted topology are defined by

$$\|f\|_{v,\ell} := \sup_{(y,\tau)\in\mathbb{R}^2\times(\tau_0,\infty)} v^{-1}(\tau)\langle y\rangle^{\ell}|f(y,\tau)|.$$

Consider

$$\begin{cases}(a+ib)\partial_\tau \phi_n(\rho,\tau) = \mathcal{L}_n\phi_n(\rho,\tau),\\ \phi_n(\rho,\tau_0) = g(\rho),\end{cases}$$

where $\tau_0 \geq 1$,

$$\mathcal{L}_n = \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho - \frac{(n-1)^2}{\rho^2} - \frac{4n}{\rho^2+1} + \frac{8}{(\rho^2+1)^2}.$$

Assume $g$ is a Schwartz function. Set $\phi_n(\rho,\tau) = \rho^{-\frac{1}{2}}A_n(\rho,\tau)$, then

$$\begin{cases}(a+ib)\partial_\tau A_n(\rho,\tau) = \tilde{\mathcal{L}}_n A_n(\rho,\tau),\\ A_n(\rho,\tau_0) = \rho^{\frac{1}{2}}g(\rho).\end{cases}$$

where $\tilde{\mathcal{L}}_n = \partial_{\rho\rho} + \frac{1}{4}\rho^{-2} - \frac{(n-1)^2}{\rho^2} - \frac{4n}{\rho^2+1} + \frac{8}{(\rho^2+1)^2}$.

Recall the generalized eigenfunctions $\Phi^n(\rho,\xi)$ with respect to $-\tilde{\mathcal{L}}_n$ is given by

$$-\tilde{\mathcal{L}}_n\Phi^n(\rho,\xi) = \xi\Phi^n(\rho,\xi).$$

We multiply it by $\Phi^n(\rho,\xi)$ and integrate by parts and get

$$\begin{cases}(a+ib)\partial_\tau \hat{A}_n(\xi,\tau) = -\xi\hat{A}_n(\xi,\tau),\\ \hat{A}_n(\xi,\tau_0) = \int_0^\infty \rho^{\frac{1}{2}}g(\rho)\Phi^n(\rho,\xi)d\rho,\end{cases}$$

where $\hat{A}_n(\xi,\tau) = \int_0^\infty A_{-1}(\rho,\tau)\Phi^n(\rho,\xi)d\rho$. Thus

$$\hat{A}_n(\xi,\tau) = e^{-(a-ib)\xi\tau}\hat{A}_n(\xi,\tau_0).$$

Taking inverse DFT, one has

$$\begin{aligned}A_n(\rho,\tau) &= \int_0^\infty \hat{A}_n(\xi,\tau)\Phi^n(\rho,\xi)\rho_n(d\xi) = \int_0^\infty e^{-(a-ib)\xi\tau}\hat{A}_n(\xi,0)\Phi^n(\rho,\xi)\rho_n(d\xi)\\ &= \int_0^\infty e^{-(a-ib)\xi\tau}\Phi^n(\rho,\xi)\int_0^\infty x^{\frac{1}{2}}g(x)\Phi^n(x,\xi)dx\rho_n(d\xi)\\ &= \int_0^\infty\int_0^\infty e^{-(a-ib)\xi\tau}\Phi^n(\rho,\xi)\Phi^n(x,\xi)\rho_n(d\xi)x^{\frac{1}{2}}g(x)dx.\end{aligned}$$

By Duhamel's principle, it holds that
(102)
$$\phi_n(\rho,\tau) = \int_{\tau_0}^{\tau}\int_0^\infty\int_0^\infty e^{-(a-ib)\xi(\tau-s)}\rho^{-\frac{1}{2}}\Phi^n(\rho,\xi)\Phi^n(x,\xi)x^{\frac{1}{2}}h_n(x,s)\rho_n(d\xi)dxds$$

gives a solution to the non-homogeneous equation with RHS $h_n$ and zero initial data.

To estimate solution in above formulation, one needs precise estimates of generalized eigenfunctions and density of spectral measure. For $n = -1$, we summarize the results in [37, Section 4.3.2] as follows.

PROPOSITION 5.1 ( [37]). *For all $\rho \geq 0$, $\xi \geq 0$, we have*

$$\left|\Phi^{-1}(\rho,\xi)\right| \lesssim \begin{cases} \rho^{\frac{5}{2}}\langle\rho\rangle^{-2} & \text{if } \rho^2\xi \leq 1 \\ \xi^{-\frac{1}{4}}\langle\xi\rangle^{-1} & \text{if } \rho^2\xi > 1 \end{cases}.$$

*$\Phi^{-1}(\rho,\xi)$ has the following expansion:*

$$\Phi^{-1}(\rho,\xi) = \Phi_0^{-1}(\rho) + \rho^{\frac{1}{2}}\sum_{j=1}^{\infty}(-\rho^2\xi)^j\Phi_j(\rho^2),$$

*which converges absolutely, where $\Phi_0^{-1}(\rho) = \frac{\rho^{\frac{5}{2}}}{1+\rho^2}$. It converges uniformly if $\rho\xi^{\frac{1}{2}}$ remains bounded. Here $\Phi_j(u) \geq 0$ are smooth functions of $u \geq 0$ satisfying*

$$\Phi_j(u) \leq \frac{1}{j!}\frac{u}{1+u}, \quad \text{for all } u \geq 0, j \geq 1,$$

*and $\Phi_1(u) \geq c_1\frac{u}{1+u}$ for all $u \geq 0$ with some absolute constant $c_1 > 0$.*

*The spectrum measure $\rho_{-1}(d\xi)$ of $-\tilde{\mathcal{L}}_{-1}$ is absolutely continuous on $\xi \geq 0$ with density*
$$\frac{d\rho_{-1}(\xi)}{d\xi} \sim \langle\xi\rangle^2.$$

Our linear theory for LLG mode $-1$ without orthogonality condition is stated as follows.

PROPOSITION 5.2. ([57, Proposition 9.8]) *Consider*

$$\begin{cases} (a+ib)\partial_\tau\phi_{-1}(\rho,\tau) = \mathcal{L}_{-1}\phi_{-1}(\rho,\tau) + h(\rho,\tau) & \text{in } (0,\infty)\times(\tau_0,\infty), \\ \phi_{-1}(\rho,\tau_0) = 0 \quad \text{in } (0,\infty). \end{cases}$$

*where $\tau_0 \geq 1$, $\|h\|_{v,\ell} < \infty$, where $\ell > \frac{3}{2}$. Then the solution $\phi_{-1} = \mathcal{T}_{-1}[h]$, where $\mathcal{T}_{-1}[h]$ is given by the linear mapping (102) with $n = -1$, satisfies the following* *estimate*

$$|\phi_{-1}(\rho,\tau)| \lesssim \|h\|_{v,\ell}\mathbf{1}_{\{\rho\le\tau^{\frac12}\}}\begin{cases} v(\tau)\tau^{1-\frac{\ell}{2}}+\tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell<2\\ v(\tau)(\ln\tau)^2+\tau^{-1}\ln\tau\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell=2\\ v(\tau)\ln\tau+\tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell>2\end{cases}$$

$$+\|h\|_{v,\ell}\mathbf{1}_{\{\rho>\tau^{\frac12}\}}\rho^{-\frac12}\begin{cases} v(\tau)\tau^{\frac54-\frac{\ell}{2}}+\tau^{\frac14-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell<2\\ v(\tau)\tau^{\frac14}\langle\ln\tau\rangle+\tau^{-\frac34}\langle\ln\tau\rangle\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell=2\\ v(\tau)\tau^{\frac14}+\tau^{-\frac34}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell>2\end{cases}.$$

Better estimates with orthogonality condition imposed can be obtained similarly, see [57, Section 9.6].

**Appendix A. Linear Theory via Blow-up Argument** In this section, we present a linear theory by *a priori estimates*, which are proved by blow-up argument. This is the reminiscent of Liapunov-Schmidt reduction method or gluing method in the elliptic concentration problems. See [14, 15, 46]. However the proofs are more involved.

Define

$$\|h\|_{a,\nu} := \sup_{(y,\tau)\in\mathbb{R}^n\times(\tau_0,\infty)} \tau^{\nu}\langle y\rangle^a |h(y,\tau)|.$$

The main results are the following.

Proposition A.1. *Consider*

$$\begin{cases}\partial_\tau\phi=\Delta\phi+pU^{p-1}(y)\phi+h(y,\tau) & \text{in } \ \mathbb{R}^n\times(\tau_0,\infty)\\ \phi(y,\tau_0)=e_0Z_0(y) & \text{in } \ \mathbb{R}^n.\end{cases}\tag{A.1}$$

*Suppose* $2<a<n-2$, $\nu<1$, $\|h\|_{2+a,\nu}<\infty$ *and*

$$\int_{\mathbb{R}^n} h(y,\tau)Z_j(y)dy=0 \qquad \text{for all} \quad \tau\in(\tau_0,\infty),\ j=1,2,\cdots,n+1.\tag{A.2}$$

*Then for* $\tau_0\ge1$, *there exists a linear mapping* $(\phi,e_0)=(\phi[h],e_0[h])$ *satisfying* (A.1) *and*

$$\int_{\mathbb{R}^n} \phi(y,\tau)Z_j(y)dy=0 \qquad \text{for all} \quad \tau\in(\tau_0,\infty),\ j=1,2,\cdots,n+1,\tag{A.3}$$

$$\langle y\rangle|\nabla\phi|+|\phi|\lesssim\tau^{-\nu}\langle y\rangle^{-a}\|h\|_{2+a,\nu},\quad |e_0[h]|\lesssim\tau_0^{-\nu}\|h\|_{2+a,\nu}.\tag{A.4}$$

Proposition A.1 is in fact a consequence of the following

PROPOSITION A.2. *Suppose* $2 < a < n-2$, $\nu < 1$, $\|h\|_{2+a,\nu} < \infty$ *and*

$$\int_{\mathbb{R}^n} h(y,\tau) Z_j(y) dy = 0 \quad \textit{for all} \quad \tau \in (\tau_0, \infty), \ j = 0, 1, \cdots, n+1. \tag{A.5}$$

*Then for* $\tau_0 \geq 1$, *there exists a unique solution* $\phi$ *of*

$$\begin{cases} \partial_\tau \phi = \Delta \phi + pU^{p-1}(y)\phi + h(y,\tau) & \textit{in} \ \ \mathbb{R}^n \times (\tau_0, \infty) \\ \phi(y, \tau_0) = 0 & \textit{in} \ \ \mathbb{R}^n \end{cases} \tag{A.6}$$

*in* $L^\infty(\mathbb{R}^n \times (\tau_0, \tilde{\tau}))$ *for all* $\tilde{\tau} > \tau_0$ *and* $\phi$ *satisfies*

$$\int_{\mathbb{R}^n} \phi(y,\tau) Z_j(y) dy = 0 \quad \textit{for all} \quad \tau > \tau_0, \ j = 0, 1, \cdots, n+1 \tag{A.7}$$

*and the estimate*

$$\langle y \rangle |\nabla \phi| + |\phi| \lesssim \tau^{-\nu} \langle y \rangle^{-a} \|h\|_{2+a,\nu}. \tag{A.8}$$

We first use Proposition A.2 to deduce Proposition A.1.

PROOF OF PROPOSITION A.1. First, set $\phi(y,\tau) = \phi_1(y,\tau) + c(\tau) Z_0(y)$. Then it suffices to consider

$$\begin{cases} \partial_\tau \phi_1 = \Delta \phi_1 + pU^{p-1}(y)\phi_1 + h_1 & \text{in} \ \ \mathbb{R}^n \times (\tau_0, \infty) \\ \phi_1(y, \tau_0) = 0 & \text{in} \ \ \mathbb{R}^n. \end{cases} \tag{A.9}$$

with $h_1 := h + c(\tau)\lambda_0 Z_0 - c'(\tau) Z_0$ where we used $\Delta Z_0 + pU^{p-1}(y) Z_0 = \lambda_0 Z_0$ with $\lambda_0 > 0$. We take

$$c'(\tau) - \lambda_0 c(\tau) = \left( \int_{\mathbb{R}^n} Z_0^2(y) dy \right)^{-1} \int_{\mathbb{R}^n} h(y,\tau) Z_0(y) dy$$

with

$$c(\tau) = -\left( \int_{\mathbb{R}^n} Z_0^2(y) dy \right)^{-1} e^{\lambda_0 \tau} \int_\tau^\infty e^{-\lambda_0 s} \int_{\mathbb{R}^n} h(y,s) Z_0(y) dy ds$$

so that $\int_{\mathbb{R}^n} h_1(y,\tau) Z_0(y) dy = 0$. Combining this with (A.2) implies (A.5). It is direct to see that

$$|c(\tau)| + |c'(\tau)| \lesssim \|h\|_{2+a,\nu} \tau^{-\nu}, \quad \|h_1\|_{2+a,\nu} \lesssim \|h\|_{2+a,\nu}.$$

By Proposition A.2, (A.9) has a unique solution $\phi_1$ satisfying

$$\int_{\mathbb{R}^n} \phi_1(y,\tau) Z_j(y) dy = 0$$

$$\text{for all} \quad \tau > \tau_0, \ j = 0, 1, \cdots, n+1, \ \langle y \rangle |\nabla \phi_1| + |\phi_1| \lesssim \tau^{-\nu} \langle y \rangle^{-a} \|h\|_{2+a,\nu}.$$

Then $\phi = \phi_1 + c(\tau) Z_0$ satisfies (A.3) and (A.4). Finally, letting $e_0 = c_0(\tau_0)$ completes the proof. □

Next, we use blow-up argument to prove Proposition A.2. Our method does not rely on the use of maximum principle, so we expect that this method is applicable to more general equations/systems in the absence of maximum principle (assuming non-degeneracy of the profile in certain sense). Typical examples are parabolic equations with complex coefficients such as

$$A_0\partial_\tau u = \Delta u + V(u, Du) + h,$$

where the complex constant $A_0$ satisfies $\mathrm{Re}(A_0) > 0$ (with dissipation). This kind of operator naturally arises, for instance, in the study of Landau-Lifshitz-Gilbert equations.

Proof of Proposition A.2. The existence and uniqueness of (A.6) are given by the classical parabolic theory. Denote

$$\|f\|_{a,\nu,\tau_1} := \sup_{(y,\tau)\in\mathbb{R}^n\times(\tau_0,\tau_1)} \tau^\nu \langle y\rangle^a |f(y,\tau)|.$$

For all $\tau > \tau_0$, by the estimate of parabolic fundamental solution (See [25]) and convolution estimate in [58, Lemma A.1, Lemma A.2], for $0 < a < n-2$, we have

$$\|\phi\|_{a,\nu,\tau} < \infty. \tag{A.10}$$

By scaling argument, we have $\|\nabla\phi\|_{1+a,\nu,\tau} < \infty$.

For $a > 2$, multiplying (A.6) by $Z_j$, $j = 0, 1, \ldots, n+1$ and integrating by parts, we obtain (A.7) by (A.5) and the initial data of (A.6).

In order to prove (A.8), it suffices to prove the following claim.

Claim: For all $\tau_1 > \tau_0$ large enough, there exists $C$ independent of $\tau_1$ such that

$$\|\phi\|_{a,\nu,\tau_1} \le C\|h\|_{2+a,\nu,\tau_1}. \tag{A.11}$$

Indeed, by taking $\tau_1 \to \infty$, (A.11) implies (A.8). To prove (A.11), we argue by contradiction. Suppose that there exist sequences $\tau_1^k \to \infty$ and $\phi_k$, $h_k$ satisfying

$$\begin{cases} \partial_\tau \phi_k = \Delta\phi_k + pU^{p-1}(y)\phi_k + h_k \quad \text{in } \mathbb{R}^n\times(\tau_0,\infty) \\ \displaystyle\int_{\mathbb{R}^n} \phi_k(y,\tau) Z_j(y)dy = 0 \quad \text{for all } \tau\in(\tau_0,\tau_1^k), \quad j = 0,1,\cdots,n+1 \\ \phi_k(y,\tau_0) = 0 \quad \text{in } \mathbb{R}^n \end{cases} \tag{A.12}$$

and

$$\|\phi_k\|_{a,\nu,\tau_1^k} = 1, \quad \|h_k\|_{2s+a,\nu,\tau_1^k} = o(1) \ \text{ where } \ o(1)\to 0 \ \text{ as } \ k\to\infty. \tag{A.13}$$

First, we claim that for any compact subset $\Omega$ in $\mathbb{R}^n$,

$$\sup_{\tau_0<\tau<\tau_1^k} \tau^\nu |\phi_k(y,\tau)| \to 0 \ \text{ uniformly in } \ \Omega. \tag{A.14}$$

Assume this is not true. Then there exist a constant $M > 0$ such that $|y_k| \le M$ and $\tau_0 < \tau_2^k < \tau_1^k$,

$$(\tau_2^k)^\nu |\phi_k(y_k, \tau_2^k)| \ge \delta_0 > 0, \tag{A.15}$$

for a constant $\delta_0 > 0$. Since $\|h_k\|_{2+a,\nu,\tau_1^k} = o(1)$, we have $\tau_2^k \to \infty$ by the same reason for getting (A.10). Without loss of generality, we assume $\tau_2^k \ge 9\tau_0$. Set

$$\tilde{\phi}_k(y,t) = (\tau_2^k)^\nu \phi_k(y, \tau_2^k + t), \quad \tilde{h}_k(y,t) = (\tau_2^k)^\nu h_k(y, \tau_2^k + t).$$

Then by (A.12), one has

$$\partial_t \tilde{\phi}_k = \Delta \tilde{\phi}_k + pU^{p-1}(y)\tilde{\phi}_k + \tilde{h}_k \text{ in } \mathbb{R}^n \times (\tau_0 - \frac{\tau_2^k}{2}, 0] \tag{A.16}$$

with

(A.17)

$$|\tilde{\phi}_k(y,\tau)| \le C(\nu)\langle y\rangle^{-a}, \quad |\tilde{h}_k(y,\tau)| \le o(1)C(\nu)\langle y\rangle^{-2-a} \text{ in } \mathbb{R}^n \times (\tau_0 - \frac{\tau_2^k}{2}, 0],$$

where $C(\nu)$ is a constant only depending on $\nu$.

By the parabolic regularity theorem, up to a subsequence, we have $\tilde{\phi}_k \to \tilde{\phi}$ in $C^1_{\rm loc}$, that is, $\tilde{\phi}_k \to \tilde{\phi}$ in $C^1$ topology on any compact subsets of $\mathbb{R}^n \times (-\infty, 0]$. Combining this with (A.15) yields

$$|\tilde{\phi}(y,\tau)| \le C(\nu)\langle y\rangle^{-a}, \quad \tilde{\phi} \ne 0. \tag{A.18}$$

By (A.16), we have

$$\tilde{\phi}_k(y,t) = \int_{\mathbb{R}^n} [4\pi(t - \tau_0 + \frac{\tau_2^k}{2})]^{-\frac{n}{2}} \exp\left(-\frac{|y-z|^2}{4(t-\tau_0+\frac{\tau_2^k}{2})}\right) \tilde{\phi}_k(z, \tau_0 - \frac{\tau_2^k}{2}) dz$$

$$+ \int_{\tau_0 - \frac{\tau_2^k}{2}}^t \int_{\mathbb{R}^n} [4\pi(t-s)]^{-\frac{n}{2}} \exp\left(-\frac{|y-z|^2}{4(t-s)}\right) \left(pU^{p-1}(z)\tilde{\phi}_k(z,s) + \tilde{h}_k(z,s)\right) dzds.$$

Then for any fixed $(y,t) \in \mathbb{R}^n \times (-\infty, 0]$, by $a > 0$, (A.17), (A.18), [54, Corollary B.4, Lemma B.5] (used for time integral $\int_{t_2}^{t_1} \cdots$ for some $t_1 \le -1$), [58, Lemma A.3] (used for Cauchy integral) and $\tilde{\phi}_k \to \tilde{\phi}$ in $C^1_{\rm loc}$, we have

$$\tilde{\phi}(y,t) = \int_{-\infty}^t \int_{\mathbb{R}^n} [4\pi(t-s)]^{-\frac{n}{2}} e^{-\frac{|y-z|^2}{4(t-s)}} pU^{p-1}(z)\tilde{\phi}(z,s) dzds. \tag{A.19}$$

Then the limiting equation reads

$$\begin{cases} \partial_\tau \tilde{\phi} = \Delta \tilde{\phi} + pU^{p-1}(y)\tilde{\phi} \quad \text{in} \quad \mathbb{R}^n \times (-\infty, 0] \\ \int_{\mathbb{R}^n} \tilde{\phi}(y,\tau) Z_j(y) dy = 0 \text{ for all } \tau \in (-\infty, 0], \ j = 0, 1, \cdots n+1 \\ |\tilde{\phi}(y,\tau)| \le C(\nu)\langle y\rangle^{-a} \text{ in } \mathbb{R}^n \times (-\infty, 0] \end{cases} \tag{A.20}$$

where we have used $a > 2$ in the orthogonality by dominated convergence theorem.

Using (A.19), [54, Corollary B.4, Lemma B.5] finitely many times for $\tau \in (-\infty, -M_0)$ with $M_0$ large and then applying [58, Lemmas A.1, A.2, A.3] in $[M_0, 0]$, we have

$$|\tilde{\phi}| \lesssim \langle y \rangle^{2-n},$$

and $\tilde{\phi}$ is smooth by the parabolic regularity theory. By scaling argument, one has

$$\langle y \rangle^{-1} |D\tilde{\phi}| + |\tilde{\phi}_\tau| + |D^2 \tilde{\phi}| \lesssim \langle y \rangle^{-n}.$$

Differentiating (A.20), we get

$$\partial_\tau \tilde{\phi}_\tau = \Delta \tilde{\phi}_\tau + pU^{p-1}(y) \tilde{\phi}_\tau, \tag{A.21}$$

and then scaling argument gives

$$\langle y \rangle^{-1} |D\tilde{\phi}_\tau| + |\tilde{\phi}_{\tau\tau}| + |D^2 \tilde{\phi}_\tau| \lesssim \langle y \rangle^{-n-2}.$$

Moreover, multiplying (A.21) by $\tilde{\phi}_\tau$ and integrating by parts, we get

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy + B(\tilde{\phi}_\tau, \tilde{\phi}_\tau) = 0,$$

where

$$B(f, f) := \int_{\mathbb{R}^n} \left( |\nabla f|^2 - pU^{p-1}(y) |f|^2 \right) dy.$$

By orthogonality in (A.20), we have $\int_{\mathbb{R}^n} \partial_\tau \tilde{\phi}(y, \tau) Z_j(y) dy = 0$ for all $\tau \in (-\infty, 0]$, $j = 0, 1, \cdots, n+1$. Then $B(\tilde{\phi}_\tau, \tilde{\phi}_\tau) \geq 0$ by $\int_{\mathbb{R}^n} \partial_\tau \tilde{\phi}(y, \tau) Z_0(y) dy = 0$ since $Z_0$ is the only eigenfucntion corresponding to the positive eigenvalue. Thus, $\partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy \leq 0$.

Multiplying (A.20) by $\tilde{\phi}_\tau$ and integrating by parts, we have

$$\int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy = -\frac{1}{2} \partial_\tau B(\tilde{\phi}, \tilde{\phi}).$$

From these relations, one has

$$\partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy \leq 0, \quad \int_{-\infty}^0 d\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy < \infty.$$

Hence $\tilde{\phi}_\tau = 0$. So $\tilde{\phi}$ is independent of $\tau$ and $\Delta \tilde{\phi} + pU^{p-1}(y) \tilde{\phi} = 0$. By the nondegeneracy of $\Delta + pU^{p-1}(y)$ (see [1, Lemma 5.2]), $\tilde{\phi}$ is a linear combination of $Z_j$, $j = 1, \cdots, n+1$. Due to the orthogonal conditions in (A.20), we must have $\tilde{\phi} \equiv 0$, which contradicts (A.18). Thus (A.14) holds.

By (A.13) and (A.14), there exists a sequence $y_k$ with $|y_k| \to \infty$ such that

$$(\tau_2^k)^\nu \langle y_k \rangle^a |\phi_k(y_k, \tau_2^k)| \ge \frac{1}{2}.$$

Set

$$\tilde{\phi}_k(z,t) := (\tau_2^k)^\nu \langle y_k \rangle^a \phi_k(y_k + |y_k| z, |y_k|^2 t + \tau_2^k).$$

Then

$$|\tilde{\phi}_k(0,0)| \ge \frac{1}{2}. \tag{A.22}$$

We reformulate (A.12) as

$$\begin{cases} \partial_t \tilde{\phi}_k = \Delta_z \tilde{\phi}_k + p|y_k|^2 U^{p-1}(y_k + |y_k|z)\tilde{\phi}_k + \tilde{h}_k(z,t) \ \text{ in } \ \mathbb{R}^n \times (\frac{\tau_0 - \tau_2^k}{|y_k|^2}, \infty) \\ \tilde{\phi}_k(\cdot, \frac{\tau_0 - \tau_2^k}{|y_k|^2}) = 0 \ \text{ in } \ \mathbb{R}^n \end{cases} \tag{A.23}$$

where

$$\tilde{h}_k(z,t) = (\tau_2^k)^\nu \langle y_k \rangle^a |y_k|^2 h_k(y_k + |y_k|z, |y_k|^2 t + \tau_2^k).$$

By (A.13), one has

$$|\tilde{h}_k(z,t)| \lesssim o(1)(\tau_2^k)^\nu \langle y_k \rangle^a |y_k|^2 \langle y_k + |y_k|z \rangle^{-2-a} \left(|y_k|^2 t + \tau_2^k\right)^{-\nu}$$

$$\sim o(1) \left(|y_k|^{-1} + |\hat{y}_k + z|\right)^{-2-a} \left((\tau_2^k)^{-1}|y_k|^2 t + 1\right)^{-\nu}, \quad \text{for} \quad (z,t) \in \mathbb{R}^n \times (\frac{\tau_0 - \tau_2^k}{|y_k|^2}, \frac{\tau_1^k - \tau_2^k}{|y_k|^2}),$$

$$\left| |y_k|^2 U^{p-1}(y_k + |y_k|z)\tilde{\phi}_k \right| \lesssim |y_k|^2 \langle y_k + |y_k|z \rangle^{-4} (\tau_2^k)^\nu \langle y_k \rangle^a \langle y_k + |y_k|z \rangle^{-a} \left(|y_k|^2 t + \tau_2^k\right)^{-\nu}$$

$$\sim |y_k|^{-2} \left(|y_k|^{-1} + |\hat{y}_k + z|\right)^{-4-a} \left((\tau_2^k)^{-1}|y_k|^2 t + 1\right)^{-\nu}, \quad \text{for} \quad (z,t) \in \mathbb{R}^n \times (\frac{\tau_0 - \tau_2^k}{|y_k|^2}, \frac{\tau_1^k - \tau_2^k}{|y_k|^2})$$

where $\hat{y}_k = y_k |y_k|^{-1}$. By (A.23), we have

$$\tilde{\phi}_k(z,t) = \int_{\frac{\tau_0 - \tau_2^k}{|y_k|^2}}^t \int_{\mathbb{R}^n} [4\pi(t-s)]^{-\frac{n}{2}} e^{-\frac{|z-w|^2}{4(t-s)}} \left(p|y_k|^2 U^{p-1}(y_k + |y_k|w)\tilde{\phi}_k(w,s) + \tilde{h}_k(w,s)\right) dwds.$$

Then

$$\left|\tilde{\phi}_k(0,0)\right| \lesssim \int_{\frac{\tau_0 - \tau_2^k}{|y_k|^2}}^0 \int_{\mathbb{R}^n} (-s)^{-\frac{n}{2}} e^{\frac{|w|^2}{4s}} \Big[ |y_k|^{-2} \left(|y_k|^{-1} + |\hat{y}_k + w|\right)^{-4-a} \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu}$$

$$+ o(1) \left(|y_k|^{-1} + |\hat{y}_k + w|\right)^{-2-a} \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} \Big] dwds.$$

Claim: Suppose that $\tau_2^k \geq 2\tau_0$, $|y_k| \geq 2$, $m > 2, n > 2, \nu < 1$, then
(A.24)

$$\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \int_{\mathbb{R}^n} (-s)^{-\frac{n}{2}} e^{\frac{|w|^2}{4s}} \left(|y_k|^{-1} + |\hat{y}_k + w|\right)^{-m} \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} dwds$$

$$\lesssim \begin{cases} 1, & \text{if } \ m < n \\ \langle \ln |y_k| \rangle, & \text{if } \ m = n \\ |y_k|^{m-n}, & \text{if } \ m > n. \end{cases}$$

Assuming (A.24), for $0 < a < n-2$, one then has

$$\left|\tilde{\phi}_k(0,0)\right| \lesssim o(1) + \begin{cases} |y_k|^{-2}, & \text{if } \ 4+a < n \\ |y_k|^{-2}\langle \ln |y_k| \rangle, & \text{if } \ 4+a = n \\ |y_k|^{2+a-n}, & \text{if } \ 4+a > n \end{cases} \to 0 \ \text{ as } \ k \to \infty$$

which contradicts (A.22).

Finally, we prove (A.24).

**Proof of** (A.24):

$$\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \int_{\mathbb{R}^n} (-s)^{-\frac{n}{2}} e^{\frac{|w|^2}{4s}} \left(|y_k|^{-1} + |\hat{y}_k + w|\right)^{-m} \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} dwds$$

$$= \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} (-s)^{-\frac{m}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4}} \left((-s)^{-\frac{1}{2}}|y_k|^{-1} + |(-s)^{-\frac{1}{2}}\hat{y}_k + x|\right)^{-m} dxds.$$

Notice for $0 < 2c_0 \leq |\vec{v}|$, we estimate the spatial integral as

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4}} (c_0 + |\vec{v}+x|)^{-m}\, dx = \left( \int_{|x|\leq \frac{|\vec{v}|}{2}} + \int_{\frac{|\vec{v}|}{2}<|x|\leq 2|\vec{v}|} + \int_{|x|>2|\vec{v}|} \right) e^{-\frac{|x|^2}{4}} (c_0 + |\vec{v}+x|)^{-m}\, dx$$

$$\sim \int_{|x|\leq \frac{|\vec{v}|}{2}} e^{-\frac{|x|^2}{4}} (c_0 + |\vec{v}|)^{-m}\, dx + \int_{\frac{|\vec{v}|}{2}<|x|\leq 2|\vec{v}|} e^{-\frac{|x|^2}{4}} (c_0 + |\vec{v}+x|)^{-m}\, dx$$

$$+ \int_{|x|>2|\vec{v}|} e^{-\frac{|x|^2}{4}} (c_0 + |x|)^{-m}\, dx$$

$$\lesssim |\vec{v}|^{-m} \int_{|x|\leq \frac{|\vec{v}|}{2}} e^{-\frac{|x|^2}{4}} dx + e^{-\frac{|\vec{v}|^2}{16}} \int_{|x+\vec{v}|\leq 3|\vec{v}|} (c_0 + |\vec{v}+x|)^{-m}\, dx + \int_{|x|>2|\vec{v}|} e^{-\frac{|x|^2}{4}} |x|^{-m} dx$$

$$\lesssim \mathbf{1}_{\{|\vec{v}|\leq 1\}} |\vec{v}|^{n-m} + \mathbf{1}_{\{|\vec{v}|>1\}} |\vec{v}|^{-m} + e^{-\frac{|\vec{v}|^2}{16}} \left( \int_0^{c_0} + \int_{c_0}^{3|\vec{v}|} \right) (c_0+r)^{-m} r^{n-1} dr$$

$$+ \mathbf{1}_{\{|\vec{v}|\leq 1\}} \begin{cases} 1, & \text{if } m<n \\ \langle \ln|\vec{v}| \rangle, & \text{if } m=n \\ |\vec{v}|^{n-m}, & \text{if } m>n \end{cases} + \mathbf{1}_{\{|\vec{v}|>1\}} e^{-\frac{|\vec{v}|^2}{2}}$$

$$\lesssim e^{-\frac{|\vec{v}|^2}{16}} \begin{cases} |\vec{v}|^{n-m}, & \text{if } m<n \\ \langle \ln(\frac{|\vec{v}|}{c_0}) \rangle, & \text{if } m=n \\ c_0^{n-m}, & \text{if } m>n \end{cases} + \mathbf{1}_{\{|\vec{v}|\leq 1\}} \begin{cases} 1, & \text{if } m<n \\ \langle \ln|\vec{v}| \rangle, & \text{if } m=n \\ |\vec{v}|^{n-m}, & \text{if } m>n \end{cases}$$

$$+ \mathbf{1}_{\{|\vec{v}|>1\}} |\vec{v}|^{-m}$$

$$\sim \mathbf{1}_{\{|\vec{v}|\leq 1\}} \begin{cases} 1, & \text{if } m<n \\ \langle \ln|\vec{v}| \rangle + \langle \ln(\frac{|\vec{v}|}{c_0}) \rangle, & \text{if } m=n \\ c_0^{n-m}, & \text{if } m>n \end{cases}$$

$$+ \mathbf{1}_{\{|\vec{v}|>1\}} \begin{cases} |\vec{v}|^{-m}, & \text{if } m<n \\ |\vec{v}|^{-m} + e^{-\frac{|\vec{v}|^2}{16}} \langle \ln(\frac{|\vec{v}|}{c_0}) \rangle, & \text{if } m=n \\ |\vec{v}|^{-m} + e^{-\frac{|\vec{v}|^2}{16}} c_0^{n-m}, & \text{if } m>n. \end{cases}$$

Thus, we obtain

$$\begin{aligned}
&\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1}|y_k|^2 s+1\right)^{-\nu}(-s)^{-\frac{m}{2}}\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4}}\left((-s)^{-\frac12}|y_k|^{-1}+|(-s)^{-\frac12}\hat{y}_k+x|\right)^{-m}dxds\\
\lesssim &\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1}|y_k|^2 s+1\right)^{-\nu}(-s)^{-\frac{m}{2}}\Bigg(\mathbf{1}_{\{s\le -1\}}\begin{cases}1, & \text{if } m<n\\ \langle \ln(-s)\rangle+\langle \ln|y_k|\rangle, & \text{if } m=n\\ \left((-s)^{-\frac12}|y_k|^{-1}\right)^{n-m}, & \text{if } m>n\end{cases}\\
&+\mathbf{1}_{\{s>-1\}}\begin{cases}(-s)^{\frac{m}{2}}, & \text{if } m<n\\ (-s)^{\frac{m}{2}}+e^{\frac{1}{16s}}\langle \ln|y_k|\rangle, & \text{if } m=n\\ (-s)^{\frac{m}{2}}+e^{\frac{1}{16s}}\left((-s)^{-\frac12}|y_k|^{-1}\right)^{n-m}, & \text{if } m>n\end{cases}\Bigg)ds\\
= &\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1}|y_k|^2 s+1\right)^{-\nu}\Bigg(\mathbf{1}_{\{s\le -1\}}\begin{cases}(-s)^{-\frac{m}{2}}, & \text{if } m<n\\ (-s)^{-\frac{n}{2}}\left(\langle \ln(-s)\rangle+\langle \ln|y_k|\rangle\right), & \text{if } m=n\\ (-s)^{-\frac{n}{2}}|y_k|^{m-n}, & \text{if } m>n\end{cases}\\
&+\mathbf{1}_{\{s>-1\}}\begin{cases}1, & \text{if } m<n\\ 1+(-s)^{-\frac{n}{2}}e^{\frac{1}{16s}}\langle \ln|y_k|\rangle, & \text{if } m=n\\ 1+(-s)^{-\frac{n}{2}}e^{\frac{1}{16s}}|y_k|^{m-n}, & \text{if } m>n\end{cases}\Bigg)ds:=\mathbf{A}.
\end{aligned}$$

If $\frac{\tau_0-\tau_2^k}{|y_k|^2}\ge -2$, for $|y_k|\ge 2$, we estimate

$$\mathbf{A}\lesssim \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1}|y_k|^2 s+1\right)^{-\nu}\begin{cases}1, & \text{if } m<n\\ \langle \ln|y_k|\rangle, & \text{if } m=n\\ |y_k|^{m-n}, & \text{if } m>n\end{cases}ds\lesssim \begin{cases}1, & \text{if } m<n\\ \langle \ln|y_k|\rangle, & \text{if } m=n\\ |y_k|^{m-n}, & \text{if } m>n\end{cases}$$

where we have used

$$\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1}|y_k|^2 s+1\right)^{-\nu}ds=\tau_2^k|y_k|^{-2}(1-\nu)^{-1}\left[1-(\tau_0(\tau_2^k)^{-1})^{1-\nu}\right]\lesssim 1$$

for $\nu<1$.

If $\frac{\tau_0-\tau_2^k}{|y_k|^2} < -2$, we have

$$\mathbf{A} = \int_{-1}^{0} \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} \begin{cases} 1, & \text{if } m < n \\ 1 + (-s)^{-\frac{n}{2}} e^{\frac{1}{16s}} \langle \ln |y_k| \rangle, & \text{if } m = n\ ds \\ 1 + (-s)^{-\frac{n}{2}} e^{\frac{1}{16s}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$+ \left( \int_{\frac{\tau_0-\tau_2^k}{2|y_k|^2}}^{-1} + \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^{\frac{\tau_0-\tau_2^k}{2|y_k|^2}} \right) \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} \begin{cases} (-s)^{-\frac{m}{2}}, & \text{if } m < n \\ (-s)^{-\frac{n}{2}} \left(\langle \ln(-s) \rangle + \langle \ln |y_k| \rangle\right), & \text{if } m = n\ ds \\ (-s)^{-\frac{n}{2}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$\lesssim (1-\nu)^{-1} \tau_2^k |y_k|^{-2} \left[ 1 - \left(1 - (\tau_2^k)^{-1}|y_k|^2\right)^{1-\nu} \right] \begin{cases} 1, & \text{if } m < n \\ \langle \ln |y_k| \rangle, & \text{if } m = n\ ds \\ |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$+ \int_{\frac{\tau_0-\tau_2^k}{2|y_k|^2}}^{-1} \begin{cases} (-s)^{-\frac{m}{2}}, & \text{if } m < n \\ (-s)^{-\frac{n}{2}} \left(\langle \ln(-s) \rangle + \langle \ln |y_k| \rangle\right), & \text{if } m = n\ ds \\ (-s)^{-\frac{n}{2}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$+ \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^{\frac{\tau_0-\tau_2^k}{2|y_k|^2}} \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} \begin{cases} (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{m}{2}}, & \text{if } m < n \\ (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{n}{2}} \left(\langle \ln(\frac{\tau_2^k-\tau_0}{|y_k|^2}) \rangle + \langle \ln |y_k| \rangle\right), & \text{if } m = n\ ds \\ (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{n}{2}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$\lesssim \begin{cases} 1, & \text{if } m < n \\ \langle \ln |y_k| \rangle, & \text{if } m = n \\ |y_k|^{m-n}, & \text{if } m > n \end{cases} + \tau_2^k |y_k|^{-2} (1-\nu)^{-1} \left[ \left( \frac{\tau_0 (\tau_2^k)^{-1}}{2} + \frac{1}{2} \right)^{1-\nu} - (\tau_0 (\tau_2^k)^{-1})^{1-\nu} \right]$$

$$\times \begin{cases} (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{m}{2}}, & \text{if } m < n \\ (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{n}{2}} \left(\langle \ln(\frac{\tau_2^k-\tau_0}{|y_k|^2}) \rangle + \langle \ln |y_k| \rangle\right), & \text{if } m = n \\ (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{n}{2}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$\lesssim \begin{cases} 1, & \text{if } m < n \\ \langle \ln |y_k| \rangle, & \text{if } m = n \\ |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

where we have used $\tau_2^k \geq 2\tau_0$, $|y_k| \geq 2$, $m > 2$ , $n > 2$, $\nu < 1$. Therefore, we conclude the validity of (A.24). □

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