On MEMS equation with fringing field

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Abstract

We consider the MEMS equation with fringing field

$$-\Delta u = \lambda(1+\delta|\nabla u|^2)(1-u)^{-2} \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

where $\lambda, \delta > 0$ and $\Omega \subset \mathbb{R}^n$ is a smooth and bounded domain. We show that when the fringing field exists (i.e. $\delta > 0$), given any $\mu > 0$, we have uniform upper bound of classical solutions u away from *the rupture level 1* for all $\lambda \geq \mu$. Moreover, there exists $\overline{\lambda}^*_\delta > 0$ such that there are at least two solutions when $\lambda \in (0, \overline{\lambda}^*_\delta)$; a unique solution exists when $\lambda = \overline{\lambda}^*_\delta$; and there is no solution when $\lambda > \overline{\lambda}^*_\delta$. This represents a dramatic change of behavior with respect to the zero fringing field case (i.e. $\delta = 0$) and confirms the simulations in [14, 11].

Key words. MEMS, rupture, fringing field, bifurcation
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1 Introduction

We consider the following elliptic equation

$$(E_\lambda) \qquad -\Delta u = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^2} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where δ, λ are positive constants, and Ω is a bounded smooth domain in $\mathbb{R}^n$ $(n \geq 2)$.

Problem (E_λ) arises in the study of electrostatic Micro-Electromechanical System (MEMS) device. We refer to [5] and the book [13] for detailed discussions on MEMS devices modeling. The parameter λ is called the voltage and the term $\delta|\nabla u|^2$ is called a fringing field (cf. [14, 11]). The eventual singular set $\{x \in \Omega, u(x) = 1\}$ is called *rupture set*. When $\delta = 0$, problem (E_λ) becomes

$$(S_\lambda) \qquad -\Delta u = \frac{\lambda}{(1-u)^2} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Recently there have been many studies on (S_λ). We summarize some of the results here:

- There exists a critical number $\overline{\lambda}^* > 0$ such that for $0 < \lambda < \overline{\lambda}^*$ problem (S_λ) has a minimal stable solution $\overline{u}_\lambda$, while for $\lambda > \overline{\lambda}^*$ there are no solutions to (S_λ) (see [6]).
- Either the solution branch stops at $\overline{\lambda}^*$ and $\lim_{\lambda\to\overline{\lambda}^*} \|\overline{u}_\lambda\|_\infty = 1$ (if Ω is a ball in $\mathbb{R}^n$ with $n \geq 8$ for example); or the solution branch bends back, we could have another critical parameter $0 < \overline{\lambda}_* < \overline{\lambda}^*$ (when Ω is a ball in $\mathbb{R}^n$ with $2 \leq n \leq 7$; or convex domain with two axes of symmetry in $\mathbb{R}^2$) such that the solution branch takes infinitely many turns and converges to a *rupture* solution of $(S_{\overline{\lambda}_*})$ (see [4, 9, 10]).
- For general strictly convex domains with $n \geq 2$, it can be shown that for $\lambda > 0$ small, the minimal solution is the unique one for (S_λ) (see [3, 16]). So we must have a family of solutions (u^k, λ^k) such that $\lim_{k\to\infty} \lambda^k = \overline{\lambda} > 0$ and $\lim_{k\to\infty} \|u^k\|_\infty = 1$.

In this short note, we show that the fringing field dramatically changes the structure of solutions of (E_λ) (see Theorem 5 below): we prove that there exists a critical parameter $\overline{\lambda}_\delta^*$ such that for $\lambda > \overline{\lambda}_\delta^*$ there are no solutions to (E_λ); for $0 < \lambda < \overline{\lambda}_\delta^*$ there are at least *two* solutions; and when $\lambda = \overline{\lambda}_\delta^*$ there exists a unique solution. Furthermore, for any fixed $\mu > 0$, all solutions to (E_λ) with $\lambda \geq \mu$ are below $C_\mu < 1$, i.e. *no ruptures* can occur by using solutions with λ tending to some $\overline{\lambda} > 0$. Our study holds for *any dimension* and confirms the numerical results obtained in [14, 11]. Here all solutions considered are *classical* solutions.

The results of this paper are also true for the generalized MEMS equation

$$(E_{\lambda,p}) \qquad -\Delta u = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^p} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

where $p > 1$.

2 A Key Transformation

To study the structure of solutions for (E_λ), we present a suitable transformation, which leads to considering a semilinear equation. More precisely, we have

Lemma 1 *Let*

$$v = \zeta_\lambda(u) = \int_0^u e^{\frac{\lambda\delta}{1-s}} ds, \quad \forall\, u \in [0,1), \tag{1}$$

then $u : \Omega \to [0,1)$ *is a solution (resp. supersolution, subsolution) of* (E_λ) *if and only if* v *is a solution (resp. supersolution, subsolution) for*

$$(F_\lambda) \qquad -\Delta v = \rho_\lambda(v), \quad v > 0 \quad in\ \Omega, \quad v = 0 \quad on\ \partial\Omega$$

where ρ_λ *is a smooth increasing function from* $\mathbb{R}_+$ *into* $(0, \infty)$*, defined by*

$$\rho_\lambda(v) = \xi_\lambda \circ \zeta_\lambda^{-1} \quad with \quad \xi_\lambda(u) = \frac{\lambda e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2}. \tag{2}$$

Proof. As ξ_λ, ζ_λ are increasing in $[0,1)$ and $\lim_{u\to 1^-} \zeta_\lambda(u) = \infty$, so is ρ_λ in $\mathbb{R}_+$. By direct calculus, $v = \zeta_\lambda(u)$ satisfies

$$-\Delta v = -e^{\frac{\lambda\delta}{1-u}} \Delta u - \frac{\lambda\delta e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2} |\nabla u|^2,$$

all conclusions are straightforward. □

Otherwise, it is not difficult to prove

Theorem 1 *Fix* $\delta > 0$*, there exists* $\overline{\lambda}_\delta^* \in (0, \infty)$ *such that for any* $\lambda < \overline{\lambda}_\delta^*$*, the equation* (E_λ) *has a minimal solution* u_λ*, while for any* $\lambda > \overline{\lambda}_\delta^*$*, no solution exists for* (E_λ)*. Moreover* $\lambda \mapsto u_\lambda$ *is increasing for* $\lambda \in (0, \overline{\lambda}_\delta^*)$*.*

Here the minimal solution means that for any solution u to (E_λ), we have $u_\lambda \le u$ in Ω.

Proof. The result is a direct consequence of the following claims:

(i) If (E_λ) is solvable with $\lambda > 0$, then $(E_{\lambda'})$ is solvable for any $\lambda' \in (0, \lambda)$.

(ii) The equation (E_λ) has no solution for λ sufficiently large.

(iii) For $\lambda > 0$ small enough, we have a solution to (E_λ).

(iv) If (E_λ) is solvable, then there exists a minimal solution u_λ.

If u is a solution to (E_λ), it is clearly a supersolution to $(E_{\lambda'})$, so $v = \zeta_\lambda(u)$ is a supersolution to $(F_{\lambda'})$ by Lemma 1. As $\rho_{\lambda'}(0) = \lambda' e^{\lambda'\delta} > 0$, 0 is always a subsolution. Moreover $\rho_{\lambda'}$ is locally Lipschitz in $\mathbb{R}_+$, so we have a solution to $(F_{\lambda'})$, which yields the claim (i).

The claim (ii) comes from the fact that any solution of (E_λ) is a supersolution for the equation (S_λ), which has no solution for large λ. Let $-\Delta\xi = 1$ in Ω and $\xi = 0$ on $\partial\Omega$, fix $c > 0$ such that $c\|\xi\|_\infty < 1$. We can check that for $c\xi$ is a supersolution of (E_λ) if $\lambda > 0$ is small enough, this leads to the claim (iii).

The last claim is due to the monotonicity of ρ_λ (cf. (4) below), ζ_λ and the monotone iteration for (F_λ) as $-\Delta v^{n+1} = \rho_\lambda(v^n)$ with Dirichlet boundary condition and $v^0 \equiv 0$. □

Remark 1 *Of course, the transformation $v = \zeta_\lambda$ is not really necessary for the above proof. Thanks to the monotonicity of function $g(u) = (1-u)^{-2}$, we can consider directly the following iteration operator $w = Tu$, the unique solution of*

$$-\Delta w = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^2} \quad in\ \Omega, \quad w = 0 \quad on\ \partial\Omega.$$

3 Stability of Minimal Solutions

The minimal solution for (E_λ) will ensure some stability properties, even though the equation (E_λ) does not have a variational structure. First, for the linearized operator of (E_λ):

$$L_\lambda\varphi = -\Delta\varphi - \frac{2\lambda(1+\delta|\nabla u|^2)}{(1-u)^3}\varphi - \frac{2\lambda\delta\nabla u\nabla\varphi}{(1-u)^2},$$

we can define the principal eigenvalue μ_1 of L_λ, associated to the Dirichlet boundary condition (cf. [12]). Then a solution u of (E_λ) is said to be stable if and only if $\mu_1(L_\lambda) \geq 0$. Another idea is to use the transformation $v = \zeta_\lambda(u)$ and the corresponding linearized operator. Following the ideas in [1], we obtain

Theorem 2 *Let $\lambda \in (0, \overline{\lambda}_\delta^*)$, the minimal solution v_λ of (F_λ) satisfies*

$$\int_\Omega |\nabla\varphi|^2 \geq \int_\Omega \rho_\lambda'(v_\lambda)\varphi^2 dx, \quad \forall\, \varphi \in H_0^1(\Omega). \tag{3}$$

Furthermore, v_λ is the unique solution of (F_λ) verifying (3) and u_λ is the unique stable solution of (E_λ).

Moreover, $u = \zeta_\lambda^{-1}(v)$ implies

$$\rho_\lambda'(v) = \left(\xi_\lambda \circ \zeta_\lambda^{-1}\right)'(v) = \frac{\xi_\lambda'}{\zeta_\lambda'} \circ \zeta_\lambda^{-1}(v) = \frac{\lambda^2\delta}{(1-u)^4} + \frac{2\lambda}{(1-u)^3} > 0. \tag{4}$$

As the minimal solution u_λ of (E_λ) is just $\zeta_\lambda^{-1}(v_\lambda)$, we conclude then

Theorem 3 *For $\lambda \in (0, \overline{\lambda}_\delta^*)$, the minimal solution u_λ is the unique solution of (E_λ) verifying the following stability condition:*

$$\int_\Omega |\nabla\varphi|^2 \geq \int_\Omega \left[\frac{\lambda^2\delta}{(1-u_\lambda)^4} + \frac{2\lambda}{(1-u_\lambda)^3}\right]\varphi^2 dx, \quad \forall\, \varphi \in H_0^1(\Omega). \tag{5}$$

4 Bifurcation and Uniform Estimate

Using the equation (F_λ) and the standard bifurcation theory of Rabinowitz (section 3 of [15]), we can say that, a solution curve (λ, v) exists in $\mathbb{R}_+ \times C(\overline{\Omega})$, it goes from $(0,0)$ to the "infinity". By Theorem 1, the only possibility is that $\|v\|_\infty$ tends to ∞. For (F_λ), when $\|v\|_\infty \to \infty$, we show that λ must tend to 0 by the following result.

Theorem 4 *For any $\mu > 0$, there exists a constant $C_\mu > 0$ such that any solution of (F_λ) with $\lambda \geq \mu$ verifies $\|v\|_\infty < C_\mu$. Consequently, there exists $c_\mu \in (0,1)$ such that any solution u of (E_λ) with $\lambda \geq \mu$ verifies $u \leq c_\mu < 1$.*

Proof. In fact, using integration by parts, we can see that

$$v = \zeta_\lambda(u) \sim \frac{(1-u)^2}{\lambda\delta} e^{\frac{\lambda\delta}{1-u}} \quad \text{as } u \to 1^-.$$

Hence for $\mu \in (0, \overline{\lambda}_\delta^*)$ fixed, there exist positive constants C, C' such that

$$Cv(\ln v)^4 \leq \rho_\lambda(v) \leq C'v(\ln v)^4 \quad \forall\ (\lambda, v) \in [\mu, \overline{\lambda}_\delta^*) \times [2, \infty).$$

We have also the uniform estimate $\rho_\lambda(v) \geq Cv + \mu$ for $(\lambda, v) \in [\mu, \overline{\lambda}_\delta^*) \times \mathbb{R}_+$, the proof of Theorem 2.1 in [2] holds and shows that there exists $C_\mu > 0$ such that $\|v\|_\infty < C_\mu < \infty$. The conclusion for u is an immediate consequence. □

An important consequence is just the uniqueness of solution for $(E_{\overline{\lambda}_\delta^*})$. We shall use the problem (F_λ). Now $v^* = \lim_{\lambda \to \overline{\lambda}_\delta^*} v_\lambda$ is a smooth solution for the limit problem $(F_{\overline{\lambda}_\delta^*})$, we claim that $\mu_1\big[-\Delta - \rho'_{\overline{\lambda}_\delta^*}(v^*)\big] = 0$. In fact, the stability of v^* (in the sense of (3)) means that $\mu_1\big[-\Delta - \rho'_{\overline{\lambda}_\delta^*}(v^*)\big] \geq 0$, while the definition of $\overline{\lambda}_\delta^*$ prevents to have $\mu_1\big[-\Delta - \rho'_{\overline{\lambda}_\delta^*}(v^*)\big] > 0$. Hence we get a positive eigenfunction φ_1 satisfying $-\Delta\varphi_1 - \rho'_{\overline{\lambda}_\delta^*}(v^*)\varphi_1 = 0$ in Ω and $\varphi_1 = 0$ on $\partial\Omega$.

If we have a solution v of $(F_{\overline{\lambda}_\delta^*})$ such that $v \neq v^*$, we know that $v \geq v^*$ as $v \geq v_\lambda$ for any $\lambda < \overline{\lambda}_\delta^*$. Let $\phi = v - v^*$, so $-\Delta\phi = \rho_{\overline{\lambda}_\delta^*}(v) - \rho_{\overline{\lambda}_\delta^*}(v^*) \geq 0$ by (4), the strong maximum principle implies that $\phi > 0$ in Ω. Remarking also that $\rho''_\lambda > 0$ in $\mathbb{R}_+$ for any $\lambda > 0$, then $-\Delta\phi - \rho'_{\overline{\lambda}_\delta^*}(v^*)\phi > 0$ in Ω. By multiplying with φ_1 and integrating by parts, we get immediately a contradiction.

Another consequence is that v^* is a bifurcation point for the solution curve, which will continue with $\|v\|_\infty$ tending to ∞ and the associated λ must go to zero. So we get at least two solutions to (F_λ) for any $\lambda \in (0, \overline{\lambda}_\delta^*)$. Coming back to u, we obtain the main theorem of this paper.

Theorem 5 *If a family of solutions $\{u^k\}$ of (E_{λ^k}) verifies $\lim_{k\to\infty} \|u^k\|_\infty = 1$, then $\lim_{k\to\infty} \lambda^k = 0$. Furthermore, $u^* = \lim_{\lambda \to \overline{\lambda}_\delta^*} u_\lambda$ is the unique solution of the limit equation $(E_{\overline{\lambda}_\delta^*})$ while for any $\lambda \in (0, \overline{\lambda}_\delta^*)$, the equation (E_λ) has at least two solutions.*

5 Estimate of $\overline{\lambda}_\delta^*$

Here we compare $\overline{\lambda}_\delta^*$ with $\overline{\lambda}^*$ in lower dimension situation.

Theorem 6 *For $n < 8$ and $\delta > 0$, we have*

$$\frac{\overline{\lambda}^*}{1 + \delta\|\nabla \overline{u}_*\|_\infty^2} \leq \overline{\lambda}_\delta^* \leq \overline{\lambda}^* \tag{6}$$

where $\overline{\lambda}^*$ is the critical value for the problem (S_λ) and $\overline{u}_*$ is the unique solution of $(S_{\overline{\lambda}^*})$.

Proof. As any solution of (E_λ) is supersolution of (S_λ), it is clear that $\overline{\lambda}_\delta^* \leq \overline{\lambda}^*$. On the other hand, when $n < 8$, $\overline{u}_*$ is a smooth function with $\|\overline{u}_*\|_\infty < 1$ (see [4]). Obviously $\overline{u}_*$ is a supersolution for (E_λ) with

$$\lambda = \frac{\overline{\lambda}^*}{1 + \delta\|\nabla \overline{u}_*\|_\infty^2},$$

so we get the lower bound. □

Therefore $\overline{\lambda}_\delta^* = \overline{\lambda}^* + O(\delta)$ in dimension two, this confirms somehow the formal result in [11] (see also another bound of $\overline{\lambda}_\delta^*$ in section 5 of [14]).

6 Remarks and Open Questions

As we have seen in Theorem 5, the introduction of fringing field basically destroys the infinite fold point structure of the basic membrane problem (S_λ) for any smooth domain.

There are still some interesting questions:

- Do we have some *weak* solutions with $\|u\|_\infty = 1$ for (E_λ)? We turn to conjecture that no *weak* solution exists for the fringing field model. In fact, using Sobolev embedding and boot-strap argument, any weak solution of (F_λ) satisfying $\rho_\lambda(v) \in L^1(\Omega)$ is indeed smooth. However, if u is a just *weak* solutions for (E_λ), it is not clear that $v = \rho_\lambda(u)$ is then a weak solution for (F_λ).

- In [11], Lindsay and Ward derived the following asymptotic behavior of $\overline{\lambda}_\delta^*$:

$$\overline{\lambda}_\delta^* = \lambda^* - C\delta + O(\delta^2) \tag{7}$$

  in the case of a unit disk or a slab in $\mathbb{R}^2$, where $C > 0$ is a constant depending on $\overline{u}_*$ of the unit disk or slab without the fringing field. Can we prove rigorously this first order expansion (7)? A key point seems to prove a uniform upper bound for v^* as δ tends to zero.

- In nice domains (disks, convex domains with two axes of symmetry in $\mathbb{R}^2$), it has been shown that for the problem (S_λ), there exists a $\overline{\lambda}_* > 0$ such that the solution branch has infinitely many turns as λ crosses $\overline{\lambda}_*$ (see [9, 10]). On the other hand, in the presence of fringing field, there are at most finitely many turns. What is the asymptotic behavior of the solutions near $\overline{\lambda}_*$ as $\delta \to 0^+$?

- It seems that there are no studies on the corresponding parabolic equation

$$u_t - \Delta u = \frac{\lambda(1 + \delta|\nabla u|^2)}{(1-u)^2}. \tag{8}$$

  What is the effect of the fringing field on (8)? Can we establish results similar to [1, 7, 8]?

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