Free and harmonic trapped spin-1 Bose-Einstein condensates in $\mathbb{R}^3$

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Abstract

We investigate physical states of spin-1 Bose-Einstein condensate in $\mathbb{R}^3$ with mean-field interaction constant c_0 and spin-exchange interaction constant c_1, two conserved quantities, the number of atoms N and the total magnetization M are involved in. Firstly, in the free case, existence and asymptotic behavior of ground states are analyzed according to the relations among c_0, c_1, N and M. Furthermore, we show that the corresponding standing wave is strongly unstable. When the atoms are trapped in a harmonic potential, we prove the existence of ground states and excited states along with some precisely asymptotics. Besides, we get that the set of ground states is stable under the associated Cauchy flow while the excited state corresponds to a strongly unstable standing wave. Our results not only show some characteristics of three-dimensional spin-1 BEC under the effect between the spin-dependent interaction and the external magnetic field, but also support some experimental observations as well as numerical results on spin-1 BEC.

Keywords: Ground State, Excited State, Spin-1 Bose-Einstein condensate, Gross-Pitaevskii system.

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1 Introduction

Bose-Einstein condensate (BEC) is a macroscopic quantum phenomenon that at very low temperature, identical bosonic particles tend to occupy their lowest quantum state and act as a single particle. BEC was first created using ultracold alkali-metal atoms in a single spin state and atoms were spatially confined with magnetic traps [2, 17]. In this situation, the spin direction of the atoms follows the magnetic field and thereby the spin degree of freedom is frozen.

BEC with spin degree of freedom, called spinor BEC, has been achieved experimentally and attracted considerable interest [25, 28, 43, 44]. In this case, ultracold atoms are confined in an optical dipole trap,

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unlike in magnetic trap, the direction of atomic spins can change due to the interparticle interaction so that all hyperfine states are active. The order parameter of a spin-F BEC has 2F+1 components that can vary over space and time, producing a very rich variety of spin textures [42, 44].

In the mean field theory, a physical state of a spin-1 BEC is described by 3 components of complex order parameter $\mathbf{\Phi}(x,t) = \big(\Phi_1(x,t), \Phi_0(x,t), \Phi_{-1}(x,t)\big)$ $(x \in \mathbb{R}^d, d = 1,2,3)$ and the time evolution of the mean field dynamics is governed by [25, 31]

$$ih\partial_t \Phi_j(x,t) = \frac{\delta E}{\delta \Phi_j^*}, \tag{1.1}$$

where h is the Planck constant, Φ_j^* denotes the complex conjugate of Φ_j. Here $E = E_{c_0,c_1}(\mathbf{\Phi})$ is defined by

$$E_{c_0,c_1}(\mathbf{\Phi}) = \int_{\mathbb{R}^d} \left(\frac{h^2}{2m}|\nabla \mathbf{\Phi}|^2 + V|\mathbf{\Phi}|^2 - \frac{c_0}{2}n^2 - \frac{c_1}{2}\big(|\mathbf{\Phi}^* F_x \mathbf{\Phi}|^2 + |\mathbf{\Phi}^* F_y \mathbf{\Phi}|^2 + |\mathbf{\Phi}^* F_z \mathbf{\Phi}|^2\big) \right) dx, \tag{1.2}$$

with m the mass, V the trapping potential, $n = |\Phi_1|^2+|\Phi_0|^2+|\Phi_{-1}|^2$, $\mathbf{\Phi}^*(x,t) = \big(\Phi_1^*(x,t), \Phi_0^*(x,t), \Phi_{-1}^*(x,t)\big)$. c_0 denotes the mean-field interaction and c_1 the spin-exchange interaction. c_0 and c_1 are both tunable in experiments. The mean-field interaction is attractive if $c_0 > 0$ and repulsive if $c_0 < 0$. The BEC system is called ferromagnetic if $c_1 > 0$ and antiferromagnetic if $c_1 < 0$. F_x, F_y, F_z are the Pauli spinor matrices

$$F_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, F_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

From (1.1)-(1.2), in the dimensionless form, the spin-1 BEC can be described by the following coupled Gross-Pitaevskii system

$$\begin{cases} i\partial_t \Phi_1(x,t) = -\Delta \Phi_1 + V(x)\Phi_1 - (c_0+c_1)|\Phi_1|^2\Phi_1 - (c_0-c_1)|\Phi_{-1}|^2\Phi_1 \\ \qquad\qquad - (c_0+c_1)|\Phi_0|^2\Phi_1 - c_1\Phi_{-1}^*|\Phi_0|^2, \\ i\partial_t \Phi_{-1}(x,t) = -\Delta \Phi_{-1} + V(x)\Phi_{-1} - (c_0+c_1)|\Phi_{-1}|^2\Phi_{-1} - (c_0-c_1)|\Phi_1|^2\Phi_{-1} \\ \qquad\qquad - (c_0+c_1)|\Phi_0|^2\Phi_{-1} - c_1\Phi_1^*|\Phi_0|^2, \\ i\partial_t \Phi_0(x,t) = -\Delta \Phi_0 + V(x)\Phi_0 - c_0|\Phi_0|^2\Phi_0 - (c_0+c_1)(|\Phi_1|^2 + |\Phi_{-1}|^2)\Phi_0 \\ \qquad\qquad - 2c_1\Phi_1\Phi_{-1}\Phi_0^* \end{cases} \tag{1.3}$$

under the following two conserved quantities

$$\int_{\mathbb{R}^d} \big(\Phi_1^2 + \Phi_{-1}^2 + \Phi_0^2\big)dx = N, \quad \int_{\mathbb{R}^d} \big(\Phi_1^2 - \Phi_{-1}^2\big)dx = M,$$

where N is the number of atoms and M denotes the total magnetization. $f_z = \Phi_1^2 - \Phi_{-1}^2$ determines the net magnetization of the spin-1 BEC system. When $f_z = 0$ (hence $M = 0$), the spin-1 BEC has no magnetism, otherwise the BEC has magnetism. Usually, a state of $f_z = 0$ will be a polar state or antiferromagnetic state, and a state of $f_z \neq 0$ may be a ferromagnetic state, a broken-axisymmetry state or others, see [13, 31] for details.

Let us recall that standing wave for (1.3) is a solution of the form $(\Phi_1(t,x), \Phi_{-1}(t,x), \Phi_0(t,x))$ with $\Phi_1(t,x) = e^{-i(\mu+\lambda)t}u_1(x)$, $\Phi_{-1}(t,x) = e^{-i(\mu-\lambda)t}u_2(x)$, $\Phi_0(t,x) = e^{-i\mu t}u_0(x)$, where μ, λ are real numbers and $(u_1, u_2, u_0) \in H^1(\mathbb{R}^d, \mathbb{R}^3)$ satisfies the system of elliptic equations

$$\begin{cases} -\Delta u_1 + V(x)u_1 = (\mu+\lambda)u_1 + (c_0+c_1)u_1^3 + (c_0-c_1)u_1u_2^2 + (c_0+c_1)u_0^2u_1 + c_1u_2u_0^2, \\ -\Delta u_2 + V(x)u_2 = (\mu-\lambda)u_2 + (c_0+c_1)u_2^3 + (c_0-c_1)u_1^2u_2 + (c_0+c_1)u_0^2u_2 + c_1u_1u_0^2, \\ -\Delta u_0 + V(x)u_0 = \mu u_0 + c_0u_0^3 + (c_0+c_1)(u_1^2+u_2^2)u_0 + 2c_1u_1u_2u_0, \end{cases} \tag{1.4}$$

along with the constraints

$$\int_{\mathbb{R}^d} (u_1^2 + u_2^2 + u_0^2)dx = N, \quad \int_{\mathbb{R}^d} (u_1^2 - u_2^2)dx = M. \tag{1.5}$$

Under certain conditions, the existence and stability of solutions to (1.4)-(1.5) have been studied by many authors, see [14–16, 26, 32, 33, 36] and the references therein. For the one-dimensional case $d = 1$, Cao, Chern and Wei in [14] proved the existence of ground states for (1.4)-(1.5) with $V(x) = 0, c_0 > 0, c_1 > 0$, by minimizing the corresponding energy under (1.5). The results of [14] have been generalized in [36] to spin-1 BEC with an external Ioffe-Pitchard magnetic field. When $d = 2$ and $c_0 > 0$, motivated by the recent works [20, 21] on two-component attractive BEC, Kong, Wang and Zhao [32] gave the existence and detailed asymptotic behavior of ground states for (1.4)-(1.5) with harmonic trapping potentials. Turning to $d = 3$, ground states for (1.4)-(1.5) were investigated by Lin and Chern in [33], where $V(x)$ is a harmonic potential with Zeeman effect and $c_0 < 0, c_1 < 0$. Furthermore, Hajaiej and Carles in [26] proved the existence and stability of ground states for (1.4)-(1.5) in $\mathbb{R}^d(d = 1, 2, 3)$, where $V(x)$ is a harmonic potential under Ioffe-Pritchard magnetic field and $c_0 < 0$, $c_1 < 0$. For numerical results on ground states and excited states of spin-1 BEC, we refer the reader to [15, 16] and the reference therein.

In $\mathbb{R}^3$, no matter when $V(x) = 0$ or $V(x) \neq 0$, for the case $c_0 > 0$, the study on solutions to (1.4)-(1.5) is absent in the literatures. In this situation, the problem becomes more difficult, one reason is that, the energy functional E_{c_0,c_1} defined by (1.2) is indefinite when $c_0 > 0$, while it is positive definite when $c_0 < 0$ and $c_1 < 0$. Moreover, E_{c_0,c_1} is always unbounded under (1.5) for any $|M| < N$ when $c_0 > 0$ and $c_0 + c_1 > 0$. Exploring physical states of spin-1 BEC relies on a good understanding of the spin-exchange interaction term. Experimentally, the existence of ground state is related to the range of c_0, c_1 and the initial data, which determines the number of atoms N and the total magnetization M.

In addition, the characteristics of BEC are different in one-dimensional (1D), 2D and 3D. For 1D system, the attractive interaction can compensate exactly for the dispersion of a wave packet, leading to an integrable, and highly stable, soliton solution. BECs with attractive interactions have interesting collapse phenomenon in 2D. When the critical number of condensate atoms exceeds the critical number which has been measured precisely and predicted by physicists, the condensate would collapse [8]. This phenomenon is caused by the decrease of the mean-field interaction energy with increasing condensate number, and it can not be balanced by the kinetic energy of atoms eventually, so the condensate tends to collapse upon itself. In 3D, for the homogeneous case, all solutions are predicted to be unstable. For an inhomogeneous condensate, however, if the nonlinearity is relatively weak, the spatial localization provided by an external trap potential can stabilize the condensate against collapse [40].

Based on these facts (two main motives) described above, in this present paper, we investigate the existence, stability and asymptotic behavior of ground states and excited states for (1.4)-(1.5) with $c_0 > 0$ in $\mathbb{R}^3$.

When $V(x)=0$, system (1.4)-(1.5) is free of the electromagnetic trap and becomes the following

$$\begin{cases} -\Delta u_1 = (\mu+\lambda)u_1 + (c_0+c_1)u_1^3 + (c_0-c_1)u_1u_2^2 + (c_0+c_1)u_0^2u_1 + c_1u_2u_0^2, \\ -\Delta u_2 = (\mu-\lambda)u_2 + (c_0+c_1)u_2^3 + (c_0-c_1)u_1^2u_2 + (c_0+c_1)u_0^2u_2 + c_1u_1u_0^2, \\ -\Delta u_0 = \mu u_0 + c_0u_0^3 + (c_0+c_1)(u_1^2+u_2^2)u_0 + 2c_1u_1u_2u_0 \end{cases} \tag{1.6}$$

with

$$\int_{\mathbb{R}^3}(u_1^2+u_2^2+u_0^2)dx = N, \quad \int_{\mathbb{R}^3}(u_1^2-u_2^2)dx = M. \tag{1.7}$$

Solutions to system (1.6)-(1.7) can be found as critical points of $I_0(u_1,u_2,u_0)$ constrained on $\mathcal{M}_0$, where

$$\begin{aligned} I_0(u_1,u_2,u_0) :&= \frac{1}{2}\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx \\ &- \frac{1}{4}\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx - c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx \\ &- \frac{1}{2}\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx, \end{aligned}$$

and

$$\mathcal{M}_0 := \Big\{(u_1,u_2,u_0)\in H^1(\mathbb{R}^3,\mathbb{R}^3)\big|\int_{\mathbb{R}^3}(u_1^2+u_2^2+u_0^2)dx = N, \ \int_{\mathbb{R}^3}(u_1^2-u_2^2)dx = M\Big\}.$$

Before introducing the main results, we recall some definitions (see also [5]).

Definition 1.1. *(i) We say that (v_1,v_2,v_0) is a ground state of (1.6)-(1.7) if*

$$I_0'|_{\mathcal{M}_0}(v_1,v_2,v_0)=0$$

and

$$I_0(v_1,v_2,v_0) = \inf\big\{I_0(u_1,u_2,u_0) \ s.t. \ I_0'|_{\mathcal{M}_0}(u_1,u_2,u_0)=0 \ and \ (u_1,u_2,u_0)\in\mathcal{M}_0\big\}.$$

(ii) We say that (w_1,w_2,w_0) is an excited state of (1.6)-(1.7) if

$$I_0'|_{\mathcal{M}_0}(w_1,w_2,w_0)=0$$

and

$$I_0(w_1,w_2,w_0) > \inf\big\{I_0(u_1,u_2,u_0) \ s.t. \ I_0'|_{\mathcal{M}_0}(u_1,u_2,u_0)=0 \ and \ (u_1,u_2,u_0)\in\mathcal{M}_0\big\}.$$

We emphasize that this definition is meaningful even if the energy I_0 is unbounded from below on $\mathcal{M}_0$. In addition, variational problems with the energy restricted on the manifold $\mathcal{M}_0$ is particularly appropriate for the study of the stability properties of the ground states, as all the energy, the number of atoms N and the total magnetization M are conserved along the flow generated by (1.3).

Definition 1.2. *(i) We say that the set $\boldsymbol{G}$ is orbitally stable if $\boldsymbol{G} \neq \emptyset$ and for any $\epsilon > 0$, there exists a $\delta > 0$ such that, provided that an initial datum $\boldsymbol{\Phi}(0) = (\Phi_1(0), \Phi_2(0), \Phi_0(0))$ for (1.3) satisfies*

$$\inf_{(u_1,u_2,u_0)\in \boldsymbol{G}} \left\|(u_1, u_2, u_0) - \big(\Phi_1(0), \Phi_2(0), \Phi_0(0)\big)\right\|_{H^1(\mathbb{R}^3,\mathbb{C}^3)} < \delta,$$

then (Φ_1, Φ_2, Φ_0) is globally defined and

$$\inf_{(u_1,u_2,u_0)\in \boldsymbol{G}} \left\|(u_1, u_2, u_0) - \big(\Phi_1(t), \Phi_2(t), \Phi_0(t)\big)\right\|_{H^1(\mathbb{R}^3,\mathbb{C}^3)} < \epsilon, \ \forall\ t > 0,$$

where $\big(\Phi_1(t), \Phi_2(t), \Phi_0(t)\big)$ is the solution to (1.3) corresponding to the initial condition $\big(\Phi_1(0), \Phi_2(0), \Phi_0(0)\big)$.
(ii) A standing wave $\big(e^{-i(\mu+\lambda)t}u_1(x), e^{-i(\mu-\lambda)t}u_2(x), e^{-i\mu t}u_0(x)\big)$ is said to be strongly unstable if for any $\epsilon > 0$ there exists

$$\big(\Phi_1(0), \Phi_2(0), \Phi_0(0)\big) \in H^1(\mathbb{R}^3, \mathbb{C}^3),$$

such that

$$\left\|(u_1, u_2, u_0) - \big(\Phi_1(0), \Phi_2(0), \Phi_0(0)\big)\right\|_{H^1(\mathbb{R}^3,\mathbb{C}^3)} < \epsilon,$$

and $\big(\Phi_1(t), \Phi_2(t), \Phi_0(t)\big)$ blows-up in finite time, namely $T_{\max} < +\infty$, where $T_{max} > 0$ is the positive maximal time of existence.

Set

$$(H1): \quad c_1 > 0,\ 0 \leq M < N \leq (2 + \sqrt{5})M,$$

$$(H2): \quad c_1 > 0,\ N > (2 + \sqrt{5})M,\ \frac{c_1}{c_0} < \frac{4N^2}{\left[(3N + M)^{\frac{3}{2}} - \sqrt{N + M}(5N + M)\right]\sqrt{N + M}} - 1,$$

$$(H3): \quad c_1 < 0,\ 0 \leq M < N,\ \frac{c_1}{c_0} > \frac{\left[\left(\frac{3N+M}{2}\right)^{\frac{3}{2}} - N^{\frac{3}{2}}\right]4\sqrt{N}}{(N + M)(5N + M)} - 1.$$

We can now state our first result regarding existence, stability and asymptotic behavior for ground states of (1.6)-(1.7).

Theorem 1. *Let $c_0 > 0$ and one of $(H1), (H2), (H3)$ hold. Then*
(i) there exists a ground state $(\bar{u}_1, \bar{u}_2, \bar{u}_0)$ of (1.6) on $\mathcal{M}_0$ with some $\mu_0, \lambda_0 \in \mathbb{R}$;
(ii)

$$I_0(\bar{u}_1, \bar{u}_2, \bar{u}_0) \to +\infty,\ \int_{\mathbb{R}^3} \big(|\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 + |\nabla \bar{u}_0|^2\big)dx \to +\infty, \quad \textit{as } N \to 0;$$

$$I_0(\bar{u}_1, \bar{u}_2, \bar{u}_0) \to 0,\ \int_{\mathbb{R}^3} \big(|\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 + |\nabla \bar{u}_0|^2\big)dx \to 0, \quad \textit{as } N \to +\infty;$$

(iii) the corresponding standing wave $\big(e^{-i(\mu_0+\lambda_0)t}\bar{u}_1(x),\ e^{-i(\mu_0-\lambda_0)t}\bar{u}_2(x),\ e^{-i\mu_0 t}\bar{u}_0(x)\big)$ is strongly unstable.

Remark 1.1. *To the best of our knowledge, this is the first theoretical result dealing with the existence, asymptotic behavior and stability/instability of standing waves for ground states of three-dimensional spin-1 BEC in both ferromagnetic and antiferromagnetic cases. Our main results in this aspect support the experimental observation that existence of ground state of spin-1 BEC is related to the range of* c_0, c_1 *and the initial data, which determines the number of atoms* N *and the total magnetization* M*, see [11]. Theorem 1 generalizes the main results of [6] from ground states of BEC to spin-1 BEC. Furthermore, these results generalize the work of [32] and [14] from ground states of spin-1 BEC in* $\mathbb{R}^2$ *and* $\mathbb{R}$ *to ground states of spin-1 BEC in* $\mathbb{R}^3$*, respectively. Theorem 1 together with the main results in [14, 32] indicate that the characteristics of spin-1 BEC are different in one-dimensional (1D), 2D and 3D. In 3D, for the homogeneous case, the authors in [32] and [40] predicted that all states of BEC are unstable. We give a complete positive answer from the aspect of ground states for spin-1 BEC.*

Remark 1.2. $(H1)$, $(H2)$ *or* $(H3)$ *in Theorem 1 ensures that the corresponding limiting problem*

$$\begin{cases} -\Delta\bar{u}_1 = (\mu+\lambda)\bar{u}_1 + (c_0+c_1)\bar{u}_1^3 + (c_0+c_1)\bar{u}_0^2\bar{u}_1, \\ -\Delta\bar{u}_0 = \mu\bar{u}_0 + c_0\bar{u}_0^3 + (c_0+c_1)\bar{u}_1^2\bar{u}_0, \\ \displaystyle\int_{\mathbb{R}^3} |\bar{u}_1|^2 dx = \frac{N+M}{2}, \quad \int_{\mathbb{R}^3} |\bar{u}_0|^2 dx = N \end{cases} \tag{1.8}$$

has a positive ground state. Indeed, the authors in [6] showed that when $c^* > 0$ *and* $c_0 + c_1 > c^*$, (1.8) *has a positive ground state solution, where*

$$c^* = \frac{(\frac{3N+M}{2})^{\frac{3}{2}}}{N(N+M)\sqrt{\min\left\{\frac{1}{\frac{M+N}{2}(c_0+c_1)^2}, \frac{1}{(N+M)c_0^2}\right\}}} - \frac{(N+M)(c_0+c_1)}{4N} - \frac{Nc_0}{N+M}. \tag{1.9}$$

By direct calculation, we can show that $c^* > 0$ *and* $c_0 + c_1 > c^*$ *are equivalent to*

$$\frac{\left[(\frac{3N+M}{2})^{\frac{3}{2}} - \frac{\sqrt{N+M}(5N+M)}{2\sqrt{2}}\right]\sqrt{\frac{N+M}{2}}}{N^2} < \frac{c_0}{c_0+c_1} < \frac{(5N+M)(N+M)}{4\sqrt{N}\left[(\frac{3N+M}{2})^{\frac{3}{2}} - N^{\frac{3}{2}}\right]},$$

which degenerates to

$$0 < \frac{c_0}{c_0+c_1} < \frac{(5N+M)(N+M)}{4\sqrt{N}\left[(\frac{3N+M}{2})^{\frac{3}{2}} - N^{\frac{3}{2}}\right]}, \quad \textit{if} \;\; \sqrt{5}-2 < \frac{M}{N} < 1.$$

All of these observations give the classification of conditions $(H1)-(H3)$ *with* $\frac{\left[(\frac{3N+M}{2})^{\frac{3}{2}} - N^{\frac{3}{2}}\right]4\sqrt{N}}{(N+M)(5N+M)} - 1 < 0$ *in* $(H3)$ *and* $\frac{4N^2}{\left[(3N+M)^{\frac{3}{2}} - \sqrt{N+M}(5N+M)\right]\sqrt{N+M}} - 1 > 0$ *in* $(H2)$*, see Proposition 2.1 for details. Based on this existence result on* (1.8)*, we can prove that the ground state energy of* (1.6) *is strictly less than that of its corresponding limiting problems. Then the compactness of the related Palais-Smale sequence follows and the ground state obtained is fully nontrivial.*

To prove Theorem 1, we adapt a minimax method and the classical Berestycki-Cazenave argument [3]. Comparing with discussing BEC system without the spin-exchange interaction term, see [6,9,20,21,38] and the references therein, we however need to overcome some extra difficulties. In order to employ the energy estimate to derive the compactness of the Palais-Smale sequences, some delicate estimates and new ideas are also needed to handle with the spin-exchange interaction term in the corresponding energy. Actually, one needs to choose some test functions skillfully, so that the ground state energy of (1.6) is strictly less than that of its corresponding limiting problems, see Lemma 3.7 for details. Indeed, it is a test to find the constraints corresponding to the limiting problems. In addition, due to the uncertainty of the sign of the spin-exchange interaction term in the corresponding energy, we need some refined calculations to show that the combining of the attractive mean-field interaction term and the spin-exchange interaction term (ferromagnetic or antiferromagnetic) in the energy functional is non-negative. This is significant in analyzing the structure of the corresponding energy functional under constraint (1.7). Finally, to obtain the L^2 convergence of the Palais-Smale sequence, it is then necessary to study carefully the signs of the multipliers (i.e $\mu+\lambda,\ \mu-\lambda,\ \mu$) in various cases and ensure that all multipliers are negative, for which we need to make full use of the fact that the ground state energy of (1.6) is strictly less than that of its corresponding limiting problems. Some of these ideas originate from a recent work [19] by Forcella, Yang, Yang and the second author in this present paper.

Next, we consider the harmonic trapped case, where a confining electromagnetic potential $V(x) = |x|^2$ is added in the system. The energy functional

$$I(u_1, u_2, u_0) := I_0(u_1, u_2, u_0) + \frac{1}{2}\int_{\mathbb{R}^3} |x|^2\big(u_1^2 + u_2^2 + u_0^2\big)dx$$

corresponding to problem (1.4) restricted to

$$\mathcal{M} := \Big\{(u_1, u_2, u_0) \in \Lambda \big| \int_{\mathbb{R}^3} \big(u_1^2 + u_2^2 + u_0^2\big)dx = N,\ \int_{\mathbb{R}^3} \big(u_1^2 - u_2^2\big)dx = M\Big\}$$

has a totally different structure. Here the working space

$$\Lambda := \Big\{(u_1, u_2, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3) \big| \int_{\mathbb{R}^3} |x|^2\big(u_1^2 + u_2^2 + u_0^2\big)dx < +\infty\Big\}, \tag{1.10}$$

is a Hilbert space equipped with the norm

$$\|(u_1, u_2, u_0)\|_\Lambda := \left(\int_{\mathbb{R}^3} \big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big)dx + \int_{\mathbb{R}^3} \big(1 + |x|^2\big)\big(u_1^2 + u_2^2 + u_0^2\big)dx\right)^{\frac{1}{2}}.$$

Motivated by [4], in order to get ground states, for any $r > 0$ and $N \le \frac{r}{3}$, we first consider the following local minimization problem

$$m_N^r := \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(r)} I(u_1, u_2, u_0),$$

where

$$B(r) := \Big\{(u_1, u_2, u_0) \in \Lambda \big| \|(u_1, u_2, u_0)\|_{\dot\Lambda}^2 \le r\Big\}$$

and

$$\|(u_1, u_2, u_0)\|_{\dot\Lambda}^2 := \int_{\mathbb{R}^3} \Big(\big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big) + |x|^2\big(u_1^2 + u_2^2 + u_0^2\big)\Big)dx.$$

Our main result in this aspect is the following

Theorem 2. *Let $0 \le M < N$, if $c_0 > 0,\ c_0 + c_1 > 0$, then*

(i) for any $r > 0$, m_N^r has a minimizer if $N \le \frac{r}{3}$;

(ii) for any $r > 0$, there exists $N_0 = N_0(r) < \frac{r}{3}$, such that for any $0 < N \le N_0$, each minimizer of m_N^r is a critical point of $I(u_1, u_2, u_0)$ restricted to $\mathcal{M}$. Moreover, there exists $N_1 \in (0, N_0]$ small enough, such that for any $0 < N < N_1$, the minimizer of m_N^r is a ground state of (1.4) on $\mathcal{M}$;

(iii) for $r > 0$ and $0 < N \le N_0$, denote

$$\mathcal{M}_N^r := \Big\{(u_1, u_2, u_0) \in \mathcal{M} \cap B(r) \big| I(u_1, u_2, u_0) = m_N^r\Big\},$$

then for any $(u_{1N}, u_{2N}, u_{0N}) \in \mathcal{M}_N^r$, there holds

$$\frac{m_N^r}{N} \to \frac{3}{2}, \qquad \frac{\|(u_{1N}, u_{2N}, u_{0N})\|_\Lambda^2}{N} \to 3, \quad \text{as } N \to 0^+.$$

Further,

$$\|(u_{1N}, u_{2N}, u_{0N}) - (l_{10}\Psi_0, l_{20}\Psi_0, l_{00}\Psi_0)\|_\Lambda^2 = O(N^2),$$

where Ψ_0 is the unique normalized positive eigenvector of the harmonic oscillator $-\Delta + |x|^2$ and

$$l_{i0} = \int_{\mathbb{R}^3} u_{iN}\Psi_0 dx, \text{ for } i = 1, 2, 0;$$

(iv) the set $\mathcal{M}_N^r$ is stable under the flow associated with problem (1.3).

Based on the ground states obtained in Theorem 2, we are able to get an excited state.

Theorem 3. *Suppose $c_0 > 0,\ c_0 + c_1 > 0$,*

(i) for any $r > 0$ and $0 \le M < N \le N_0$, there exists an excited state $(\hat{u}_1, \hat{u}_2, \hat{u}_0)$ of (1.4) on $\mathcal{M}$, with some $\hat{\mu}, \hat{\lambda} \in \mathbb{R}$;

(ii) the corresponding standing wave $\Big(e^{-i(\hat{\mu}+\hat{\lambda})t}\hat{u}_1,\ e^{-i(\hat{\mu}-\hat{\lambda})t}\hat{u}_2,\ e^{-i\hat{\mu}t}\hat{u}_0\Big)$ is strongly unstable.

Remark 1.3. *The signs of these ground and excited state solutions obtained in Theorem 1-Theorem 3 are positive if $c_1 > 0$, while $(u_1, u_2, u_0) \in (\mathbb{R}^-, \mathbb{R}^+, \mathbb{R}^+)$ or $(u_1, u_2, u_0) \in (\mathbb{R}^+, \mathbb{R}^-, \mathbb{R}^+)$ when $c_1 < 0$. Indeed, for any $(u_1, u_2, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3)$, the spin-exchange interaction term and energy functional satisfy*

$$-\int_{\mathbb{R}^3} |u_1||u_2||u_0|^2 dx \le \int_{\mathbb{R}^3} u_1 u_2 u_0^2 dx \le \int_{\mathbb{R}^3} |u_1||u_2||u_0|^2 dx$$

and

$$I_0(u_1, u_2, u_0) \ge \begin{cases} I_0(|u_1|, |u_2|, |u_0|), & c_1 > 0, \\ I_0(-|u_1|, |u_2|, |u_0|) = I_0(|u_1|, -|u_2|, |u_0|), & c_1 < 0, \end{cases}$$

then for any approximate solution sequence $\{(u_{1n}, u_{2n}, u_{0n})\}$, we have $\{(|u_{1n}|, |u_{2n}|, |u_{0n}|)\}$ is also an approximate solution sequence when $c_1 > 0$ and $\{(-|u_{1n}|, |u_{2n}|, |u_{0n}|)\}$ or $\{(|u_{1n}|, -|u_{2n}|, |u_{0n}|)\}$ is also an approximate solution sequence when $c_1 < 0$.

Remark 1.4. *Theorem 3 together with Theorem 2 yield the multiplicity of standing waves for problem (1.3) and correspond to the numerical results established in [15]. This indicates that the introduction of an external harmonic trapping potential enriches the solutions set of (1.3). Theorem 2 (iii) shows that the standing waves of problem (1.3) associated to the set $\mathcal{M}_N^r$ behave like the first eigenvector of the* *harmonic oscillator* $-\Delta+|x|^2$ *for small* N*. This exact localization also benefits from the introduction of the harmonic trapping potential in problem* (1.3)*. Item (iv) of Theorem 2 indicates that the introduction of the trapping potential* $|x|^2$ *leads to a stabilization of system* (1.3)*, as problem* (1.3) *without the term* $|x|^2u$ *has at least one unstable standing wave and whose all solutions are predicted to blow up in finite time, see [32, 40]. In this sense, the influence of the trapping potential term* $|x|^2u$ *on system* (1.3) *is important. In 3D, the authors in [40] described that for an inhomogeneous condensate, however, if the nonlinearity is relatively weak, the spatial localization provided by an external trap potential can stabilize the condensate against collapse, our results are consistent with this phenomenon.*

Remark 1.5. *Under the conditions of (ii) in Theorem 2, we show that*

$$\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{r}{4})} I(u_1,u_2,u_0) < \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap\big(B(r)\setminus B(\frac{r}{2})\big)} I(u_1,u_2,u_0), \tag{1.11}$$

see Lemma 4.6. This inequality is crucial for us to prove that each minimizer of m_N^r *is a critical point of* $I(u_1,u_2,u_0)$ *restricted to* $\mathcal{M}$*. This is motivated by [4], where the following similar strict inequality was obtained,*

$$\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{Nr}{2})} I(u_1,u_2,u_0) < \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap\big(B(r)\setminus B(Nr)\big)} I(u_1,u_2,u_0). \tag{1.12}$$

However, a necessary condition $r\geq 6$ *must be imposed in* (1.12) *to ensure that* $\mathcal{M}\cap B(\frac{Nr}{2})\neq\emptyset$*. Indeed, if* $(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{Nr}{2})$*, then by* (4.3)*,* $3N\leq\|(u_1,u_2,u_0)\|_\Lambda^2\leq\frac{Nr}{2}$*, which implies* $r\geq 6$*. That is, the proof in [4] does not apply for all* $r>0$*. This observation was first pointed out by Luo, see Remark 1.5 of [37]. Instead of* (1.12)*, the author in [37] proved an inequality similar to* (1.11) *by assuming that* $\mathcal{M}\cap\big(B(r)\setminus B(\frac{r}{2})\big)\neq\emptyset$*. However, this condition may not obviously hold for small* $N>0$*. We give a detailed proof in Lemma 4.6. See [18] for a similar description.*

Remark 1.6. *In addition, the condition* $0\leq M<N$ *in Theorem 2 and Theorem 3 can be broadened to* $0\leq|M|<N$*, while it fails in Theorem 1. Indeed, in the proof of Theorem 1, condition* $M\geq 0$ *ensures that the Lagrange multipliers are all negative, which is vital in recovering the* L^2 *strong convergence of the related Palais-Smale sequences. However, when a harmonic trapping potential is involved, the compactness issue become simple.*

Remark 1.7. *The results in Theorem 2 and Theorem 3 can be extended with slightly modifications to the case where the trapping potential* $|x|^2$ *is replaced by a more general harmonic trapping potential* $V(x)=a_1|x_1|^2+a_2|x_2|^2+a_3|x_3|^2, a_i>0 (i=1,2,3)$*, which is called the anisotropy factors of the trap in quantum physics and trapping frequency of the ith-direction in mathematics, see [10, 23, 24].*

Notations. In the paper, we use the following notations. $L^p=L^p(\mathbb{R}^3)$ with norm $\|f\|_{L^p(\mathbb{R}^3)}=\|f\|_{L^p}$, $H^1(\mathbb{R}^3)$ is the usual Sobolev space, with $H^1(\mathbb{R}^3,\mathbb{C}^3)$ or $H^1(\mathbb{R}^3,\mathbb{R}^3)$ for vector valued functions, $H^1(\mathbb{R}^3,\mathbb{C})$ or $H^1(\mathbb{R}^3)=H^1(\mathbb{R}^3,\mathbb{R})$ for scalar functions. $H^{-1}(\mathbb{R}^3)$ denotes the dual space of $H^1(\mathbb{R}^3)$. Re and Im are for the real and imaginary part of a complex number, and $\overline{z}$ stands for the complex conjugate of z.

The paper is organized as follows. In Section 2, we introduce some preliminary results. In Section 3, we focus on the free case and prove Theorem 1. In Section 4, we deal with the harmonic trapped case and prove Theorem 2. Finally, Theorem 3 will be proved in Section 5.

2 Preliminaries

In this section, we give some preliminaries which are useful for the rest of the paper. First, we give some compact embedding results.

Lemma 2.1. *(Compact embedding)*
(1). *(Willem [45]) Denote the radially symmetric subspace of* $H^1(\mathbb{R}^3,\mathbb{R}^3)$ *as*

$$H_r := \Big\{(u_1,u_2,u_0)\in H^1(\mathbb{R}^3,\mathbb{R}^3)\Big|\ u_i(x)=u_i(|x|),\ \forall\ x\in\mathbb{R}^3,\ i=1,2,0\Big\},$$

then $H_r \hookrightarrow\hookrightarrow L^p(\mathbb{R}^3)\times L^p(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ *for any* $p\in(2,6)$.
(2). *(Pankov [39]) The embedding* $\Lambda\hookrightarrow L^p(\mathbb{R}^3)\times L^p(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ *is compact for any* $p\in[2,6)$. *Here* Λ *is defined at (1.10).*

We now recall the scale field equation

$$\begin{cases} -\Delta u+u=u^3, \quad x\in\mathbb{R}^3,\\ u>0 \quad \text{and} \quad u\in H^1(\mathbb{R}^3). \end{cases} \tag{2.1}$$

From [30], there exists a unique positive radial solution $Q(x)\in H^1(\mathbb{R}^3)$ for (2.1). Further, $Q(x)$ satisfies the Pohozaev identity $\int_{\mathbb{R}^3}|\nabla Q|^2dx+3\int_{\mathbb{R}^3}Q^2dx=\frac{3}{2}\int_{\mathbb{R}^3}Q^4dx$. Then, we get

$$\int_{\mathbb{R}^3}|\nabla Q|^2dx=3\int_{\mathbb{R}^3}Q^2dx,\qquad \int_{\mathbb{R}^3}Q^4dx=4\int_{\mathbb{R}^3}Q^2dx. \tag{2.2}$$

In the following, we always denote

$$C_0:=\int_{\mathbb{R}^3}Q^2dx. \tag{2.3}$$

For any fixed $c_0+c_1>0$, we consider the following equation

$$\begin{cases} -\Delta u=(\mu+\lambda)u+(c_0+c_1)u^3,\\ u\in H^1(\mathbb{R}^3) \quad \text{and} \quad \displaystyle\int_{\mathbb{R}^3}|u|^2dx=\frac{N+M}{2}. \end{cases} \tag{2.4}$$

By scaling, we can see that there exists a unique positive solution $w_1(x)$ to (2.4) with

$$w_1(x)=\Big(\frac{-(\lambda+\mu)}{c_0+c_1}\Big)^{\frac{1}{2}}Q\Big(\big(-(\lambda+\mu)\big)^{\frac{1}{2}}x\Big),$$

where $\lambda+\mu=-\Big(\frac{C_0}{(c_0+c_1)\frac{N+M}{2}}\Big)^2$. Furthermore, w_1 satisfies the Pohozaev identity

$$\int_{\mathbb{R}^3}|\nabla w_1|^2dx=\frac{3}{4}(c_0+c_1)\int_{\mathbb{R}^3}w_1^4dx. \tag{2.5}$$

It follows that the corresponding energy is

$$\begin{aligned} m_1\left(\sqrt{\frac{N+M}{2}},c_0+c_1\right) := I_1(w_1) :&= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla w_1|^2dx-\frac{1}{4}(c_0+c_1)\int_{\mathbb{R}^3}w_1^4dx\\ &=\frac{1}{8}(c_0+c_1)\int_{\mathbb{R}^3}w_1^4dx>0. \end{aligned} \tag{2.6}$$

Similarly, for any fixed $c_0 > 0$, there exists a unique positive solution $w_0(x)$ for the following equation

$$\begin{cases} -\Delta u = \mu u + c_0 u^3, \\ u \in H^1(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3} |u|^2 dx = N. \end{cases} \tag{2.7}$$

Moreover, the corresponding energy is

$$m_2(\sqrt{N}, c_0) := I_2(w_0) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w_0|^2 dx - \frac{1}{4} c_0 \int_{\mathbb{R}^3} w_0^4 dx = \frac{1}{8} c_0 \int_{\mathbb{R}^3} w_0^4 dx > 0. \tag{2.8}$$

For $\tau \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^3)$, we introduce an operator $\tau \star u$ as $(\tau \star u)(x) = e^{\frac{3\tau}{2}} u(e^\tau x)$. Then it is easy to get the following lemma, so we omit the details of the proof.

Lemma 2.2. *There holds*

$$\sup_{\tau \in \mathbb{R}} I_1(\tau \star w_1) = I_1(w_1) = m_1. \tag{2.9}$$

For fixed $c_0 > 0$, we consider the following system

$$\begin{cases} -\Delta \bar{u}_1 = (\mu + \lambda)\bar{u}_1 + (c_0 + c_1)\bar{u}_1^3 + (c_0 + c_1)\bar{u}_0^2 \bar{u}_1, \\ -\Delta \bar{u}_0 = \mu \bar{u}_0 + c_0 \bar{u}_0^3 + (c_0 + c_1)\bar{u}_1^2 \bar{u}_0, \\ \int_{\mathbb{R}^3} |\bar{u}_1|^2 dx = \frac{N+M}{2}, \quad \int_{\mathbb{R}^3} |\bar{u}_0|^2 dx = N. \end{cases} \tag{2.10}$$

Proposition 2.1. *Let $c_0 > 0$ and one of $(H1) - (H3)$ holds, then (2.10) has a positive ground state solution.*

Proof. From [6], we know that when $c^* > 0$ and $c_0 + c_1 > c^*$, (2.10) has a positive ground state solution, where

$$c^* = \frac{\left(\frac{3N+M}{2}\right)^{\frac{3}{2}}}{N(N+M)\sqrt{\min\left\{\frac{1}{\frac{M+N}{2}(c_0+c_1)^2}, \frac{1}{(N+M)c_0^2}\right\}}} - \frac{(N+M)(c_0+c_1)}{4N} - \frac{Nc_0}{N+M}. \tag{2.11}$$

By direct calculation, we can show that $c^* > 0$ and $c_0 + c_1 > c^*$ are equivalent to the following conditions

$$\frac{\left[\left(\frac{3N+M}{2}\right)^{\frac{3}{2}} - \frac{\sqrt{N+M}(5N+M)}{2\sqrt{2}}\right]\sqrt{\frac{N+M}{2}}}{N^2} < \frac{c_0}{c_0+c_1} < \min\left\{\frac{\left[\left(\frac{3N+M}{2}\right)^{\frac{3}{2}} - \left(\frac{N+M}{2}\right)^{\frac{3}{2}}\right]\sqrt{\frac{N+M}{2}}}{N^2}, \frac{\sqrt{N+M}}{\sqrt{2N}}\right\}$$

or

$$\frac{4\sqrt{N}\left[\left(\frac{3N+M}{2}\right)^{\frac{3}{2}} - N^{\frac{3}{2}}\right]}{(5N+M)(N+M)} < \frac{c_0+c_1}{c_0} < \min\left\{\frac{\sqrt{N}\left[\left(\frac{3N+M}{2}\right)^{\frac{3}{2}} - N^{\frac{3}{2}}\right]}{\left(\frac{N+M}{2}\right)^2}, \frac{\sqrt{2N}}{\sqrt{N+M}}\right\}.$$

Let $M = kN,\ k \in (0,1)$, by direct analysis, we can deduce that for any $k \in (0,1)$,

$$\left[\left(\frac{3+k}{2}\right)^{\frac{3}{2}} - \frac{\sqrt{k+1}(5+k)}{2\sqrt{2}}\right]\sqrt{\frac{1+k}{2}} < \sqrt{\frac{1+k}{2}} < \left[\left(\frac{3+k}{2}\right)^{\frac{3}{2}} - \left(\frac{1+k}{2}\right)^{\frac{3}{2}}\right]\sqrt{\frac{1+k}{2}},$$

and

$$\frac{4\left[(\frac{3+k}{2})^{\frac{3}{2}}-1\right]}{(1+k)(5+k)}<\sqrt{\frac{2}{k+1}}<\frac{(\frac{3+k}{2})^{\frac{3}{2}}-1}{(\frac{1+k}{2})^{2}}.$$

Therefore,

$$\left[\Big(\frac{3+k}{2}\Big)^{\frac{3}{2}}-\frac{\sqrt{k+1}(5+k)}{2\sqrt{2}}\right]\sqrt{\frac{1+k}{2}}<\frac{c_0}{c_0+c_1}\leq\sqrt{\frac{1+k}{2}},$$

or

$$\sqrt{\frac{1+k}{2}}<\frac{c_0}{c_0+c_1}<\frac{(1+k)(5+k)}{4\left[(\frac{3+k}{2})^{\frac{3}{2}}-1\right]}.$$

Thus,

$$\left[\Big(\frac{3+k}{2}\Big)^{\frac{3}{2}}-\frac{\sqrt{k+1}(5+k)}{2\sqrt{2}}\right]\sqrt{\frac{1+k}{2}}<\frac{c_0}{c_0+c_1}<\frac{(1+k)(5+k)}{4\left[(\frac{3+k}{2})^{\frac{3}{2}}-1\right]}.$$

Since $(\frac{3+k}{2})^{\frac{3}{2}}-\frac{\sqrt{k+1}(5+k)}{2\sqrt{2}}\geq 0$ for any $k\in(0,\sqrt{5}-2]$ and $(\frac{3+k}{2})^{\frac{3}{2}}-\frac{\sqrt{k+1}(5+k)}{2\sqrt{2}}<0$ for any $k\in(\sqrt{5}-2,1)$, then

$$0<\frac{c_0}{c_0+c_1}<\frac{(1+k)(5+k)}{4\left[(\frac{3+k}{2})^{\frac{3}{2}}-1\right]},\quad\text{if}\quad k\in(\sqrt{5}-2,1),$$

and

$$\left[\Big(\frac{3+k}{2}\Big)^{\frac{3}{2}}-\frac{\sqrt{k+1}(5+k)}{2\sqrt{2}}\right]\sqrt{\frac{1+k}{2}}<\frac{c_0}{c_0+c_1}<\frac{(1+k)(5+k)}{4\left[(\frac{3+k}{2})^{\frac{3}{2}}-1\right]},\quad\text{if}\quad k\in(0,\sqrt{5}-2].$$

Therefore, for any $0\leq M<N$, we have

$$\frac{\left[(\frac{3N+M}{2})^{\frac{3}{2}}-\frac{\sqrt{N+M}(5N+M)}{2\sqrt{2}}\right]\sqrt{\frac{N+M}{2}}}{N^2}<\frac{\sqrt{N+M}}{\sqrt{2N}}<\frac{\left[(\frac{3N+M}{2})^{\frac{3}{2}}-(\frac{N+M}{2})^{\frac{3}{2}}\right]\sqrt{\frac{N+M}{2}}}{N^2},$$

$$\frac{4\sqrt{N}\big[(\frac{3N+M}{2})^{\frac{3}{2}}-N^{\frac{3}{2}}\big]}{(5N+M)(N+M)}<\frac{\sqrt{2N}}{\sqrt{N+M}}<\frac{\sqrt{N}\big[(\frac{3N+M}{2})^{\frac{3}{2}}-N^{\frac{3}{2}}\big]}{(\frac{N+M}{2})^2},$$

and

$$\frac{\left[(\frac{3N+M}{2})^{\frac{3}{2}}-\frac{\sqrt{N+M}(5N+M)}{2\sqrt{2}}\right]\sqrt{\frac{N+M}{2}}}{N^2}<\frac{c_0}{c_0+c_1}<\frac{(5N+M)(N+M)}{4\sqrt{N}\big[(\frac{3N+M}{2})^{\frac{3}{2}}-N^{\frac{3}{2}}\big]},$$

which degenerates to

$$0<\frac{c_0}{c_0+c_1}<\frac{(5N+M)(N+M)}{4\sqrt{N}\big[(\frac{3N+M}{2})^{\frac{3}{2}}-N^{\frac{3}{2}}\big]},\quad\text{if}\quad\frac{M}{N}\in(\sqrt{5}-2,1).$$

Thus, if $c_0>0$ and one of $(H1)-(H3)$ holds, then (2.10) has a positive ground state solution. □

Denote the positive ground state solution of (2.10) obtained in Proposition 2.1 as (v_1, v_0), then (v_1, v_0) satisfies the Pohozaev identity

$$\begin{aligned}&\int_{\mathbb{R}^3} |\nabla v_1|^2 dx + \int_{\mathbb{R}^3} |\nabla v_0|^2 dx \\ =& \frac{3}{4}(c_0+c_1)\int_{\mathbb{R}^3} v_1^4 dx + \frac{3}{4}c_0 \int_{\mathbb{R}^3} v_0^4 dx + \frac{3}{2}(c_0+c_1)\int_{\mathbb{R}^3} v_1^2 v_0^2 dx.\end{aligned} \tag{2.12}$$

Denote the corresponding energy as

$$\begin{aligned}m_3 := J(v_1, v_0) :=& \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla v_1|^2 + |\nabla v_0|^2\big) dx - \frac{1}{4}\int_{\mathbb{R}^3} \Big((c_0+c_1)v_1^4 + c_0 v_0^4\Big) dx \\ &- \frac{1}{2}\big(c_0+c_1\big)\int_{\mathbb{R}^3} v_1^2 v_0^2 dx > 0.\end{aligned} \tag{2.13}$$

Similar to (2.9), we can deduce that $\sup_{\tau\in\mathbb{R}} J(\tau \star v_1, \tau \star v_0) = J(v_1, v_0) = m_3$.

For any $u \in H^1(\mathbb{R}^3)$, by Lemma 2.4 in [6], u satisfies the classical Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^3} u^4 dx \le \frac{4\sqrt{3}}{9C_0}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^{\frac{3}{2}} \cdot \Big(\int_{\mathbb{R}^3} u^2 dx\Big)^{\frac{1}{2}}. \tag{2.14}$$

For any $(u_1, u_2, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3)$, there also holds the similar inequality.

Lemma 2.3. *For $(u_1, u_2, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3)$, there holds*

$$\int_{\mathbb{R}^3} \big(u_0^2+u_1^2+u_2^2\big)^2 dx \le C_*\Big(\int_{\mathbb{R}^3} \big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big) dx\Big)^{\frac{3}{2}} \cdot \Big(\int_{\mathbb{R}^3} \big(u_1^2+u_2^2+u_0^2\big) dx\Big)^{\frac{1}{2}}, \tag{2.15}$$

where $C_* = \frac{4\sqrt{3}}{9C_0}$.

Proof. Consider the minimization problem:

$$\Omega = \inf_{(0,0,0)\neq(u_1,u_2,u_0)\in H^1(\mathbb{R}^3,\mathbb{R}^3)} F(u_1, u_2, u_0),$$

where

$$F(u_1, u_2, u_0) = \frac{\Big(\int_{\mathbb{R}^3} \big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big) dx\Big)^{\frac{3}{2}} \Big(\int_{\mathbb{R}^3} \big(u_1^2+u_2^2+u_0^2\big) dx\Big)^{\frac{1}{2}}}{\int_{\mathbb{R}^3} \big(u_0^2+u_1^2+u_2^2\big)^2 dx}.$$

To obtain (2.15), it is sufficient to show $\Omega = \frac{3\sqrt{3}C_0}{4}$. Let $Q(x)$ be the unique positive solution to (2.1) and set

$$(u_1, u_2, u_0) = \Big(\frac{Q}{\sqrt{3}}, \frac{Q}{\sqrt{3}}, \frac{Q}{\sqrt{3}}\Big),$$

then by (2.2),

$$F(u_1, u_2, u_0) = \frac{\Big(\int_{\mathbb{R}^3} |\nabla Q|^2 dx\Big)^{\frac{3}{2}} \Big(\int_{\mathbb{R}^3} Q^2 dx\Big)^{\frac{1}{2}}}{\int_{\mathbb{R}^3} Q^4 dx} = \frac{3\sqrt{3}C_0}{4}.$$

By direct calculation, for arbitrary $(u_1, u_2, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3)$, there holds

$$\Big|\nabla\sqrt{u_1^2+u_2^2+u_0^2}\Big|^2 \le |\nabla u_0|^2 + |\nabla u_1|^2 + |\nabla u_2|^2,$$

therefore, by (2.14),

$$\begin{aligned}F(u_1,u_2,u_0) &\geq \frac{\left(\int_{\mathbb{R}^3}\left(|\nabla\sqrt{u_1^2+u_2^2+u_0^2}|^2\right)dx\right)^{\frac{3}{2}}\left(\int_{\mathbb{R}^3}(u_1^2+u_2^2+u_0^2)dx\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^3}(u_0^2+u_1^2+u_2^2)^2dx}\\&=\frac{\left(\int_{\mathbb{R}^3}\left(|\nabla\sqrt{u_1^2+u_2^2+u_0^2}|^2\right)dx\right)^{\frac{3}{2}}\left(\int_{\mathbb{R}^3}\left(\sqrt{u_1^2+u_2^2+u_0^2}\right)^2dx\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^3}\left(\sqrt{u_1^2+u_2^2+u_0^2}\right)^4dx}\\&\geq\frac{3\sqrt{3}C_0}{4}.\end{aligned}$$

Thus, we get $\Omega=\frac{3\sqrt{3}C_0}{4}$. □

In the end, we list a well known Liouville type result, which is crucial for us to determine the signs of Lagrange multipliers.

Lemma 2.4. *(Ikoma [27]) Suppose $0<p\leq\frac{n}{n-2}$ when $n\geq 3$ and $0<p<\infty$ when $n=1,2$. Let $u\in L^p(\mathbb{R}^n)$ be a smooth function and satisfy $-\Delta u\geq 0$ in $\mathbb{R}^n$. Then $u\equiv 0$ holds.*

3 Proof of Theorem 1

In this section, we are devoted to studying the existence, stability, and asymptotic results of solutions to (1.6), i.e. the following system

$$\begin{cases}-\Delta u_1=(\mu+\lambda)u_1+(c_0+c_1)u_1^3+(c_0-c_1)u_1u_2^2+(c_0+c_1)u_0^2u_1+c_1u_2u_0^2,\\-\Delta u_2=(\mu-\lambda)u_2+(c_0+c_1)u_2^3+(c_0-c_1)u_1^2u_2+(c_0+c_1)u_0^2u_2+c_1u_1u_0^2,\\-\Delta u_0=\mu u_0+c_0u_0^3+(c_0+c_1)(u_1^2+u_2^2)u_0+2c_1u_1u_2u_0.\end{cases}\tag{3.1}$$

Recall

$$\mathcal{M}_0:=\Big\{(u_1,u_2,u_0)\in H^1(\mathbb{R}^3,\mathbb{R}^3)\big|\int_{\mathbb{R}^3}(u_1^2+u_2^2+u_0^2)dx=N,\ \int_{\mathbb{R}^3}(u_1^2-u_2^2)dx=M\Big\},$$

we set $\mathcal{M}_{0,r}:=H_r\cap\mathcal{M}_0$. By the Palais' principle of symmetric criticality (Theorem 1.28 in [45]), the critical point of I_0 on $\mathcal{M}_{0,r}$ is actually a critical point of I_0 on $\mathcal{M}_0$. Therefore, we always work in a radial setting throughout this section.

Now, we show that $I_0(u_1,u_2,u_0)$ has a mountain pass structure on $\mathcal{M}_0$. For $l>0$, we define

$$A_l:=\Big\{(u_1,u_2,u_0)\in\mathcal{M}_0\big|\int_{\mathbb{R}^3}\left(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\right)dx\leq l\Big\}$$

and

$$B_l:=\Big\{(u_1,u_2,u_0)\in\mathcal{M}_0\big|\int_{\mathbb{R}^3}\left(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\right)dx=l\Big\}.$$

Lemma 3.1. *Suppose $c_0>0$ and $c_0+c_1>0$, then there exist $l_1>0$ small enough and*

$$l_2:=\left(\frac{1}{3\max\left\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\right\}C_*N^{\frac{1}{2}}}\right)^2,$$

such that

$$I_0(u_1,u_2,u_0)>0 \quad \textit{for } (u_1,u_2,u_0)\in A_{l_1}$$

and

$$\inf_{B_{l_2}} I_0(u_1,u_2,u_0)>\sup_{A_{l_1}} I_0(u_1,u_2,u_0),$$

where C_* *is defined in* (2.15).

Proof. For any $(u_1,u_2,u_0)\in\mathcal{M}_0$, by Hölder's inequality, we get

$$\int_{\mathbb{R}^3} u_1u_2u_0^2dx \le \Big(\int_{\mathbb{R}^3} u_0^4dx\Big)^{\frac12}\cdot\Big(\int_{\mathbb{R}^3} u_1^2u_2^2dx\Big)^{\frac12} \le \frac12\int_{\mathbb{R}^3} u_0^4dx+\frac12\int_{\mathbb{R}^3} u_1^2u_2^2dx. \tag{3.2}$$

By (2.15), we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\big(u_1^4+u_2^4+u_0^4\big)dx \le \int_{\mathbb{R}^3}\big(u_1^2+u_2^2+u_0^2\big)^2dx\\
&\le C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\Big)^{\frac32}\cdot\Big(\int_{\mathbb{R}^3}\big(u_0^2+u_1^2+u_2^2\big)dx\Big)^{\frac12}\\
&= C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\Big)^{\frac32}N^{\frac12}.
\end{aligned}\tag{3.3}$$

If $c_0>0$ and $c_1>0$, then together (3.2) with (3.3), we can see that

$$\begin{aligned}
&\frac14\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx+c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\\
&+\frac12\int_{\mathbb{R}^3}\Big((c_0-c_1)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx\\
&\le\frac{c_0+c_1}{4}\int_{\mathbb{R}^3}\big(u_1^4+u_2^4\big)dx+\frac{c_0}{4}\int_{\mathbb{R}^3}u_0^4dx+\frac{c_1}{2}\int_{\mathbb{R}^3}u_0^4dx+\frac{c_0}{2}\int_{\mathbb{R}^3}u_1^2u_2^2dx\\
&+\frac{c_0+c_1}{2}\int_{\mathbb{R}^3}(u_1^2+u_2^2)u_0^2dx\\
&\le\frac{c_0+c_1}{4}\int_{\mathbb{R}^3}\big(u_1^4+u_2^4\big)dx+\frac{c_0}{4}\int_{\mathbb{R}^3}u_0^4dx+\frac{c_1}{2}\int_{\mathbb{R}^3}u_0^4dx+\frac{c_0}{4}\int_{\mathbb{R}^3}\big(u_1^4+u_2^4\big)dx\\
&+\frac{c_0+c_1}{4}\int_{\mathbb{R}^3}\big(u_1^4+u_2^4+2u_0^4\big)dx\\
&\le\Big(\frac{3c_0}{4}+c_1\Big)C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\Big)^{\frac32}N^{\frac12}.
\end{aligned}\tag{3.4}$$

If $c_0>0$ and $c_1<0$, by (3.3) and direct calculation, we have

$$\begin{aligned}
&\frac14\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx+c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\\
&+\frac12\int_{\mathbb{R}^3}\Big((c_0-c_1)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx\\
&=\frac{c_0}{4}\int_{\mathbb{R}^3}\big(u_1^2+u_2^2+u_0^2\big)^2dx+\frac{c_1}{2}\int_{\mathbb{R}^3}\big(u_1+u_2\big)^2u_0^2dx+\frac{c_1}{4}\int_{\mathbb{R}^3}\big(u_1^2-u_2^2\big)^2dx\\
&\le\frac{c_0}{4}\int_{\mathbb{R}^3}\big(u_1^2+u_2^2+u_0^2\big)^2dx\le\frac{c_0}{4}C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\Big)^{\frac32}N^{\frac12}.
\end{aligned}\tag{3.5}$$

Combining (3.4) and (3.5), for $c_0 > 0$ and $c_0 + c_1 > 0$, there holds

$$\begin{aligned}
&\frac{1}{4}\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx + c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx \\
&+\frac{1}{2}\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx \\
&\le \max\Big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\Big\}C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\Big)^{\frac{3}{2}}N^{\frac{1}{2}},
\end{aligned} \tag{3.6}$$

which yields that

$$\begin{aligned}
I_0(u_1,u_2,u_0) \ge\ & \frac{1}{2}\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx \\
& - \max\Big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\Big\}C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\Big)^{\frac{3}{2}}N^{\frac{1}{2}}.
\end{aligned} \tag{3.7}$$

Let

$$f(x) := \frac{1}{2}x - \max\Big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\Big\}C_*N^{\frac{1}{2}}x^{\frac{3}{2}}.$$

Then it is easy to see that the function $f(x)$ has a unique maximum point x_0 with

$$x_0 := \left(\frac{1}{3\max\Big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\Big\}C_*N^{\frac{1}{2}}}\right)^2,$$

and further $f(x) > 0$ when $0 < x \le l_1 < x_0$. It follows from (3.7) that

$$I_0(u_1,u_2,u_0) \ge f\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\Big) > 0, \quad \text{for } (u_1,u_2,u_0)\in A_{l_1}.$$

Now, we take arbitrary $(u_1,u_2,u_0)\in A_{l_1}$ and $(v_1,v_2,v_0)\in B_{l_2}$, that is

$$\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx \le l_1 \text{ and } \int_{\mathbb{R}^3}\big(|\nabla v_1|^2+|\nabla v_2|^2+|\nabla v_0|^2\big)dx = l_2.$$

For $c_0 > 0$ and $c_1 > 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx + 4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx \\
&+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx \\
&= c_0\int_{\mathbb{R}^3}\big(u_1^2+u_2^2+u_0^2\big)^2dx + 2c_1\int_{\mathbb{R}^3}\big(u_1+u_2\big)^2u_0^2dx + c_1\int_{\mathbb{R}^3}\big(u_1^2-u_2^2\big)^2dx \ge 0.
\end{aligned} \tag{3.8}$$

Again by Hölder's inequality,

$$2\int_{\mathbb{R}^3}u_1u_2u_0^2dx \le \frac{1}{2}\int_{\mathbb{R}^3}u_0^4dx + 2\int_{\mathbb{R}^3}u_1^2u_2^2dx,$$

then for $c_0 > 0$ and $c_1 < 0$, we have

$$4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx \ge c_1\int_{\mathbb{R}^3}u_0^4dx + 4c_1\int_{\mathbb{R}^3}u_1^2u_2^2dx.$$

It follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\\
&+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx \qquad (3.9)\\
\ge&\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx+c_1\int_{\mathbb{R}^3}u_0^4dx+4c_1\int_{\mathbb{R}^3}u_1^2u_2^2dx+2\big(c_0-c_1\big)\int_{\mathbb{R}^3}u_1^2u_2^2dx\\
=&\,(c_0+c_1)\Big\{\int_{\mathbb{R}^3}(u_1^4+u_2^4+u_0^4)dx+2\int_{\mathbb{R}^3}u_1^2u_2^2dx\Big\}\ge 0,
\end{aligned}
$$

as $c_0+c_1>0$. Hence, from (3.8) and (3.9), we conclude for $c_0>0$ and $c_0+c_1>0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\\
&+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx\ge 0.
\end{aligned}
\tag{3.10}
$$

Then by (3.6), we get

$$
\begin{aligned}
&I_0(v_1,v_2,v_0)-I_0(u_1,u_2,u_0)\\
=&\frac{1}{2}\Big\{\int_{\mathbb{R}^3}(|\nabla v_1|^2+|\nabla v_2|^2+|\nabla v_0|^2)dx-\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2)dx\Big\}\\
&+\frac{1}{4}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx-\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(v_1^4+v_2^4\big)+c_0v_0^4\Big)dx)\Big\}\\
&+\frac{1}{2}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx\\
&-\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)v_1^2v_2^2+\big(c_0+c_1\big)(v_1^2+v_2^2)v_0^2\Big)dx\Big\}+\Big(c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx-c_1\int_{\mathbb{R}^3}v_1v_2v_0^2dx\Big)\\
\ge&\frac{1}{2}(l_2-l_1)-\frac{1}{4}\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(v_1^4+v_2^4\big)+c_0v_0^4\Big)dx\\
&-\frac{1}{2}\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)v_1^2v_2^2+\big(c_0+c_1\big)(v_1^2+v_2^2)v_0^2\Big)dx-c_1\int_{\mathbb{R}^3}v_1v_2v_0^2dx\\
\ge&\frac{1}{2}(l_2-l_1)-\max\Big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\Big\}C_*N^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}(|\nabla v_1|^2+|\nabla v_2|^2+|\nabla v_0|^2)dx\Big)^{\frac{3}{2}}\\
=&\frac{1}{2}(l_2-l_1)-\max\Big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\Big\}C_*N^{\frac{1}{2}}l_2^{\frac{3}{2}}=f(l_2)-\frac{1}{2}l_1>0,
\end{aligned}
$$

when we choose l_1 small and $l_2=x_0$ is the maximum point of $f(x)$. Then

$$
\inf_{B_{l_2}}I_0(u_1,u_2,u_0)>\sup_{A_{l_1}}I_0(u_1,u_2,u_0).
$$

Therefore, we complete the proof. □

For $l>0$, we denote

$$
D_l=\Big\{(u_1,u_2,u_0)\in\mathcal{M}_0\Big|\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2)dx\ge 3l,\ I_0(u_1,u_2,u_0)\le 0\Big\},
$$

then similar to Lemma 4.5 in [6], A_{l_1} and D_{l_1} are connected sets on $\mathcal{M}_0$. Hence, the functional I_0 possesses a mountain pass structure on $\mathcal{M}_0$. For simplicity, we just denote A_{l_1}, B_{l_2} and D_{l_1} as A, B, D respectively. Define the path Γ_0 as

$$\Gamma_0 := \Big\{h \in C\big([0,1], \mathcal{M}_{0,r}\big)\big| h(0) \in A, h(1) \in D\Big\},$$

then we get

$$\sigma_0 := \inf_{h\in\Gamma_0} \max_{t\in[0,1]} I_0\big(h(t)\big) \geq \inf_{B} I_0(u_1, u_2, u_0) > 0. \tag{3.11}$$

Next, we give some properties of σ_0. Before this, we provide some important Lemmas. Define the Pohozaev manifold of system (3.1) as

$$\mathcal{P}_0 := \Big\{(u_1, u_2, u_0) \in \mathcal{M}_0\big|\ P_0(u_1, u_2, u_0) = 0\Big\}, \tag{3.12}$$

with

$$\begin{aligned} P_0(u_1, u_2, u_0) = &\int_{\mathbb{R}^3} \big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big)dx - \frac{3}{4}\Big\{ \int_{\mathbb{R}^3} \Big(\big(c_0 + c_1\big)\big(u_1^4 + u_2^4\big) + c_0 u_0^4\Big)dx \\ &+ 2\int_{\mathbb{R}^3} \Big(\big(c_0 - c_1\big)u_1^2u_2^2 + \big(c_0 + c_1\big)(u_1^2 + u_2^2)u_0^2\Big)dx + 4c_1 \int_{\mathbb{R}^3} u_1u_2u_0^2dx\Big\}, \end{aligned}$$

then it is easy to get following Lemmas.

Lemma 3.2. *Suppose $(u_1, u_2, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ is a solution of (3.1), then $P_0(u_1, u_2, u_0) = 0$.*

Lemma 3.3. *$I_0(u_1, u_2, u_0)$ is bounded from below and coercive on $\mathcal{P}_0$. Moreover, there exists a positive constant C, such that $I_0(u_1, u_2, u_0) \geq C$ for any $(u_1, u_2, u_0) \in \mathcal{P}_0$.*

For $\tau \in \mathbb{R}$ and $(u_1, u_2, u_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3)$, we introduce an operator $\tau \star (u_1, u_2, u_0)$ as

$$\big(\tau \star (u_1, u_2, u_0)\big)(x) = e^{\frac{3\tau}{2}}\big(u_1(e^\tau x), u_2(e^\tau x), u_0(e^\tau x)\big), \tag{3.13}$$

then it is easy to see that

$$\int_{\mathbb{R}^3} \big|\big(\tau \star (u_1, u_2, u_0)\big)\big|^2 dx = \int_{\mathbb{R}^3} \big(u_1^2 + u_2^2 + u_0^2\big)dx.$$

Define

$$\Psi(\tau) := I_0\big(\tau \star (u_1, u_2, u_0)\big),$$

then we give following lemma.

Lemma 3.4. *Suppose $c_0 > 0$ and $c_0 + c_1 > 0$, then for any $(u_1, u_2, u_0) \in \mathcal{M}_0$, there exists a unique $\tau_0 \in \mathbb{R}$, such that $\tau_0 \star (u_1, u_2, u_0) \in \mathcal{P}_0$ and further*

$$\Psi(\tau_0) = \max_{\tau\in\mathbb{R}} \Psi(\tau) > 0.$$

Moreover, if $P_0(u_1, u_2, u_0) = 0$, then $\Psi''(0) < 0$.

Proof. For any $(u_1,u_2,u_0)\in\mathcal{M}_0$, by direct calculation, we get

$$\begin{aligned}\Psi(\tau)=&\frac{1}{2}e^{2\tau}\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\\&-\frac{1}{4}e^{3\tau}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\\&+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_1^2u_2^2+(c_0+c_1)(u_1^2+u_2^2)u_0^2\Big)dx\Big\}\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}\frac{d}{d\tau}\Psi(\tau)=&e^{2\tau}\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\\&-\frac{3}{4}e^{3\tau}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\\&+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_1^2u_2^2+(c_0+c_1)(u_1^2+u_2^2)u_0^2\Big)dx\Big\}.\end{aligned}\tag{3.15}$$

By (3.10), we get for any $(u_1,u_2,u_0)\in\mathcal{M}_0$,

$$\lim_{\tau\to-\infty}\frac{d}{d\tau}\Psi(\tau)=0^+\quad\text{and}\quad\lim_{\tau\to+\infty}\frac{d}{d\tau}\Psi(\tau)=-\infty.\tag{3.16}$$

By (3.15), we can see that there exists a unique $\tau_0\in\mathbb{R}$, such that

$$\frac{d}{d\tau}\Psi(\tau)\big|_{\tau=\tau_0}=P_0\big(\tau_0\star(u_1,u_2,u_0)\big)=0.$$

Thus, $\tau_0\star(u_1,u_2,u_0)\in\mathcal{P}_0$. Moreover, τ_0 is the unique maximum point of $\Psi(\tau)$, that is

$$\Psi(\tau_0)=\max_{\tau\in\mathbb{R}}\Psi(\tau)>0.$$

Since $\tau\star(u_1,u_2,u_0)\in\mathcal{P}_0$ if and only if $\Psi'(\tau)=0$. If $(u_1,u_2,u_0)\in\mathcal{P}_0$, then $\tau=0$ is the maximum point, we have that $\Psi''(0)\le 0$. We claim that $\Psi''(0)<0$. Assume by contradiction, that is $\Psi'(0)=\Psi''(0)=0$, then necessarily $\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx=0$, which is not possible because $(u_1,u_2,u_0)\in\mathcal{M}_0$. Thus, we get $\Psi''(0)<0$. □

Lemma 3.5. *$\mathcal{P}_0$ is a C^1 submanifold in $\mathcal{M}_0$ with codimension 3.*

Proof. By (3.12), it is easy to see that $\mathcal{P}_0$ is defined by $P_0(u_1,u_2,u_0)=0$, $G(u_1,u_2,u_0)=0$, $F(u_1,u_2)=0$, where

$$G(u_1,u_2,u_0)=\int_{\mathbb{R}^3}\big(u_1^2+u_2^2+u_0^2\big)dx-N,\quad F(u_1,u_2)=\int_{\mathbb{R}^3}\big(u_1^2-u_2^2\big)dx-M.$$

Since $P_0(u_1,u_2,u_0)$, $G(u_1,u_2,u_0)$, $F(u_1,u_2)$ are class of C^1, we only need to check that

$$d\big(P_0(u_1,u_2,u_0),G(u_1,u_2,u_0),F(u_1,u_2)\big):H^1(\mathbb{R}^3,\mathbb{R}^3)\to\mathbb{R}^3\ \text{ is surjective.}$$

If this is not true, $dP_0(u_1,u_2,u_0)$ has to be linearly dependent from $dG(u_1,u_2,u_0)$ and $dF(u_1,u_2)$ i.e. there exist $\nu_1,\nu_2\in\mathbb{R}$ such that

$$\begin{cases} \int_{\mathbb{R}^3}\big(2\nabla u_1\nabla\varphi+2(\nu_1+\nu_2)u\varphi\big)dx \\ \qquad = 3\int_{\mathbb{R}^3}\big((c_0+c_1)u_1^3\varphi+(c_0-c_1)u_1u_2^2\varphi+(c_0+c_1)u_0^2u_1\varphi+c_1u_2u_0^2\varphi\big)dx, \\ \int_{\mathbb{R}^3}\big(2\nabla u_2\nabla\psi+2(\nu_1-\nu_2)v\psi\big)dx \\ \qquad = 3\int_{\mathbb{R}^3}\big((c_0+c_1)u_2^3\psi+(c_0-c_1)u_1^2u_2\psi+(c_0+c_1)u_0^2u_2\psi+c_1u_1u_0^2\psi\big)dx, \\ \int_{\mathbb{R}^3}(2\nabla u_0\nabla\zeta+2\nu_1u_0\zeta)dx \\ \qquad = 3\int_{\mathbb{R}^3}\big(c_0u_0^3\zeta+(c_0+c_1)(u_1^2+u_2^2)u_0\zeta+2c_1u_1u_2u_0\zeta\big)dx, \end{cases}$$

for every $(\varphi,\psi,\zeta)\in H$, so

$$\begin{cases} -2\Delta u_1+2(\nu_1+\nu_2)u_1=3(c_0+c_1)u_1^3+3(c_0-c_1)u_1u_2^2+3(c_0+c_1)u_0^2u_1+3c_1u_2u_0^2, \\ -2\Delta u_2+2(\nu_1-\nu_2)u_2=3(c_0+c_1)u_2^3+3(c_0-c_1)u_1^2u_2+3(c_0+c_1)u_0^2u_2+3c_1u_1u_0^2, \\ -2\Delta u_0+2\nu_1u_0=3c_0u_0^3+3(c_0+c_1)(u_1^2+u_2^2)u_0+6c_1u_1u_2u_0. \end{cases}$$

The Pohozaev identity for above system is

$$\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx-\frac{9}{8}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx$$
$$+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx+4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\Big\}=0.$$

Therefore, $\Psi''(0)=0$, which contradicts with $\Psi''(0)<0$. Hence, $\mathcal{P}_0$ is a natural constraint. □

Lemma 3.6. *Suppose $c_0>0$ and $c_0+c_1>0$, then there holds*

$$\sigma_0=\inf_{(u_1,u_2,u_0)\in\mathcal{P}_0}I_0(u_1,u_2,u_0)=\inf_{(u_1,u_2,u_0)\in\mathcal{M}_0}\max_{\tau\in\mathbb{R}}I_0\big(\tau\star(u_1,u_2,u_0)\big), \tag{3.17}$$

where σ_0 is defined in (3.11).

Proof. We prove (3.17) in the following two steps:
Step 1: There holds

$$m_*:=\inf_{(u_1,u_2,u_0)\in\mathcal{M}_0}\max_{\tau\in\mathbb{R}}I_0\big(\tau\star(u_1,u_2,u_0)\big)=\inf_{(u_1,u_2,u_0)\in\mathcal{P}_0}I_0(u_1,u_2,u_0). \tag{3.18}$$

Indeed, by Lemma 3.3,

$$m_0:=\inf_{(u_1,u_2,u_0)\in\mathcal{P}_0}I_0(u_1,u_2,u_0)\ge C>0$$

is well defined. On the one hand, for any $(u_1,u_2,u_0)\in\mathcal{P}_0\subset\mathcal{M}_0$, by Lemma 3.4, there exists a unique $\tau_0\in\mathbb{R}$, such that $\tau_0\star(u_1,u_2,u_0)\in\mathcal{P}_0$ and

$$I_0\big(\tau_0\star(u_1,u_2,u_0)\big)=\max_{\tau\in\mathbb{R}}I_0\big(\tau\star(u_1,u_2,u_0)\big)\ge\inf_{(v_1,v_2,v_0)\in\mathcal{M}_0}\max_{\tau\in\mathbb{R}}I_0\big(\tau\star(v_1,v_2,v_0)\big).$$

Then

$$m_0=\inf_{(u_1,u_2,u_0)\in\mathcal{P}_0}I_0(u_1,u_2,u_0)\ge\inf_{(u_1,u_2,u_0)\in\mathcal{M}_0}\max_{\tau\in\mathbb{R}}I_0\big(\tau\star(u_1,u_2,u_0)\big)=m_*.$$

On the other hand, for any $(u_1,u_2,u_0)\in\mathcal{M}_0$, again by Lemma 3.4, there exists a unique $\tau_0\in\mathbb{R}$, such that $\tau_0\star(u_1,u_2,u_0)\in\mathcal{P}_0$ and further

$$\max_{\tau\in\mathbb{R}} I_0\big(\tau\star(u_1,u_2,u_0)\big)=I_0\big(\tau_0\star(u_1,u_2,u_0)\big)\geq\inf_{(u_1,u_2,u_0)\in\mathcal{P}_0} I_0(u_1,u_2,u_0)=m_0,$$

which follows that

$$m_*=\inf_{(u_1,u_2,u_0)\in\mathcal{M}_0}\max_{\tau\in\mathbb{R}} I_0\big(\tau\star(u_1,u_2,u_0)\big)\geq m_0.$$

Therefore, we have proved (3.18).

Step 2: There holds

$$\sigma_0=\inf_{h\in\Gamma_0}\max_{t\in[0,1]} I_0\big(h(t)\big)=m_0=m_*.$$

We first show that for any $h=(h_1,h_2,h_0)\in\Gamma_0$, $h\big([0,1]\big)\cap\mathcal{P}_0\neq\emptyset$. In fact, for any $h\in\Gamma_0$, by (3.6), we get

$$\begin{aligned}
&P_0\big(h_1(0),h_2(0),h_0(0)\big)=\int_{\mathbb{R}^3}\big(|\nabla h_1(0)|^2+|\nabla h_2(0)|^2+|\nabla h_0(0)|^2\big)dx\\
&-\frac{3}{4}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(h_1(0)^4+h_2(0)^4\big)+c_0h_0(0)^4\Big)dx+4c_1\int_{\mathbb{R}^3}h_1(0)h_2(0)h_0(0)^2dx\\
&+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)h_1(0)^2h_2(0)^2+\big(c_0+c_1\big)(h_1(0)^2+h_2(0)^2)h_0(0)^2\Big)dx\Big\}\\
\geq&\int_{\mathbb{R}^3}\big(|\nabla h_1(0)|^2+|\nabla h_2(0)|^2+|\nabla h_0(0)|^2\big)dx\\
&-3\max\Big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\Big\}C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla h_1(0)|^2+|\nabla h_2(0)|^2+|\nabla h_0(0)|^2\big)dx\Big)^{\frac{3}{2}}N^{\frac{1}{2}}>0,
\end{aligned}\tag{3.19}$$

since $h(0)\in A_{l_1}$ and l_1 is small. Again by $h=(h_1,h_2,h_0)\in\Gamma_0$ and (3.10), we have

$$\begin{aligned}
&P_0\big(h_1(1),h_2(1),h_0(1)\big)=\int_{\mathbb{R}^3}\big(|\nabla h_1(1)|^2+|\nabla h_2(1)|^2+|\nabla h_0(1)|^2\big)dx\\
&-\frac{3}{4}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(h_1(1)^4+h_2(1)^4\big)+c_0h_0(1)^4\Big)dx+4c_1\int_{\mathbb{R}^3}h_1(1)h_2(1)h_0(1)^2dx\\
&+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)h_1(1)^2h_2(1)^2+\big(c_0+c_1\big)(h_1(1)^2+h_2(1)^2)h_0(1)^2\Big)dx\Big\}\\
=&2I_0\big(h_1(1),h_2(1),h_0(1)\big)-\frac{1}{4}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(h_1(1)^4+h_2(1)^4\big)+c_0h_0(1)^4\Big)dx\\
&+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)h_1(1)^2h_2(1)^2+\big(c_0+c_1\big)(h_1(1)^2+h_2(1)^2)h_0(1)^2\Big)dx\\
&+4c_1\int_{\mathbb{R}^3}h_1(1)h_2(1)h_0(1)^2dx\Big\}\leq 2I_0\big(h_1(1),h_2(1),h_0(1)\big)<0.
\end{aligned}$$

Together with (3.19), there exists $t_0\in(0,1)$, such that $P_0\big(h(t_0)\big)=0$, that is $h\big([0,1]\big)\cap\mathcal{P}_0\neq\emptyset$. It implies that

$$\sigma_0\geq\inf_{(u_1,u_2,u_0)\in\mathcal{P}_0} I_0(u_1,u_2,u_0)=m_0.\tag{3.20}$$

For $(u_1,u_2,u_0)\in\mathcal{M}_0$, denote (u_1^*,u_2^*,u_0^*) as the Schwarz symmetrization rearrangement of (u_1,u_2,u_0), then by [35], we have

$$\int_{\mathbb{R}^3}\big(|\nabla u_1^*|^2+|\nabla u_2^*|^2+|\nabla u_0^*|^2\big)dx\le\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx$$

and

$$\int_{\mathbb{R}^3}|u_i^*|^p dx=\int_{\mathbb{R}^3}|u_i|^p dx,\quad \forall p\in[1,\infty),\quad i=1,2,0.$$

It implies that

$$I_0\big(\tau\star(u_1^*,u_2^*,u_0^*)\big)\le I_0\big(\tau\star(u_1,u_2,u_0)\big). \tag{3.21}$$

Since

$$\lim_{\tau\to-\infty}I_0\big(\tau\star(u_1^*,u_2^*,u_0^*)\big)=0^+\quad\text{and}\quad\lim_{\tau\to+\infty}I_0\big(\tau\star(u_1^*,u_2^*,u_0^*)\big)=-\infty,$$

then it is easy to see that there exist sufficiently negative τ_1 and sufficiently large τ_2, such that $\tau_1\star(u_1^*,u_2^*,u_0^*)\in A$ and $\tau_2\star(u_1^*,u_2^*,u_0^*)\in D$. Let

$$\bar h_0(t):=\big((1-t)\tau_1+t\tau_2\big)\star(u_1^*,u_2^*,u_0^*),\quad\text{for } t>0,$$

then $\bar h_0(0)\in A$ and $\bar h_0(1)\in D$, which implies that $\bar h_0\in\Gamma_0$. Hence, by (3.21), we get

$$\max_{\tau\in\mathbb{R}}I_0\big(\tau\star(u_1,u_2,u_0)\big)\ge\max_{t>0}I_0\big(\bar h_0(t)\big)\ge\max_{t\in[0,1]}I_0\big(\bar h_0(t)\big)\ge\sigma_0.$$

It follows that

$$m_*=\inf_{(u_1,u_2,u_0)\in\mathcal{M}_0}\max_{\tau\in\mathbb{R}}I_0\big(\tau\star(u_1,u_2,u_0)\big)\ge\sigma_0.$$

Together with (3.18), (3.20), we get $\sigma_0=m_0=m_*$. Therefore, we complete the proof. □

Let $w_1,\ w_0$ and (v_1,v_0) be the ground state solution to (2.4), (2.7) and (2.10) respectively. Denote the corresponding energy as $m_1(\sqrt{\frac{N+M}{2}},c_0+c_1):=I_1(w_1)$, $m_2(\sqrt{N},c_0):=I_2(w_0)$ and $m_3:=J(v_1,v_0)$, which have been defined in (2.6), (2.8) and (2.13).

In the following lemma, we give an estimation of the minimax value σ_0.

Lemma 3.7. *Suppose $c_0,\ c_1$ and $M,\ N$ satisfy the conditions in Theorem 1, then there holds*

$$\sigma_0<m_3<\min\Big\{m_1\Big(\sqrt{\frac{N+M}{2}},c_0+c_1\Big),m_2\big(\sqrt{N},c_0\big)\Big\}.$$

Proof. For any $b>0$, define

$$\mathrm{T}_b:=\Big\{u\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}u^2dx=b\Big\}.$$

From Proposition 2.1, we know that m_3 can be achieved by $(v_1,v_0)\in\mathrm{T}_{\frac{M+N}{2}}\times\mathrm{T}_N$. By the standard decay result, we get that

$$v_1(x)=O\Big(|x|^{-1}e^{-\sqrt{-(\mu+\lambda)}|x|}\Big),\quad v_0(x)=O\Big(|x|^{-1}e^{-\sqrt{-\mu}|x|}\Big),\quad\text{as } |x|\to\infty$$

and $|v_1(x)| \le \tilde{C}$, $|v_0(x)| \le \hat{C}$ in $\mathbb{R}^3$, for some $\tilde{C},\ \hat{C} > 0$. We now take

$$u^*(x) = \begin{cases} \frac{\theta\varphi(x)}{|x|^k}, & c_1 > 0, \\ \frac{-\theta\varphi(x)}{|x|^k}, & c_1 < 0, \end{cases} \tag{3.22}$$

where $\theta > 0$, $0 < k < \frac{1}{2}$ and

$$\varphi(x) \in C_c^\infty\big(B_2(0)\big), \quad 0 \le \varphi(x) \le 1, \quad \varphi(x) \equiv 1 \text{ in } B_1(0).$$

Then we can choose a proper θ, such that $(v_1, u^*, v_0) \in \mathcal{M}_0$. Thus, for any $\tau \in \mathbb{R}$, we get $(v_1, \tau \star u^*, v_0) \in \mathcal{M}_0$. By Lemma 3.4, there exists $\tau_0 \in \mathbb{R}$, such that $\tau_0 \star (v_1, \tau \star u^*, v_0) \in \mathcal{P}_0$, then we have

$$\begin{aligned} &\int_{\mathbb{R}^3} (|\nabla v_1|^2 + |\nabla(\tau \star u^*)|^2 + |\nabla v_0|^2) dx \\ = &\frac{3e^{\tau_0}}{4} \int_{\mathbb{R}^3} \Big((c_0 + c_1)\big(v_1^4 + (\tau \star u^*)^4\big) + c_0 v_0^4 \Big) dx + 3c_1 e^{\tau_0} \int_{\mathbb{R}^3} v_1 (\tau \star u^*) v_0^2 dx \\ &+ \frac{3e^{\tau_0}}{2} \int_{\mathbb{R}^3} \Big((c_0 - c_1) v_1^2 (\tau \star u^*)^2 + (c_0 + c_1)\big(v_1^2 + (\tau \star u^*)^2\big) v_0^2 \Big) dx. \end{aligned}$$

Thus, by (2.12), we get $e^{\tau_0} \to 1$ as $\tau \to -\infty$. Moreover, we can see that as $\tau \to -\infty$,

$$\int_{\mathbb{R}^3} (\tau \star u^*)^2 v_0^2 dx = \theta^2 e^{(3-2k)\tau}\big(C_2 + o(1)\big) > \frac{C_2 \theta^2}{2} e^{(3-2k)\tau} \tag{3.23}$$

and

$$\int_{\mathbb{R}^3} v_1^2 (\tau \star u^*)^2 dx = \theta^2 e^{(3-2k)\tau}\big(\bar{C}_2 + o(1)\big), \tag{3.24}$$

for some $C_2,\ \bar{C}_2 > 0$. In addition, there also exists a $\tilde{C}_2 > 0$, such that as $\tau \to -\infty$,

$$\int_{\mathbb{R}^3} v_1 (\tau \star u^*) v_0^2 dx = \theta e^{(\frac{3}{2}-k)\tau}\big(\tilde{C}_2 + o(1)\big) > \frac{\tilde{C}_2 \theta}{2} e^{(\frac{3}{2}-k)\tau}, \tag{3.25}$$

for $c_1 > 0$. Therefore, we conclude from (3.23)-(3.25) that if $c_1 > 0$,

$$\begin{aligned} \sigma_0 &= \inf_{(u_1,u_2,u_0) \in \mathcal{P}_0} I_0(u_1, u_2, u_0) \le I_0\big(\tau_0 \star (v_1, \tau \star u^*, v_0)\big) \\ &= \frac{e^{2\tau_0}}{2} \int_{\mathbb{R}^3} (|\nabla v_1|^2 + |\nabla v_0|^2) dx - \frac{e^{3\tau_0}}{4} \int_{\mathbb{R}^3} \Big((c_0 + c_1) v_1^4 + c_0 v_0^4 \Big) dx \\ &\quad - \frac{e^{3\tau_0}}{2} \int_{\mathbb{R}^3} (c_0 + c_1) v_1^2 v_0^2 dx + \frac{e^{2(\tau+\tau_0)}}{2} \int_{\mathbb{R}^3} |\nabla u^*|^2 dx - \frac{e^{3(\tau+\tau_0)}}{4} \int_{\mathbb{R}^3} (c_0 + c_1)(u^*)^4 dx \\ &\quad - \frac{e^{3\tau_0}}{2} \int_{\mathbb{R}^3} \Big((c_0 - c_1) v_1^2 (\tau \star u^*)^2 + (c_0 + c_1)(\tau \star u^*)^2 v_0^2 \Big) dx - c_1 e^{3\tau_0} \int_{\mathbb{R}^3} v_1 (\tau \star u^*) v_0^2 dx \\ &\le \frac{e^{2\tau_0}}{2} \int_{\mathbb{R}^3} (|\nabla v_1|^2 + |\nabla v_0|^2) dx - \frac{e^{3\tau_0}}{4} \int_{\mathbb{R}^3} \Big((c_0 + c_1) v_1^4 + c_0 v_0^4 \Big) dx \\ &\quad - \frac{e^{3\tau_0}}{2} \int_{\mathbb{R}^3} (c_0 + c_1) v_1^2 v_0^2 dx + \frac{e^{2(\tau+\tau_0)}}{2} \int_{\mathbb{R}^3} |\nabla u^*|^2 dx - \frac{e^{3(\tau+\tau_0)}}{4} \int_{\mathbb{R}^3} (c_0 + c_1)(u^*)^4 dx \\ &\quad - \frac{e^{3\tau_0}}{2} (c_0 - c_1) \theta^2 e^{(3-2k)\tau}\big(\bar{C}_2 + o(1)\big) - \frac{e^{3\tau_0}}{4} (c_0 + c_1) C_2 \theta^2 e^{(3-2k)\tau} - c_1 e^{3\tau_0} \frac{\tilde{C}_2 \theta}{2} e^{(\frac{3}{2}-k)\tau} \\ &< I_0(v_1, 0, v_0) = J(v_1, v_0) = m_3, \quad \text{as } \tau \to -\infty. \end{aligned} \tag{3.26}$$

Since $v_0,\ v_1>0$, by (3.22), we obtain $c_1\int_{\mathbb{R}^3}v_1(\tau\star u^*)v_0^2dx\geq 0$, for $c_1<0$. Then if $c_1<0$,

$$\begin{aligned}\sigma_0\leq&\frac{e^{2\tau_0}}{2}\int_{\mathbb{R}^3}(|\nabla v_1|^2+|\nabla v_0|^2)dx-\frac{e^{3\tau_0}}{4}\int_{\mathbb{R}^3}\Big((c_0+c_1)v_1^4+c_0v_0^4\Big)dx\\&-\frac{e^{3\tau_0}}{2}\int_{\mathbb{R}^3}(c_0+c_1)v_1^2v_0^2dx+\frac{e^{2(\tau+\tau_0)}}{2}\int_{\mathbb{R}^3}|\nabla u^*|^2dx-\frac{e^{3(\tau+\tau_0)}}{4}\int_{\mathbb{R}^3}(c_0+c_1)(u^*)^4dx\\&-\frac{e^{3\tau_0}}{2}\int_{\mathbb{R}^3}\Big((c_0-c_1)v_1^2(\tau\star u^*)^2+(c_0+c_1)(\tau\star u^*)^2v_0^2\Big)dx-c_1e^{3\tau_0}\int_{\mathbb{R}^3}v_1(\tau\star u^*)v_0^2dx\\\leq&\frac{e^{2\tau_0}}{2}\int_{\mathbb{R}^3}(|\nabla v_1|^2+|\nabla v_0|^2)dx-\frac{e^{3\tau_0}}{4}\int_{\mathbb{R}^3}\Big((c_0+c_1)v_1^4+c_0v_0^4\Big)dx\\&-\frac{e^{3\tau_0}}{2}\int_{\mathbb{R}^3}(c_0+c_1)v_1^2v_0^2dx+\frac{e^{2(\tau+\tau_0)}}{2}\int_{\mathbb{R}^3}|\nabla u^*|^2dx-\frac{e^{3(\tau+\tau_0)}}{4}\int_{\mathbb{R}^3}(c_0+c_1)(u^*)^4dx\\&-\frac{e^{3\tau_0}}{2}(c_0-c_1)\theta^2e^{(3-2k)\tau}\big(\bar{C}_2+o(1)\big)-\frac{e^{3\tau_0}}{4}(c_0+c_1)C_2\theta^2e^{(3-2k)\tau}+c_1e^{3\tau_0}\frac{\tilde{C}_2\theta}{2}e^{(\frac{3}{2}-k)\tau}\\<&I_0(v_1,0,v_0)=J(v_1,v_0)=m_3,\quad\text{as }\tau\to-\infty.\end{aligned}\tag{3.27}$$

Together (3.26) with (3.27), we get that for $c_1>0$ or $c_1<0$,

$$\sigma_0<J(v_1,v_0)=m_3.$$

From the proof of Theorem 1.2 in [6], it is easy to see that under the condition of Theorem 1, we have $m_3<\min\Big\{m_1\Big(\sqrt{\frac{N+M}{2}},c_0+c_1\Big),m_2\big(\sqrt{N},c_0\big)\Big\}$. Therefore

$$\sigma_0<m_3<\min\Big\{m_1\Big(\sqrt{\frac{N+M}{2}},c_0+c_1\Big),m_2\big(\sqrt{N},c_0\big)\Big\}.$$

□

In the following Proposition, we prove the existence of bounded Palais-Smale sequence for $I_0(u_1,u_2,u_0)$ restricted to $\mathcal{M}_{0,r}$ at level σ_0.

Proposition 3.1. *Suppose $c_0>0$ and $c_0+c_1>0$, then there exists a Palais-Smale sequence $\{(u_{1n},u_{2n},u_{0n})\}\subset\mathcal{M}_{0,r}$ for I_0 at level σ_0. In addition, $\{(u_{1n},u_{2n},u_{0n})\}$ is bounded in $H^1(\mathbb{R}^3,\mathbb{R}^3)$ and satisfies*

$$\begin{aligned}&P_0(u_{1n},u_{2n},u_{0n})\\=&\int_{\mathbb{R}^3}(|\nabla u_{1n}|^2+|\nabla u_{2n}|^2+|\nabla u_{0n}|^2)dx-\frac{3}{4}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_{1n}^4+u_{2n}^4\big)+c_0u_{0n}^4\Big)dx\\&+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_{1n}^2u_{2n}^2+(c_0+c_1)(u_{1n}^2+u_{2n}^2)u_{0n}^2\Big)dx+4c_1\int_{\mathbb{R}^3}u_{1n}u_{2n}u_{0n}^2dx\Big\}\\\to&\,0,\quad\textit{as }n\to\infty.\end{aligned}\tag{3.28}$$

Proof. Let $\tilde{I}_0\big(\tau,(u_1,u_2,u_0)\big)=I_0\big(\tau\star(u_1,u_2,u_0)\big)$ and

$$\tilde{\Gamma}_0=\Big\{\tilde{h}\in C\big([0,1],\mathbb{R}\times\mathcal{M}_{0,r}\big)\big|\ \tilde{h}(0)=\big(0,h(0)\big),\tilde{h}(1)=\big(0,h(1)\big),h(0)\in A,h(1)\in D\Big\},$$

then it is easy to see that

$$\tilde{\sigma}_0 = \inf_{\tilde{h}\in\tilde{\Gamma}_0} \max_{t\in[0,1]} \tilde{I}_0\big(\tilde{h}(t)\big) = \sigma_0.$$

Take a sequence $\{g_n\} := \big\{(g_{1n}, g_{2n}, g_{0n})\big\} \subset \Gamma_0$, such that

$$\max_{t\in[0,1]} I_0\big(g_n(t)\big) \le \sigma_0 + \frac{1}{n}. \tag{3.29}$$

Let $\tilde{g}_n := (0, g_n) \in \tilde{\Gamma}_0$, then we get

$$\max_{t\in[0,1]} \tilde{I}_0\big(\tilde{g}_n(t)\big) = \max_{t\in[0,1]} I_0\big(g_n(t)\big) \le \sigma_0 + \frac{1}{n} = \tilde{\sigma}_0 + \frac{1}{n}.$$

By Lemma 2.3 in [29], there exists a sequence $\big\{\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big)\big\} \subset \mathbb{R}\times\mathcal{M}_{0,r}$, such that

(1). $\lim_{n\to+\infty} \tilde{I}_0\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big) = \tilde{\sigma}_0 = \sigma_0$;

(2). $\lim_{n\to+\infty} |\tau_n| + dist\big((\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n}), g_n(t)\big) = 0$;

(3). let $E := \mathbb{R}\times H_r$ and E^{-1} denote the dual space of E, then there holds

$$\Big\|\tilde{I}_0'|_{\mathbb{R}\times\mathcal{M}_0}\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big)\Big\|_{E^{-1}} \le 2\sqrt{\frac{1}{n}}.$$

That is,

$$\big|\big\langle \tilde{I}_0'\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big), \big(\tau, (u_1^*, u_2^*, u_0^*)\big)\big\rangle\big| \le 2\sqrt{\frac{1}{n}}\big\|\big(\tau, (u_1^*, u_2^*, u_0^*)\big)\big\|_E,$$

for all $\big(\tau, (u_1^*, u_2^*, u_0^*)\big) \in \tilde{T}_{\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big)}$, where

$$\tilde{T}_{\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big)} := \Big\{\big(\tau, (u_1^*, u_2^*, u_0^*)\big) \in E \Big| \int_{\mathbb{R}^3} \Big(u_1^*\tilde{u}_{1n} + u_2^*\tilde{u}_{2n} + u_0^*\tilde{u}_{0n}\Big)dx = 0, \int_{\mathbb{R}^3} \Big(u_1^*\tilde{u}_{1n} - u_2^*\tilde{u}_{2n}\Big)dx = 0\Big\}.$$

Let

$$(u_{1n}, u_{2n}, u_{0n}) := (\tau_n \star \tilde{u}_{1n}, \tau_n \star \tilde{u}_{2n}, \tau_n \star \tilde{u}_{0n}),$$

then by point (1),

$$I_0(u_{1n}, u_{2n}, u_{0n}) = I_0(\tau_n \star \tilde{u}_{1n}, \tau_n \star \tilde{u}_{2n}, \tau_n \star \tilde{u}_{0n}) = \tilde{I}_0\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big) \to \sigma_0, \quad \text{as } n \to \infty. \tag{3.30}$$

Further, by direct calculation, we can obtain from (3) that

$$\begin{aligned}
&\big\langle \tilde{I}_0'\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big), \big(1, (0,0,0)\big)\big\rangle \\
&= e^{2\tau_n}\int_{\mathbb{R}^3} \big(|\nabla\tilde{u}_{1n}|^2 + |\nabla\tilde{u}_{2n}|^2 + |\nabla\tilde{u}_{0n}|^2\big)dx - \frac{3}{4}e^{3\tau_n}\Big\{\int_{\mathbb{R}^3} \Big(\big(c_0+c_1\big)\big(\tilde{u}_{1n}^4 + \tilde{u}_{2n}^4\big) + c_0\tilde{u}_{0n}^4\Big)dx \\
&\quad + 2\int_{\mathbb{R}^3} \Big(\big(c_0-c_1\big)\tilde{u}_{1n}^2\tilde{u}_{2n}^2 + \big(c_0+c_1\big)\big(\tilde{u}_{1n}^2 + \tilde{u}_{2n}^2\big)\tilde{u}_{0n}^2\Big)dx + 4c_1\int_{\mathbb{R}^3} \tilde{u}_{1n}\tilde{u}_{2n}\tilde{u}_{0n}^2 dx\Big\} \\
&= P_0\big(\tau_n \star (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big) \to 0, \quad \text{as } n \to \infty,
\end{aligned} \tag{3.31}$$

that is $P_0(u_{1n}, u_{2n}, u_{0n}) \to 0$ as $n \to \infty$.

Next, we are going to show that $I'_0\big|_{\mathcal{M}_0}(u_{1n}, u_{2n}, u_{0n}) \to 0$ as $n \to \infty$. For this, it is sufficient to show that there exists a constant $C > 0$, such that

$$\big|\langle I'_0(u_{1n}, u_{2n}, u_{0n}), (u_1^*, u_2^*, u_0^*)\rangle\big| \le \frac{C}{\sqrt{n}}\|(u_1^*, u_2^*, u_0^*)\|, \tag{3.32}$$

for all

$$\begin{aligned}(u_1^*, u_2^*, u_0^*) \in T_{(u_{1n},u_{2n},u_{0n})} := \Big\{(u_1^*, u_2^*, u_0^*) \in H_r\Big| \int_{\mathbb{R}^3} \Big(u_1^* u_{1n} + u_2^* u_{2n} + u_0^* u_{0n}\Big)dx = 0,\\ \int_{\mathbb{R}^3} \Big(u_1^* u_{1n} - u_2^* u_{2n}\Big)dx = 0\Big\}.\end{aligned}$$

For any $(u_1^*, u_2^*, u_0^*) \in T_{(u_{1n},u_{2n},u_{0n})}$, we set $(\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_0^*) = (-\tau_n) \star (u_1^*, u_2^*, u_0^*)$, then

$$\big(0, (\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_0^*)\big) \in \tilde{T}_{(\tau_n,(\tilde{u}_{1n},\tilde{u}_{2n},\tilde{u}_{0n}))}$$

and

$$\big\langle \tilde{I}'_0\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big), \big(0, (\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_0^*)\big)\big\rangle = \langle I'_0(u_{1n}, u_{2n}, u_{0n}), (u_1^*, u_2^*, u_0^*)\rangle.$$

By point (2), we may assume that $\tau_n \to 0$ as $n \to \infty$. For n sufficiently large, we have

$$\begin{aligned}&\big\|\big(0, (\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_0^*)\big)\big\|_E^2 = \|(\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_0^*)\|^2\\ &= \int_{\mathbb{R}^3} \big(|\nabla \tilde{u}_1^*|^2 + |\nabla \tilde{u}_2^*|^2 + |\nabla \tilde{u}_0^*|^2\big)dx + \int_{\mathbb{R}^3} \big((\tilde{u}_1^*)^2 + (\tilde{u}_2^*)^2 + (\tilde{u}_0^*)^2\big)dx\\ &= e^{-2\tau_n} \int_{\mathbb{R}^3} \big(|\nabla u_1^*|^2 + |\nabla u_2^*|^2 + |\nabla u_0^*|^2\big)dx + \int_{\mathbb{R}^3} \big((u_1^*)^2 + (u_2^*)^2 + (u_0^*)^2\big)dx\\ &\le 2\|(u_1^*, u_2^*, u_0^*)\|^2.\end{aligned}$$

Then by point (3), we get

$$\begin{aligned}&\big|\langle I'_0(u_{1n}, u_{2n}, u_{0n}), (u_1^*, u_2^*, u_0^*)\rangle\big| = \big|\big\langle \tilde{I}'_0\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big), \big(0, (\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_0^*)\big)\big\rangle\big|\\ &\le 2\sqrt{\frac{1}{n}}\big\|\big(0, (\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_0^*)\big)\big\|_E \le 2\sqrt{\frac{2}{n}}\|(u_1^*, u_2^*, u_0^*)\|,\end{aligned}$$

thus we get (3.32). Together with (3.30), (3.31), $\{(u_{1n}, u_{2n}, u_{0n})\}$ is a Palais-Smale sequence for I_0 restricted to $\mathcal{M}_{0,r}$. Moreover, direct calculation gives

$$\begin{aligned}I_0(u_{1n}, u_{2n}, u_{0n}) &= I_0(u_{1n}, u_{2n}, u_{0n}) - \frac{1}{3}P_0(u_{1n}, u_{2n}, u_{0n}) + o(1)\\ &= \frac{1}{6}\int_{\mathbb{R}^3} \big(|\nabla u_{1n}|^2 + |\nabla u_{2n}|^2 + |\nabla u_{0n}|^2\big)dx.\end{aligned}$$

Since $\{(u_{1n}, u_{2n}, u_{0n})\} \subset \mathcal{M}_{0,r}$ and $I_0(u_{1n}, u_{2n}, u_{0n}) \to \sigma_0$ as $n \to \infty$, then we get the boundedness of $\{(u_{1n}, u_{2n}, u_{0n})\}$ in $H^1(\mathbb{R}^3, \mathbb{R}^3)$. Therefore, we complete the proof. □

By Proposition 3.1, we get a bounded Palais-Smale sequence $\{(u_{1n}, u_{2n}, u_{0n})\}$ for I_0 at level σ_0 with $P_0(u_{1n}, u_{2n}, u_{0n}) = o(1)$. Therefore, there exists $(\bar{u}_1, \bar{u}_2, \bar{u}_0) \in H_r$, such that up to a subsequence, as $n \to +\infty$,

$$\begin{cases}(u_{1n}, u_{2n}, u_{0n}) \rightharpoonup (\bar{u}_1, \bar{u}_2, \bar{u}_0), & \text{in } H_r.\\ (u_{1n}, u_{2n}, u_{0n}) \to (\bar{u}_1, \bar{u}_2, \bar{u}_0), & \text{in } L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3),\ \forall t \in (2, 2^*).\\ (u_{1n}, u_{2n}, u_{0n}) \to (\bar{u}_1, \bar{u}_2, \bar{u}_0), & a.e. \text{ in } \mathbb{R}^3.\end{cases} \tag{3.33}$$

Further, since $I_0'\big|_{\mathcal{M}_0}(u_{1n}, u_{2n}, u_{0n}) \to 0$, there are two sequences $\{\lambda_n\},\ \{\mu_n\} \subset \mathbb{R}$, such that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(\nabla u_{1n}\cdot\nabla\phi_1 + \nabla u_{2n}\cdot\nabla\phi_2 + \nabla u_{0n}\cdot\nabla\phi_0\big)dx \\
&- \int_{\mathbb{R}^3} \Big((c_0+c_1)\big(u_{1n}^3\phi_1 + u_{2n}^3\phi_2\big) + c_0 u_{0n}^3\phi_0\Big)dx - (c_0-c_1)\int_{\mathbb{R}^3}\Big(u_{1n}\phi_1 u_{2n}^2 + u_{1n}^2 u_{2n}\phi_2\Big)dx \\
&- (c_0+c_1)\int_{\mathbb{R}^3}\Big(u_{1n}\phi_1 u_{0n}^2 + u_{1n}^2 u_{0n}\phi_0 + u_{2n}\phi_2 u_{0n}^2 + u_{2n}^2 u_{0n}\phi_0\Big)dx \qquad (3.34)\\
&- c_1\int_{\mathbb{R}^3}\Big(\phi_1 u_{2n}u_{0n}^2 + u_{1n}\phi_2 u_{0n}^2 + 2u_{1n}u_{2n}u_{0n}\phi_0\Big)dx \\
=\ &\mu_n\int_{\mathbb{R}^3}\Big(u_{1n}\phi_1 + u_{2n}\phi_2 + u_{0n}\phi_0\Big)dx + \lambda_n\int_{\mathbb{R}^3}(u_{1n}\phi_1 - u_{2n}\phi_2)dx + o(1) \\
=\ &(\mu_n+\lambda_n)\int_{\mathbb{R}^3} u_{1n}\phi_1 dx + (\mu_n-\lambda_n)\int_{\mathbb{R}^3} u_{2n}\phi_2 dx + \mu_n\int_{\mathbb{R}^3} u_{0n}\phi_0 dx + o(1),
\end{aligned}$$

for any $(\phi_1, \phi_2, \phi_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3)$.

Lemma 3.8. *$\{\lambda_n\}$ and $\{\mu_n\}$ are bounded sequences in $\mathbb{R}$. In addition, up to subsequence, at least one of $\{\lambda_n\}$ and $\{\mu_n\}$ converges to a negative value.*

Proof. Choose $(\phi_1, \phi_2, \phi_0) = (u_{1n}, 0, 0)$ in (3.34), we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla u_{1n}|^2 dx - (c_0+c_1)\int_{\mathbb{R}^3} u_{1n}^4 dx - (c_0-c_1)\int_{\mathbb{R}^3} u_{1n}^2 u_{2n}^2 dx \\
&- (c_0+c_1)\int_{\mathbb{R}^3} u_{1n}^2 u_{0n}^2 dx - c_1\int_{\mathbb{R}^3} u_{1n}u_{2n}u_{0n}^2 dx = (\mu_n+\lambda_n)\int_{\mathbb{R}^3} u_{1n}^2 dx + o(1).
\end{aligned} \tag{3.35}$$

We choose $(\phi_1, \phi_2, \phi_0) = (0, u_{2n}, 0)$ and $(0, 0, u_{0n})$ respectively in (3.34), then

$$\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla u_{2n}|^2 dx - (c_0+c_1)\int_{\mathbb{R}^3} u_{2n}^4 dx - (c_0-c_1)\int_{\mathbb{R}^3} u_{1n}^2 u_{2n}^2 dx \\
&- (c_0+c_1)\int_{\mathbb{R}^3} u_{2n}^2 u_{0n}^2 dx - c_1\int_{\mathbb{R}^3} u_{1n}u_{2n}u_{0n}^2 dx = (\mu_n-\lambda_n)\int_{\mathbb{R}^3} u_{2n}^2 dx + o(1)
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla u_{0n}|^2 dx - c_0\int_{\mathbb{R}^3} u_{0n}^4 dx - (c_0+c_1)\int_{\mathbb{R}^3} u_{1n}^2 u_{0n}^2 dx \\
&- (c_0+c_1)\int_{\mathbb{R}^3} u_{2n}^2 u_{0n}^2 dx - 2c_1\int_{\mathbb{R}^3} u_{1n}u_{2n}u_{0n}^2 dx = \mu_n\int_{\mathbb{R}^3} u_{0n}^2 dx + o(1).
\end{aligned} \tag{3.37}$$

Since $\big\{(u_{1n}, u_{2n}, u_{0n})\big\}$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R}^3)$, it is easy to see that $\{\lambda_n\},\ \{\mu_n\}$ are bounded sequences in $\mathbb{R}$. We may assume that $\lambda_n \to \lambda_0$ and $\mu_n \to \mu_0$ as $n \to \infty$. Moreover, since $(u_{1n}, u_{2n}, u_{0n}) \in$

$\mathcal{M}_{0,r}$, we then get from (3.35)-(3.37) that

$$\begin{aligned}\mu_n N+\lambda_n M&=\mu_n\int_{\mathbb{R}^3}(u_{1n}^2+u_{2n}^2+u_{0n}^2)dx+\lambda_n\int_{\mathbb{R}^3}(u_{1n}^2-u_{2n}^2)dx\\&=(\mu_n+\lambda_n)\int_{\mathbb{R}^3}u_{1n}^2dx+(\mu_n-\lambda_n)\int_{\mathbb{R}^3}u_{2n}^2dx+\mu_n\int_{\mathbb{R}^3}u_{0n}^2dx\\&=\int_{\mathbb{R}^3}\big(|\nabla u_{1n}|^2+|\nabla u_{2n}|^2+|\nabla u_{0n}|^2\big)dx\\&\quad-\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_{1n}^4+u_{2n}^4\big)+c_0u_{0n}^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_{1n}u_{2n}u_{0n}^2dx\\&\quad+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_{1n}^2u_{2n}^2+\big(c_0+c_1\big)(u_{1n}^2+u_{2n}^2)u_{0n}^2\Big)dx\Big\}+o(1).\end{aligned}\tag{3.38}$$

Together with (3.28), that is $P_0(u_{1n},u_{2n},u_{0n})=o(1)$ as $n\to\infty$, we can deduce that

$$\begin{aligned}\mu_n N+\lambda_n M=&-\frac{1}{4}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_{1n}^4+u_{2n}^4\big)+c_0u_{0n}^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_{1n}u_{2n}u_{0n}^2dx\\&+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_{1n}^2u_{2n}^2+\big(c_0+c_1\big)(u_{1n}^2+u_{2n}^2)u_{0n}^2\Big)dx\Big\}+o(1).\end{aligned}\tag{3.39}$$

Again by (3.28), we obtain that

$$\begin{aligned}I_0(u_{1n},u_{2n},u_{0n})=&\frac{1}{8}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_{1n}^4+u_{2n}^4\big)+c_0u_{0n}^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_{1n}u_{2n}u_{0n}^2dx\\&+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_{1n}^2u_{2n}^2+\big(c_0+c_1\big)(u_{1n}^2+u_{2n}^2)u_{0n}^2\Big)dx\Big\}+o(1),\end{aligned}$$

from (3.39), we get

$$\mu_n N+\lambda_n M=-2I_0(u_{1n},u_{2n},u_{0n})+o(1)\to-2\sigma_0,\quad\text{as } n\to\infty.\tag{3.40}$$

Therefore, we get $\mu_0N+\lambda_0M=-2\sigma_0<0$, which implies that one of $\mu_0,\ \lambda_0$ is negative. □

Lemma 3.9. *Suppose $\mu_0+\lambda_0<0$, $\mu_0-\lambda_0<0$ and $\mu_0<0$, then $(u_{1n},u_{2n},u_{0n})\to(\bar{u}_1,\bar{u}_2,\bar{u}_0)$ is strongly in $H^1(\mathbb{R}^3,\mathbb{R}^3)$.*

Proof. Suppose $\mu_0<0$, we prove that $u_{0n}\to\bar{u}_0$ is strongly in $H^1(\mathbb{R}^3)$. By (3.33), it is easy to see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\big(u_{1n}^2+u_{2n}^2\big)u_{0n}^2dx=\int_{\mathbb{R}^3}\big(\bar{u}_1^2+\bar{u}_2^2\big)\bar{u}_0^2dx\text{ and }\lim_{n\to\infty}\int_{\mathbb{R}^3}u_{1n}u_{2n}u_{0n}^2dx=\int_{\mathbb{R}^3}\bar{u}_1\bar{u}_2\bar{u}_0^2dx.$$

Take $(\phi_1,\phi_2,\phi_0)=(0,0,u_{0n}-\bar{u}_0)$ in (3.34), we then deduce that

$$\int_{\mathbb{R}^3}\nabla u_{0n}\cdot\nabla(u_{0n}-\bar{u}_0)dx=\mu_0\int_{\mathbb{R}^3}u_{0n}(u_{0n}-\bar{u}_0)dx+o(1).\tag{3.41}$$

Since $u_{0n}\rightharpoonup\bar{u}_0$ in $H^1(\mathbb{R}^3)$, then for any $\phi\in H^1(\mathbb{R}^3)$,

$$\begin{aligned}&\int_{\mathbb{R}^3}\nabla\bar{u}_0\cdot\nabla\phi dx-c_0\int_{\mathbb{R}^3}\bar{u}_0^3\phi dx-\big(c_0+c_1\big)\int_{\mathbb{R}^3}\big(\bar{u}_1^2+\bar{u}_2^2\big)\bar{u}_0\phi dx\\&-2c_1\int_{\mathbb{R}^3}\bar{u}_1\bar{u}_2\bar{u}_0\phi dx-\mu_0\int_{\mathbb{R}^3}\bar{u}_0\phi dx=0.\end{aligned}\tag{3.42}$$

Take $\phi = u_{0n} - \bar{u}_0$ in (3.42), we get $\int_{\mathbb{R}^3} \nabla \bar{u}_0 \cdot \nabla(u_{0n} - \bar{u}_0)dx = \mu_0 \int_{\mathbb{R}^3} \bar{u}_0(u_{0n} - \bar{u}_0)dx$. Thus, together with (3.41), we obtain $\int_{\mathbb{R}^3} |\nabla(u_{1n} - \bar{u}_0)|^2 dx - \mu_0 \int_{\mathbb{R}^3}(u_{1n} - \bar{u}_0)^2 dx = o(1)$, it yields that $\{u_{0n}\}$ is strongly convergent to $\bar{u}_0$ in $H^1(\mathbb{R}^3)$. Similarly, if $\mu_0 + \lambda_0 < 0$, we can show that $u_{1n} \to \bar{u}_1$ is strongly in $H^1(\mathbb{R}^3)$ and if $\mu_0 - \lambda_0 < 0$, $u_{2n} \to \bar{u}_2$ is strongly in $H^1(\mathbb{R}^3)$. Therefore, we complete the proof. □

Proof of Theorem 1. (i) By (3.33), we get $(\bar{u}_1, \bar{u}_2, \bar{u}_0)$ satisfies the following system

$$\begin{cases} -\Delta \bar{u}_1 = (\mu_0 + \lambda_0)\bar{u}_1 + (c_0 + c_1)\bar{u}_1^3 + (c_0 - c_1)\bar{u}_1 \bar{u}_2^2 + (c_0 + c_1)\bar{u}_0^2 \bar{u}_1 + c_1 \bar{u}_2 \bar{u}_0^2, \\ -\Delta \bar{u}_2 = (\mu_0 - \lambda_0)\bar{u}_2 + (c_0 + c_1)\bar{u}_2^3 + (c_0 - c_1)\bar{u}_1^2 \bar{u}_2 + (c_0 + c_1)\bar{u}_0^2 \bar{u}_2 + c_1 \bar{u}_1 \bar{u}_0^2, \\ -\Delta \bar{u}_0 = \mu_0 \bar{u}_0 + c_0 \bar{u}_0^3 + (c_0 + c_1)(\bar{u}_1^2 + \bar{u}_2^2)\bar{u}_0 + 2c_1 \bar{u}_1 \bar{u}_2 \bar{u}_0. \end{cases} \tag{3.43}$$

Next, we show that $\mu_0 + \lambda_0$, $\mu_0 - \lambda_0$ and μ_0 are all negative. By (3.38) and (3.40), we can see that

$$(\mu_n + \lambda_n)\int_{\mathbb{R}^3} u_{1n}^2 dx + (\mu_n - \lambda_n)\int_{\mathbb{R}^3} u_{2n}^2 dx + \mu_n \int_{\mathbb{R}^3} u_{0n}^2 dx \to -2\sigma_0 < 0, \quad \text{as } n \to \infty,$$

which implies one of $\mu_0 + \lambda_0$, $\mu_0 - \lambda_0$ and μ_0 is negative. By Lemma 3.8, there exist five possibilities:

Case 1: $\lambda_0 < 0,\ \mu_0 < 0,\ \mu_0 - \lambda_0 < 0$ and $\mu_0 + \lambda_0 < 0$;

Case 2: $\lambda_0 < 0,\ \mu_0 < 0,\ \mu_0 - \lambda_0 \geq 0$ and $\mu_0 + \lambda_0 < 0$;

Case 3: $\lambda_0 < 0,\ \mu_0 \geq 0,\ \mu_0 - \lambda_0 \geq 0$ and $\mu_0 + \lambda_0 < 0$;

Case 4: $\lambda_0 \geq 0,\ \mu_0 < 0,\ \mu_0 - \lambda_0 < 0$ and $\mu_0 + \lambda_0 < 0$;

Case 5: $\lambda_0 \geq 0,\ \mu_0 < 0,\ \mu_0 - \lambda_0 < 0$ and $\mu_0 + \lambda_0 \geq 0$.

We now argue by contradiction to rule out Case 2, Case 3 and Case 5. Suppose Case 2 holds, i.e. $\lambda_0 < 0,\ \mu_0 < 0,\ \mu_0 - \lambda_0 \geq 0$ and $\mu_0 + \lambda_0 < 0$. From the proof of Lemma 3.9, we get that both $u_{1n} \to \bar{u}_1$ and $u_{0n} \to \bar{u}_0$ are strongly in $H^1(\mathbb{R}^3)$. In addition, by (3.43) and Lemma 2.4, we get $\bar{u}_2 \equiv 0$. By the structure of system (3.43), we have $\bar{u}_1 \equiv 0$ or $\bar{u}_0 \equiv 0$. If $\bar{u}_1 = \bar{u}_2 \equiv 0$, then $\bar{u}_0$ satisfies

$$\begin{cases} -\Delta \bar{u}_0 = \mu_0 \bar{u}_0 + c_0 \bar{u}_0^3, \\ \int_{\mathbb{R}^3} |\bar{u}_0|^2 dx = N. \end{cases}$$

Lemma 2.3 in [22] shows that $m_1(b, c_0 + c_1)$ and $m_2(b, c_0)$ is strictly decreasing with respect to b. So we have

$$\begin{aligned} \sigma_0 &= \lim_{n\to\infty} I_0(u_{1n}, u_{2n}, u_{0n}) \\ &= \lim_{n\to\infty} \frac{1}{8}\Big\{ \int_{\mathbb{R}^3} \Big((c_0 + c_1)\big(u_{1n}^4 + u_{2n}^4\big) + c_0 u_{0n}^4\Big)dx + 4c_1 \int_{\mathbb{R}^3} u_{1n} u_{2n} u_{0n}^2 dx \\ &\quad + 2\int_{\mathbb{R}^3} \Big(\big(c_0 - c_1\big)u_1^2 u_2^2 + \big(c_0 + c_1\big)(u_1^2 + u_2^2)u_0^2\Big)dx\Big)dx\Big\} \\ &= \frac{1}{8}c_0 \int_{\mathbb{R}^3} |\bar{u}_0|^4 dx = m_2\big(\sqrt{N}, c_0\big), \end{aligned}$$

which contradicts to Lemma 3.7.

If $\bar{u}_0 = \bar{u}_2 \equiv 0$, then $\bar{u}_1$ satisfies

$$\begin{cases} -\Delta \bar{u}_1 = (\mu_0 + \lambda_0)\bar{u}_1 + (c_0 + c_1)\bar{u}_1^3, \\ \int_{\mathbb{R}^3} |\bar{u}_1|^2 dx = \dfrac{M + N}{2}. \end{cases}$$

We have

$$\begin{aligned}\sigma_0 &= \lim_{n\to\infty} I_0(u_{1n}, u_{2n}, u_{0n})\\ &= \lim_{n\to\infty} \frac{1}{8}\Big\{ \int_{\mathbb{R}^3} \Big((c_0+c_1)\big(u_{1n}^4 + u_{2n}^4\big) + c_0 u_{0n}^4\Big) dx + 4c_1 \int_{\mathbb{R}^3} u_{1n}u_{2n}u_{0n}^2 dx\\ &\quad + 2\int_{\mathbb{R}^3} \Big(\big(c_0 - c_1\big) u_1^2 u_2^2 + \big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2 \Big) dx \Big) dx \Big\}\\ &= \frac{1}{8}(c_0+c_1)\int_{\mathbb{R}^3} |\bar{u}_1|^4 dx = m_1\Big(\sqrt{\frac{N+M}{2}}, c_0+c_1\Big),\end{aligned}$$

which contradicts to Lemma 3.7 as well. Hence, Case 2 is impossible.

Suppose Case 3 holds, i.e. $\lambda_0 < 0,\ \mu_0 \geq 0,\ \mu_0 - \lambda_0 \geq 0$ and $\mu_0 + \lambda_0 < 0$. From the proof of Lemma 3.9, we get $u_{1n} \to \bar{u}_1$ is strongly in $H^1(\mathbb{R}^3)$. Moreover, by (3.43) and Lemma 2.4, it is easy to see that $\bar{u}_2 = \bar{u}_0 \equiv 0$. Therefore, $\bar{u}_1$ satisfies

$$\begin{cases} -\Delta \bar{u}_1 = (\mu_0+\lambda_0)\bar{u}_1 + (c_0+c_1)\bar{u}_1^3,\\ \displaystyle\int_{\mathbb{R}^3} |\bar{u}_1|^2 dx \leq \frac{N+M}{2}. \end{cases} \tag{3.44}$$

Thus, we obtain

$$\begin{aligned}\sigma_0 &= \lim_{n\to\infty} I_0(u_{1n}, u_{2n}, u_{0n})\\ &= \lim_{n\to\infty} \frac{1}{8}\Big\{ \int_{\mathbb{R}^3} \Big((c_0+c_1)\big(u_{1n}^4 + u_{2n}^4\big) + c_0 u_{0n}^4\Big) dx + 4c_1 \int_{\mathbb{R}^3} u_{1n}u_{2n}u_{0n}^2 dx\\ &\quad + 2\int_{\mathbb{R}^3} \Big(\big(c_0 - c_1\big) u_1^2 u_2^2 + \big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2 \Big) dx \Big) dx \Big\}\\ &= \frac{1}{8}(c_0+c_1)\int_{\mathbb{R}^3} |\bar{u}_1|^4 dx = m_1(\|\bar{u}_1\|_{L^2}, c_0+c_1) \geq m_1\Big(\sqrt{\frac{N+M}{2}}, c_0+c_1\Big),\end{aligned}$$

which contradicts to Lemma 3.7. Hence, Case 3 can not happen.

Suppose Case 5 holds, i.e. $\lambda_0 \geq 0,\ \mu_0 < 0,\ \mu_0 - \lambda_0 < 0$ and $\mu_0 + \lambda_0 \geq 0$. From the proof of Lemma 3.9, we get that both $u_{2n} \to \bar{u}_2$ and $u_{0n} \to \bar{u}_0$ are strongly in $H^1(\mathbb{R}^3)$. In addition, by (3.43) and Lemma 2.4, we get $\bar{u}_1 \equiv 0$. By the structure of system (3.43), we have $\bar{u}_2 \equiv 0$ or $\bar{u}_0 \equiv 0$. If $\bar{u}_1 = \bar{u}_2 \equiv 0$, then $\bar{u}_0$ satisfies

$$\begin{cases} -\Delta \bar{u}_0 = \mu_0\bar{u}_0 + c_0\bar{u}_0^3,\\ \displaystyle\int_{\mathbb{R}^3} |\bar{u}_0|^2 dx = N - M, \end{cases}$$

so we have

$$\begin{aligned}\sigma_0 &= \lim_{n\to\infty} I_0(u_{1n}, u_{2n}, u_{0n})\\ &= \lim_{n\to\infty} \frac{1}{8}\Big\{ \int_{\mathbb{R}^3} \Big((c_0+c_1)\big(u_{1n}^4 + u_{2n}^4\big) + c_0 u_{0n}^4\Big) dx + 4c_1 \int_{\mathbb{R}^3} u_{1n}u_{2n}u_{0n}^2 dx\\ &\quad + 2\int_{\mathbb{R}^3} \Big(\big(c_0 - c_1\big) u_1^2 u_2^2 + \big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2 \Big) dx \Big) dx \Big\}\\ &= \frac{1}{8}c_0\int_{\mathbb{R}^3} |\bar{u}_0|^4 dx = m_2\big(\sqrt{N-M}, c_0\big) \geq m_2\big(\sqrt{N}, c_0\big),\end{aligned}$$

which contradicts to Lemma 3.7.

If $\bar{u}_1 = \bar{u}_0 \equiv 0$, then $\bar{u}_2$ satisfies

$$\begin{cases} -\Delta \bar{u}_2 = (\mu_0 + \lambda_0)\bar{u}_2 + (c_0 + c_1)\bar{u}_2^3, \\ \int_{\mathbb{R}^3} |\bar{u}_2|^2 dx = \frac{N-M}{2}. \end{cases}$$

Thus, we obtain

$$\begin{aligned} \sigma_0 &= \lim_{n\to\infty} I_0(u_{1n}, u_{2n}, u_{0n}) \\ &= \lim_{n\to\infty} \frac{1}{8}\Big\{ \int_{\mathbb{R}^3} \Big((c_0 + c_1)\big(u_{1n}^4 + u_{2n}^4\big) + c_0 u_{0n}^4 \Big) dx + 4c_1 \int_{\mathbb{R}^3} u_{1n}u_{2n}u_{0n}^2 dx \\ &\quad + 2\int_{\mathbb{R}^3} \Big((c_0 - c_1)u_1^2u_2^2 + \big(c_0 + c_1\big)(u_1^2 + u_2^2)u_0^2 \Big) dx \Big) dx \Big\} \\ &= \frac{1}{8}(c_0 + c_1) \int_{\mathbb{R}^3} |\bar{u}_2|^4 dx = m_1\Big(\sqrt{\frac{N-M}{2}}, c_0 + c_1\Big) > m_1\Big(\sqrt{\frac{N+M}{2}}, c_0 + c_1\Big), \end{aligned}$$

which also contradicts to Lemma 3.7. Hence, Case 5 can not happen. Therefore, we get that $\mu_0 + \lambda_0$, $\mu_0 - \lambda_0$ and μ_0 are all negative. By Lemma 3.9, $(u_{1n}, u_{2n}, u_{0n}) \to (\bar{u}_1, \bar{u}_2, \bar{u}_0)$ is strongly in $H^1(\mathbb{R}^3, \mathbb{R}^3)$. Then $(\bar{u}_1, \bar{u}_2, \bar{u}_0) \in \mathcal{M}_0$ is a solution to (1.6). Further, from Lemma 3.2 and Lemma 3.6, we can see that $(\bar{u}_1, \bar{u}_2, \bar{u}_0)$ is a ground state solution of (1.6). Hence, we complete the proof of Theorem 1 (i).

(ii) Let $(\bar{u}_1, \bar{u}_2, \bar{u}_0) \in \mathcal{M}_0$ be the ground state solution for (1.6) obtained in Theorem 1 (i), then $P_0(\bar{u}_1, \bar{u}_2, \bar{u}_0) = 0$, that is

$$\begin{aligned} &\int_{\mathbb{R}^3} \big(|\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 + |\nabla \bar{u}_0|^2\big) dx - \frac{3}{4}\Big\{ \int_{\mathbb{R}^3} \Big((c_0 + c_1)\big(\bar{u}_1^4 + \bar{u}_2^4\big) + c_0 \bar{u}_0^4 \Big) dx \\ &+ 2\int_{\mathbb{R}^3} \Big((c_0 - c_1)\bar{u}_1^2\bar{u}_2^2 + \big(c_0 + c_1\big)(\bar{u}_1^2 + \bar{u}_2^2)\bar{u}_0^2 \Big) dx + 4c_1 \int_{\mathbb{R}^3} \bar{u}_1\bar{u}_2\bar{u}_0^2 dx \Big\} = 0. \end{aligned} \tag{3.45}$$

Together with (3.6), we get

$$\begin{aligned} &\frac{1}{3}\int_{\mathbb{R}^3} \big(|\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 + |\nabla \bar{u}_0|^2\big) dx \\ &\le \max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\} C_*\Big(\int_{\mathbb{R}^3} \big(|\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 + |\nabla \bar{u}_0|^2\big) dx\Big)^{\frac{3}{2}} N^{\frac{1}{2}}, \end{aligned}$$

then

$$\int_{\mathbb{R}^3} \big(|\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 + |\nabla \bar{u}_0|^2\big) dx \ge \Big(\frac{4}{3\max\big\{c_0, 3c_0 + 4c_1\big\} C_* N^{\frac{1}{2}}}\Big)^2.$$

Thus,

$$\int_{\mathbb{R}^3} \big(|\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 + |\nabla \bar{u}_0|^2\big) dx \to +\infty, \quad \text{as } N \to 0^+.$$

Further, by (3.45), we get

$$I_0(\bar{u}_1, \bar{u}_2, \bar{u}_0) = \frac{1}{6}\int_{\mathbb{R}^3} \big(|\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 + |\nabla \bar{u}_0|^2\big) dx \to +\infty, \quad \text{as } N \to 0^+.$$

Now, we consider the asymptotic behavior of $(\bar{u}_1, \bar{u}_2, \bar{u}_0)$ as $N \to +\infty$. Define

$$u_{1N}(x) := \sqrt{\frac{M}{C_0}}(\tau \star Q)(x), \quad u_{2N}(x) :\equiv 0, \quad u_{0N}(x) := \sqrt{\frac{N-M}{C_0}}(\tau \star Q)(x),$$

where $Q(x)$ and C_0 are defined in (2.1) and (2.3), then it is easy to see that $(u_{1N}, u_{2N}, u_{0N}) \in \mathcal{M}_0$ and further by direct calculation and (2.2), we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(|\nabla u_{1N}|^2 + |\nabla u_{2N}|^2 + |\nabla u_{0N}|^2\big)dx \\
=& \frac{M}{C_0}e^{2\tau}\int_{\mathbb{R}^3}|\nabla Q|^2 dx + \frac{N-M}{C_0}e^{2\tau}\int_{\mathbb{R}^3}|\nabla Q|^2 dx = \frac{N}{C_0}e^{2\tau}\int_{\mathbb{R}^3}|\nabla Q|^2 dx = 3Ne^{2\tau}.
\end{aligned}$$

Moreover, by (2.2) and (3.5), we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_{1N}^4 + u_{2N}^4\big) + c_0 u_{0N}^4\Big)dx + 4c_1\int_{\mathbb{R}^3} u_{1N}u_{2N}u_{0N}^2 dx \\
&+ 2\int_{\mathbb{R}^3}\Big(\big(c_0 - c_1\big)u_{1N}^2 u_{2N}^2 + \big(c_0+c_1\big)(u_{1N}^2 + u_{2N}^2)u_{0N}^2\Big)dx \\
=& c_0\int_{\mathbb{R}^3}\big(u_{1N}^2 + u_{2N}^2 + u_{0N}^2\big)^2 dx + 2c_1\int_{\mathbb{R}^3}\big(u_{1N} + u_{2N}\big)^2 u_{0N}^2 dx + c_1\int_{\mathbb{R}^3}\big(u_{1N}^2 - u_{2N}^2\big)^2 dx \\
=& c_0\int_{\mathbb{R}^3}\Big(\frac{M}{C_0}e^{3\tau}Q^2(\tau x) + \frac{N-M}{C_0}e^{3\tau}Q^2(\tau x)\Big)^2 dx \\
&+ 2c_1\int_{\mathbb{R}^3}\Big(\frac{M}{C_0}e^{3\tau}Q^2(\tau x)\cdot\frac{N-M}{C_0}e^{3\tau}Q^2(\tau x)\Big)dx + c_1\int_{\mathbb{R}^3}\Big(\frac{M}{C_0}e^{3\tau}Q^2(\tau x)\Big)^2 dx \\
=& c_0\int_{\mathbb{R}^3}\Big(\frac{N}{C_0}\Big)^2 e^{6\tau}Q^4(\tau x)dx + c_1\int_{\mathbb{R}^3}\Big(\frac{M}{C_0}\Big)^2 e^{6\tau}Q^4(\tau x)dx \\
&+ 2c_1\int_{\mathbb{R}^3}\frac{M(N-M)}{C_0^2}e^{6\tau}Q^4(\tau x)dx \\
=& \frac{c_0N^2 + c_1M^2}{C_0^2}e^{3\tau}\int_{\mathbb{R}^3}Q^4(x)dx + \frac{2c_1M(N-M)}{C_0^2}e^{3\tau}\int_{\mathbb{R}^3}Q^4(x)dx \\
=& \frac{4(c_0N^2 + c_1M^2)}{C_0}e^{3\tau} + \frac{8c_1M(N-M)}{C_0}e^{3\tau} = \frac{4\big(c_0N^2 + c_1M^2 + 2c_1M(N-M)\big)}{C_0}e^{3\tau}.
\end{aligned}$$

Choose

$$\tau = \ln\Big(\frac{NC_0}{c_0N^2 + c_1M^2 + 2c_1M(N-M)}\Big),$$

then

$$\begin{aligned}
&\int_{\mathbb{R}^3}\big(|\nabla u_{1N}|^2 + |\nabla u_{2N}|^2 + |\nabla u_{0N}|^2\big)dx - \frac{3}{4}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_{1N}^4 + u_{2N}^4\big) + c_0u_{0N}^4\Big)dx \\
&+ 4c_1\int_{\mathbb{R}^3}u_{1N}u_{2N}u_{0N}^2 dx + 2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_{1N}^2u_{2N}^2 + \big(c_0+c_1\big)(u_{1N}^2 + u_{2N}^2)u_{0N}^2\Big)dx\Big\} \\
=& 3Ne^{2\tau} - \frac{3\big(c_0N^2 + c_1M^2 + 2c_1M(N-M)\big)}{C_0}e^{3\tau} = 0.
\end{aligned}$$

It yields that

$$
\begin{aligned}
&I_0(u_{1N},u_{2N},u_{0N})\\
&=\frac{1}{2}\int_{\mathbb{R}^3}\big(|\nabla u_{1N}|^2+|\nabla u_{2N}|^2+|\nabla u_{0N}|^2\big)dx-\frac{1}{4}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_{1N}^4+u_{2N}^4\big)+c_0u_{0N}^4\Big)dx\\
&\quad+4c_1\int_{\mathbb{R}^3}u_{1N}u_{2N}u_{0N}^2dx+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_{1N}^2u_{2N}^2+(c_0+c_1)(u_{1N}^2+u_{2N}^2)u_{0N}^2\Big)dx\Big\}\\
&=\frac{1}{6}\int_{\mathbb{R}^3}\big(|\nabla u_{1N}|^2+|\nabla u_{2N}|^2+|\nabla u_{0N}|^2\big)dx\\
&=\frac{Ne^{2\tau}}{2}=\frac{N}{2}\Big(\frac{NC_0}{c_0N^2+c_1M^2+2c_1M(N-M)}\Big)^2\to 0,\quad \text{as } N\to+\infty.
\end{aligned}
$$

Hence, we conclude that

$$
0<I_0(\bar u_1,\bar u_2,\bar u_0)=\inf_{(u_1,u_2,u_0)\in\mathcal{P}_0}I_0(u_1,u_2,u_0)\le I_0(u_{1N},u_{2N},u_{0N})\to 0,\quad \text{as } N\to+\infty.
$$

Moreover,

$$
\int_{\mathbb{R}^3}\big(|\nabla\bar u_1|^2+|\nabla\bar u_2|^2+|\nabla\bar u_0|^2\big)dx=6I_0(\bar u_1,\bar u_2,\bar u_0)\to 0,\quad \text{as } N\to+\infty.
$$

Therefore, we complete the proof. □

For any $(u_1,u_2,u_0)\in H^1(\mathbb{R}^3,\mathbb{R}^3)$, we introduce the function

$$
\begin{aligned}
\eta(\tau):=&\frac{\tau^2}{2}\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx-\frac{\tau^3}{4}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx\\
&+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_1^2u_2^2+(c_0+c_1)(u_1^2+u_2^2)u_0^2\Big)dx+4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\Big\},\quad \text{for } \tau>0.
\end{aligned}
$$

Notice that $\eta(\tau)=I_0\big(\ln\tau\star(u_1,u_2,u_0)\big)$. It is clear that for any $(u_1,u_2,u_0)\in H^1(\mathbb{R}^3,\mathbb{R}^3)$, there exists a unique critical point $\tau_0>0$ for $\eta(\tau)$, which is a strict maximum and $\ln\tau_0\star(u_1,u_2,u_0)\in\mathcal{P}_0$.

Lemma 3.10. *Let $m=\inf_{(u_1,u_2,u_0)\in\mathcal{P}_0}I_0(u_1,u_2,u_0)$, then*

$$
P_0(u_1,u_2,u_0)<0\Rightarrow P_0(u_1,u_2,u_0)\le I_0(u_1,u_2,u_0)-m.
$$

Proof. By direct calculation, we get $P_0(u_1,u_2,u_0)=\eta'(1)$. Thus the condition $P_0(u_1,u_2,u_0)<0$ implies that $\tau_0<1$. For $(u_1,u_2,u_0)\in\mathcal{M}_0$, we have

$$
\begin{aligned}
\eta''(\tau)=&\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\\
&-\frac{3\tau}{2}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\\
&+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_1^2u_2^2+(c_0+c_1)(u_1^2+u_2^2)u_0^2\Big)dx\Big\},
\end{aligned}
$$

then there exists a unique $\tau_1\in\mathbb{R}$, such that $\eta''(\tau_1)=0$. Since $\eta'(\tau_0)=0$, we have

$$
\begin{aligned}
\eta''(\tau_0)=&-\frac{3}{4}\tau_0\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx\\
&+2\int_{\mathbb{R}^3}\Big((c_0-c_1)u_1^2u_2^2+(c_0+c_1)(u_1^2+u_2^2)u_0^2\Big)dx\Big\}<0.
\end{aligned}
$$

Then it is easy to see that $\eta(\tau)$ is concave in $(\tau_0, +\infty)$. As a consequence,

$$I_0(u_1, u_2, u_0) = \eta(1) \geq \eta(\tau_0) + (1 - \tau_0)\eta'(1) \geq \eta(\tau_0) + P_0(u_1, u_2, u_0) \geq m + P_0(u_1, u_2, u_0),$$

thus $P_0(u_1, u_2, u_0) \leq I_0(u_1, u_2, u_0) - m$. $\square$

Proof of Theorem 1. (iii) Suppose $\bar{\mathbf{u}} := (\bar{u}_1, \bar{u}_2, \bar{u}_0)$ be the solution obtained in (i) of Theorem 1. Let $\bar{\mathbf{u}}_\tau := \tau \star \bar{\mathbf{u}} := \tau \star (\bar{u}_1, \bar{u}_2, \bar{u}_0)$, since $\bar{\mathbf{u}} \in \mathcal{P}_0$, then for every $\tau > 0$, $P_0(\tau \star \bar{u}_1, \tau \star \bar{u}_2, \tau \star \bar{u}_0) < 0$. Let $\mathbf{\Phi}^\tau = (\Phi_1^\tau, \Phi_2^\tau, \Phi_0^\tau)$ be the solution of system (1.3) with initial datum $\bar{\mathbf{u}}_\tau$ defined on the maximal interval $(T_{\min}, T_{\max})$. By continuity of P_0, provided $|t|$ is sufficiently small we have $P_0(\mathbf{\Phi}^\tau(t)) < 0$. Therefore, by Lemma 3.10 and recalling that the energy is conserved along trajectories of system (1.3), we have

$$P_0(\mathbf{\Phi}^\tau(t)) \leq I_0(\mathbf{\Phi}^\tau(t)) - m = I_0(\bar{\mathbf{u}}_\tau) - m =: -\delta < 0,$$

for any such t, and by continuity again we infer that $P_0(\mathbf{\Phi}^\tau(t)) < -\delta < 0$ for every $t \in (T_{\min}, T_{\max})$. To obtain a contradiction, we define

$$f_\tau(t) := \int_{\mathbb{R}^3} |x|^2 \big((\Phi_1^\tau(t,x))^2 + (\Phi_2^\tau(t,x))^2 + (\Phi_0^\tau(t,x))^2 \big) dx,$$

then

$$f'_\tau(t) = 2\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} |x|^2 \overline{\Phi^\tau}_j(t,x) i\partial_t \Phi_j^\tau(t,x) dx = 4\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \overline{\Phi^\tau}_j(t,x) x \cdot \triangledown \Phi_j^\tau(t,x) dx.$$

Thus

$$f''_\tau(t) = 4\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \Big(\partial_t \overline{\Phi^\tau}_j(t,x) x \cdot \triangledown \Phi_j^\tau(t,x) + \overline{\Phi^\tau}_j(t,x) x \cdot \triangledown \partial_t \Phi_j^\tau(t,x) \Big) dx.$$

Since

$$\begin{aligned}
&4\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \overline{\Phi^\tau}_j(t,x) x \cdot \triangledown \partial_t \Phi_j^\tau(t,x) dx = 4\sum_{j=0}^{2}\sum_{k=1}^{3} \operatorname{Im} \int_{\mathbb{R}^3} \overline{\Phi^\tau}_j(t,x) x_k \cdot \partial_k \partial_t \Phi_j^\tau(t,x) dx \\
&= -4\sum_{j=0}^{2}\sum_{k=1}^{3} \operatorname{Im} \int_{\mathbb{R}^3} \partial_t \Phi_j^\tau(t,x) \partial_k (\overline{\Phi^\tau}_j(t,x) x_k) dx \\
&= -4 \left(\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \partial_t \Phi_j^\tau(t,x) x \cdot \triangledown \overline{\Phi^\tau}_j(t,x) dx + 3\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \partial_t \Phi_j^\tau(t,x) \overline{\Phi^\tau}_j(t,x) dx \right),
\end{aligned}$$

we have

$$\begin{aligned}
f''_\tau(t) &= 4\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \Big(\partial_t \overline{\Phi^\tau}_j(t,x) x \cdot \triangledown \Phi_j^\tau(t,x) + \overline{\Phi^\tau}_j(t,x) x \cdot \triangledown \partial_t \Phi_j^\tau(t,x) \Big) dx \\
&= -4 \left(\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \partial_t \Phi_j^\tau(t,x) 2x \cdot \triangledown \overline{\Phi^\tau}_j(t,x) dx + 3\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \partial_t \Phi_j^\tau(t,x) \overline{\Phi^\tau}_j(t,x) dx \right).
\end{aligned}$$

From the virial identity (see Lemma 3.7 in [26]), we get

$$
\begin{aligned}
f_\tau''(t) = 8\int_{\mathbb{R}^3} &\big(|\nabla\Phi_1^\tau|^2+|\nabla\Phi_2^\tau|^2+|\nabla\Phi_0^\tau|^2\big)dx \\
&-6c_0\int_{\mathbb{R}^3}\Big((\Phi_1^\tau)^2+(\Phi_2^\tau)^2+(\Phi_0^\tau)^2\Big)^2dx-6c_1\Big\{4\int_{\mathbb{R}^3}\mathrm{Re}\overline{\Phi}_1^\tau\overline{\Phi}_2^\tau(\Phi_0^\tau)^2dx \\
&+\int_{\mathbb{R}^3}\Big(\big((\Phi_1^\tau)^2-(\Phi_2^\tau)^2\big)^2+2\big((\Phi_1^\tau)^2+(\Phi_2^\tau)^2\big)(\Phi_0^\tau)^2\Big)dx\Big\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
P_0(\mathbf{\Phi}^\tau(t)) = \int_{\mathbb{R}^3} &\big(|\nabla\Phi_1^\tau|^2+|\nabla\Phi_2^\tau|^2+|\nabla\Phi_0^\tau|^2\big)dx \\
&-\frac{3}{4}c_0\int_{\mathbb{R}^3}\Big((\Phi_1^\tau)^2+(\Phi_2^\tau)^2+(\Phi_0^\tau)^2\Big)^2dx-\frac{3}{4}c_1\Big\{4\int_{\mathbb{R}^3}\mathrm{Re}\overline{\Phi}_1^\tau\overline{\Phi}_2^\tau(\Phi_0^\tau)^2dx \\
&+\int_{\mathbb{R}^3}\Big(\big((\Phi_1^\tau)^2-(\Phi_2^\tau)^2\big)^2+2\big((\Phi_1^\tau)^2+(\Phi_2^\tau)^2\big)(\Phi_0^\tau)^2\Big)dx\Big\},
\end{aligned}
$$

we have $f_\tau''(t) = 8P_0(\mathbf{\Phi}^\tau(t)) < -8\delta < 0$, and as a consequence

$$
0 \le f_\tau(t) \le -\delta t^2 + O(t), \quad \text{for all } t\in(-T_{\min}, T_{\max}).
$$

Since the right hand side becomes negative for $|t|$ sufficiently large, it is necessary that both $T_{\min}$ and $T_{\max}$ are bounded. This proves that, for a sequence of initial data arbitrarily close to $\bar{\mathbf{u}}$, we have blow-up in finite time, implying orbital instability. □

4 Proof of Theorem 2

In this section, we are going to investigate the existence, stability, and asymptotic behavior of solutions to (1.4) with $V(x) = |x|^2$. We first recall some notations. Define

$$
\Lambda := \Big\{(u_1,u_2,u_0)\in H^1(\mathbb{R}^3,\mathbb{R}^3)\big|\int_{\mathbb{R}^3}|x|^2\big(u_1^2+u_2^2+u_0^2\big)dx < +\infty\Big\},
$$

then Λ is a Hilbert space equipped with the norm

$$
\|(u_1,u_2,u_0)\|_\Lambda := \Big(\int_{\mathbb{R}^3}\Big(\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)+(1+|x|^2)\big(u_1^2+u_2^2+u_0^2\big)\Big)dx\Big)^{\frac{1}{2}}.
$$

Denote

$$
\mathcal{M}(N) := \Big\{(u_1,u_2,u_0)\in\Lambda\big|\int_{\mathbb{R}^3}\big(u_1^2+u_2^2+u_0^2\big)dx = N,\ \int_{\mathbb{R}^3}\big(u_1^2-u_2^2\big)dx = M\Big\},
$$

then the corresponding energy functional to (1.4) is defined as

$$
\begin{aligned}
I(u_1,u_2,u_0) :&= I_0(u_1,u_2,u_0) + \frac{1}{2}\int_{\mathbb{R}^3} |x|^2\big(u_1^2+u_2^2+u_0^2\big)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx + \frac{1}{2}\int_{\mathbb{R}^3} |x|^2\big(u_1^2+u_2^2+u_0^2\big)dx \\
&\quad - \frac{1}{4}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx + 4c_1\int_{\mathbb{R}^3} u_1u_2u_0^2dx \\
&\quad + 2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx\Big\},
\end{aligned}
$$

for $(u_1,u_2,u_0)\in\mathcal{M}(N)$. The solutions of (1.4) can be found as critical points of $I(u_1,u_2,u_0)$ restricted to $\mathcal{M}(N)$. For simplicity, we just denote $\mathcal{M}(N)$ as $\mathcal{M}$ in the following. Define the Pohozaev manifold of system (1.4) as

$$
\mathcal{P} := \Big\{(u_1,u_2,u_0)\in\mathcal{M}\Big|\ P(u_1,u_2,u_0)=0\Big\}, \tag{4.1}
$$

where

$$
\begin{aligned}
P(u_1,u_2,u_0) :&= \int_{\mathbb{R}^3} \big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx - \int_{\mathbb{R}^3} |x|^2\big(u_1^2+u_2^2+u_0^2\big)dx \\
&\quad - \frac{3}{4}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx + 4c_1\int_{\mathbb{R}^3} u_1u_2u_0^2dx \\
&\quad + 2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx\Big\}.
\end{aligned}
$$

By the similar argument as the proof of Lemma 3.5, we have

Lemma 4.1. *$\mathcal{P}$ is a C^1 submanifold in $\mathcal{M}$ with codimension 3.*

Lemma 4.2. *Suppose $(u_1,u_2,u_0)\in\mathcal{M}$ is a solution of (1.4), then $P(u_1,u_2,u_0)=0$ and further $I(u_1,u_2,u_0)>0$.*

Proof. Since (u_1,u_2,u_0) is a solution of (1.4), we get that (u_1,u_2,u_0) satisfies the following Pohozaev identity

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx + 5\int_{\mathbb{R}^3} |x|^2\big(u_1^2+u_2^2+u_0^2\big)dx \\
&- \frac{3}{2}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx \\
&+ 2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx + 4c_1\int_{\mathbb{R}^3} u_1u_2u_0^2dx\Big\} \\
&= 3\Big((\mu+\lambda)\int_{\mathbb{R}^3} u_1^2dx + (\mu-\lambda)\int_{\mathbb{R}^3} u_2^2dx + \mu\int_{\mathbb{R}^3} u_0^2dx\Big).
\end{aligned} \tag{4.2}
$$

Multiplying the three equations in (1.4) by $u_1,\ u_2,\ u_0$ and integrating by parts respectively, we then obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big)dx + \int_{\mathbb{R}^3} |x|^2\big(u_1^2 + u_2^2 + u_0^2\big)dx \\
&- \Big\{ \int_{\mathbb{R}^3} \Big(\big(c_0 + c_1\big)\big(u_1^4 + u_2^4\big) + c_0 u_0^4\Big)dx \\
&+ 2\int_{\mathbb{R}^3} \Big(\big(c_0 - c_1\big)u_1^2u_2^2 + \big(c_0 + c_1\big)(u_1^2 + u_2^2)u_0^2\Big)dx + 4c_1 \int_{\mathbb{R}^3} u_1u_2u_0^2dx\Big\} \\
&= \Big((\mu + \lambda)\int_{\mathbb{R}^3} u_1^2dx + (\mu - \lambda)\int_{\mathbb{R}^3} u_2^2dx + \mu\int_{\mathbb{R}^3} u_0^2dx\Big).
\end{aligned}
$$

Together with (4.2), we have $P(u_1, u_2, u_0) = 0$. It implies that

$$
\begin{aligned}
I(u_1, u_2, u_0) &= \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big)dx + \frac{1}{2}\int_{\mathbb{R}^3} |x|^2\big(u_1^2 + u_2^2 + u_0^2\big)dx \\
&- \frac{1}{3}\Big\{ \int_{\mathbb{R}^3} \big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big)dx - \int_{\mathbb{R}^3} |x|^2\big(u_1^2 + u_2^2 + u_0^2\big)dx\Big\} \\
&= \frac{1}{6}\int_{\mathbb{R}^3} \big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big)dx + \frac{5}{6}\int_{\mathbb{R}^3} |x|^2\big(u_1^2 + u_2^2 + u_0^2\big)dx > 0.
\end{aligned}
$$

□

Now, we prove a local minima structure for $I(u_1, u_2, u_0)$ on $\mathcal{M}$. Define

$$
\|(u_1, u_2, u_0)\|_\Lambda^2 := \int_{\mathbb{R}^3} \Big(\big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big) + |x|^2\big(u_1^2 + u_2^2 + u_0^2\big)\Big)dx
$$

and for any $r > 0$, let

$$
B(r) := \Big\{(u_1, u_2, u_0) \in \Lambda \Big| \|(u_1, u_2, u_0)\|_\Lambda^2 \le r\Big\}.
$$

Lemma 4.3. *(Antonelli et al [1]) The pure point spectrum of the harmonic oscillator* $-\Delta + |x|^2$ *is*

$$
\sigma(-\Delta + |x|^2) = \big\{\xi_k = 3 + 2k,\ k \in \mathbb{N}\big\},
$$

and the corresponding eigenfunctions are given by Hermite functions (denoted by Ψ_k*, associated to* ξ_k*), which form an orthogonal basis of* $L^2(\mathbb{R}^3)$*.*

By Lemma 4.3, for any $(u_1, u_2, u_0) \in \Lambda$, there holds

$$
\int_{\mathbb{R}^3} \Big(\big(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\big) + |x|^2\big(u_1^2 + u_2^2 + u_0^2\big)\Big)dx \ge 3\int_{\mathbb{R}^3} \big(u_1^2 + u_2^2 + u_0^2\big)dx. \tag{4.3}
$$

Lemma 4.4. *Suppose* $c_0 > 0$ *and* $c_0 + c_1 > 0$*, then for any* $r > 0$*, it holds*

$$
\mathcal{M} \cap B(r) \ne \emptyset, \quad \text{if } N \le \frac{r}{3}, \tag{4.4}
$$

and further $I(u_1, u_2, u_0)$ *is bounded from below on* $\mathcal{M} \cap B(r)$*.*

Proof. For any $r>0$, by Lemma 4.3, it is easy to see that $\Big(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \in \mathcal{M}$. Moreover, if $N \le \frac{r}{3}$,

$$\begin{aligned}
&\Big\|\Big(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big)\Big\|_{\dot\Lambda}^2 \\
&= N\int_{\mathbb{R}^3}\Big(|\nabla\Psi_0|^2 + |x|^2{\Psi_0}^2\Big)dx = N\|(\Psi_0,0,0)\|_{\dot\Lambda}^2 = 3N \le r.
\end{aligned}$$

Hence, $\Big(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \in \mathcal{M}\cap B(r)$. For any $(u_1,u_2,u_0)\in\mathcal{M}\cap B(r)$, by (3.6), we get

$$\begin{aligned}
I(u_1,u_2,u_0) &\ge \frac{1}{2}\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx + \frac{1}{2}\int_{\mathbb{R}^3}|x|^2\big(u_1^2+u_2^2+u_0^2\big)dx \\
&\quad - \max\Big\{\frac{c_0}{4}, \frac{3c_0}{4}+c_1\Big\}C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_0|^2\big)dx\Big)^{\frac{3}{2}}N^{\frac{1}{2}} \\
&\ge -\max\Big\{\frac{c_0}{4}, \frac{3c_0}{4}+c_1\Big\}C_*\Big(\|(u_1,u_2,u_0)\|_{\dot\Lambda}^2\Big)^{\frac{3}{2}}N^{\frac{1}{2}} \ge -\max\Big\{\frac{c_0}{4}, \frac{3c_0}{4}+c_1\Big\}C_* r^{\frac{3}{2}}N^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we have proved that $I(u_1,u_2,u_0)$ is bounded from below on $\mathcal{M}\cap B(r)$. □

For any $r>0$ and $N\le\frac{r}{3}$, we consider the following local minimization problem:

$$m_N^r := \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(r)} I(u_1,u_2,u_0).$$

By Lemma 4.4, m_N^r is well defined.

Lemma 4.5. *Suppose $c_0>0$ and $c_0+c_1>0$, then for any $r>0$, there exists $\tilde N=\tilde N(r)$, such that*

$$m_N^r = \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{r}{2})} I(u_1,u_2,u_0), \quad \text{for } N\le\tilde N. \tag{4.5}$$

Proof. For any $r>0$, if $\mathcal{M}\cap\big(B(r)\setminus B(\frac{r}{2})\big)=\emptyset$, then it is easy to see that (4.5) holds.

If $\mathcal{M}\cap\big(B(r)\setminus B(\frac{r}{2})\big)\ne\emptyset$, then for any $(u_1,u_2,u_0)\in\mathcal{M}\cap\big(B(r)\setminus B(\frac{r}{2})\big)$ and

$$N \le \Big(\frac{1}{4\max\{c_0, 3c_0+4c_1\}C_* r^{\frac{1}{2}}}\Big)^2,$$

we have

$$\begin{aligned}
I(u_1,u_2,u_0) &= \frac{1}{2}\|(u_1,u_2,u_0)\|_{\dot\Lambda}^2 - \frac{1}{4}\Big\{\int_{\mathbb{R}^3}\Big((c_0+c_1)\big(u_1^4+u_2^4\big)+c_0u_0^4\Big)dx \\
&\quad + 4c_1\int_{\mathbb{R}^3}u_1u_2u_0^2dx + 2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_1^2u_2^2+\big(c_0+c_1\big)(u_1^2+u_2^2)u_0^2\Big)dx\Big\} \\
&\ge \frac{1}{2}\|(u_1,u_2,u_0)\|_{\dot\Lambda}^2 - \max\Big\{\frac{c_0}{4}, \frac{3c_0}{4}+c_1\Big\}C_*\|(u_1,u_2,u_0)\|_{\dot\Lambda}^3N^{\frac{1}{2}} \\
&\ge \frac{r}{4} - \max\Big\{\frac{c_0}{4}, \frac{3c_0}{4}+c_1\Big\}C_* r^{\frac{3}{2}}N^{\frac{1}{2}} \ge \frac{3}{16}r.
\end{aligned}$$

For any $r>0$, by (4.4),

$$\mathcal{M}\cap B\Big(\frac{r}{4}\Big)\ne\emptyset, \quad \text{if } N\le\frac{r}{12}.$$

For any $(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{r}{4})$, we have

$$I(u_1,u_2,u_0)\le\frac{1}{2}\|(u_1,u_2,u_0)\|_{\Lambda}^2\le\frac{r}{8}<\frac{3r}{16}\le\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap\big(B(r)\setminus B(\frac{r}{2})\big)}I(u_1,u_2,u_0).$$

Take

$$\tilde N:=\min\left\{\left(\frac{1}{4\max\{c_0,3c_0+4c_1\}C_*r^{\frac12}}\right)^2,\frac{r}{12}\right\},$$

then we conclude that for $0<N\le\tilde N$,

$$m_N^r\le\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{r}{4})}I(u_1,u_2,u_0)<\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap\big(B(r)\setminus B(\frac{r}{2})\big)}I(u_1,u_2,u_0).\tag{4.6}$$

Therefore, we complete the proof. $\square$

In [4], the following strict inequality was obtained,

$$\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{Nr}{2})}I(u_1,u_2,u_0)<\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap\big(B(r)\setminus B(Nr)\big)}I(u_1,u_2,u_0).\tag{4.7}$$

However, a necessary condition $r\ge 6$ must be imposed in this case to ensure that $\mathcal{M}\cap B(\frac{Nr}{2})\ne\emptyset$. Indeed, if $(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{Nr}{2})$, then by (4.3),

$$3N\le\|(u_1,u_2,u_0)\|_{\Lambda}^2\le\frac{Nr}{2},$$

which implies $r\ge 6$. That is, the proof in [4] does not apply for all $r>0$. This observation was pointed out in Remark 1.5 of [37]. We give a detailed proof of the following lemma.

Lemma 4.6. *Suppose $c_0>0$ and $c_0+c_1>0$, then for any $r>0$, there exists $N_0=N_0(r)$, such that $N\le N_0$,*

$$\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{r}{4})}I(u_1,u_2,u_0)<\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap\big(B(r)\setminus B(\frac{r}{2})\big)}I(u_1,u_2,u_0).\tag{4.8}$$

Proof. We first show that $\mathcal{M}\cap\big(B(r)\setminus B(\frac{r}{2})\big)\ne\emptyset$ for small N. For any $\tau>0$ and $(u_1,u_2,u_0)\in H^1(\mathbb{R}^3,\mathbb{R}^3)$, let $\tau\star(u_1,u_2,u_0)$ be the operation defined in (3.13), then by Lemma 4.4,

$$(U_1,U_2,U_0):=\tau\star\left(\sqrt{\frac{N+M}{2}}\Psi_0,\sqrt{\frac{N-M}{2}}\Psi_0,0\right)\in\mathcal{M},$$

and by direct calculation, we get

$$\big\|(U_1,U_2,U_0)\big\|_{\Lambda}^2=e^{2\tau}N\int_{\mathbb{R}^3}|\nabla\Psi_0|^2dx+e^{-2\tau}N\int_{\mathbb{R}^3}|x|^2{\Psi_0}^2dx.$$

Denote

$$D_1:=\int_{\mathbb{R}^3}|\nabla\Psi_0|^2dx,\quad D_2:=\int_{\mathbb{R}^3}|x|^2{\Psi_0}^2dx,$$

then it is obvious that

$$e^{2\tau}D_1+e^{-2\tau}D_2\ge2\sqrt{D_1D_2}.$$

Hence for any $r > 0$, if we choose

$$N \le \frac{3r}{8\sqrt{D_1 D_2}},$$

then there exists $\tau > 0$, such that $\big\|(U_1, U_2, U_0)\big\|_{\dot\Lambda}^2 = \frac{3}{4}r$, that is $(U_1, U_2, U_0) \in \mathcal{M} \cap \big(B(r) \setminus B(\frac{r}{2})\big)$. Let

$$N_0 := \min\Big\{\tilde{N}, \frac{3r}{8\sqrt{D_1 D_2}}\Big\},$$

we conclude (4.8) from the proof Lemma 4.5. □

Lemma 4.7. *Suppose $c_0 > 0$ and $c_0 + c_1 > 0$, then for any $r > 0$ and $0 < N \le N_0$, there holds*

$$m_N^r < \frac{3N}{2}.$$

Proof. From the proof of Lemma 4.4, we get $\Big(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \in \mathcal{M} \cap B(r)$. Thus

$$\begin{aligned} m_N^r = \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(r)} I(u_1, u_2, u_0) &\le I\Big(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \\ < \frac{N}{2}\int_{\mathbb{R}^3}\Big(|\nabla\Psi_0|^2 + |x|^2{\Psi_0}^2\Big)dx &= \frac{N}{2}\|(\Psi_0, 0, 0)\|_{\dot\Lambda}^2 = \frac{3N}{2}. \end{aligned}$$

□

Proof of Theorem 2. (i) For any $r > 0$ and $0 < N \le \frac{r}{3}$, suppose $\{(u_{1n}, u_{2n}, u_{0n})\} \subset \mathcal{M} \cap B(r)$ is a minimizing sequence for m_N^r, i.e. $I(u_{1n}, u_{2n}, u_{0n}) \to m_N^r$ as $n \to \infty$. Then

$$\|(u_{1n}, u_{2n}, u_{0n})\|_{\Lambda}^2 = \|(u_{1n}, u_{2n}, u_{0n})\|_{\dot\Lambda}^2 + \|(u_{1n}, u_{2n}, u_{0n})\|_{L^2}^2 \le r + N,$$

which implies that $\{(u_{1n}, u_{2n}, u_{0n})\}$ is bounded in Λ. Therefore, there exists $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in \Lambda$, such that up to a subsequence, as $n \to \infty$,

$$\begin{cases} (u_{1n}, u_{2,n}, u_{0,n}) \rightharpoonup (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0), & \text{in } \Lambda. \\ (u_{1n}, u_{2,n}, u_{0,n}) \to (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0), & \text{in } L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3),\ \forall t \in [2, 2^*). \\ (u_{1n}, u_{2,n}, u_{0,n}) \to (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0), & a.e. \text{ in } \mathbb{R}^3. \end{cases}$$

Then we get $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in \mathcal{M} \cap B(r)$. Further, by the lower semi-continuity of the norm in Λ, there holds

$$m_N^r \le I(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \le \lim_{n\to\infty} I(u_{1n}, u_{2n}, u_{0n}) = m_N^r.$$

It yields that $I(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) = m_N^r$. Hence, m_N^r has at least one minimizer for any $r > 0$ and $N \le \frac{r}{3}$.
(ii) For any $r > 0$ and $0 < N \le N_0$, by (4.8), we can see that $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in B(\frac{r}{2})$, which follows that $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)$ stays away from the boundary of $B(r)$. Thus, $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)$ is indeed a critical point of $I(u_1, u_2, u_0)$ restricted to $\mathcal{M}$ and further $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)$ is a weak solution for (1.4) with some constants $\tilde{\mu}$, $\tilde{\lambda}$ as Lagrange multipliers.

Next, we show that $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)$ is a ground state solution for (1.4) as N small by contradiction. Let $N_n := \min\{\frac{1}{n}, N_0\}$, suppose there exists $r_0 > 0$ and $\{(v_{1n}, v_{2n}, v_{0n})\} \subset \mathcal{M}(N_n)$, such that

$$I'|_{\mathcal{M}}(v_{1n}, v_{2n}, v_{0n}) = 0 \quad \text{and} \quad I(v_{1n}, v_{2n}, v_{0n}) < m_{N_n}^{r_0}.$$

Then by Lemma 4.2, we get $P(v_{1n}, v_{2n}, v_{0n}) = 0$ and further by Lemma 4.7,

$$
\begin{aligned}
I(v_{1n}, v_{2n}, v_{0n}) &= \frac{1}{6}\int_{\mathbb{R}^3} \big(|\nabla v_{1n}|^2 + |\nabla v_{2n}|^2 + |\nabla v_{0n}|^2\big)dx + \frac{5}{6}\int_{\mathbb{R}^3} |x|^2\big(v_{1n}^2 + v_{2n}^2 + v_{0n}^2\big)dx \\
&< m_{N_n}^{r_0} \to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

It implies that

$$
\|(v_{1n}, v_{2n}, v_{0n})\|_{\Lambda}^2 = \int_{\mathbb{R}^3} \big(|\nabla v_{1n}|^2 + |\nabla v_{2n}|^2 + |\nabla v_{0n}|^2\big)dx + \int_{\mathbb{R}^3} |x|^2\big(v_{1n}^2 + v_{2n}^2 + v_{0n}^2\big)dx \to 0,
$$

then $(v_{1n}, v_{2n}, v_{0n}) \in \mathcal{M}(N_n) \cap B(r_0)$. We can see that $I(v_{1n}, v_{2n}, v_{0n}) \geq m_{N_n}^{r_0}$, which is a contradiction. Therefore, $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)$ is a ground state solution of (1.4).

(iii) By Lemma 4.7, for any $r > 0$ and $0 < N \leq N_0$, there holds

$$
m_N^r < \frac{3N}{2}. \tag{4.9}
$$

Denote

$$
\mathcal{M}_N^r := \Big\{(u_1, u_2, u_0) \in \mathcal{M} \cap B(r) \Big| I(u_1, u_2, u_0) = m_N^r\Big\}.
$$

Suppose $(u_{1N}, u_{2N}, u_{0N}) \in \mathcal{M}_N^r$, by Lemma 4.2, we can see that

$$
\begin{aligned}
&m_N^r = I(u_{1N}, u_{2N}, u_{0N}) \\
&= \frac{1}{6}\int_{\mathbb{R}^3} \big(|\nabla u_{1N}|^2 + |\nabla u_{2N}|^2 + |\nabla u_{0N}|^2\big)dx + \frac{5}{6}\int_{\mathbb{R}^3} |x|^2\big(u_{1N}^2 + u_{2N}^2 + u_{0N}^2\big)dx \\
&\geq \frac{1}{6}\Big\{\int_{\mathbb{R}^3} \big(|\nabla u_{1N}|^2 + |\nabla u_{2N}|^2 + |\nabla u_{0N}|^2\big)dx + \int_{\mathbb{R}^3} |x|^2\big(u_{1N}^2 + u_{2N}^2 + u_{0N}^2\big)dx\Big\} \\
&= \frac{1}{6}\|(u_{1N}, u_{2N}, u_{0N})\|_{\Lambda}^2,
\end{aligned}
$$

that is $\|(u_{1N}, u_{2N}, u_{0N})\|_{\Lambda}^2 \leq 6m_N^r$. Together with (4.9), we have

$$
\Big(\frac{\|(u_{1N}, u_{2N}, u_{0N})\|_{\Lambda}^2}{N}\Big)^{\frac{3}{2}} \leq \Big(\frac{6m_N^r}{N}\Big)^{\frac{3}{2}} < \Big(\frac{6 \cdot \frac{3N}{2}}{N}\Big)^{\frac{3}{2}} = 27. \tag{4.10}
$$

Then by (3.6), we get

$$
\begin{aligned}
&\frac{1}{N}\Big\{\frac{1}{4}\int_{\mathbb{R}^3} \Big((c_0 + c_1)\big(u_{1N}^4 + u_{2N}^4\big) + c_0 u_{0N}^4\Big)dx + c_1\int_{\mathbb{R}^3} u_{1N}u_{2N}u_{0N}^2 dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} \Big(\big(c_0 - c_1\big)u_{1N}^2 u_{2N}^2 + \big(c_0 + c_1\big)(u_{1N}^2 + u_{2N}^2)u_{0N}^2\Big)dx\Big\} \\
&\leq \max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_*\Big(\int_{\mathbb{R}^3} \big(|\nabla u_{1N}|^2 + |\nabla u_{2N}|^2 + |\nabla u_{0N}|^2\big)dx\Big)^{\frac{3}{2}} N^{-\frac{1}{2}} \\
&\leq \max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_*\|(u_{1N}, u_{2N}, u_{0N})\|_{\Lambda}^3 N^{-\frac{1}{2}} = \max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_*\Big(\frac{\|(u_{1N}, u_{2N}, u_{0N})\|_{\Lambda}^2}{N}\Big)^{\frac{3}{2}} N \\
&< 27\max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_* N \to 0, \quad \text{as } N \to 0^+,
\end{aligned}
$$

which implies that

$$\begin{aligned}\lim_{N\to 0^+}\frac{1}{N}\Big\{\frac{1}{4}\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_{1N}^4+u_{2N}^4\big)+c_0u_{0N}^4\Big)dx+c_1\int_{\mathbb{R}^3}u_{1N}u_{2N}u_{0N}^2dx\\ +\frac{1}{2}\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_{1N}^2u_{2N}^2+\big(c_0+c_1\big)(u_{1N}^2+u_{2N}^2)u_{0N}^2\Big)dx\Big\}=0.\end{aligned}\tag{4.11}$$

Since $I'|_{\mathcal{M}}(u_{1N},u_{2N},u_{0N})=0$, there exist two sequences $\{\mu_N\},\ \{\lambda_N\}\subset\mathbb{R}$, such that

$$\begin{aligned}&\mu_NN+\lambda_NM\\ &=\int_{\mathbb{R}^3}\big(|\nabla u_{1N}|^2+|\nabla u_{2N}|^2+|\nabla u_{0N}|^2\big)dx+\int_{\mathbb{R}^3}|x|^2\big(u_{1N}^2+u_{2N}^2+u_{0N}^2\big)dx\\ &\quad-\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_{1N}^4+u_{2N}^4\big)+c_0u_{0N}^4\Big)dx+4c_1\int_{\mathbb{R}^3}u_{1N}u_{2N}u_{0N}^2dx\\ &\quad+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_{1N}^2u_{2N}^2+\big(c_0+c_1\big)(u_{1N}^2+u_{2N}^2)u_{0N}^2\Big)dx\Big\}.\end{aligned}\tag{4.12}$$

Then by (4.3) and (4.11), we obtain

$$\lim_{N\to 0^+}\frac{\mu_NN+\lambda_NM}{N}\geq 3.\tag{4.13}$$

By (4.9) and (4.12), we can see that

$$\begin{aligned}&\mu_NN+\lambda_NM=\|(u_{1N},u_{2N},u_{0N})\|_\Lambda^2-\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_{1N}^4+u_{2N}^4\big)+c_0u_{0N}^4\Big)dx\\ &\quad+4c_1\int_{\mathbb{R}^3}u_{1N}u_{2N}u_{0N}^2dx+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_{1N}^2u_{2N}^2+\big(c_0+c_1\big)(u_{1N}^2+u_{2N}^2)u_{0N}^2\Big)dx\Big\}\\ &=2I(u_{1N},u_{2N},u_{0N})-\frac{1}{2}\Big\{\int_{\mathbb{R}^3}\Big(\big(c_0+c_1\big)\big(u_{1N}^4+u_{2N}^4\big)+c_0u_{0N}^4\Big)dx\\ &\quad+4c_1\int_{\mathbb{R}^3}u_{1N}u_{2N}u_{0N}^2dx+2\int_{\mathbb{R}^3}\Big(\big(c_0-c_1\big)u_{1N}^2u_{2N}^2+\big(c_0+c_1\big)(u_{1N}^2+u_{2N}^2)u_{0N}^2\Big)dx\Big\}\\ &\leq 2I(u_{1N},u_{2N},u_{0N})=2m_N^r<2\cdot\frac{3}{2}N=3N.\end{aligned}$$

Hence, together with (4.13), we get $\lim_{N\to 0^+}\frac{\mu_NN+\lambda_NM}{N}=3$. Further, we can deduce from (4.11) and (4.12) that

$$\lim_{N\to 0^+}\frac{\|(u_{1N},u_{2N},u_{0N})\|_\Lambda^2}{N}=\lim_{N\to 0^+}\frac{2I(u_{1N},u_{2N},u_{0N})}{N}=\lim_{N\to 0^+}\frac{2m_N^r}{N}=3.$$

Next, we show as $N\to 0^+$, there holds

$$\|(u_{1N},u_{2N},u_{0N})-(l_{10}\Psi_0,l_{20}\Psi_0,l_{00}\Psi_0)\|_\Lambda^2=O(N^2),\tag{4.14}$$

where $l_{i0}=\int_{\mathbb{R}^3}u_{iN}\Psi_0dx,$ for $i=1,2,0$. Set $l_{ik}=\int_{\mathbb{R}^3}u_{iN}\Psi_kdx,$ for $i=1,2,0$, then

$$(u_{1N},u_{2N},u_{0N})=\Big(\sum_{k=0}^\infty l_{1k}\Psi_k,\sum_{k=0}^\infty l_{2k}\Psi_k,\sum_{k=0}^\infty l_{0k}\Psi_k\Big).$$

Moreover, we can conclude

$$N = \|(u_{1N}, u_{2N}, u_{0N})\|_{L^2}^2 = \sum_{k=0}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2)\|\Psi_k\|_{L^2}^2 = \sum_{k=0}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) \tag{4.15}$$

and $\|(u_{1N}, u_{2N}, u_{0N})\|_{\dot\Lambda}^2 = \sum_{k=0}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2)\|\Psi_k\|_{\dot\Lambda}^2 = \sum_{k=0}^{\infty}\xi_k(l_{1k}^2 + l_{2k}^2 + l_{0k}^2)$. By (3.6) and (4.10), we get

$$\begin{aligned}
&m_N^r = I(u_{1N}, u_{2N}, u_{0N}) \\
&= \frac{1}{2}\|(u_{1N}, u_{2N}, u_{0N})\|_{\dot\Lambda}^2 - \Big\{\frac{1}{4}\int_{\mathbb{R}^3}\Big(\big(c_0 + c_1\big)\big(u_{1N}^4 + u_{2N}^4\big) + c_0 u_{0N}^4\Big)dx + c_1\int_{\mathbb{R}^3} u_{1N}u_{2N}u_{0N}^2 dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3}\Big(\big(c_0 - c_1\big)u_{1N}^2 u_{2N}^2 + \big(c_0 + c_1\big)(u_{1N}^2 + u_{2N}^2)u_{0N}^2\Big)dx\Big\} \\
&\geq \frac{1}{2}\|(u_{1N}, u_{2N}, u_{0N})\|_{\dot\Lambda}^2 - \max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_{1N}|^2 + |\nabla u_{2N}|^2 + |\nabla u_{0N}|^2\big)dx\Big)^{\frac{3}{2}}N^{\frac{1}{2}} \\
&\geq \frac{1}{2}\|(u_{1N}, u_{2N}, u_{0N})\|_{\dot\Lambda}^2 - \max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_*\|(u_{1N}, u_{2N}, u_{0N})\|_{\dot\Lambda}^3 N^{\frac{1}{2}} \\
&\geq \frac{1}{2}\sum_{k=0}^{\infty}\xi_k(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) - 27\max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_* N^2 \\
&= \frac{1}{2}\sum_{k=0}^{\infty}(\xi_k - \xi_0)(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) + \frac{1}{2}\sum_{k=0}^{\infty}\xi_0(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) - 27\max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_* N^2.
\end{aligned}$$

Then by (4.9) and (4.15),

$$\begin{aligned}
(\xi_1 - \xi_0)\sum_{k=1}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) &\leq \sum_{k=1}^{\infty}(\xi_k - \xi_0)(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) \\
&\leq 54\max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_* N^2 - 3N + m_N^r < 54\max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_* N^2,
\end{aligned}$$

which yields that

$$\sum_{k=1}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) \leq \frac{54\max\big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\big\}C_* N^2}{\xi_1 - \xi_0}.$$

Thus

$$\begin{aligned}
\sum_{k=1}^{\infty}\xi_k(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) &= \sum_{k=1}^{\infty}(\xi_k - \xi_0)(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) + \xi_0\sum_{k=1}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) \\
&\leq 54\max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_* N^2 + \xi_0 \cdot \frac{54\max\big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\big\}C_* N^2}{\xi_1 - \xi_0} \\
&= \frac{\xi_1}{\xi_1 - \xi_0} \cdot 54\max\Big\{\frac{c_0}{4}, \frac{3c_0}{4} + c_1\Big\}C_* N^2.
\end{aligned}$$

For $N \to 0^+$, we can see that

$$\begin{aligned}
&\|(u_{1N}, u_{2N}, u_{0N}) - (l_{10}\Psi_0, l_{20}\Psi_0, l_{00}\Psi_0)\|_{\dot\Lambda}^2 = \Big\|\Big(\sum_{k=1}^{\infty}l_{1k}\Psi_k, \sum_{k=1}^{\infty}l_{2k}\Psi_k, \sum_{k=1}^{\infty}l_{0k}\Psi_k\Big)\Big\|_{\dot\Lambda}^2 \\
&= \sum_{k=1}^{\infty}\xi_k(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) = O(N^2)
\end{aligned}$$

and

$$\|(u_{1N},u_{2N},u_{0N})-(l_{10}\Psi_0,l_{20}\Psi_0,l_{00}\Psi_0)\|_{L^2}^2=\left\|\left(\sum_{k=1}^{\infty}l_{1k}\Psi_k,\sum_{k=1}^{\infty}l_{2k}\Psi_k,\sum_{k=1}^{\infty}l_{0k}\Psi_k\right)\right\|_{L^2}^2$$
$$=\sum_{k=1}^{\infty}(l_{1k}^2+l_{2k}^2+l_{0k}^2)=O(N^2).$$

Therefore, it is obvious that (4.14) holds. We complete the proof of (iii) in Theorem 2. □

Next, we will show that the set $\mathcal{M}_N^r$ is orbitally stable under the flow of (1.3). To this end, we need the following global well-posedness result.

Lemma 4.8. *For any $r>0$, $(u_1(0),u_2(0),u_0(0))$ in Λ be such that $\|(u_1(0),u_2(0),u_0(0))\|_{\Lambda}^2\le r$. Then there exists $N_0=N_0(r)>0$ sufficiently small such that for all $0<N<N_0$, if $\big(u_1(0),u_2(0),u_0(0)\big)\in\mathcal{M}$, then the corresponding solution to (1.3) exists globally in time.*

Proof. The proof is based on the following continuity argument: Let $I\subset\mathbb{R}$ be a time interval and $X:I\to[0,+\infty)$ be a continuous function satisfying for every $t\in I$, $X(t)\le a+b(X(t))^{\theta}$, for some constants $a,b,\theta>0$. Assume that for some $t_0\in I$, $X(t_0)\le 2a,\ b<2^{-\theta}a^{1-\theta}$. Then for every $t\in I$, we have $X(t)\le 2a$. Observe by the uncertainty principle (see e.g. [47]) that

$$\int_{\mathbb{R}^3}\big(u_1^2(0)+u_2^2(0)+u_0^2(0)\big)dx$$
$$\le\frac{2}{3}\left(\int_{\mathbb{R}^3}\big(|\nabla u_1(0)|^2+|\nabla u_2(0)|^2+|\nabla u_0(0)|^2\big)dx\right)^{\frac12}\left(\int_{\mathbb{R}^3}|x|^2\big(u_1^2(0)+u_2^2(0)+u_0^2(0)\big)dx\right)^{\frac12}$$
$$\le\|(u_1(0),u_2(0),u_0(0))\|_{\Lambda}^2\le r.$$

By (3.6), we have

$$\big|I(u_1(0),u_2(0),u_0(0))\big|\le\frac12\|(u_1(0),u_2(0),u_0(0))\|_{\Lambda}^2$$
$$+\max\Big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\Big\}C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1(0)|^2+|\nabla u_2(0)|^2+|\nabla u_0(0)|^2\big)dx\Big)^{\frac32}N^{\frac12}\le C(r),$$

for some constant $C(r)$ depending only on r. Similarly, by the conservation of mass and energy, we have

$$\|(u_1(t),u_2(t),u_0(t))\|_{\Lambda}^2\le 2|I(u_1(0),u_2(0),u_0(0))|$$
$$+2\max\Big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\Big\}C_*\Big(\int_{\mathbb{R}^3}\big(|\nabla u_1(t)|^2+|\nabla u_2(t)|^2+|\nabla u_0(t)|^2\big)dx\Big)^{\frac32}N^{\frac12}.$$

Set $X(t)=\|(u_1(t),u_2(t),u_0(t))\|_{\Lambda}^2$, $a=2|I(u_1,u_2,u_0)|+\frac12\|(u_1,u_2,u_0)\|_{\Lambda}^2$ and $b=2\max\big\{\frac{c_0}{4},\frac{3c_0}{4}+c_1\big\}C_*N^{\frac12}$. We see that $X(t)\le a+b(X(t))^{\frac32}$ for all t in the existence time. Since $X(0)=\|(u_1,u_2,u_0)\|_{\Lambda}^2\le 2a$ and a is bounded from above by some constant depending only on r, we apply the above continuity argument with $b<2^{-\frac32}a^{-\frac12}$ to get $X(t)\le 2a$ for all t in the existence time. This shows that for $\int_{\mathbb{R}^3}\big(u_1^2(0)+u_2^2(0)+u_0^2(0)\big)dx=N$ is sufficiently small depending only on r, the corresponding solution to (1.3) has bounded norm. The local theory implies that the solution exists globally in time. □

Proof of Theorem 2. (iv) From Lemma 4.8, we know that if $(u_1(0), u_2(0), u_0(0)) \in \mathcal{M}$, then the corresponding solution to (1.3) exists globally in time. Suppose that there exists a $\epsilon_0 > 0$, a sequence of initial data $\big(u_{1n}(0,\cdot), u_{2n}(0,\cdot), u_{0n}(0,\cdot)\big) \subset \Lambda$ and a sequence $\{t_n\} \subset \mathbb{R}$ such that the solution (u_{1n}, u_{2n}, u_{0n}) of problem (1.4) with initial data $\big(u_{1n}(0,\cdot), u_{2n}(0,\cdot), u_{0n}(0,\cdot)\big)$ satisfies

$$\inf_{(u_1,u_2,u_0)\in\mathcal{M}_N^r} \big\|(u_1,u_2,u_0) - \big(u_{1n}(0,\cdot), u_{2n}(0,\cdot), u_{0n}(0,\cdot)\big)\big\|_\Lambda < \frac{1}{n}$$

and

$$\inf_{(u_1,u_2,u_0)\in\mathcal{M}_N^r} \big\|(u_1,u_2,u_0) - \big(u_{1n}(t_n,\cdot), u_{2n}(t_n,\cdot), u_{0n}(t_n,\cdot)\big)\big\|_\Lambda > \epsilon_0.$$

Without loss of generality, we may assume that $\big\{\big(u_{1n}(0,\cdot), u_{2n}(0,\cdot), u_{0n}(0,\cdot)\big)\big\} \subset \mathcal{M}$, we claim that $\big\{\big(u_{1n}(t_n,\cdot), u_{2n}(t_n,\cdot), u_{0n}(t_n,\cdot)\big)\big\} \subset B(r)$. Indeed, if $\big\{\big(u_{1n}(t_n,\cdot), u_{2n}(t_n,\cdot), u_{0n}(t_n,\cdot)\big)\big\} \subset \Lambda\backslash B(r)$, then by the continuity there exists $\bar{t}_n \in [0, t_n]$ such that $\{(u_{1n}(\bar{t}_n,\cdot), u_{2n}(\bar{t}_n,\cdot), u_{0n}(t_n,\cdot))\} \subset \partial B(r)$. Hence by the conversation laws of the energy and mass (see [12]), Lemma 4.5 and (4.8), we see that

$$\begin{aligned} &I\big(u_{1n}(0,\cdot), u_{2n}(0,\cdot), u_{0n}(0,\cdot)\big) = I\big(u_{1n}(\bar{t}_n,\cdot), u_{2n}(\bar{t}_n,\cdot), u_{0n}(\bar{t}_n,\cdot)\big) \\ \geq &\inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap\big(B(r)\backslash B(\frac{r}{2})\big)} I(u_1,u_2,u_0) > \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{r}{4})} I(u_1,u_2,u_0) \geq m_N^r, \end{aligned}$$

which contradicts

$$\inf_{(u_1,u_2,u_0)\in\mathcal{M}_N^r} \big\|(u_1,u_2,u_0) - \big(u_{1n}(0,\cdot), u_{2n}(0,\cdot), u_{0n}(0,\cdot)\big)\big\|_\Lambda \to 0, \quad \text{as } n \to +\infty.$$

Then $\big\{\big(u_{1n}(t_n,\cdot), u_{2n}(t_n,\cdot), u_{0n}(t_n,\cdot)\big)\big\}$ is a minimizing sequence of m_N^r. Similarly to the proof of Theorem 2 (i), there exists $(v_1,v_2,v_0) \in \mathcal{M}_N^r$ such that $\big(u_{1n}(t_n,\cdot), u_{2n}(t_n,\cdot), u_{0n}(t_n,\cdot)\big) \to (v_1,v_2,v_0)$ in Λ, which contradicts

$$\inf_{(u_1,u_2,u_0)\in\mathcal{M}_N^r} \big\|(u_1,u_2,u_0) - \big(u_{1n}(t_n,\cdot), u_{2n}(t_n,\cdot), u_{0n}(t_n,\cdot)\big)\big\|_\Lambda > \epsilon_0.$$

Therefore, we complete the proof of (iv) in Theorem 2. □

5 Proof of Theorem 3

For any $r > 0$ and $0 < N \leq N_0$, suppose $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in \mathcal{M} \cap B(r)$ is the solution of (1.4) obtained in Theorem 2 (ii), then we see that $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in B(\frac{r}{2})$. By (3.10), we get for any $(u_1,u_2,u_0) \in \mathcal{M}$, there holds $\lim_{\tau\to+\infty} I\big(\tau \star (u_1,u_2,u_0)\big) = -\infty$. Hence there exists a large $\tau_1 > 0$, such that

$$\big\|\big(\tau_1 \star (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)\big)\big\|_{\dot{\Lambda}}^2 > r \quad \text{and} \quad I\big(\tau_1 \star (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)\big) < 0.$$

We now define a path as

$$\Gamma := \Big\{g \in C\big([0,1], \mathcal{M}\big) \big| g(0) = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0),\ g(1) = \tau_1 \star (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)\Big\},$$

then for $t \in [0,1]$, it is easy to see that $g(t) := \big((1-t) + t\tau_1\big) \star (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in \Gamma$, that is $\Gamma \neq \emptyset$. Hence, the minimax value

$$\sigma := \inf_{g\in\Gamma} \max_{t\in[0,1]} I\big(g(t)\big)$$

is well defined. Further, we can deduce

$$\sigma > \max\Big\{I(\tilde{u}_1,\tilde{u}_2,\tilde{u}_0), I\big(\tau_1 \star (\tilde{u}_1,\tilde{u}_2,\tilde{u}_0)\big)\Big\} > 0. \tag{5.1}$$

Indeed, for any $g \in \Gamma$, we have $g(0) = (\tilde{u}_1,\tilde{u}_2,\tilde{u}_0) \in B(\frac{r}{2})$ and $g(1) = \tau_1 \star (\tilde{u}_1,\tilde{u}_2,\tilde{u}_0)$ with $\big\|\big(\tau_1 \star (\tilde{u}_1,\tilde{u}_2,\tilde{u}_0)\big)\big\|_{\dot{\Lambda}}^2 > r$, then there exists $t_0 \in (0,1)$, such that $g(t_0) \in \partial B(r)$. Then by (4.5) and (4.8), we get

$$\begin{aligned}
&\max_{t\in[0,1]} I\big(g(t)\big) \geq I\big(g(t_0)\big) \geq \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap\big(B(r)\backslash B(\frac{r}{2})\big)} I(u_1,u_2,u_0)\\
&> \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{r}{4})} I(u_1,u_2,u_0) \geq \inf_{(u_1,u_2,u_0)\in\mathcal{M}\cap B(\frac{r}{2})} I(u_1,u_2,u_0)\\
&= m_N^r = I(\tilde{u}_1,\tilde{u}_2,\tilde{u}_0) > 0 > I\big(\tau_1 \star (\tilde{u}_1,\tilde{u}_2,\tilde{u}_0)\big),
\end{aligned}$$

which implies (5.1).

Lemma 5.1. *Suppose $c_0 > 0$ and $c_0 + c_1 > 0$, then for any $r > 0$ and $0 < N \leq N_0$, there exists a bounded Palais-Smale sequence $\{(u_{1n},u_{2n},u_{0n})\}$ for I restricted to $\mathcal{M}$ at level σ. In addition,*

$$\begin{aligned}
P(u_{1n},u_{2n},u_{0n}) = &\int_{\mathbb{R}^3} \big(|\nabla u_{1n}|^2 + |\nabla u_{2n}|^2 + |\nabla u_{0n}|^2\big)dx\\
&- \frac{3}{4}\Big\{\int_{\mathbb{R}^3} \Big(\big(c_0+c_1\big)\big(u_{1n}^4 + u_{2n}^4\big) + c_0 u_{0n}^4\Big)dx + 4c_1 \int_{\mathbb{R}^3} u_{1n}u_{2n}u_{0n}^2 dx\\
&+ 2\int_{\mathbb{R}^3} \Big(\big(c_0-c_1\big)u_{1n}^2u_{2n}^2 + \big(c_0+c_1\big)(u_{1n}^2+u_{2n}^2)u_{0n}^2\Big)dx\Big\}\\
&- \int_{\mathbb{R}^3} |x|^2\big(u_{1n}^2 + u_{2n}^2 + u_{0n}^2\big)dx = o(1), \quad \text{as } n \to \infty.
\end{aligned}$$

Proof. The existence of Palais-Smale sequence $\{(u_{1n},u_{2n},u_{0n})\}$ for I at level σ with $P(u_{1n},u_{2n},u_{0n}) = o(1)$ is similar to the proof of Proposition 3.1, we omit the details here. We only show $\{(u_{1n},u_{2n},u_{0n})\} \subset \mathcal{M}$ is bounded in Λ. Indeed, direct calculation gives

$$\begin{aligned}
&I(u_{1n},u_{2n},u_{0n}) = I(u_{1n},u_{2n},u_{0n}) - \frac{1}{3}P(u_{1n},u_{2n},u_{0n}) + o(1)\\
&= \frac{1}{6}\int_{\mathbb{R}^3} \big(|\nabla u_{1n}|^2 + |\nabla u_{2n}|^2 + |\nabla u_{0n}|^2\big)dx + \frac{5}{6}\int_{\mathbb{R}^3} |x|^2\big(u_{1n}^2 + u_{2n}^2 + u_{0n}^2\big)dx + o(1).
\end{aligned}$$

Since $\{(u_{1n},u_{2n},u_{0n})\} \subset \mathcal{M}$ and $I(u_{1n},u_{2n},u_{0n}) \to \sigma$ as $n \to \infty$, then we get the boundedness of $\{(u_{1n},u_{2n},u_{0n})\}$ in Λ. Therefore, we complete the proof. □

Lemma 5.2. *Suppose $c_0 > 0$ and $c_0+c_1 > 0$, for any $r > 0$ and $0 < N \leq N_0$, let $\{(u_{1n},u_{2n},u_{0n})\} \subset \mathcal{M}$ be the Palais-Smale sequence obtained in Proposition 5.1, then there exists $(\hat{u}_1,\hat{u}_2,\hat{u}_0) \in \mathcal{M}$, such that $(u_{1n},u_{2n},u_{0n}) \to (\hat{u}_1,\hat{u}_2,\hat{u}_0)$ is strongly in Λ.*

Proof. By Lemma 2.1 and Proposition 5.1, there exists $(\hat{u}_1,\hat{u}_2,\hat{u}_0) \in \Lambda$, such that up to a subsequence, as $n \to +\infty$,

$$\begin{cases}
(u_{1n},u_{2n},u_{0n}) \rightharpoonup (\hat{u}_1,\hat{u}_2,\hat{u}_0), & \text{in } \Lambda.\\
(u_{1n},u_{2n},u_{0n}) \to (\hat{u}_1,\hat{u}_2,\hat{u}_0), & \text{in } L^t(\mathbb{R}^3)\times L^t(\mathbb{R}^3)\times L^t(\mathbb{R}^3),\ \forall t \in [2,2^*).\\
(u_{1n},u_{2n},u_{0n}) \to (\hat{u}_1,\hat{u}_2,\hat{u}_0), & a.e. \text{ in } \mathbb{R}^3.
\end{cases} \tag{5.2}$$

Since $I'\big|_{\mathcal{M}}(u_{1n}, u_{2,n}, u_{0,n}) \to 0$, then there exist two sequences $\{\mu_n\}$, $\{\lambda_n\} \subset \mathbb{R}$, such that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left(\nabla u_{1n} \cdot \nabla \phi_1 + \nabla u_{2n} \cdot \nabla \phi_2 + \nabla u_{0n} \cdot \nabla \phi_0\right) dx \\
&- \int_{\mathbb{R}^3} \Big((c_0 + c_1)\big(u_{1n}^3 \phi_1 + u_{2n}^3 \phi_2\big) + c_0 u_{0n}^3 \phi_0 \Big) dx - (c_0 - c_1) \int_{\mathbb{R}^3} \Big(u_{1n} \phi_1 u_{2n}^2 + u_{1n}^2 u_{2n} \phi_2 \Big) dx \\
&- (c_0 + c_1) \int_{\mathbb{R}^3} \Big(u_{1n} \phi_1 u_{0n}^2 + u_{1n}^2 u_{0n} \phi_0 + u_{2n} \phi_2 u_{0n}^2 + u_{2n}^2 u_{0n} \phi_0 \Big) dx \\
&- c_1 \int_{\mathbb{R}^3} \Big(\phi_1 u_{2n} u_{0n}^2 + u_{1n} \phi_2 u_{0n}^2 + 2 u_{1n} u_{2n} u_{0n} \phi_0 \Big) dx + \int_{\mathbb{R}^3} |x|^2 \Big(u_{1n} \phi_1 + u_{2n} \phi_2 + u_{0n} \phi_0 \Big) dx \\
= &(\mu_n + \lambda_n) \int_{\mathbb{R}^3} u_{1n} \phi_1 dx + (\mu_n - \lambda_n) \int_{\mathbb{R}^3} u_{2n} \phi_2 dx + \mu_n \int_{\mathbb{R}^3} u_{0n} \phi_0 dx + o(1),
\end{aligned}
\tag{5.3}
$$

for any $(\phi_1, \phi_2, \phi_0) \in \Lambda$. Since $\{(u_{1n}, u_{2,n}, u_{0,n})\} \subset \mathcal{M}$ is bounded in Λ by Proposition 5.1, take $(\phi_1, \phi_2, \phi_0) = (u_{1n}, u_{2n}, u_{0n})$ in (5.3), then it is easy to see that we get $\{\mu_n\}$, $\{\lambda_n\}$ are two bounded sequences in $\mathbb{R}$. Suppose that $\mu_n \to \hat{\mu}$, $\lambda_n \to \hat{\lambda}$ as $n \to \infty$. Take $(\phi_1, \phi_2, \phi_0) = (u_{1n} - \hat{u}_1, u_{2n} - \hat{u}_2, u_{0n} - \hat{u}_0)$ in (5.3), then we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left(\nabla u_{1n} \cdot \nabla (u_{1n} - \hat{u}_1) + \nabla u_{2n} \cdot \nabla (u_{2n} - \hat{u}_2) + \nabla u_{0n} \cdot \nabla (u_{0n} - \hat{u}_0)\right) dx \\
&- \int_{\mathbb{R}^3} \Big((c_0 + c_1)\big(u_{1n}^3 (u_{1n} - \hat{u}_1) + u_{2n}^3 (u_{2n} - \hat{u}_2)\big) + c_0 u_{0n}^3 (u_{0n} - \hat{u}_0) \Big) dx \\
&- (c_0 - c_1) \int_{\mathbb{R}^3} \Big(u_{1n}(u_{1n} - \hat{u}_1) u_{2n}^2 + u_{1n}^2 u_{2n} (u_{2n} - \hat{u}_2) \Big) dx \\
&- (c_0 + c_1) \int_{\mathbb{R}^3} \Big(u_{1n}(u_{1n} - \hat{u}_1) u_{0n}^2 + (u_{1n}^2 + u_{2n}^2) u_{0n} (u_{0n} - \hat{u}_0) + u_{2n}(u_{2n} - \hat{u}_2) u_{0n}^2 \Big) dx \\
&- c_1 \int_{\mathbb{R}^3} \Big((u_{1n} - \hat{u}_1) u_{2n} u_{0n}^2 + u_{1n}(u_{2n} - \hat{u}_2) u_{0n}^2 + 2 u_{1n} u_{2n} u_{0n} (u_{0n} - \hat{u}_0) \Big) dx \\
&+ \int_{\mathbb{R}^3} |x|^2 \Big(u_{1n}(u_{1n} - \hat{u}_1) + u_{2n}(u_{2n} - \hat{u}_2) + u_{0n}(u_{0n} - \hat{u}_0) \Big) dx \\
= &(\mu_n + \lambda_n) \int_{\mathbb{R}^3} u_{1n}(u_{1n} - \hat{u}_1) dx + (\mu_n - \lambda_n) \int_{\mathbb{R}^3} u_{2n}(u_{2n} - \hat{u}_2) dx + \mu_n \int_{\mathbb{R}^3} u_{0n}(u_{0n} - \hat{u}_0) dx + o(1).
\end{aligned}
\tag{5.4}
$$

By (5.2), we get $(\hat{u}_1, \hat{u}_2, \hat{u}_0)$ satisfies (1.4). Thus using $(u_{1n} - \hat{u}_1, u_{2n} - \hat{u}_2, u_{0n} - \hat{u}_0)$ as a text function in (1.4), we then obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(\nabla \hat{u}_1 \cdot \nabla (u_{1n} - \hat{u}_1) + \nabla \hat{u}_2 \cdot \nabla (u_{2n} - \hat{u}_2) + \nabla \hat{u}_0 \cdot \nabla (u_{0n} - \hat{u}_0)\big) dx \\
&- \int_{\mathbb{R}^3} \Big((c_0 + c_1) \big(\hat{u}_1^3 (u_{1n} - \hat{u}_1) + \hat{u}_2^3 (u_{2n} - \hat{u}_2) \big) + c_0 \hat{u}_0^3 (u_{0n} - \hat{u}_0) \Big) dx \\
&- (c_0 - c_1) \int_{\mathbb{R}^3} \Big(\hat{u}_1 (u_{1n} - \hat{u}_1) \hat{u}_2^2 + \hat{u}_1^2 \hat{u}_2 (u_{2n} - \hat{u}_2) \Big) dx \\
&- (c_0 + c_1) \int_{\mathbb{R}^3} \Big(\hat{u}_1 (u_{1n} - \hat{u}_1) \hat{u}_0^2 + (\hat{u}_1^2 + \hat{u}_2^2) \hat{u}_0 (u_{0n} - \hat{u}_0) + \hat{u}_2 (u_{2n} - \hat{u}_2) \hat{u}_0^2 \Big) dx \\
&- c_1 \int_{\mathbb{R}^3} \Big((u_{1n} - \hat{u}_1) \hat{u}_2 \hat{u}_0^2 + \hat{u}_1 (u_{2n} - \hat{u}_2) \hat{u}_0^2 + 2 \hat{u}_1 \hat{u}_2 \hat{u}_0 (u_{0n} - \hat{u}_0) \Big) dx \\
&+ \int_{\mathbb{R}^3} |x|^2 \Big(\hat{u}_1 (u_{1n} - \hat{u}_1) + \hat{u}_2 (u_{2n} - \hat{u}_2) + \hat{u}_0 (u_{0n} - \hat{u}_0) \Big) dx \\
= {}& (\hat{\mu} + \hat{\lambda}) \int_{\mathbb{R}^3} \hat{u}_1 (u_{1n} - \hat{u}_1) dx + (\hat{\mu} - \hat{\lambda}) \int_{\mathbb{R}^3} \hat{u}_2 (u_{2n} - \hat{u}_2) dx + \hat{\mu} \int_{\mathbb{R}^3} \hat{u}_0 (u_{0n} - \hat{u}_0) dx.
\end{aligned}$$

Together with (5.2), (5.4), we can see that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(|\nabla (u_{1n} - \hat{u}_1)|^2 + |\nabla (u_{2n} - \hat{u}_2)|^2 + |\nabla (u_{0n} - \hat{u}_0)|^2 \big) dx \\
&+ \int_{\mathbb{R}^3} |x|^2 \Big(|u_{1n} - \hat{u}_1|^2 + |u_{2n} - \hat{u}_2|^2 + |u_{0n} - \hat{u}_0|^2 \Big) dx = o(1),
\end{aligned}$$

which gives

$$\int_{\mathbb{R}^3} \big(|\nabla u_{1n}|^2 + |\nabla u_{2n}|^2 + |\nabla u_{0n}|^2 \big) dx \to \int_{\mathbb{R}^3} \big(|\nabla \hat{u}_1|^2 + |\nabla \hat{u}_2|^2 + |\nabla \hat{u}_0|^2 \big) dx, \quad \text{as } n \to \infty$$

and

$$\int_{\mathbb{R}^3} |x|^2 \big(u_{1n}^2 + u_{2n}^2 + u_{0n}^2 \big) dx \to \int_{\mathbb{R}^3} |x|^2 \big(\hat{u}_1^2 + \hat{u}_2^2 + \hat{u}_0^2 \big) dx, \quad \text{as } n \to \infty.$$

Therefore, we get the strong convergence of $(u_{1n}, u_{2n}, u_{0n}) \to (\hat{u}_1, \hat{u}_2, \hat{u}_0)$ in Λ as $n \to \infty$. □

By the similar argument as the proof of Lemma 3.10, we have following lemma.

Lemma 5.3. *Let $\hat{m} = \inf_{(u_1,u_2,u_0)\in\mathcal{P}} I(u_1, u_2, u_0)$, then*

$$P(u_1, u_2, u_0) < 0 \Rightarrow P(u_1, u_2, u_0) \le I(u_1, u_2, u_0) - \hat{m}.$$

Proof of Theorem 3. (i) By Lemma 5.1 and Lemma 5.2, $(\hat{u}_1, \hat{u}_2, \hat{u}_0) \in \mathcal{M}$ is a mountain pass type solution to (1.4). Moreover, by (5.1),

$$\sigma > \max \big\{ I(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0), I\big(\tau_1 \star (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)\big) \big\} > 0.$$

Therefore, we complete the proof of (i) in Theorem 3.

(ii) Let $(\hat{u}_1, \hat{u}_2, \hat{u}_0)$ be the solution obtained in (i) of Theorem 3. Let $\hat{\mathbf{u}}_\tau := \tau \star \hat{\mathbf{u}} := \tau \star (\hat{u}_1, \hat{u}_2, \hat{u}_0)$, since $\hat{\mathbf{u}} \in \mathcal{P}$, then for every $\tau > 0$, $P(\tau \star \hat{u}_1, \tau \star \hat{u}_2, \tau \star \hat{u}_0) < 0$. Let $\mathbf{\Phi}^\tau = (\Phi_1^\tau, \Phi_2^\tau, \Phi_0^\tau)$ be the solution of system (1.3) with initial datum $\hat{\mathbf{u}}_\tau$ defined on the maximal interval $(T_{\min}, T_{\max})$. By continuity of

P, provided $|t|$ is sufficiently small we have $P(\mathbf{\Phi}^\tau(t)) < 0$. Therefore, by Lemma 5.3 and recalling that the energy is conserved along trajectories of system (1.3), we have

$$P(\mathbf{\Phi}^\tau(t)) \le I(\mathbf{\Phi}^\tau(t)) - \hat{m} = I(\hat{\mathbf{u}}_\tau) - \hat{m} =: -\delta < 0,$$

for any such t, and by continuity again we infer that $P(\mathbf{\Phi}^\tau(t)) < -\delta < 0$ for every $t \in (T_{\min}, T_{\max})$. To obtain a contradiction, we define

$$f_\tau(t) := \int_{\mathbb{R}^3} |x|^2 \big((\Phi_1^\tau(t,x))^2 + (\Phi_2^\tau(t,x))^2 + (\Phi_0^\tau(t,x))^2 \big) dx,$$

from the proof of (iii) of Theorem 1, we have

$$\begin{aligned} f_\tau''(t) &= 4 \sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \Big(\partial_t \overline{\Phi^\tau}_j(t,x) x \cdot \bigtriangledown \Phi_j^\tau(t,x) + \overline{\Phi^\tau}_j(t,x) x \cdot \bigtriangledown \partial_t \Phi_j^\tau(t,x) \Big) dx \\ &= -4 \left(\sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \partial_t \Phi_j^\tau(t,x) 2x \cdot \bigtriangledown \overline{\Phi^\tau}_j(t,x) dx + 3 \sum_{j=0}^{2} \operatorname{Im} \int_{\mathbb{R}^3} \partial_t \Phi_j^\tau(t,x) \overline{\Phi^\tau}_j(t,x) dx \right). \end{aligned}$$

From the virial identity (see Lemma 3.7 in [26]), we have

$$\begin{aligned} f_\tau''(t) &= 8 \int_{\mathbb{R}^3} \big(|\nabla \Phi_1^\tau|^2 + |\nabla \Phi_2^\tau|^2 + |\nabla \Phi_0^\tau|^2 \big) dx - 8 \int_{\mathbb{R}^3} |x|^2 \big((\Phi_1^\tau)^2 + (\Phi_2^\tau)^2 + (\Phi_0^\tau)^2 \big) dx \\ &\quad - 6c_0 \int_{\mathbb{R}^3} \Big((\Phi_1^\tau)^2 + (\Phi_2^\tau)^2 + (\Phi_0^\tau)^2 \Big)^2 dx - 6c_1 \Big\{ 4 \int_{\mathbb{R}^3} \operatorname{Re} \overline{\Phi}_1^\tau \overline{\Phi}_2^\tau (\Phi_0^\tau)^2 dx \\ &\quad + \int_{\mathbb{R}^3} \Big(\big((\Phi_1^\tau)^2 - (\Phi_2^\tau)^2 \big)^2 + 2 \big((\Phi_1^\tau)^2 + (\Phi_2^\tau)^2 \big) (\Phi_0^\tau)^2 \Big) dx \Big\}. \end{aligned}$$

Since

$$\begin{aligned} P(\mathbf{\Phi}^\tau(t)) &= \int_{\mathbb{R}^3} \big(|\nabla \Phi_1^\tau|^2 + |\nabla \Phi_2^\tau|^2 + |\nabla \Phi_0^\tau|^2 \big) dx - \int_{\mathbb{R}^3} |x|^2 \big((\Phi_1^\tau)^2 + (\Phi_2^\tau)^2 + (\Phi_0^\tau)^2 \big) dx \\ &\quad - \frac{3}{4} c_0 \int_{\mathbb{R}^3} \Big((\Phi_1^\tau)^2 + (\Phi_2^\tau)^2 + (\Phi_0^\tau)^2 \Big)^2 dx - \frac{3}{4} c_1 \Big\{ 4 \int_{\mathbb{R}^3} \operatorname{Re} \overline{\Phi}_1^\tau \overline{\Phi}_2^\tau (\Phi_0^\tau)^2 dx \\ &\quad + \int_{\mathbb{R}^3} \Big(\big((\Phi_1^\tau)^2 - (\Phi_2^\tau)^2 \big)^2 + 2 \big((\Phi_1^\tau)^2 + (\Phi_2^\tau)^2 \big) (\Phi_0^\tau)^2 \Big) dx \Big\}, \end{aligned}$$

we have $f_\tau''(t) = 8P(\mathbf{\Phi}^\tau(t)) < -8\delta < 0$, and as a consequence

$$0 \le f_\tau(t) \le -\delta t^2 + O(t), \quad \text{for all } t \in (-T_{\min}, T_{\max}).$$

Since the right hand side becomes negative for $|t|$ sufficiently large, it is necessary that both $T_{\min}$ and $T_{\max}$ are bounded. This proves that, for a sequence of initial data arbitrarily close to $\hat{\mathbf{u}}$, we have blow-up in finite time, implying instability. Therefore, we complete the proof of (ii) in Theorem 3. □

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References

[1] P. Antonelli, R. Carles, J. Silva, Scattering for nonlinear Schrödinger equation under partial harmonic confinement, *Commun. Math. Phys.* 334 (2015) 367–396.

[2] M. Anderson, J. Ensher, M. Matthews, C. Wieman, C. Cornell, Observation of Bose- Einstein condensation in a dilute atomic vapor, *Science* 269 (1995) 198–201.

[3] H. Berestycki, T. Cazenave, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéarires, *C. R. Acad. Sci. Paris, Seire I* 293 (1981) 489–492.

[4] J. Bellazzini, N. Boussaïd, L. Jeanjean, N. Visciglia, Existence and Stability of Standing Waves for Supercritical NLS with a Partial Confinement, *Commun. Math. Phys.* 353 (2017) 229–251.

[5] J. Bellazzini, L. Jeanjean, On dipolar quantum gases in the unstable regime, *SIAM J. Math. Anal.* 48 (2016) 2028–2058.

[6] T. Bartsch, L. Jeanjean, N. Soave, Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$, *J. Math. Pures Appl.* 106 (2016) 583–614.

[7] T. Bartsch, N. Soave, A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems, *J. Funct. Anal.* 272 (2017) 4998–5037.

[8] C. Bradley, C. Sackett, R. Hulet, Bose-Einstein condensation of lithium: observation of limited condensate number, *Phys. Rev. Lett.* 78 (1997) 985–989.

[9] T. Bartsch, X.X. Zhong, W.M. Zou, Normalized solutions for a coupled Schrödinger system, *Math. Ann.* 380 (2021) 1713–1740.

[10] M. Blau, J. Hartong, B. Rollier, Geometry of Schrödinger space-times, global coordinates, and harmonic trapping, *J. High Energy Phys.* 027 (2009) 17.

[11] M.-S. Chang, Q. Qin, W. Zhang, Y. Li, Michael S. Chapman, Coherent spinor dynamics in a spin-1 Bose condensate, *Nat. Phys.* 1 (2005) 111-116.

[12] T. Cazenave, Semilinear Schrödinger equations, In Courant Lecture Notes in Mathematics, American Mathematical Society, Courant Institute of Mathematical Sciences, 2003.

[13] I- L. Chern, C.F. Chou, T. Shieh, Ground-state patterns and phase diagram of spin-1 Bose-Einstein condensates in uniform magnetic field, *Phys. D* 388(2019) 73–86.

[14] D.M. Cao, I- L. Chern, J. C. Wei, On ground state of spinor Bose-Einstein, *Nonlinear Differ. Equ. Appl.* 18 (2011) 427–445.

[15] J.H. Chen, I- L. Chern, W.C. Wang, Exploring ground states and excited states of spin-1 Bose-Einstein condensates by continuation methods, *J. Comput. Phys.* 230 (2011) 2222–2236.

[16] J.H. Chen, I- L. Chern, W.C. Wang, A Complete Study of the Ground state phase diagrams of spin-1 Bose-Einstein condensates in a magnetic field via continuation Methods, *J Sci Comput* 64 (2015) 35–54.

[17] K. Davis, M. Mewes, M. Andrews, N. van Druten, D. Durfee, D. Kurn, W. Ketterle, Bose-Einstein condensation in a gas of sodium atoms, *Phys. Rev. Lett.* 75 (1995) 3969–3973.

[18] V.D. Dinh, On the instability of standing waves for 3D dipolar Bose-Einstein condensates, *Phys. D* 419 (2021) Paper No. 132856.

[19] L. Forcella, X. Luo, T. Yang, X.L. Yang, Standing waves for a Schödinger system with three waves interaction, *arXiv:* 2210.07643 (2022).

[20] Y.J. Guo, S. Li, J.C. Wei, X.Y. Zeng, Ground states of two-component attractive Bose-Einstein condensates I: Existence and uniqueness, *J. Funct. Anal.* 276 (2019) 183–230.

[21] Y.J. Guo, S. Li, J.C. Wei, X.Y. Zeng, Ground states of two-component attractive Bose-Einstein condensates II: Semi-trivial limit behavior, *Trans. Amer. Math. Soc.* 371 (2019) 6903–6948.

[22] H. Hajaiej, C.A. Stuart, On the variational approach to the stability of standing waves for the nonlinear Schödinger equation, *Adv. Nonlinear Stud.* 4 (2004) 469–501.

[23] Z. Hani, L. Thomann, Asymptotic behavior of the nonlinear Schödinger equation with harmonic trapping, *Comm. Pure Appl. Math.* 69 (2016) 1727–1776.

[24] P. Lushnikov, Collapse of Bose-Einstein condensates with dipole-dipole interactions, *Phys. Rev. A* 82 (2002) 357–364.

[25] T.L. Ho, Spinor Bose-Einstein condensates in optical traps, *Phys. Rev. Lett.* 81 (1998) 742–745.

[26] H. Hajaiej, R. Carles, On the spin-1 Bose-Einstein condensates in the presence of Ioffe-Pritchard magnetic field, *Commun. Contemp. Math.* 18 (2016) 1550062, 18 pp.

[27] N. Ikoma, Compactness of minimizing sequences in nonlinear Schrödinger systems under multi-constraint conditions, *Adv. Nonlinear Stud.* 14 (2014) 115–136.

[28] T. Isoshima, K. Machida, T. Ohmi, Spin-domain formation in spinor Bose-Einstein condensation, *Phys. Rev. A* 60 (1999) 4857.

[29] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, Nonlinear Anal. 28 (1997) 1633–1659.

[30] M.K. Kwong, Uniqueness of positive solutions of $-\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, *Arch. Ration. Mech. Anal.* 105 (1989) 243–266.

[31] Y. Kawaguchi, M. Ueda, Spinor Bose-Einstein condensates, *Phys. Rep.* 520 (2012) 253–381.

[32] Y.Z. Kong, Q.X. Wang, D. Zhao, Ground states of spin-1 BEC with attractive mean-field interaction trapped in harmonic potential in $\mathbb{R}^2$, *Calc. Var. Partial Differential Equations* 60 (2021) 152.

[33] L.R. Lin, I-L. Chern, Bifurcation between 2-component and 3-component ground states of spin-1 Bose-Einstein condensates in uniform magnetic fields, *arXiv:* 1302.0279 (2013).

[34] L. Stefan, A note on Berestycki-Cazenave's classical instability result for nonlinear Schrödinger equations, *Adv. Nonlinear Stud.* 8 (2008) 455–463.

[35] E. Lieb, M. Loss, Analysis. American Mathematical Society, Providence, 2001.

[36] W. Luo, Z.X. Lü, Z.H. Liu, On the ground state of spin-1 Bose-Einstein condensates with an external Ioffe-Pitchard magnetic field, *Bull. Aust. Math. Soc.* 86 (2012) 356–369.

[37] X. Luo, Stability and multiplicity of standing waves for the inhomogeneous NLS equation with a harmonic potential, *Nonlinear Anal. Real World Appl.* 45 (2019) 688–703.

[38] N.V. Nguyen, Z.Q. Wang, Existence and stability of a two-parameter family of solitary waves for a 2-coupled nonlinear Schrödinger system, *Discrete Contin. Dyn. Syst.* 36 (2016) 1005-1021.

[39] A. Pankov, Introduction to spectral theory of Schrödinger operators. Giessen University, 2001.

[40] P. Ruprecht, M. Holland, K. Burnett, M. Edwards, Time-dependent solution of the nonlinear Schrödinger equation for Bose-condensed trapped neutral atoms, *Phys. Rev. A* 51 (1995) 4704–4711.

[41] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: The Sobolev critical case, *J. Funct. Anal.* 279 (2020) 108610.

[42] L. Sadler, J. Higbie, S. Leslie, M. Vengalattore, D. Stamper-Kurn, Spontaneous symmetry breaking in a quenched ferromagnetic spinor Bose-Einstein condensate, *Nature* 443 (2006) 312–315.

[43] D. Stamper-Kurn, M. Andrews, A. Chikkatur, S. Inouye, H. Miesner, J. Stenger, W. Ketterle, Optical confinement of a Bose-Einstein condensate, *Phys. Rev. Lett.* 80 (1998) 2027–2030.

[44] J. Stenger, S. Inouye, D. Stamper-Kurn, H. Miesner, A. Chikkatur, W. Ketterle, Spin domains in ground-state Bose-Einstein condensates, *Nature* 396 (1998) 345–348.

[45] M. Willem, Minimax theorems. Birkhä user Boston, Boston, 1996.

[46] J.C. Wei, Y.Z. Wu, Normalized solutions for Schrödinger equations with critical Sobolev exponent and mixed nonlinearities, *J. Funct. Anal.* 283 (2022) 109574.

[47] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.* 87 (1982) 567–576.