LECTURE NOTES ON THE PARABOLIC GLUING METHOD

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Abstract. Singularity formation for evolution equations has attracted much attention in recent years. In this lecture note, we will introduce some recent progress on the *parabolic gluing method* and its applications in investigating the mechanism of singularity formation for parabolic flows. Several model problems will be revisited to illustrate the ideas.

Contents

1. Introduction

Singularity formation for evolution equations has attracted much attention in recent years, probably because of the connection to the possible singularity or global regularity for the incompressible Navier-Stokes equation in $\mathbb{R}^3$, a Clay Millennium Problem, as well as the motivations from geometric flows (Ricci flow and mean curvature flow). As a matter of fact, the resolution of Poincare's conjecture (another Clay Millennium problem) by G. Perelman [78, 79] is a manifesto of the importance of the analysis of singularity formulation in evolution equations. Many equations, such as Fujita equations and harmonic map heat flows, which certainly have their own interest and significance, might be regarded as testing fields for the analysis of singularity formation in evolution equations. In this survey, we shall report some recent development on the *parabolic gluing method* and its applications in constructing finite- and infinite-time blow-up solutions to various evolution equations.

In the elliptic context, the *inner–outer gluing method* was developed by del Pino, Kowalczyk, and Wei [23, 24] to investigate concentration on higher dimensional sets such as curves and surfaces. The climax of this method is the resolution of De Giorgi's Conjecture in dimensions greater than 8 [24]. Since then, many new phenomena and features have been found in the Allen-Cahn equations, critical or supercritical elliptic problems, and other settings. Its parabolic analogue, motivated by a recent surge of interests in the singularity formation in evolution settings, was developed by Dávila, del Pino, Musso, and Wei to investigate finite- and infinite-time blow-up solutions for energy critical heat equations and heat flow of harmonic maps [17, 22]. Later, the gluing method was generalized and applied to a wider class of evolution equations. Without being exhaustive, these include Euler equations and related fluid equations, geometric flows, parabolic equations, and systems arising from mathematical biology and physics such as Keller-Segel systems, nematic liquid crystal flow, the LLG equation, and others. We refer the reader to [19, 20, 21, 26, 27, 63, 88, 99] and the references therein.

Let us first explain some of the ideas and recent progress in the development of the gluing method in an abstract fashion and then give two specific examples to better illustrate the ideas. Roughly speaking, the parabolic gluing method is a refined type of perturbative argument. Our aim is to construct solutions exhibiting singular asymptotic behavior near some concentration points as $t \to T$ or $t \to +\infty$. The construction starts with a well-chosen blow-up profile, usually driven by energy concentration. Then one looks for a perturbation that consists of inner and outer parts, where the inner part captures the heart of the singularity formation and the outer part handles all the external noises. This leads to a coupled inner–outer gluing system involving the inner and outer solutions and the (typically scaling and translation) parameter functions. The full system is then solved by a fixed point argument provided that one can obtain suitable linear theories for inner and outer problems as well as the reduced problems that determine the dynamics of parameters. The linear theories are designed such that the full system is decoupled or less coupled, namely the gluing procedure can be implemented, and it usually involves careful and rather precise choices of weighted topologies in a pointwise sense for solution spaces. On the other hand, the linearization for the inner problem is surely not invertible in the presence of an infinitesimal generator of rigid motions, and thus for an inner solution with sufficient decay to exist, orthogonality conditions are required ensuring the development of the linear theory. These orthogonalities in turn determine the dynamics of parameters, yielding the desired blow-up speed and location.

A typical first approximate solution is the steady state invariant under rescaling and translation. These invariances naturally imply kernels in the linearized operator. We call the case where all the kernels decay sufficiently fast the L^2 case, while the case with slowly decaying kernels ($\notin L^2(\mathbb{R}^n)$) is called the non-L^2 case. Usually the non-L^2 case happens in lower dimensions, and under such circumstances, well chosen nonlocal corrections are needed in order to improve the spatial decay. The new error terms introduced by nonlocal corrections enter the orthogonality condition (at the corresponding mode) as leading order, leading to a certain integro-differential operator in the reduced equation. This global feature has been observed, for instance, in [18, 22, 26, 100]. Techniques such as the Laplace transform and Riemann-Liouville type can be applied for certain cases, but for the other threshold cases, one has to take advantage of the Hölder regularity inherited from the outer problem to control the nonlocal operator.

In general, the development of the linear theory for the outer problem is more straightforward compared to the one for the inner problem. In parabolic settings, the maximum principle, or the direct and careful use of Duhamel's formula, can be employed. However, the design of the weighted space can be more delicate in the absence of the maximum principle. The gluing method in such set-ups has been developed and applied equally well. See [19, 20, 21, 99] for recent progress on the incompressible Euler equations and LLG equation.

For the inner problem, solutions with sufficient decay in space-time can only be expected with orthogonality conditions imposed, and careful choice of initial data might be needed if instability is present for the corresponding linearized operator, resulting in codimension stability. There are various techniques and tools available, and the spectral information plays a key role. A refined version, that gets less deteriorated in the innermost region, can be achieved by the re-gluing process, namely another inner–outer gluing procedure. The distorted Fourier transform also turns out to be a powerful tool in the gluing method and is present in [99]. This is motivated by [42] on the spectral analysis of the Schrödinger operator and [59, 60, 61, 62] on the singularity formation for wave equations and wave maps.

In the rest of this lecture note, to illustrate the ideas and techniques in the parabolic gluing method, we plan to revisit several models

- finite time blow-up for the Fujita equation with critical exponent in $\mathbb{R}^5$ (L^2 case) and in $\mathbb{R}^4$ (non-L^2 case);
- finite time blow-up for the harmonic map heat flow and the Landau-Lifshitz-Gilbert equation in $\mathbb{R}^2$;
- long-term dynamics for the 1-equivariant harmonic map heat flow, Fujita equation in $\mathbb{R}^4$, and the Keller-Segel system in $\mathbb{R}^2$.

In the appendices, different methods for linear theories will be presented.

2. Fujita equation: a brief introduction

Let us start with a brief introduction to singularity formation for the Fujita equation,

$$u_t = \Delta u + |u|^{p-1}u \ \text{ in } \ \Omega \times (0,T), \tag{2.1}$$

where Ω is the entire space $\mathbb{R}^n$ or a smooth domain in $\mathbb{R}^n$ and $0 < T \le +\infty$. This semilinear heat equation with $p > 1$ has been widely studied since Fujita's celebrated work [39]. The Fujita equation might be the one of the most simple-looking semilinear parabolic equations. However, rich and sophisticated phenomena arise, and those are intimately related to the power nonlinearity in a rather precise manner. Much literature has been devoted to studying this problem concerning the singularity formation. For a comprehensive survey in the literature, we refer the readers to the book of Quittner and Souplet [82].

For the finite time blow-up, the solution u is said to be *type I* if

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty < +\infty,$$

and *type II* if

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty = +\infty.$$

Type I blow-up is more generic and similar to that of the ODE $u_t = u^p$, while type II blow-up, where the Laplacian dominates, is much more difficult to detect. In particular, two different types of blow-up phenomena in problem (4.6) depend sensitively on the power nonlinearity. For instance, it is known after a series of works, including [43, 44], that type I is the only way possible if $p < p_s$ in the case that Ω is $\mathbb{R}^n$ or a convex domain, where p_s is the critical Sobolev exponent

$$p_s := \begin{cases} \frac{n+2}{n-2} & \text{if } n \ge 3,\\ +\infty & \text{if } n = 1,2.\end{cases}$$

The critical exponent p_s is special in various ways. For the energy critical case $p = p_s$, in the positive radial and monotonically decreasing class, Filippas, Herrero and Velázquez [36] excluded the possibility of type II blow-up for $n \ge 3$, and Matano and Merle [70, Theorem 1.7] removed the monotone assumption and obtained the same result. Wang and Wei [98] generalized the result to the non-radial positive class in higher dimensions $n \ge 7$. For $p < p_s$, finite time type I blow-up solution was found and its stability was studied in [74]. For the critical case $p = p_s$ in $\mathbb{R}^n$ with $n \ge 7$, classification results were proved near the ground state of the energy critical heat equation in [14]. In the aspect of type II blow-ups, the first example was discovered by Herrero-Velázquez [54, 55], for $p > p_{JL}$ where p_{JL} is the Joseph-Lundgren exponent [57]

$$p_{JL} = \begin{cases} 1 + \frac{4}{n-4-2\sqrt{n-1}} & \text{if } n \ge 11,\\ +\infty, & \text{if } n \le 10.\end{cases}$$

See, for instance, [12, 15, 28, 75, 87] and references therein for more results on existence and construction of type II blow-ups. For the critical case $p = p_s$ in dimensions $n = 3, 4, 5, 6$, sign-changing type II blow-up solutions were conjectured to exist, via formal matched asymptotic analysis, by Filippas, Herrero and Velazquez [36] and have been rigorously constructed recently in [25, 29, 31, 51, 52, 65, 86].

In view of the results mentioned above, regarding finite-time blow-up for positive solutions to the Fujita equation (4.6), we mention three interesting open questions/Conjectures.

Conjecture 1. For $3 \le n \le 6$ and $p = \frac{n+2}{n-2}$, all positive finite time blow-ups to (4.6) are Type I.

Conjecture 2. For $n \ge 7$ and $p = \frac{n+2}{n-2}$, all (sign-changing) finite time blow-ups to (4.6) are Type I.

Conjecture 3. For $\frac{n+2}{n-2} < p < p_{JL}(n)$ and $p \ne \frac{n-m+2}{n-m-2}$, all finite time blow-ups to (4.6) are Type I.

On the other hand, infinite time blow-ups for $p = p_s$ have also received some attention recently. In dimensions $n \geq 3$, Galaktionov and King [40] investigated positive, radially symmetric, infinite time blow-up solutions for problem (4.6) in the case of the unit ball with Dirichlet boundary condition. See also [92, Theorem 1.4] for the case that the domain is convex and symmetric. In the non-radial setting, the positive infinite time blow-up solutions for problem (4.6) with zero Dirichlet boundary condition and $n \geq 5$ was constructed in [17], where the role of the Green's function in the bubbling phenomenon was studied, in parallel to the seminal works [3] and [4] in elliptic settings. See also [30] for the construction based on non-degenerate sign-changing profile and [27, 91] for the bubble towers in higher dimensions at forward and backward time infinity. Infinite time blow-ups for the lower dimensions $n = 3, 4$ have been constructed in [26, 100] confirming a conjecture by Fila and King [35].

2.1. L^2 **case: critical Fujita equation in** $\mathbb{R}^5$**.** The first example is the type II singularity for Fujita equation with critical exponent in $\mathbb{R}^5$.

The first step is to find a suitable blow-up profile whose natural choice is the steady state. We recall that all positive entire solutions of the equation

$$\Delta u + |u|^{\frac{4}{n-2}} u = 0 \quad \text{in } \mathbb{R}^n$$

are given by the family of *Aubin-Talenti bubbles*

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\mu}\right) \tag{2.2}$$

where

$$U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{n-2}{4}}.$$

The solutions we construct do change sign, and look at main order near the blow-up points as one of the bubbles (2.2) with time dependent parameters and $\mu(t) \to 0$ as $t \to T$. Thus we consider the equation

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u \quad \text{in } \Omega \times (0,T), \\ u(\cdot, 0) = u_0 \quad \text{in } \Omega \end{cases} \tag{2.3}$$

in the case $n = 5$, $p = 7/3$. Let us fix arbitrary points $q_1, q_2, \dots q_k \in \Omega$. We consider a smooth function $Z_0^* \in L^\infty(\Omega)$ with the property that

$$Z_0^*(q_j) < 0 \ \text{ for all } \ j = 1, \dots, k.$$

The sign condition is required to ensure the existence of desired blow-up dynamics.

Theorem 1 ([25])**.** *Let $n = 5$. For each $T > 0$ sufficiently small there exists an initial condition u_0 such that the solution of problem (2.3) blows up at time T exactly at the k points $q_1, \dots, q_k$. It looks at main order as*

$$u(x,t) = \sum_{j=1}^{k} U_{\mu_j(t),\xi_j(t)}(x) + Z_0^*(x) + \theta(x,t)$$

where

$$\mu_j(t) \to 0, \quad \xi_j(t) \to q_j \quad \text{as} \quad t \to T,$$

and $\|\theta\|_{L^\infty} \leq T^a$ for some $a > 0$. More precisely, for numbers $\beta_j > 0$ we have

$$\mu_j(t) \ = \ \beta_j (T-t)^2 \, (1 + o(1)),$$

We observe that in particular, the solution constructed in Theorem 7 is type II since

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^5)} \sim (T-t)^{-3} \gg (T-t)^{-3/4}.$$

For notational simplicity we shall only sketch the proof in the single-bubble case $k = 1$. The general case requires relatively minor changes.

- **Ansatzes and error estimates.**

We fix a point $q \in \Omega$. Let us consider a function Z_0^* smooth in $\bar{\Omega}$ with $Z_0^* = 0$ on $\partial\Omega$. We assume in addition that

$$Z_0^*(q) < 0. \tag{2.4}$$

We let $Z^*(x,t)$ be the unique solution of the initial-boundary value problem

$$\begin{cases} \partial_t Z^* = \Delta Z^* \quad \text{in } \ \Omega\times(0,\infty), \\ \ \ Z^* = 0 \quad \text{on } \partial\Omega\times(0,\infty), \quad Z^*(\cdot,0) = Z_0^* \quad \text{in } \Omega. \end{cases} \tag{2.5}$$

We consider functions $\xi(t)\to q$, and parameters $\mu(t)\to 0$ as $t\to T$. We look for a solution of the form

$$u(x,t) = U_{\mu(t),\xi(t)}(x) + Z^*(x,t) + \varphi(x,t) \tag{2.6}$$

with a remainder φ consisting of inner and outer parts

$$\varphi(x,t) = \mu^{-\frac{n-2}{2}}\phi\,(y,t)\,\eta_R(y) + \psi(x,t), \quad y = \frac{x-\xi(t)}{\mu(t)} \tag{2.7}$$

where

$$\eta_R(y) = \eta_0\left(\frac{|y|}{R}\right)$$

and $\eta_0(s)$ is a smooth cut-off function with $\eta_0(s)=1$ for $s<1$ and $=0$ for $s>1$.

Let us define the error of u as

$$S(u) = -u_t + \Delta u + u^p.$$

Then

$$\begin{aligned} S(U_{\lambda,\xi}+Z^*+\varphi) \ &= \ -\varphi_t + \Delta\varphi + pU_{\mu,\xi}^{p-1}(\varphi+Z^*) + \mu^{-\frac{n+2}{2}}E + N(Z^*+\varphi) \\ &= \ \eta_R\mu^{-\frac{n+2}{2}}\big[-\mu^2\phi_t + \Delta_y\phi + pU(y)^{p-1}[\phi + \mu^{\frac{n-2}{2}}(Z^*+\psi)] + E\,\big] \\ &\quad -\psi_t + \Delta_x\psi + p\mu^{-2}(1-\eta_R)U(y)^{p-1}(Z^*+\psi) + A[\phi] \\ &\quad + B[\phi] + \mu^{-\frac{n+2}{2}}E(1-\eta_R) + N(Z^*+\varphi) \end{aligned}$$

where

$$E(y,t) \ := \mu\dot\mu[y\cdot\nabla U(y) + \frac{n-2}{2}U(y)] + \mu\dot\xi\cdot\nabla U(y), \tag{2.8}$$

$$N_{\mu,\xi}(Z) := |U_{\mu,\xi}+Z|^{p-1}(U_{\mu,\xi}+Z) - U_{\mu,\xi}^p - pU_{\mu,\xi}^{p-1}Z,$$

$$A[\phi] := \mu^{-\frac{n+2}{2}}\left\{\,\Delta_y\eta_R\phi + 2\nabla_y\eta_R\nabla_y\phi\right\},$$

$$B[\phi] := \mu^{-\frac{n}{2}}\left\{\,\dot\mu\big[y\cdot\nabla_y\phi + \frac{n-2}{2}\phi\big]\eta_R + \dot\xi\cdot\nabla_y\phi\,\eta_R + \big[\dot\mu y\cdot\nabla_y\eta_R + \dot\xi\cdot\nabla_y\eta_R\big]\,\phi\right\}$$

and we have used $U_{\mu,\xi}^{p-1}\varphi = \mu^{-2}U(y)^{p-1}\varphi$. Thus, we will have a solution if the pair $(\phi(y,t),\psi(x,t))$ solves the following inner–outer gluing system

$$\mu^2\phi_t \ = \ \Delta_y\phi + pU(y)^{p-1}\phi + H(\psi,\mu,\xi) \quad \text{in } B_{2R}(0)\times(0,T) \tag{2.9}$$

$$\begin{cases} \quad \psi_t \ &= \Delta_x\psi + G(\phi,\psi,\mu,\xi) \quad \text{in } \Omega\times(0,T) \\ \quad \psi \ &= -\,U_{\mu,\xi} \quad \text{on } \partial\Omega\times(0,T), \\ \psi(\cdot,0) \ &= 0 \quad \text{in } \Omega \end{cases} \tag{2.10}$$

where

$$\begin{aligned} H(\psi,\mu,\xi)(y,t) \ &:= \mu^{\frac{n-2}{2}}pU(y)^{p-1}(Z^*(\xi+\mu y,t) + \psi(\xi+\mu y,t)) + E(y,t), \\ G(\phi,\psi,\mu,\xi)(x,t) \ &:= p\mu^{-2}(1-\eta_R)U(y)^{p-1}(Z^*+\psi) + A[\phi] \ + B[\phi] \\ &\qquad + \mu^{-\frac{n+2}{2}}E(1-\eta_R) + N(Z^*+\varphi), \quad y = \frac{x-\xi}{\mu}. \end{aligned} \tag{2.11}$$

- **Formal derivation of μ and ξ.**

Next we do a formal consideration that allows us to identify the parameters $\mu(t)$ and $\xi(t)$ at main order. Leaving aside smaller order terms, the inner problem (2.9) is approximately an equation of the form

$$\begin{aligned} &\mu^2\phi_t = \Delta_y\phi + pU(y)^{p-1}\phi + h(y,t) \quad \text{in } \mathbb{R}^n\times(0,T) \\ &\qquad \phi(y,t)\to 0 \quad \text{as} \quad |y|\to\infty \end{aligned} \tag{2.12}$$

with

$$
\begin{aligned}
h(y,t) \;&= \mu\dot{\mu}\,(U(y) + y\cdot\nabla U(y)) + p\mu^{\frac{n-2}{2}}U(y)^{p-1}Z_0^*(q)\\
&+\; \mu\dot{\xi}\cdot\nabla U(y) + p\mu^{\frac{n}{2}}U(y)^{p-1}\nabla Z_0^*(q)\cdot y.
\end{aligned} \tag{2.13}
$$

The condition of spatial decay in y for the inner problem solution ϕ mitigates the effect of ϕ in the outer problem (2.10), making at main order (2.9) and (2.10) decoupled.

Roughly speaking, for $n \geq 5$, the elliptic equation

$$
\begin{aligned}
L[\phi] := \Delta_y\phi + pU(y)^{p-1}\phi = g(y) \quad \text{in } \mathbb{R}^n\\
\phi(y) \to 0 \quad \text{as} \quad |y| \to \infty,
\end{aligned}
$$

with $g(y) = O((1+|y|)^{-2-a})$ and $0 < a < 1$, is solved by $\phi = O((1+|y|)^{-a})$ provided that

$$
\int_{\mathbb{R}^n} g(y)Z_i(y)\,dy \;=\; 0 \;\text{ for all }\; i = 1,\dots,n+1,
$$

where

$$
Z_i(y) = \partial_i U(y), \quad i = 1,\dots,n, \quad Z_{n+1}(y) = \frac{n-2}{2}U(y) + y\cdot\nabla U(y).
$$

These are in fact all bounded solutions of the linearized equation $L[Z] = 0$.

It seems reasonable to get an approximation to a solution of equation (2.12) (valid up to large $|y|$) by solving the elliptic equation

$$
\begin{aligned}
\Delta_y\phi + pU(y)^{p-1}\phi + h(y,t) = 0 \quad \text{in } \mathbb{R}^n\times(0,T)\\
\phi(y,t) \to 0 \quad \text{as} \quad |y| \to \infty,
\end{aligned}
$$

which we can indeed do under the orthogonality conditions

$$
\int_{\mathbb{R}^n} h(y,t)Z_i(y)\,dy \;=\; 0 \;\text{ for all }\; i = 1,\dots,n+1, \quad t\in[0,T). \tag{2.14}
$$

These orthogonalities imply the dynamics of the parameters. Indeed, integrating against $Z_{n+1}(y)$ we get

$$
\int_{\mathbb{R}^n} h(y,t)Z_{n+1}(y)\,dy \;=\; \mu\dot{\mu}(t)\int_{\mathbb{R}^n} Z_{n+1}^2 dy - \frac{n-2}{2}\mu(t)^{\frac{n-2}{2}}Z_0^*(q)\int_{\mathbb{R}^n} U^p dy.
$$

Clearly, one sees from above that integrability issues arise for lower dimensional cases, yielding nonlocal/global features, which we shall discuss in next Section for the case $n = 4$. This quantity is zero if and only if for a certain explicit constant $\beta_n > 0$

$$
\dot{\mu}(t) = -\beta_n|Z_0^*(q)|\mu(t)^{\frac{n-4}{2}}, \quad \mu(T) = 0,
$$

and thus for $n = 5$

$$
\mu_*(t) = \alpha(T-t)^2, \quad \alpha = \frac{1}{4}\beta_n^2|Z_0^*(q)|^2. \tag{2.15}
$$

In a similar way, the remaining n relations in (2.14) lead us to $\dot{\xi}(t) = \mu(t)^{\frac{n-2}{2}}b$ for a certain vector b. Hence $\dot{\xi}(t) = O(T-t)^3$ and

$$
\xi(t) = q + O(T-t)^2.
$$

To solve the actual inner problem (2.12), even assuming orthogonalities (2.14) is not sufficient, and further constraints are needed. Indeed, let us recall that the operator L has a positive radially symmetric bounded eigenfunction Z_0 associated to the only positive eigenvalue λ_0 to the problem

$$
L[\phi] = \lambda_0\phi, \quad \phi\in L^\infty(\mathbb{R}^n).
$$

It is known that λ_0 is a simple eigenvalue and

$$
Z_0(y) \sim |y|^{-\frac{n-1}{2}}e^{-\sqrt{\lambda_0}\,|y|} \quad \text{as} \quad |y| \to \infty.
$$

Let us write

$$
p(t) = \int_{\mathbb{R}^n}\phi(y,t)Z_0(y)\,dy, \quad q(t) = \int_{\mathbb{R}^n} h(y,t)Z_0(y)\,dy.
$$

One expects instability produced by Z_0 along the flow without restriction on the initial data. Then we compute

$$
\mu(t)^2\dot{p}(t) - \lambda_0 p(t) = q(t).
$$

Since $\mu(t) \sim (T-t)^{-2}$, then $p(t)$ will have exponential growth in time $p(t) \sim e^{\frac{c}{T-t}}$ unless

$$p(t) = e^{\int_0^t \frac{d\tau}{\mu^2(\tau)}} \int_t^T e^{-\int_0^s \frac{d\tau}{\mu^2(\tau)}} \mu(s)^{-2} q(s)\, ds$$

This relation imposes a linear constraint on the initial data $\phi(y,0)$ to the desired solution $\phi(y,t)$ to (2.9)

$$\int_{\mathbb{R}^n} \phi(y,0) Z_0(y)\, dy \;=\int_0^T e^{-\int_0^s \frac{d\tau}{\mu^2(\tau)}} \mu(s)^{-2} \int_{\mathbb{R}^n} h(y,s) Z_0(y)\, dy\, ds. \tag{2.16}$$

For this reason, we impose an initial data for the inner problem along Z_0-direction to get rid of such instability.

• **The linear theories.**

The outer problem (2.10) in linear version is actually simpler than its counterpart (2.19), corresponding just to the standard heat equation with nearly singular right hand sides and zero initial and boundary conditions. Thus we consider the problem

$$\begin{cases} \psi_t &= \Delta_x \psi + g(x,t) \quad \text{in } \Omega \times (0,T) \\ \psi &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ \psi(\cdot,0) &= 0 \quad \text{in } \Omega. \end{cases} \tag{2.17}$$

The class of right hand sides g that we want to take are naturally controlled by the following norms. Let $0 < a < 1$, $q \in \Omega$ and $\mu_0(t) = (T-t)^2$. We define the norms $\|g\|_{o*}$ and $\|\psi\|_o$ to be respectively the least numbers K_1 and K_2 such that for all $(x,t) \in \Omega \times [0,T)$,

$$\left.\begin{aligned} |g(x,t)| &\le K_1 \left[\frac{1}{\mu_0(t)^2} \frac{1}{1+|y|^{2+a}} + 1\right] \\ |\psi(x,t)| &\le K_2 \left[\frac{1}{1+|y|^a} + T^{\frac{3}{2}a}\right] \end{aligned}\right., \qquad y = \frac{x-q}{\mu_0(t)}.$$

Then the following estimate holds.

Lemma 2.1. ([25, Lemma 4.2]) *There exists a constant C such that for all sufficiently small $T > 0$ and any g with $\|g\|_o < +\infty$, the unique solution $\psi = \mathcal{T}^{out}[g]$ of problem (3.34) satisfies the estimate*

$$\|\psi\|_{o*} \le C\|g\|_o. \tag{2.18}$$

The proof can be carried out either by barriers or Duhamel's representation.

The inner problem (2.19) in linear version is actually harder and more delicate than its counterpart (2.10). In order to deal with the inner problem (2.9), we need to solve a linear problem like (2.12) restricted to a large ball B_{2R} where orthogonality conditions like (2.14) are assumed and the initial condition of the solution depends on a scalar parameter which is part of the unknown, connected with constraint (2.16). We construct a solution (ϕ,ℓ) which defines a linear operator of functions $h(y,t)$ defined on

$$\mathcal{D}_{2R} = B_{2R} \times (0,T)$$

to the initial value problem

$$\begin{aligned} \mu^2 \phi_t &= \Delta_y \phi + pU(y)^{p-1}\phi + h(y,t) \quad \text{in } \mathcal{D}_{2R} \\ \phi(y,0) &= \ell Z_0(y) \quad \text{in } B_{2R}, \end{aligned} \tag{2.19}$$

for some constant ℓ, under the orthogonality conditions

$$\int_{B_{2R}} h(y,t)\, Z_i(y)\, dy = 0 \;\text{ for all }\; i = 1,\dots,Z_{n+1},\; t \in [0,T). \tag{2.20}$$

We impose on the parameter function μ the following constraints, which are motivated on the discussion earlier: let us write

$$\mu_0(t) = (T-t)^2.$$

For some positive constants α and β (to be fixed later), we impose

$$\alpha\mu_0(t) \;\le\; \mu(t) \;\le\; \beta\mu_0(t) \;\text{ for all }\; t \in [0,T].$$

Let us fix numbers $0 < a < 1$ and $\nu > 0$. We will consider functions h satisfying

$$|h(y,t)| \lesssim \frac{\mu_0(t)^\nu}{1+|y|^{2+a}} \quad \text{in } \mathcal{D}_{2R}.$$

The formal analysis of the previous section would make us hope to find a solution to (2.19) such that

$$|\phi(y,t)| \lesssim \frac{\mu_0(t)^\nu}{1+|y|^{a}} \quad \text{in } \mathcal{D}_{2R}.$$

We will find a solution so that a somewhat worse bound for $\phi(y,t)$ in space variable is found but coinciding with the expected behavior in the gluing regime $|y| \sim R$. Let us define the following norms. We let $\|h\|_{2+a,\nu}$ be the least number K such that

$$|h(y,t)| \ \le\ K\frac{\mu_0(t)^\nu}{1+|y|^{2+a}} \quad \text{in } \mathcal{D}_{2R} \tag{2.21}$$

and let $\|\phi\|_{*a,\nu}$ be the least number K with

$$|\phi(y,t)| \ \le\ K\mu_0(t)^\nu\frac{R^{n+1-a}}{1+|y|^{n+1}} \quad \text{in } \mathcal{D}_{2R}. \tag{2.22}$$

We observe that $\|\phi\|_{*a,\nu} \le \|\phi\|_{a,\nu}$.

The following is the key linear result associated to the inner problem.

Lemma 2.2. ([17, Proposition 7.1],[25, Lemma 4.1]) *There is a $C > 0$ such For all sufficiently large $R > 0$ and any h with $\|h\|_{2+a,\nu} < +\infty$ that satisfies relations (2.20) there exist linear operators*

$$\phi = \mathcal{T}_\mu^{in}[h], \quad \ell = \ell[h]$$

which solve Problem (2.19) and define linear operators of h with

$$|\ell[h]| + \|(1+|y|)\nabla_y\phi\|_{*a,\nu} \ +\ \|\phi\|_{*a,\nu} \ \le\ C\,\|h\|_{\nu,2+a}.$$

The proof is by using the self-similar variables (y,τ) with

$$\tau = \tau_0 + \int_0^t \mu(s)^{-2}ds.$$

Expressing $\phi = \phi(y,\tau)$, problem (2.19) becomes

$$\begin{aligned} \phi_\tau &= \Delta_y\phi + pU(y)^{p-1}\phi + h(y,\tau) \quad \text{in } B_{2R}\times(\tau_0,\infty),\\ \phi(y,0) &= \ell Z_0(y) \quad \text{in } B_{2R}. \end{aligned}$$

Then finding solution with sufficient decay in space-time consisting of three steps:

Step 1: solving an elliptic equation by orthogonality;
Step 2: solving a parabolic equation with slower decay by a spectrum gap estimate (see Lemma 2.3) and energy estimates, then improving the pointwise estimate;
Step 3: acting the linearized operator on both sides of the parabolic equation in Step 2 yields a desired solution.

In fact, for the higher dimensional case $n \ge 5$, blow-up argument can be employed to show a more refined version of the linear theory for the inner problem. The result obtained via blow-up argument turns out to be exactly what we have discussed formally before, and there is no loss of R's in the innermost region. We give a detailed proof in Appendix B. Since we do not use maximum principle in the blow-up argument, this argument is rather general and flexible and can be applied to a larger class of equations.

• **Proof of Theorem 7: fixed point argument.**

With the above preliminaries we are now ready to carry out the proof of Theorem 7 for the case $k = 1$. We want to find a tuple $\vec{p} = (\phi,\psi,\mu,\xi)$ solving the inner–outer gluing system (2.9)-(2.10) so that a desired blow-up solution u is constructed. This is achieved by formulating the problem as a fixed point problem for $\vec{p}$ in a small region of a suitable Banach space.

We first set up inner problem. For a function $h(y,t)$ defined in $\mathcal{D}_{2R}$, we write

$$c_j[h](t) = \frac{\int_{B_{2R}} h(y,t)\,Z_j(y)\,dy}{\int_{B_{2R}} |Z_j(y)|^2 dy}$$

so that the function

$$\bar h(y,t) = h(y,t) - \sum_{j=1}^{n+1} c_j[h](t)\, Z_j(y)$$

satisfies

$$\int_{B_{2R}} \bar h(y,t)\, Z_j(y)\, dy \ = \ 0 \ \text{ for all } \ j=1,\dots,n+1, \quad t\in[0,T)$$

which makes the result of Lemma 2.2 applicable to the equation

$$\begin{cases} \mu^2\phi_t \ = \ \Delta_y\phi + pU(y)^{p-1}\phi + \bar H(\psi,\mu,\xi) & \text{in } \mathcal{D}_{2R} \\ \phi(\cdot,0) = \ell\, Z_0 \quad \text{in } B_{2R} \end{cases} \tag{2.23}$$

where

$$\bar H(\psi,\mu,\xi) = H(\psi,\mu,\xi) - \sum_{j=1}^{n+1} c_j[H(\psi,\mu,\xi)]\, Z_j$$

and $H(\psi,\mu,\xi)$ is defined in (2.11). Using Lemma 2.2, we find a solution to (2.23) if the following equation is satisfied

$$\phi = \mathcal{T}^{in}_\mu[\, \bar H(\psi,\mu,\xi)\,] =: \mathcal{F}_1(\phi,\psi,\mu,\xi). \tag{2.24}$$

Then the inner equation (2.9) is satisfied if in addition we have

$$c_j[H(\psi,\mu,\xi)] = 0 \ \text{ for all } \ j=1,\dots,n+1. \tag{2.25}$$

In addition, the outer equation (2.10) is satisfied provided

$$\psi = \mathcal{T}^{out}[\, G(\phi,\psi,\mu,\xi)\,] =: \mathcal{F}_2(\phi,\psi,\mu,\xi). \tag{2.26}$$

where the operator $G(\phi,\psi,\mu,\xi)$ is defined in (2.11). We will solve system (2.23)-(2.25)-(2.26) using a degree-theoretical argument.

For $\lambda\in[0,1]$, we define the homotopy

$$\begin{aligned} H_\lambda(\psi,\mu,\xi)(y,t) \ &= \mu^{\frac{n-2}{2}} pU(y)^{p-1} Z_0^*(q) + \mu\dot\mu Z_{n+1}(y) + \mu\sum_{j=1}^{n} \dot\xi_j Z_j(y) \\ &\quad + \lambda\,\mu^{\frac{n-2}{2}} pU(y)^{p-1}(Z^*(\xi+\mu y,t) - Z_0^*(q) + \psi(\xi+\mu y,t)\,), \end{aligned}$$

and consider the system of equations

$$\begin{cases} \ \phi \ = \mathcal{T}^{in}_\mu[\, H_\lambda(\psi,\mu,\xi) - \displaystyle\sum_{j=1}^{n+1} c_j[H_\lambda(\psi,\mu,\xi)]\, Z_j\,] \\ c_j[H_\lambda(\psi,\mu,\xi)] \ = \ 0 \ \text{ for all } \ j=1,\dots,n+1, \\ \ \psi \ = \mathcal{T}^{out}[\, \lambda G(\phi,\psi,\mu,\xi)\,]. \end{cases} \tag{2.27}$$

We observe that for $\lambda=1$ this problem precisely corresponds to the system (2.23)-(2.25)-(2.26) that we want to solve.

It is convenient to write

$$\mu(t) = \mu_*(t) + \mu^{(1)}(t), \quad \xi(t) = q + \xi^{(1)}(t), \quad t\in[0,T]$$

where $\mu_*(t)$ is defined in (2.15), and $\mu^{(1)}(T)=0$, $\xi^{(1)}(t)=0$.

We assume that we have a solution $(\phi,\psi,\mu^{(1)},\xi^{(1)})$ to system (2.27) with

$$\begin{cases} |\dot\mu^{(1)}(t)| + |\dot\xi^{(1)}(t)| \ \le \delta_0 \\ \quad \|\phi\|_{*a,\nu} + \|\psi\|_\infty \ \le \delta_1 \end{cases} \tag{2.28}$$

where δ_0,δ_1 are small positive constants to be adjusted later. We will also assume that Z^* is sufficiently small but fixed independently of T, i.e., $\|Z^*\|_\infty \ll 1$.

The function $\mu_*(t)$ solves the equation

$$\dot\mu_*(t)\int_{\mathbb{R}^N} Z_{n+1}^2 dy + \mu_*(t)^{\frac{n-4}{2}} Z_0^*(q)\int_{\mathbb{R}^n} pU^{p-1} Z_{n+1} dy = 0. \tag{2.29}$$

The equation

$$c_{n+1}(H_\lambda(\psi,\mu_*+\mu_1,\xi))(t)=0,\quad t\in[0,T) \tag{2.30}$$

which corresponds to

$$\begin{aligned} 0 \;=\; & \dot\mu(t)\Big(\int_{B_{2R}} Z_{n+1}^2 dy\Big) \;+\; \mu(t)^{\frac{n-4}{2}} Z_0^*(q)\int_{B_{2R}} pU^{p-1}Z_{n+1}dy \\ & +\; \lambda\,\mu(t)^{\frac{n-4}{2}}\int_{\mathbb{R}^n} pU(y)^{p-1}(Z^*(\xi(t)+\mu(t)y,t)-Z_0^*(q)+\psi(\xi(t)+\mu(t)y,t)\,)Z_{n+1}(y)\,dy \end{aligned}$$

can be written as

$$\dot\mu(t)+\beta\mu(t)^{\frac{n-4}{2}}=\mu(t)^{\frac{n-4}{2}}(\delta_R+\lambda\theta(\psi,\xi,\mu_1))$$

for a suitable number $\beta>0$, $\delta_R=O(R^{-2})$ and the operator θ satisfies

$$|\theta(\psi,\xi,\mu_1)|\;\le\; C\;(T+\|\psi\|_\infty)$$

for some constant C. From (2.29), the equation for μ_1 can then be written, in the "linearized" form, as

$$\dot\mu_1+\frac{\gamma}{T-t}\mu_1=(T-t)g_0(\psi,\mu,\xi)$$

for a suitable $\gamma>0$, where

$$|g_0(\psi,\xi,\mu^{(1)},\lambda)(t)|\;\le\; C\,(\|\psi\|_\infty+T+R^{-2}).$$

The linear problem

$$\dot\mu+\frac{\gamma}{T-t}\mu=(T-t)g(t),\quad \mu_1(T)=0$$

can be uniquely solved by the following operator in g

$$\mu(t)=\mathcal{T}^0[g](t):=-(T-t)^{-\gamma}\int_t^T(T-s)^{\gamma+1}g_0(s)\,ds.$$

It defines a linear operator on g with estimates

$$\|(T-t)^{-1}\dot\mu\|_\infty+\|(T-t)^{-2}\mu\|_\infty\le C\|g_0\|_\infty.$$

Equation (2.30) then becomes

$$\mu^{(1)}(t)=\mathcal{T}^{(0)}[\,g_0(\psi,\xi,\mu^{(1)},\lambda)\,](t)\;\text{ for all }\; t\in[0,T)$$

and we get

$$\|(T-t)^{-1}\dot\mu^{(1)}\|_\infty+\|(T-t)^{-2}\mu^{(1)}\|_\infty\;\le\; C\,(\|\psi\|_\infty+T+R^{-2}). \tag{2.31}$$

Similarly, equations

$$c_j[H_\lambda(\psi,\mu,\xi)]\;=\;0\;\text{ for all }\; j=1,\dots,n,$$

can be written in vector form as

$$\xi^{(1)}(t)=\mathcal{T}^{(1)}[g_1(\psi,\mu_1,\xi_1)](t)\;\text{ for all }\; t\in[0,T), \tag{2.32}$$

where

$$\mathcal{T}^{(1)}[g]:=\int_t^T(T-s)g(s)\,ds$$

and

$$|g_1(\psi,\xi,\mu^{(1)},\lambda)(t)|\le C\,(\|\psi\|_\infty+T).$$

From equation (2.32), we thus find

$$\|(T-t)^{-1}\dot\xi^{(1)}\|_\infty+\|(T-t)^{-2}\xi^{(1)}\|_\infty\;\le\; C\,(\|\psi\|_\infty+T). \tag{2.33}$$

On the other hand, we have

$$|H(\psi,\mu,\xi)(y,t)|\;\le\; C\frac{\mu(t)^{\frac{n-2}{2}}}{1+|y|^4}(\|\psi\|_\infty+\|Z^*\|_\infty)+\frac{\mu\dot\mu}{1+|y|^{n-2}}+\frac{\mu|\dot\xi|}{1+|y|^{n-1}}$$

and thus

$$|H(\psi,\mu,\xi)(y,t)|\;\le\; C\frac{\mu_0(t)^{\frac{n-2}{2}}}{1+|y|^{2+a}}\big(\|\psi\|_\infty+\|Z^*\|_\infty\big)$$

for $0 < a < 1$. From the first equation in (2.27) and Lemma 2.2, we obtain

$$\|\phi\|_{*a,\nu} \ \le \ C\big(\|\psi\|_\infty + \|Z^*\|_\infty\big), \quad \nu = \frac{n-2}{2}. \tag{2.34}$$

with the $\|\cdot\|_{*a,\nu}$-norm defined in (2.22). Next we consider the last equation in (2.27). We recall that

$$\begin{aligned} G(\phi,\psi,\mu,\xi)(x,t) \ &= p\mu^{-2}(1-\eta_R)U(y)^{p-1}(Z^*+\psi) + A[\phi] \ + B[\phi] \\ &\quad + \mu^{-\frac{n+2}{2}}E(1-\eta_R) + N(Z^* + \mu^{-\frac{n-2}{2}}\eta_R\phi + Z^* + \psi), \\ E(y,t) \ &= \ \mu\dot\mu[y\cdot\nabla U(y) + \frac{n-2}{2}U(y)] + \mu\dot\xi\cdot\nabla U(y), \\ A[\phi] &= \mu^{-\frac{n+2}{2}}\left\{\Delta_y\eta_R\phi + 2\nabla_y\eta_R\nabla_y\phi\right\}, \\ B[\phi] &= \mu^{-\frac{n}{2}}\left\{\dot\mu\big[y\cdot\nabla_y\phi + \frac{n-2}{2}\phi\big]\eta_R + \dot\xi\cdot\nabla_y\phi\,\eta_R + \big[\dot\mu y\cdot\nabla_y\eta_R + \dot\xi\cdot\nabla_y\eta_R\big]\,\phi\right\}. \end{aligned}$$

Let us consider for example the error terms

$$g_1(x,t) = \mu^{-2}(1-\eta_R)U^{p-1}(Z^*+\psi), \quad g_2(x,t) = \mu^{-\frac{n+2}{2}}E(1-\eta_R).$$

We see that

$$|g_1(x,t)| \ \le \frac{1}{R^{2-\sigma}}\mu^{-2}\frac{C}{1+|y|^{2+\sigma}}(\|Z^*\|_\infty + \|\psi\|_\infty)$$

and

$$|g_2(x,t)| \ \le \ \frac{1}{\mu^2}\Big[\frac{1}{|y|^{n-2}}\mu^{-\frac{n-2}{2}}(|\mu\dot\mu| + |\mu\dot\xi|) \ \le \ \frac{1}{R^{3-\sigma}}\mu^{-2}\frac{C}{1+|y|^{2+\sigma}}.$$

Let us now estimate the term $A[\phi]$. Let us choose $\sigma = \frac{a}{2}$, where a is the number in the definition of $\|\phi\|_{*a,\nu}$. We have

$$\begin{aligned} \big|A[\phi](x,t)\big| \ &\le \mu^{-2}\frac{1}{R^2}\frac{1}{1+R^{-2-\sigma}|y|^{2+\sigma}}\,\mu^{-\frac{n-2}{2}}\sup_{R<|y|<2R}(|\phi| + |y||\nabla\phi|) \\ &\le \mu^{-2}\frac{R^{-\frac{a}{2}}}{1+|y|^{2+\sigma}}\|\phi\|_{*a,\frac{n-2}{2}} \end{aligned}$$

and similarly,

$$\big|B[\phi](x,t)\big| \ \le C\mu^{-2}[\mu\dot\mu + \mu|\dot\xi|]\frac{R^{n+1-a}}{1+|y|^{n+1}}\|\phi\|_{*a,\frac{n-2}{2}} \le C\mu^{-2}\frac{\mu^{\frac{3}{2}}R^{n+1-a}}{1+|y|^{2+\sigma}}\|\phi\|_{*a,\frac{n-2}{2}}.$$

Now for some $\sigma > 0$ we have

$$\begin{aligned} \big|N(Z^* + \mu^{-\frac{n-2}{2}}\eta_R\phi + Z^* + \psi)\big| \ &\le C\mu^{-2}\frac{\mu^\sigma}{1+|y|^{2+\sigma}}(\|\phi\|_{*a,\frac{n-2}{2}}R^{n+1-a} + \|Z_*\|_\infty + \|\psi\|_\infty)^2 \\ &\quad + C(\|Z_*\|_\infty + \|\psi\|_\infty)^p. \end{aligned}$$

According to the above estimates, it follows by Lemma 2.1 that

$$\|\psi\|_\infty \ \le \ CT^{\sigma'}\|Z_*\|_\infty + R^{-\sigma'}\|\phi\|_{*a,\frac{n-2}{2}}. \tag{2.35}$$

Combining (2.34) and (2.35) and then using (2.31)-(2.33), we finally get

$$\left\{\begin{aligned} \|\psi\|_\infty \ &\le CT^{\sigma'}\|Z_*\|_\infty \\ \|\phi\|_{*a,\frac{n-2}{2}} \ &\le C\|Z_*\|_\infty \\ \|(T-t)^{-1}\dot\xi^{(1)}\|_\infty + \|(T-t)^{-2}\xi^{(1)}\|_\infty \ &\le C\left(T^{\sigma'}(\|Z_*\|_\infty + 1) + R^{-2}\right) \\ \|(T-t)^{-1}\dot\mu^{(1)}\|_\infty + \|(T-t)^{-2}\mu^{(1)}\|_\infty \ &\le C\,T^{\sigma'}(\|Z_*\|_\infty + 1). \end{aligned}\right. \tag{2.36}$$

We write System (2.27) in the form

$$\left\{\begin{aligned} \phi \ &= \mathcal{T}^{in}_\mu[\,\bar H_\lambda(\mathcal{T}^{out}[\,\lambda G(\phi,\psi,\mu,\xi),\mu,\xi)] \\ \psi \ &= \mathcal{T}^{out}[\,\lambda G(\phi,\psi,\mu,\xi)\,] \\ \mu^{(1)} \ &= \mathcal{T}^{(0)}[\,\tilde g_0(\psi,\xi^{(1)},\mu^{(1)},\lambda)\,] \\ \xi^{(1)} \ &= \mathcal{T}^{(1)}[\tilde g_1(\psi,\mu^{(1)},\xi^{(1)},\lambda)]. \end{aligned}\right. \tag{2.37}$$

Here, we can write

$$\tilde{g}_0(\psi, \xi^{(1)}, \mu^{(1)}, \lambda) = c_R^1 \int_{B_{2R}} H_\lambda(\mathcal{T}^{out}[\lambda G(\phi, \psi, \mu, \xi)], \mu, \xi) Z_{n+1}(y) dy$$
$$\tilde{g}_1(\psi, \xi^{(1)}, \mu^{(1)}, \lambda) = c_R^2 \int_{B_{2R}} H_\lambda(\mathcal{T}^{out}[\lambda G(\phi, \psi, \mu, \xi)], \mu, \xi) \nabla U(y) dy$$

for suitable positive constants c_R^ℓ, $\ell = 0, 1$. We fix an arbitrarily small $\varepsilon > 0$ and consider the problem defined only up to time $t = T - \varepsilon$.

$$\begin{cases} \phi &= \mathcal{T}_\mu^{in}[\bar{H}_\lambda(\mathcal{T}^{out}[\lambda G(\phi, \psi, \mu, \xi), \mu, \xi)], \quad (y,t) \in \bar{B}_{2R} \times [0, T-\varepsilon] \\ \psi &= \mathcal{T}^{out}[\lambda G(\phi, \psi, \mu, \xi)], \quad (x,t) \in \bar{\Omega} \times [0, T-\varepsilon] \\ \mu^{(1)} &= \mathcal{T}_\varepsilon^{(0)}[\tilde{g}_0(\psi, \xi^{(1)}, \mu^{(1)}, \lambda)], \quad t \in [0, T-\varepsilon] \\ \xi^{(1)} &= \mathcal{T}_\varepsilon^{(1)}[\tilde{g}_1(\psi, \mu^{(1)}, \xi^{(1)}, \lambda)], \quad t \in [0, T-\varepsilon] \end{cases} \tag{2.38}$$

where

$$\mathcal{T}_\varepsilon^0[g](t) := -(T-t)^{-\gamma} \int_t^{T-\varepsilon} (T-s)^{\gamma+1} g_0(s)\, ds, \quad \mathcal{T}_\varepsilon^{(1)}[g] := \int_t^{T-\varepsilon} (T-s) g(s)\, ds.$$

The key is that the operators in the right hand side of (2.38) are compact when we regard them as defined in the space of functions

$$(\phi, \psi, \mu^{(1)}, \xi^{(1)}) \in X_1 \times X_2 \times X_3 \times X_4$$

with their respective norms defined as

$$X^1 = \{\phi \ / \ \phi \in C(B_{2R} \times [0, T-\varepsilon]), \ \nabla_y \phi \in C(B_{2R} \times [0, T-\varepsilon])\}, \quad \|\phi\|_{X_1} = \|\phi\|_\infty + \|\nabla_y \phi\|_\infty$$
$$X^2 = \{\psi \ / \ \phi \in C(\bar{\Omega} \times [0, T - ve])\}, \quad \|\psi\|_{X_2} = \|\psi\|_\infty$$
$$X^3 = \{\mu^{(1)} \ / \ \mu^{(1)} \in C^1[0, T-\varepsilon]\}, \quad \|\mu^{(1)}\|_{X_3} = \|\mu^{(1)}\|_\infty + \|\dot{\mu}^{(1)}\|_\infty$$
$$X^4 = \{\xi^{(1)} \ / \ \xi^{(1)} \in C^1[0, T-\varepsilon]\}, \quad \|\xi^{(1)}\|_{X_4} = \|\xi^{(1)}\|_\infty + \|\dot{\xi}^{(1)}\|_\infty.$$

Compactness on bounded sets of all the operators involved in the above expression is a direct consequence of the Hölder estimate for the operator $\mathcal{T}^{out}$ and Arzela-Ascoli's theorem. On the other hand, the a priori estimate we obtained for $\varepsilon = 0$ holds equally well, uniformly on arbitrary small $\varepsilon > 0$.

Leray Schauder degree applies in a suitable ball $\mathcal{B}$ that contains the origin in this space: essentially one slightly bigger than that defined by relations (2.36), which amounts to a choice of the parameters δ_0 and δ_1 in (2.28). In fact, the homotopy connects with the identity at $\lambda = 0$, and hence the total degree in the region defined by relations (2.36) is equal to 1. The existence of a solution to the approximate problem satisfying bounds (2.36) then follows. Finally, a standard diagonal argument yields a solution to the original problem with the desired asymptotics. The proof of Theorem 7 for the case $k = 1$ is concluded.

The general case of k distinct points $q_1, \ldots, q_k$ is actually identical: in that case we have k inner problems and one outer problem with analogous properties. We look for a solution of the form

$$u(x,t) = \sum_{j=1}^k U_{\mu_j, \xi_j}(x) + Z^*(x,t) + \mu_j^{-\frac{n-2}{2}} \phi(y_j, t) \eta_R(y_j) + \psi(x,t), \quad y_j = \frac{x - \xi_j}{\mu_j}, \tag{2.39}$$

where Z^* solves heat equation with initial condition Z_0^* which is chosen so that (2.4) holds at all concentration points, namely $Z_0^*(q_j) < 0$, and $\xi_j(T) = q_j$, $\mu_j(T) = 0$.

A string of fixed point problems (with essentially decoupled equations associated at each point) then appears and can be solved in the same way. We omit the details. □

2.2. **Non-L^2 case: critical Fujita equation in $\mathbb{R}^4$.** In this section, we consider the finite time blow-up for case $n = 4$ and $p = 3$. In what follows we let Ω be a smooth bounded domain in $\mathbb{R}^4$ or $\Omega = \mathbb{R}^4$ and consider the equation

$$\begin{cases} u_t = \Delta u + u^3 \quad \text{in } \ \Omega \times (0, T), \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{2.40}$$

Let us fix arbitrary points $q_1, q_2, \dots q_k \in \Omega$. We consider a smooth function $Z_0^* \in L^\infty(\Omega)$ with the property that

$$Z_0^*(q_j) < 0 \ \text{ for all } \ j = 1, \dots, k.$$

Theorem 2 ([31]). *For each $T > 0$ sufficiently small, there exists an initial condition u_0 such that the solution to problem (4.3) blows up at time T exactly at the k points $q_1, \dots, q_k$. The solution is of the sharply scaled form*

$$u(x, t) = \sum_{j=1}^{k} U_{\lambda_j(t), \xi_j(t)}(x) + Z_0^*(x) + \theta(x, t)$$

where

$$\lambda_j(t) \to 0, \quad \xi_j(t) \to q_j \quad \text{as} \quad t \to T,$$

and $\|\theta\|_{L^\infty} \le T^a$ for some $a > 0$. More precisely,

$$\lambda_j(t) \sim \frac{T - t}{|\log(T - t)|^2} \quad \text{as} \quad t \to T.$$

We observe that the solution constructed in Theorem 2 is type II. The result in Theorem 2 is the exact analog of Theorem 7 in dimension 5. We follow the same general approach of the *parabolic gluing method*. However, substantial differences and difficulties arise, due to the fact that the equation that determines $\lambda(t)$ involves a delicate nonlocal integro-differential operator. In dimension 5, the dynamics of $\lambda(t)$ is found in a much more direct way by just solving an ODE. This nonlocal effect is related to the slower decay of the linear generator of dilations of the Aubin-Talenti bubbles in lower dimensions, i.e., $Z_5 \notin L^2(\mathbb{R}^4)$ (see (2.43)). A very similar difficulty was already encountered in the work [22] on blow-up in the harmonic map flow, where such nonlocal operator appeared in the reduced complex system at mode 0. The similarity between these problems in the presence of symmetries had already been noticed in [83, 86].

- **Approximation and correction.**

We first choose a proper approximate solution to (4.3) and compute its error. We consider the case $k = 1$ for simplicity and mention the minor changes for the general multi-bubble case when needed. We define the error operator

$$\mathcal{S}(u) := -u_t + \Delta u + u^3.$$

Recall that the Aubin-Talenti bubble

$$U(y) = \frac{\alpha_0}{1 + |y|^2} \tag{2.41}$$

solves the Yamabe problem

$$\Delta_y U + U^3 = 0 \ \text{ in } \ \mathbb{R}^4,$$

where $\alpha_0 = 2\sqrt{2}$. It is well-known that the linearized operator around the bubble

$$L_0(\phi) := \Delta\phi + 3U^2\phi \tag{2.42}$$

is non-degenerate in the sense that all bounded solutions to $L_0(\phi) = 0$ are the linear combination of

$$Z_i(y) := \partial_{y_i} U(y), \ i = 1, 2, 3, 4, \ Z_5(y) := U(y) + \nabla U(y) \cdot y. \tag{2.43}$$

Our first approximation is chosen as

$$U_{\lambda(t), \xi(t)} = \lambda^{-1}(t) U\left(\frac{x - \xi(t)}{\lambda(t)}\right),$$

where $\lambda(t)$ and $\xi(t)$ are scaling and translation parameter functions to be adjusted later. Direct computations yield

$$\begin{aligned} S(U_{\lambda(t), \xi(t)}) = -\partial_t U_{\lambda(t), \xi(t)} &= \lambda^{-2}(t)\dot{\lambda}(t)\left(-\frac{\alpha_0}{1 + |y|^2} + \frac{2\alpha_0}{(1 + |y|^2)^2}\right) \\ &\quad + \lambda^{-2}(t)\nabla_y U(y) \cdot \dot{\xi}(t), \end{aligned} \tag{2.44}$$

where $y=\frac{x-\xi(t)}{\lambda(t)}$. Observe that the slow decaying error in (4.9) is

$$\mathcal{E}_0=-\frac{\alpha_0\dot{\lambda}(t)}{\lambda^2(t)+\rho^2}\approx-\frac{\alpha_0\dot{\lambda}(t)}{\rho^2}\notin L^2(\mathbb{R}^4),$$

where $\rho:=|x-\xi(t)|$. In order to improve the approximation, we consider

$$\partial_t u_1=\Delta u_1+\mathcal{E}_0 \text{ in } \mathbb{R}^4\times(0,T). \tag{2.45}$$

By similar computations as in [22][1], a solution to (2.45) is given explicitly by

$$u_1=-\alpha_0\int_{-T}^{t}\dot{\lambda}(s)k(\rho,t-s)ds,$$

where

$$k(\rho,t):=\frac{1-e^{-\frac{\rho^2}{4t}}}{\rho^2}. \tag{2.46}$$

We regularize the above u_1 and choose a correction Ψ_0 to be

$$\Psi_0(x,t)=-\alpha_0\int_{-T}^{t}\dot{\lambda}(s)k(\zeta(\rho,t),t-s)ds, \tag{2.47}$$

where

$$\zeta(\rho,t)=\sqrt{\rho^2+\lambda^2(t)}.$$

Then the new error produced by Ψ_0 is given by

$$\begin{aligned}
&\partial_t\Psi_0-\Delta\Psi_0-\mathcal{E}_0\\
&=\alpha_0\left[\frac{y\cdot\dot{\xi}-\dot{\lambda}(t)}{(1+|y|^2)^{1/2}}\right]\int_{-T}^{t}\dot{\lambda}(s)k_\zeta(\zeta,t-s)ds\\
&\quad+\frac{\alpha_0}{\lambda(t)(1+|y|^2)^{3/2}}\int_{-T}^{t}\dot{\lambda}(s)\left[-\zeta k_{\zeta\zeta}(\zeta,t-s)+k_\zeta(\zeta,t-s)\right]ds\\
&:=\mathcal{R}[\lambda].
\end{aligned} \tag{2.48}$$

It is thus reasonable to choose the corrected approximation as

$$u^*=U_{\lambda(t),\xi(t)}+\Psi_0$$

and its error is

$$\begin{aligned}
\mathcal{S}(u^*)&=\mathcal{S}(U_{\lambda(t),\xi(t)})-\mathcal{E}_0+(U_{\lambda(t),\xi(t)}+\Psi_0)^3-U^3_{\lambda(t),\xi(t)}\\
&=\mathcal{K}[\lambda,\xi]+(U_{\lambda(t),\xi(t)}+\Psi_0)^3-U^3_{\lambda(t),\xi(t)},
\end{aligned}$$

where $\mathcal{K}[\lambda,\xi]$ is defined as

$$\mathcal{K}[\lambda,\xi]:=\frac{2\alpha_0\lambda^{-2}(t)\dot{\lambda}(t)}{(1+|y|^2)^2}+\lambda^{-2}(t)\nabla U(y)\cdot\dot{\xi}(t)-\mathcal{R}[\lambda] \tag{2.49}$$

with $\mathcal{R}[\lambda]$ given in (2.48).

• **Formulating the inner–outer gluing system.**

We look for solution of the following form

$$u=u^*+\mathtt{w},$$

where $\mathtt{w}$ is a small perturbation consisting of inner and outer parts

$$\mathtt{w}=\varphi_{\rm in}+\varphi_{\rm out},\ \varphi_{\rm in}=\lambda^{-1}(t)\eta_R\phi(y,t),\ \varphi_{\rm out}=\psi(x,t)+Z^*(x,t).$$

Here the cut-off function is defined by

$$\eta_R=\eta_{R(t)}(x,t)=\eta\left(\frac{|x-\xi(t)|}{\lambda(t)R(t)}\right)$$

[1]See Section 17 in the full version available at https://personal.math.ubc.ca/~jcwei/hmf-2018-08-16.pdf

where the smooth cut-off function $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ for $s > 2$, and Z^* satisfies

$$\begin{cases} Z_t^* = \Delta_x Z^* & \text{in } \Omega \times (0,T), \\ Z^*(\cdot,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ Z^*(\cdot,0) = Z_0^* & \text{in } \Omega. \end{cases}$$

Denote

$$B_{2R} = \{x \in \Omega : |x - \xi(t)| \le 2\lambda R\}, \ \mathcal{D}_{2R} = B_{2R} \times (0,T),$$

and $\Psi^* = \psi + Z^*$. Then u is a solution to the original problem (4.3) if

- ϕ solves the **inner problem**

$$\lambda^2 \phi_t = \Delta_y \phi + 3U^2(y)\phi + \mathcal{H}(\phi,\psi,\lambda,\xi) \ \text{ in } \mathcal{D}_{2R} \tag{2.50}$$

where

$$\begin{aligned} \mathcal{H}(\phi,\psi,\lambda,\xi)(y,t) := {} & 3\lambda U^2(y)[\Psi_0 + \psi + Z^*](\lambda y + \xi, t) \\ & + \lambda\left[\dot\lambda(\nabla_y\phi \cdot y + \phi) + \nabla_y \phi \cdot \dot\xi\right] \\ & + \lambda^3 \mathcal{N}(\mathtt{w}) + \lambda^3 \mathcal{K}[\lambda,\xi] \end{aligned} \tag{2.51}$$

with $\mathcal{K}[\lambda,\xi]$ defined in (2.49), and

$$\mathcal{N}(\mathtt{w}) := (U_{\lambda,\xi} + \Psi_0 + \mathtt{w})^3 - U_{\lambda,\xi}^3 - 3U_{\lambda,\xi}^2(\Psi_0 + \mathtt{w}). \tag{2.52}$$

- ψ solves the **outer problem**

$$\psi_t = \Delta\psi + \mathcal{G}(\phi,\psi,\lambda,\xi) \ \text{ in } \Omega \times (0,T) \tag{2.53}$$

with

$$\begin{aligned} \mathcal{G}(\phi,\psi,\lambda,\xi) := {} & 3\lambda^{-2}(1-\eta_R)U^2(y)(\Psi_0 + \psi + Z^*) \\ & + \lambda^{-3}\left[(\Delta_y \eta_R)\phi + 2\nabla_y \eta_R \cdot \nabla_y \phi - \lambda^2 \phi \partial_t \eta_R\right] \\ & + (1-\eta_R)\mathcal{K}[\lambda,\xi] + (1-\eta_R)\mathcal{N}(\mathtt{w}). \end{aligned} \tag{2.54}$$

- **The choices of parameters.**

We choose the leading orders $\lambda_*(t)$, $\xi_*(t)$ of the parameter functions $\lambda(t)$ and $\xi(t)$. As mentioned earlier, a good inner solution can be found provided approximately the following orthogonality conditions

$$\int_{\mathbb{R}^4} \mathcal{H}(\phi,\psi,\lambda,\xi) Z_j(y)\, dy = 0 \ \text{ for all } j = 1,\cdots,5, \ t \in (0,T) \tag{2.55}$$

are satisfied. Here Z_j are the kernel functions (c.f. (2.43)) of the linearized operator L_0 defined in (2.42). Basically, the scaling and translation parameters $\lambda(t)$ and $\xi(t)$ at main order will be derived from the orthogonality conditions (2.55).

Singling out the leading term $\mathcal{H}_*$ of $\mathcal{H}$ and computing

$$\int_{\mathbb{R}^4} \mathcal{H}_*[\lambda,\xi,\Psi^*] Z_\ell(y)\, dy = 0 \ \text{ for } \ \ell = 1,\cdots,4$$

with

$$\begin{aligned} \mathcal{H}_*[\lambda,\xi,\Psi^*] := {} & 3\lambda U^2(y)[\Psi_0 + \Psi^*](\lambda y + \xi, t) + \lambda^3 \mathcal{K}[\lambda,\xi] \\ = {} & 3\lambda U^2(y)[\Psi_0 + \Psi^*](\lambda y + \xi, t) + \frac{2\alpha_0 \lambda(t)\dot\lambda(t)}{(1+|y|^2)^2} + \lambda(t)\nabla U(y) \cdot \dot\xi(t) \\ & - \frac{\alpha_0 \lambda^2(t)}{(1+|y|^2)^{3/2}} \int_{-T}^t \dot\lambda(s)\left[-\zeta k_{\zeta\zeta}(\zeta, t-s) + k_\zeta(\zeta, t-s)\right] ds \\ & - \alpha_0 \lambda^3(t)\left[\frac{y \cdot \dot\xi - \dot\lambda(t)}{(1+|y|^2)^{1/2}}\right] \int_{-T}^t \dot\lambda(s) k_\zeta(\zeta, t-s)\, ds \end{aligned}$$

imply that

$$\dot\xi_\ell = o(1),$$

where $\Psi^* = \psi + Z^*$ and $o(1) \to 0$ as $t \nearrow T$. So the choice of $\xi(t)$ at main order is

$$\xi(t) = q,$$

where q is a prescribed point in Ω.

The dynamics for $\lambda(t)$ from

$$\int_{\mathbb{R}^4} \mathcal{H}_*[\lambda, \xi, \Psi^*] Z_5(y) dy = 0$$

turns out to be more involved due to the non-local/global correction, and the reduced problem involves the following integro-differential operator

$$c_* \int_{-T}^{t-\lambda^2(t)} \frac{\dot{\lambda}(s)}{t-s} ds = -3c_0[Z_0^*(q) + \psi(q, 0)] + o(1), \tag{2.56}$$

where

$$c_0 := \int_{\mathbb{R}^4} U^2(y) Z_5(y) dy < 0.$$

Here careful calculations are needed for nonlocal terms, and detailed derivation can be found in [31, Section 4]. Since $\lambda(t)$ decreases to 0 as $t \nearrow T$, we impose

$$a_* := Z_0^*(q) + \psi(q, 0) < 0.$$

Now we claim that a good choice of $\lambda(t)$ at main order is

$$\dot{\lambda}(t) = -\frac{c}{|\log(T-t)|^2}, \tag{2.57}$$

where $c > 0$ is a constant to be determined later. Indeed, we get by substituting

$$\begin{aligned}
\int_{-T}^{t-\lambda^2(t)} \frac{\dot{\lambda}(s)}{t-s} ds &= \int_{-T}^{t-(T-t)} \frac{\dot{\lambda}(s)}{t-s} ds + \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot{\lambda}(t)}{t-s} ds - \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot{\lambda}(t) - \dot{\lambda}(s)}{t-s} ds \\
&= \int_{-T}^{t-(T-t)} \frac{\dot{\lambda}(s)}{t-s} ds + \dot{\lambda}(t)(\log(T-t) - 2\log\lambda(t)) \\
&\quad - \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot{\lambda}(t) - \dot{\lambda}(s)}{t-s} ds \\
&\approx \int_{-T}^{t} \frac{\dot{\lambda}(s)}{T-s} ds - \dot{\lambda}(t)\log(T-t) := \beta(t).
\end{aligned}$$

By (2.57), we then get

$$\log(T-t)\frac{d\beta}{dt}(t) = \frac{d}{dt}\left(-\log^2(T-t)\dot{\lambda}(t)\right) = 0,$$

which means $\beta(t)$ is a constant. Thus, equation (2.56) can be approximately solved for

$$\dot{\lambda}(t) = -\frac{c}{|\log(T-t)|^2}$$

with the constant c chosen as

$$-c \int_{-T}^{T} \frac{ds}{(T-s)|\log(T-s)|^2} = \kappa_*,$$

where $\kappa_* := -\frac{3c_0 a_*}{c_*}$. At main order, we obtain

$$\dot{\lambda}(t) = \kappa_* \dot{\lambda}_*(t)$$

with

$$\dot{\lambda}_*(t) = -\frac{|\log T|}{|\log(T-t)|^2}.$$

By imposing $\lambda_*(T) = 0$, we obtain

$$\lambda_*(t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2}(1 + o(1)) \quad \text{as} \quad t \nearrow T.$$

• **Linear theories.**

We start from the the outer problem (2.53) and consider

$$\begin{cases} \psi_t = \Delta\psi + f, & \text{in } \Omega \times (0, T), \\ \psi = 0, & \text{on } \partial\Omega \times (0, T), \\ \psi(x, 0) = 0, & \text{in } \Omega, \end{cases} \tag{2.58}$$

where the non-homogeneous term f in (2.58) is assumed to be bounded with respect to the weights appearing in the outer problem (2.53). Define the weights

$$\begin{cases} \varrho_1 := \lambda_*^{\nu-3}(t)R^{-2-\alpha}(t)\chi_{\{|x-\xi(t)|\le 2\lambda_* R\}} \\ \varrho_2 := \frac{\lambda_*^{\nu_2}}{|x-\xi(t)|^2}\chi_{\{|x-\xi(t)|\ge \lambda_* R\}} \\ \varrho_3 := 1 \end{cases} \tag{2.59}$$

where we choose $R(t) = \lambda_*^{-\beta}(t)$ for $\beta \in (0, 1/2)$. We define the norms

$$\|f\|_{**} := \sup_{(x,t)\in\Omega\times(0,T)} \left(\sum_{i=1}^{3} \varrho_i(x,t)\right)^{-1} |f(x,t)|, \tag{2.60}$$

$$\begin{aligned} \|\psi\|_* := &\frac{\lambda_*^{1-\nu}(0)R^{\alpha}(0)}{|\log T|}\|\psi\|_{L^\infty(\Omega\times(0,T))} + \frac{\lambda_*^{2-\nu}(0)R^{1+\alpha}(0)}{|\log T|}\|\nabla\psi\|_{L^\infty(\Omega\times(0,T))} \\ &+ \sup_{(x,t)\in\Omega\times(0,T)}\left[\frac{\lambda_*^{1-\nu}(t)R^{\alpha}(t)}{|\log(T-t)|}|\psi(x,t)-\psi(x,T)|\right] \\ &+ \sup_{(x,t)\in\Omega\times(0,T)}\left[\frac{\lambda_*^{2-\nu}(t)R^{1+\alpha}(t)|}{|\log(T-t)|}|\nabla\psi(x,t)-\nabla\psi(x,T)|\right] \\ &+ \sup_{\Omega\times I_T}\frac{\lambda_*^{2\gamma+1-\nu}(t_2)R^{2\gamma+\alpha}(t_2)}{(t_2-t_1)^{\gamma}}|\psi(x,t_2)-\psi(x,t_1)|, \end{aligned} \tag{2.61}$$

where $\nu, \alpha, \gamma \in (0, 1)$, and the last supremum is taken over

$$\Omega \times I_T = \left\{(x, t_1, t_2) : \ x \in \Omega,\ 0 \le t_1 \le t_2 \le T,\ t_2 - t_1 \le \frac{1}{10}(T - t_2)\right\}.$$

For problem (2.58), we have the following estimates.

Proposition 2.1. ([22, Appendix A],[31, Proposition 1]) *Let ψ be the solution to problem (2.58) with $\|f\|_{**} < +\infty$. Then it holds that*

$$\|\psi\|_* \lesssim \|f\|_{**}.$$

Proposition 2.1 is established by estimating carefully Duhamel's formula with different right hand sides.

To solve the inner problem (2.50), we consider the associated linear problem

$$\lambda^2\phi_t = \Delta_y\phi + 3U^2(y)\phi + h(y,t) \ \text{ in } \ \mathcal{D}_{2R}. \tag{2.62}$$

Recall that the linearized operator $L_0 = \Delta + 3U^2$ has only one positive eigenvalue μ_0 such that

$$L_0(Z_0) = \mu_0 Z_0,\ Z_0 \in L^\infty(\mathbb{R}^4),$$

where the corresponding eigenfunction Z_0 is radially symmetric with the asymptotic behavior

$$Z_0(y) \sim |y|^{-3/2}e^{-\sqrt{\mu_0}|y|} \ \text{ as } \ |y| \to +\infty.$$

Similar to the discussion in previous section, such instability reflects in a careful choice of the initial data to ensure a well-behaved solution. Therefore, we consider the associated linear Cauchy problem of the inner problem (2.50)

$$\begin{cases} \lambda^2\phi_t = \Delta_y\phi + 3U^2(y)\phi + h(y,t), & \text{in } \mathcal{D}_{2R}, \\ \phi(y,0) = e_0 Z_0(y), & \text{in } B_{2R(0)}, \end{cases} \tag{2.63}$$

where $R = R(t) = \lambda_*^{-\beta}(t)$ for $\beta \in (0, 1/2)$. On the other hand, the parabolic operator $-\lambda^2\partial_t + L_0$ is certainly not invertible since all the time independent elements in the 5 dimensional kernel of L_0 (see (2.43)) also belong to the kernel of $-\lambda^2\partial_t + L_0$. In order to construct solution to (2.63) with suitable space-time decay, we expect some orthogonality conditions to hold. We shall construct a solution (ϕ, e_0) to problem (2.63) under the orthogonality conditions

$$\int_{B_{2R}} h(y,t)Z_\ell(y)dy = 0 \ \text{ for } \ \ell = 1, \cdots, 5,\ t \in (0,T). \tag{2.64}$$

Define

$$\|h\|_{\nu,2+a} := \sup_{(y,t)\in\mathcal{D}_{2R}} \lambda_*^{-\nu}(t)(1+|y|^{2+a})\left[|h(y,t)| + (1+|y|)|\nabla h(y,t)|\right]. \tag{2.65}$$

The construction of such solution is achieved by decomposing the equation into different spherical harmonic modes. Consider an orthonormal basis $\{\Theta_i\}_{i=0}^{\infty}$ made up of spherical harmonics in $L^2(\mathbb{S}^3)$, i.e.

$$\Delta_{\mathbb{S}^3}\Theta_i + \lambda_i \Theta_i = 0 \quad \text{in} \quad \mathbb{S}^3$$

with $0 = \lambda_0 < \lambda_1 = \cdots = \lambda_4 = 3 < \lambda_5 \leq \cdots$. More precisely, $\Theta_0(y) = a_0,\ \Theta_i(y) = a_1 y_i,\ i = 1, \cdots, 4$ for two constants a_0, a_1 and

$$\lambda_i = i(2+i) \quad \text{with multiplicity} \quad \frac{(3+i)!}{6i!} \quad \text{for} \quad i \geq 0.$$

For $h \in L^2(\mathcal{D}_{2R})$, we decompose

$$h(y,t) = \sum_{j=0}^{\infty} h_j(r,t)\Theta_j(y/r),\ r = |y|,\ h_j(r,t) = \int_{\mathbb{S}^3} h(r\theta,t)\Theta_j(\theta)d\theta$$

and write $h = h^0 + h^1 + h^{\perp}$ with

$$h^0 = h_0(r,t),\ h^1 = \sum_{j=1}^{4} h_j(r,t)\Theta_j,\ h^{\perp} = \sum_{j=5}^{\infty} h_j(r,t)\Theta_j.$$

Also, we decompose $\phi = \phi^0 + \phi^1 + \phi^{\perp}$ in a similar form. Then looking for a solution to problem (2.63) is equivalent to finding the pairs $(\phi^0, h^0),\ (\phi^1, h^1),\ (\phi^{\perp}, h^{\perp})$ in each mode.

The key linear result for the inner problem is stated as follows.

Proposition 2.2. *Let the constants $a, \nu, \nu_1 \in (0,1)$, $a_1 \in (1,2)$. For $T > 0$ sufficiently small and any $h(y,t)$ satisfying $\|h\|_{\nu,2+a} < +\infty$, $\|h^1\|_{\nu_1,2+a_1} < +\infty$ and the orthogonality conditions (2.64), there exists a pair (ϕ, e_0) solving (2.63), and $(\phi, e_0) = (\phi[h], e_0[h])$ defines a linear operator of $h(y,t)$ that satisfies the estimates*

$$\begin{aligned} &|\phi(y,t)| + (1+|y|)|\nabla\phi(y,t)| \\ \lesssim\ & \frac{\lambda_*^{\nu}(t)R^{\delta}}{1+|y|^a}\|h^0\|_{\nu,2+a} + \frac{\lambda_*^{\nu_1}(t)}{1+|y|^{a_1}}\|h^1\|_{\nu_1,2+a_1} + \frac{\lambda_*^{\nu}(t)}{1+|y|^a}\|h^{\perp}\|_{\nu,2+a} \end{aligned}$$

and

$$|e_0[h]| \lesssim \|h\|_{\nu,2+a}$$

where $0 \leq \delta < 1$ is small.

In the proof of Proposition 2.2, mode 0 and higher modes can be carried out in a similar manner as in [100, Proposition 7.1] via a careful re-gluing process. The rougher version can be found in [17, Proposition 7.1]. Mode 1 is obainted by blow-up argument. The restriction $a_1 \in (1,2)$ is required to guarantee the integrability in the blow up argument at translation mode 1.

If we define the norm

$$\|\phi^0\|_{*,\nu,a,\delta} := \sup_{(y,t)\in\mathcal{D}_{2R}} \lambda_*^{-\nu}(t)R^{-\delta}(1+|y|^a)\left[|\phi^0(y,t)| + (1+|y|)|\nabla\phi^0(y,t)|\right], \tag{2.66}$$

then Proposition 2.2 implies that

$$\|\phi^0\|_{*,\nu,a,\delta} \lesssim \|h^0\|_{\nu,2+a}.$$

We shall use the norm (2.66) when we solve the inner–outer gluing system.

The following spectrum gap plays a crucial role in the proof of the above Proposition, In fact, we have

Lemma 2.3. *For all sufficiently large R and all radially symmetric $\varphi \in H_0^1(B_R)$ with $\int_{B_{2R}} \varphi Z_0 = 0$, there exists a positive constant γ independent of R such that*

$$\int_{B_{2R}} \left(|\nabla\varphi|^2 - pU^{p-1}\varphi^2\right) \geq \gamma \begin{cases} \dfrac{1}{R^2}\displaystyle\int_{B_{2R}} \varphi^2, & \textit{for} \quad n=3, \\ \dfrac{1}{R^2\log R}\displaystyle\int_{B_{2R}} \varphi^2, & \textit{for} \quad n=4, \\ \dfrac{1}{R^{n-2}}\displaystyle\int_{B_{2R}} \varphi^2, & \textit{for} \quad n\geq 5. \end{cases}$$

Similar estimates for the linearization of harmonic map equation around degree 1 bubble are derived in [99, Lemma 9.2].

• Solving the inner–outer gluing system.

Our aim now is to find a solution $(\phi, \psi, \lambda, \xi)$ to the inner–outer gluing system such that the desired blow-up solution is constructed. We shall solve the inner–outer gluing system in the function space $\mathcal{X}$ defined in (2.97). We first make some assumptions about the parameter functions. Write

$$\lambda_*(t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2}$$

and assume that for some numbers $c_1, c_2 > 0$,

$$c_1|\dot{\lambda}_*(t)| \le |\dot{\lambda}(t)| \le c_2|\dot{\lambda}_*(t)| \quad \text{for all} \quad t \in (0,T).$$

For given $\|\phi^0\|_{*,\nu,a,\delta}$, $\|\phi^1\|_{\nu_1,a_1}$, $\|\phi^\perp\|_{\nu,a}$, $\|\psi\|_*$, $\|Z^*\|_\infty$, $\|\lambda\|_F$, $\|\xi\|_G$ bounded, we first estimate right hand sides $\mathcal{G}(\phi,\psi,\lambda,\xi)$ and $\mathcal{H}(\phi,\psi,\lambda,\xi)$ in the inner and outer problems. Here the above norms are defined in (2.66), (2.65), (2.61), (2.95) and (2.96).

The outer problem: estimates of $\mathcal{G}$.

Consider the outer problem

$$\psi_t = \Delta\psi + \mathcal{G}(\phi,\psi,\lambda,\xi) \quad \text{in } \Omega \times (0,T)$$

where

$$\begin{aligned}\mathcal{G}(\phi,\psi,\lambda,\xi) := {}& 3\lambda^{-2}(1-\eta_R)U^2(y)(\Psi_0 + \psi + Z^*)\\ &+ \lambda^{-3}\left[(\Delta_y\eta_R)\phi + 2\nabla_y\eta_R\cdot\nabla_y\phi - \lambda^2\phi\partial_t\eta_R\right]\\ &+ (1-\eta_R)\mathcal{K}[\lambda,\xi] + (1-\eta_R)\mathcal{N}(\mathtt{w})\end{aligned}$$

with $\mathcal{K}[\lambda,\xi]$ and $\mathcal{N}(\mathtt{w})$ defined in (2.49) and (2.52) respectively.

In order to apply the linear theory Proposition 2.1, we estimate all the terms in $\mathcal{G}(\phi,\psi,\lambda,\xi)$ in the $\|\cdot\|_{**}$-norm, defined in (2.60). Direct computations imply that for a fixed number $\epsilon_0 > 0$

$$\begin{aligned}\|\mathcal{G}\|_{**} \lesssim T^{\epsilon_0}\big(&\|\psi\|_* + \|Z^*\|_\infty + \|\phi^0\|_{*,\nu,a,\delta} + \|\phi^1\|_{\nu_1,a_1} + \|\phi^\perp\|_{\nu,a}\\ &+ \|\lambda\|_\infty + \|\xi\|_G + 1\big)\end{aligned} \tag{2.67}$$

if the parameters are chosen in the following range

$$\begin{aligned}&\nu - 1 + \beta(2+\alpha) - \nu_2 > 0, \quad 2\beta - \nu_2 > 0, \quad 0 < \alpha + \delta < a < 1,\\ &\beta + \upsilon - \nu_2 > 0, \quad 2\nu_1 - \nu + \beta(2a_1 - \alpha) > 0, \quad \nu_2 < 1,\\ &2\nu - \nu_2 - 1 + 2\alpha\beta > 0, \quad \nu - \beta(\alpha + 2\delta - 2a) > 0.\end{aligned} \tag{2.68}$$

The inner problem: Estimate of $\mathcal{H}$.

Consider the inner problem

$$\lambda^2\phi_t = \Delta_y\phi + 3U^2(y)\phi + \mathcal{H}(\phi,\psi,\lambda,\xi) \quad \text{in } \mathcal{D}_{2R}$$

where

$$\begin{aligned}\mathcal{H}(\phi,\psi,\lambda,\xi)(y,t) := {}& 3\lambda U^2(y)[\Psi_0 + \psi + Z^*](\lambda y + \xi, t)\\ &+ \lambda\left[\dot{\lambda}(\nabla_y\phi\cdot y + \phi) + \nabla_y\phi\cdot\dot{\xi}\right]\\ &+ \lambda^3\mathcal{N}(\mathtt{w}) + \lambda^3\mathcal{K}[\lambda,\xi]\end{aligned}$$

with $\mathcal{N}(\mathtt{w})$ and $\mathcal{K}[\lambda,\xi]$ defined in (2.52) and (2.49).

From the linear theory, we know that for $\mathcal{H} = \mathcal{H}^0 + \mathcal{H}^1 + \mathcal{H}^\perp$ satisfying

$$\|\mathcal{H}^0\|_{\nu,2+a}, \ \|\mathcal{H}^1\|_{\nu_1,2+a_1}, \ \|\mathcal{H}^\perp\|_{\nu,2+a} < +\infty,$$

there exists a solution $(\phi^0, \phi^1, \phi^\perp, c^0, c^\ell)$ $(\ell = 1, \cdots, 4)$ solving the projected inner problems

$$\begin{cases}\lambda^2\phi^0_t = \Delta_y\phi^0 + 3U^2(y)\phi^0 + \mathcal{H}^0(\phi,\psi,\lambda,\xi) + c^0 Z_5 & \text{in } \mathcal{D}_{2R},\\ \phi^0(\cdot,0) = 0 & \text{in } B_{2R},\end{cases} \tag{2.69}$$

$$\begin{cases} \lambda^2\phi^1_t = \Delta_y\phi^1 + 3U^2(y)\phi^1 + \mathcal{H}^1(\phi,\psi,\lambda,\xi) + \sum\limits_{\ell=1}^{4} c^\ell Z_\ell & \text{in } \mathcal{D}_{2R}, \\ \phi^1(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{2.70}$$

$$\begin{cases} \lambda^2\phi^\perp_t = \Delta_y\phi^\perp + 3U^2(y)\phi^\perp + \mathcal{H}^\perp(\phi,\psi,\lambda,\xi) & \text{in } \mathcal{D}_{2R}, \\ \phi^\perp(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{2.71}$$

and the inner solution $\phi[\mathcal{H}] = \phi^0[\mathcal{H}^0] + \phi^1[\mathcal{H}^1] + \phi^\perp[\mathcal{H}^\perp]$ with proper space-time decay can be obtained for the inner–outer gluing to be carried out. We have the following estimate for some fixed $\epsilon_0 > 0$

$$\begin{aligned} \|\mathcal{H}\|_{\nu,2+a} \lesssim T^{\epsilon_0}\Big(&\|\phi^0\|_{*,\nu,a,\delta} + \|\phi^1\|_{\nu_1,a_1} + \|\phi^\perp\|_{\nu,a} + \|\psi\|_* + \|Z^*\|_\infty \\ &+ \|\lambda\|_\infty + \|\xi\|_G + 1\Big) \end{aligned} \tag{2.72}$$

provided

$$\begin{aligned} &0<\nu<1, \quad 1-\beta(2+\frac{a}{2})>0, \quad 1+\nu_1-\nu-\beta(2+a-a_1)>0, \\ &1-2\beta>0, \quad \nu-\beta(4-a)>0, \quad 2\nu_1-\nu>0, \\ &2-\nu-a\beta>0, \quad \nu-\beta(a-2\alpha)>0, \quad 2-\nu-\beta(1+a)>0, \\ &1-\beta(\delta+2)>0, \quad \nu-2\delta\beta>0. \end{aligned} \tag{2.73}$$

Similar computations give that for some fixed $\epsilon_0 > 0$

$$\|\mathcal{H}^1\|_{\nu_1,2+a_1} \lesssim T^{\epsilon_0}\left(\|\phi^1\|_{\nu_1,a_1} + \|\psi\|_* + \|Z^*\|_\infty + \|\lambda\|_\infty + \|\xi\|_G + 1\right) \tag{2.74}$$

provided

$$\begin{aligned} &0<\nu_1<1, \quad \nu-\nu_1+\alpha\beta>0, \quad 2-\nu_1-a_1\beta>0, \\ &2\nu-\nu_1+2\alpha\beta-a_1\beta>0, \quad 1-\nu_1-\beta(a_1-1)>0. \end{aligned}$$

• **The parameter problems.**

From (2.69)–(2.71), it remains to adjust the parameter functions $\lambda(t)$, $\xi(t)$ such that

$$c^0[\lambda,\xi,\Psi^*]=0, \ c^\ell[\lambda,\xi,\Psi^*]=0, \ \ell=1,\cdots,4, \ \forall\, t\in(0,T),$$

where

$$c^0[\lambda,\xi,\Psi^*] = -\frac{\int_{B_{2R}}\mathcal{H}Z_5dy}{\int_{B_{2R}}|Z_5|^2dy}, \tag{2.75}$$

$$c^\ell[\lambda,\xi,\Psi^*] = -\frac{\int_{B_{2R}}\mathcal{H}Z_\ell dy}{\int_{B_{2R}}|Z_\ell|^2dy} \quad \text{for} \quad \ell=1,\cdots,4. \tag{2.76}$$

It turns out that we can easily achieve at the translation mode (2.76), but the scaling mode (2.75) is more delicate.

We first consider the reduced equation for $\xi(t)$. Observe that (2.76) is equivalent to

$$\int_{B_{2R}}\mathcal{H}(\phi,\psi,\lambda,\xi)(y,t)Z_\ell(y)dy=0, \quad \text{for all } t\in(0,T), \ \ell=1,\cdots,4.$$

Write $\Psi^*=\psi+Z^*$ and $\xi(t)=(\xi_1(t),\cdots,\xi_4(t))$. Then for $\ell=1,\cdots,4$,

$$\int_{B_{2R}}\mathcal{H}(\phi,\psi,\lambda,\xi)(y,t)Z_\ell(y)dy=0$$

give that

$$\dot{\xi}_\ell = b_\ell[\lambda,\xi,\phi,\Psi^*], \tag{2.77}$$

where

$$b_\ell[\lambda,\xi,\phi,\Psi^*] = \int_{B_{2R}}\left(\mathcal{H}[\lambda,\xi,\phi,\Psi^*](y,t) - \lambda U_{y_\ell}(y)\dot{\xi}_\ell\right)Z_\ell(y)dy.$$

Furthermore, the size of $b_\ell[\lambda,\xi,\phi,\Psi^*]$ can be controlled by similarly estimating $\mathcal{H}$. Next, we analyze the reduced problem (2.77), which defines operators Ξ_ℓ ($\ell=1,\cdots,4$) that return the solutions ξ_ℓ ($\ell=1,\cdots,4$) respectively. Here we write

$$\Xi=(\Xi_1,\Xi_2,\Xi_3,\Xi_4) \tag{2.78}$$

and $\xi(t) = q + \xi^1(t)$ where $q = (q_1, \cdots, q_4)$ is a prescribed point in Ω. We shall solve $\xi^1(t)$ under the norm

$$\|\xi\|_G = \|\xi\|_{L^\infty(0,T)} + \sup_{t\in(0,T)} \lambda_*^{-\upsilon}(t)|\dot{\xi}(t)|$$

for some fixed $\upsilon \in (0,1)$. From (2.77), we have

$$|\xi_\ell(t)| \le |q_\ell| + \|b_\ell[\lambda, \xi, \phi, \Psi^*]\|_{L^\infty(0,T)} (T-t).$$

Therefore, we obtain

$$\|\Xi_\ell\|_G \le |q_\ell| + (T-t)^{-\upsilon} \|b_\ell[\lambda, \xi, \phi, \Psi^*]\|_{L^\infty(0,T)}. \tag{2.79}$$

Since the reduced problem of $\lambda(t)$ is essentially the same as the real part of the reduced problem at mode 0 in [22], we shall follow the strategy and logic in [22, Section 8].

Direct computations show that (2.75) gives a non-local integro-differential equation

$$\int_{-T}^{t} \frac{\dot{\lambda}(s)}{t-s}\Gamma\left(\frac{\lambda^2(t)}{t-s}\right) ds + \mathbf{c_0}\dot{\lambda} = a[\lambda, \xi, \Psi^*](t) + \mathsf{a}_r[\lambda, \xi, \phi, \Psi^*](t), \tag{2.80}$$

where $\mathbf{c_0} = 2\alpha_0 \int_{\mathbb{R}^4} \frac{Z_5(y)}{(1+|y|^2)^2} dy$,

$$a[\lambda, \xi, \Psi^*] = -\int_{B_{2R}} 3U^2(y)\left(\Psi_0 + \Psi^*\right) Z_5(y) dy, \tag{2.81}$$

and the remainder term $\mathsf{a}_r[\lambda, \xi, \phi, \Psi^*](t)$ turns out to be of smaller order and is controlled by

$$\begin{aligned}
&|\mathsf{a}_r[\lambda, \xi, \phi, \Psi^*](t)| \\
\lesssim\ & \lambda_*^\nu R^\delta \left(|\dot{\lambda}_*||\log(T-t)| + |\dot{\xi}|\right) \|\phi^0\|_{*,\nu,a,\delta} \\
&+ \lambda_*^\nu \left(|\dot{\lambda}_*|R^{2-a_1} + |\dot{\xi}|\right) \|\phi^1\|_{\nu_1,a_1} + \lambda_*^\nu \left(|\dot{\lambda}_*|R^{2-a} + |\dot{\xi}|R^{1-a}\right) \|\phi^\perp\|_{\nu,a} \\
&+ \lambda_*^{2\nu-1}R^{2\delta}\|\phi^0\|^2_{*,\nu,a,\delta} + \lambda_*^{2\nu_1-1}\|\phi^1\|^2_{\nu_1,a_1} \\
&+ \lambda_*^{2\nu-1}R^{2-2a}\|\phi^\perp\|^2_{\nu,a} + \lambda_*|\dot{\lambda}_*|^2|\log(T-t)|^3 + \lambda_*|\log(T-t)|\|Z^*\|^2_\infty \\
&+ \lambda_*(t)|\log(T-t)|\lambda_*^{2\nu-2}(0)R^{-2\alpha}(0)|\log T|^2\|\psi\|^2_*.
\end{aligned}$$

To solve $\lambda(t)$, we introduce the following norms

- $\|\cdot\|_{\Theta,l}$-norm

$$\|f\|_{\Theta,l} := \sup_{t\in[0,T]} \frac{|\log(T-t)|^l}{(T-t)^\Theta}|f(t)|,$$

  where $f \in C([-T,T];\mathbb{R})$ with $f(T) = 0$, and $\Theta \in (0,1)$, $l \in \mathbb{R}$.
- $[\,\cdot\,]_{\gamma,m,l}$-seminorm

$$[g]_{\gamma,m,l} := \sup_{I_T} \frac{|\log(T-t)|^l}{(T-t)^m(t-s)^\gamma}|g(t)-g(s)|,$$

  where $I_T = \left\{-T \le s \le t \le T : t-s \le \frac{1}{10}(T-t)\right\}$, $g \in C([-T,T];\mathbb{R})$ with $g(T) = 0$ and $0 < \gamma < 1$, $m > 0$, $l \in \mathbb{R}$.

Also, we define

$$\mathcal{B}_0[\lambda](t) := \int_{-T}^{t} \frac{\dot{\lambda}(s)}{t-s}\Gamma\left(\frac{\lambda^2(t)}{t-s}\right) ds + \mathbf{c_0}\dot{\lambda} \tag{2.82}$$

and write

$$c^0[\mathcal{H}] = \frac{\mathcal{B}_0[\lambda] - (a[\lambda, \xi, \Psi^*] + \mathsf{a}_r[\lambda, \xi, \phi, \Psi^*])}{\int_{B_{2R}} |Z_5(y)|^2 dy}. \tag{2.83}$$

We invoke a key proposition proved in [22, Proposition 6.5] concerning the solvability of $\lambda(t)$.

Proposition 2.3. *Let $\omega, \Theta \in (0, \frac{1}{2})$, $\gamma \in (0,1)$, $m \le \Theta - \gamma$ and $l \in \mathbb{R}$. If $a(t)$ satisfies $a(T) < 0$ with $1/C \le |a(T)| \le C$ for some constant $C > 1$, and*

$$T^\Theta|\log T|^{1+c-l}\|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \le C_1 \tag{2.84}$$

for some $c > 0$, then there exist two operators $\mathcal{P}$ and $\mathcal{R}_0$ such that $\lambda = \mathcal{P}[a] : [-T,T] \to \mathbb{R}$ satisfies

$$\mathcal{B}_0[\lambda](t) = a(t) + \mathcal{R}_0[a](t) \tag{2.85}$$

with

$$|\mathcal{R}_0[a](t)| \lesssim \left(T^{\frac{1}{2}+c} + T^{\Theta} \frac{\log|\log T|}{|\log T|} \|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \right) \frac{(T-t)^{m+(1+\omega)\gamma}}{|\log(T-t)|^l}.$$

When applying Proposition 2.3, the Hölder property is essentially inherited from regularity of the outer solution, and this is one of the reasons that we work in the weighted space (2.61).

- **The fixed point formulation.**

We first transform the inner–outer problems (2.50), (2.53) into the problems of finding solutions $(\psi, \phi^0, \phi^1, \phi^\perp, \lambda, \xi)$ solving the following *inner–outer gluing system*

$$\begin{cases} \psi_t = \Delta\psi + \mathcal{G}(\phi^0 + \phi^1 + \phi^\perp, \psi + Z^*, \lambda, \xi), & \text{in } \Omega \times (0,T), \\ \psi = 0, & \text{on } \partial\Omega \times (0,T), \\ \psi(x,0) = 0, & \text{in } \Omega, \end{cases} \tag{2.86}$$

$$\begin{cases} \lambda^2 \phi^0_t = \Delta_y \phi^0 + 3U^2(y)\phi^0 + \mathcal{H}^0(\phi,\psi,\lambda,\xi) + \tilde{c}^0[\mathcal{H}^0] Z_5 & \text{in } \mathcal{D}_{2R}, \\ \phi^0(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{2.87}$$

$$\begin{cases} \lambda^2 \phi^1_t = \Delta_y \phi^1 + 3U^2(y)\phi^1 + \mathcal{H}^1(\phi,\psi,\lambda,\xi) + \sum_{\ell=1}^{4} c^\ell[\mathcal{H}^1] Z_\ell & \text{in } \mathcal{D}_{2R}, \\ \phi^1(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{2.88}$$

$$\begin{cases} \lambda^2 \phi^\perp_t = \Delta_y \phi^\perp + 3U^2(y)\phi^\perp + \mathcal{H}^\perp(\phi,\psi,\lambda,\xi) + c^0_*[\lambda,\xi,\Psi^*] Z_5 & \text{in } \mathcal{D}_{2R}, \\ \phi^\perp(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{2.89}$$

$$c^0[\mathcal{H}](t) - \tilde{c}^0[\lambda,\xi,\Psi^*](t) = 0 \quad \text{for all} \quad t \in (0,T), \tag{2.90}$$

$$c^1[\mathcal{H}](t) = 0 \quad \text{for all} \quad t \in (0,T), \tag{2.91}$$

where $\mathcal{G}$ is defined in (2.54), $\mathcal{H}^0$, $\mathcal{H}^1$, $\mathcal{H}^\perp$ are the projections of $\mathcal{H}$ (see (2.51)) on different modes. It is direct to see that if $(\psi, \phi^0, \phi^1, \phi^\perp, \lambda, \xi)$ satisfies the system (2.86)–(2.91), then

$$\Psi^* = \psi + Z^*, \;\; \phi = \phi^0 + \phi^1 + \phi^\perp$$

solve the inner–outer problems (2.50), (2.53) and thus the desired blow-up solution is obtained.

The inner–outer gluing system (2.86)–(2.91) can be then formulated as a fixed point problem for operators we will describe below.

We first define the following function spaces

$$\begin{aligned} X_{\phi^0} &:= \left\{ \phi^0 \in L^\infty(\mathcal{D}_{2R}) : \; \nabla_y \phi^0 \in L^\infty(\mathcal{D}_{2R}), \; \|\phi^0\|_{*,\nu,a,\delta} < +\infty \right\}, \\ X_{\phi^1} &:= \left\{ \phi^1 \in L^\infty(\mathcal{D}_{2R}) : \; \nabla_y \phi^1 \in L^\infty(\mathcal{D}_{2R}), \; \|\phi^1\|_{\nu_1,a_1} < +\infty \right\}, \\ X_{\phi^\perp} &:= \left\{ \phi^\perp \in L^\infty(\mathcal{D}_{2R}) : \; \nabla_y \phi^\perp \in L^\infty(\mathcal{D}_{2R}), \; \|\phi^\perp\|_{\nu,a} < +\infty \right\}, \\ X_\psi &:= \left\{ \psi \in L^\infty(\Omega \times (0,T)) : \; \|\psi\|_* < +\infty \right\}. \end{aligned} \tag{2.92}$$

In order to introduce the space for the parameter function $\lambda(t)$, we recall from (2.82) that the integral operator $\mathcal{B}_0$ takes the following approximate form

$$\mathcal{B}_0[\lambda] = \int_{-T}^{t-\lambda_*^2(t)} \frac{\dot\lambda(s)}{t-s} ds + O(\|\dot\lambda\|_\infty).$$

Proposition 2.3 defines an approximate inverse operator $\mathcal{P}$ of the integral operator $\mathcal{B}_0$ such that for a satisfying (2.84), $\lambda := \mathcal{P}[a]$ satisfies

$$\mathcal{B}_0[\lambda] = a + \mathcal{R}_0[a] \;\; \text{in} \;\; [-T,T],$$

where $\mathcal{R}_0[a]$ is a small remainder. Also, the proof as in [22, Proposition 6.6] implies a refined decomposition

$$\mathcal{P}[a] = \lambda_{0,\kappa} + \mathcal{P}_1[a] \tag{2.93}$$

with

$$\lambda_{0,\kappa} := \kappa |\log T| \int_t^T \frac{1}{|\log(T-s)|^2} ds, \;\; t \le T,$$

$\kappa = \kappa[a] \in \mathbb{R}$, and the function $\lambda_1 = \mathcal{P}_1[a]$ satisfies

$$\|\lambda_1\|_{*,3-\iota} \lesssim |\log T|^{1-\iota} \log^2(|\log T|) \tag{2.94}$$

for $0 < \iota < 1$, where the $\|\cdot\|_{*,3-\iota}$-norm is defined by

$$\|f\|_{*,k} := \sup_{t\in[-T,T]} |\log(T-t)|^k |\dot f(t)|.$$

Therefore, we define

$$X_\lambda := \{\lambda_1 \in C^1([-T,T]) : \lambda_1(T) = 0, \|\lambda_1\|_{*,3-\iota} < \infty\}.$$

Here by (κ,λ_1) we represent λ in the form

$$\lambda = \lambda_{0,\kappa} + \lambda_1,$$

and from [22, Proposition 6.6], one can write the norm

$$\|\lambda\|_F = |\kappa| + \|\lambda_1\|_{*,3-\iota}. \tag{2.95}$$

For the translation parameter function $\xi(t)$, we write $\xi(t) = q + \xi^1(t)$ and define the following space for $\xi^1(t)$

$$X_\xi = \left\{\xi \in C^1((0,T);\mathbb{R}^4),\ \dot\xi(T) = 0,\ \|\xi\|_G < +\infty\right\}$$

with

$$\|\xi\|_G = \|\xi\|_{L^\infty(0,T)} + \sup_{t\in(0,T)} \lambda_*^{-\upsilon}(t)|\dot\xi(t)| \tag{2.96}$$

for some fixed $\upsilon \in (0,1)$.

Define

$$\mathcal{X} = X_{\phi^0} \times X_{\phi^1} \times X_{\phi^\perp} \times X_\psi \times \mathbb{R} \times X_\lambda \times X_\xi. \tag{2.97}$$

We will solve the inner–outer gluing system in a closed ball $\mathcal{B}$ in which

$$(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) \in \mathcal{X}$$

satisfies

$$\begin{cases} \|\phi^0\|_{*,\nu,a,\delta} + \|\phi^1\|_{\nu_1,a_1} + \|\phi^\perp\|_{\nu,a} \le 1 \\ \|\psi\|_* \le 1 \\ |\kappa-\kappa_0| \le |\log T|^{-1/2} \\ \|\lambda_1\|_{*,3-\iota} \le C|\log T|^{1-\iota}\log^2(|\log T|) \\ \|\xi\|_G \le 1 \end{cases} \tag{2.98}$$

for some large and fixed constant C, where $\kappa_0 = Z_0^*(0)$. The inner–outer gluing system (2.86)–(2.91) can be formulated as the following fixed point problem. We define an operator $\mathcal{F}$ which returns the solution from $\mathcal{B}$ to $\mathcal{X}$

$$\begin{aligned} \mathcal{F} : \mathcal{B} \subset \mathcal{X} &\to \mathcal{X} \\ v &\mapsto \mathcal{F}(v) = (\mathcal{F}_{\phi^0}(v), \mathcal{F}_{\phi^1}(v), \mathcal{F}_{\phi^\perp}(v), \mathcal{F}_\psi(v), \mathcal{F}_\kappa(v), \mathcal{F}_{\lambda_1}(v), \mathcal{F}_\xi(v)) \end{aligned}$$

with

$$\begin{aligned} \mathcal{F}_{\phi^0}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_0(\mathcal{H}^0[\lambda,\xi,\Psi^*]) \\ \mathcal{F}_{\phi^1}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_1(\mathcal{H}^1[\lambda,\xi,\Psi^*]) \\ \mathcal{F}_{\phi^\perp}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_\perp\left(\mathcal{H}^\perp[\lambda,\xi,\Psi^*] + c_*^0[\lambda,\xi,\Psi^*]Z_5\right) \\ \mathcal{F}_\psi(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_\psi\left(\mathcal{G}(\phi^0+\phi^1+\phi^\perp,\Psi^*,\lambda,\xi)\right) \\ \mathcal{F}_\kappa(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \kappa\left[a^0[\lambda,\xi,\Psi^*]\right] \\ \mathcal{F}_{\lambda_1}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{P}_1\left[a^0[\lambda,\xi,\Psi^*]\right] \\ \mathcal{F}_\xi(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \Xi(\phi^0,\phi^1,\phi^\perp,\psi,\lambda,\xi). \end{aligned} \tag{2.99}$$

Here $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_\perp$ are the operators given in Proposition 2.2 which solve different modes of the inner problems (2.87)–(2.89). The operator $\mathcal{T}_\psi$ defined by Proposition 2.1 deals with the outer problem (2.86). Operators $\kappa[a]$, $\mathcal{P}_1$ and Ξ handle the equations for λ and ξ which are defined in Proposition 2.3, (2.93) and (2.78), respectively.

- **Choices of constants.**

We list all the constraints of the constants β, α, a, a_1, ν, ν_1, ν_2, δ which are sufficient for the inner–outer gluing scheme to work.

First, we indicate all the parameters used in different norms.

- $R(t) = \lambda_*^{-\beta}(t)$ with $\beta \in (0,1/2)$.

- The norm for ϕ^0 solving mode 0 of the inner problem (2.87) is $\|\cdot\|_{*,\nu,a,\delta}$ which is defined in (2.66), where we require that $\nu, a \in (0,1)$ and $\delta \geq 0$ small enough.
- The norm for ϕ^1 solving modes 1 to 4 of the inner problem (2.88) is $\|\cdot\|_{\nu_1,a_1}$ which is defined in (2.65), where we require that $\nu_1 \in (0,1)$ and $a_1 \in (1,2)$.
- The norm for $\phi^\perp$ solving higher modes ($j \geq 5$) of the inner problem (2.89) is $\|\cdot\|_{\nu,a}$ which is defined in (2.65), where $\nu, a \in (0,1)$.
- The norm for ψ solving the outer problem (2.86) is $\|\cdot\|_*$ which is defined in (2.61), while the $\|\cdot\|_{**}$-norm for the right hand side of the outer problem (2.86) is defined in (2.60). Here we require that $\nu, \alpha, \nu_2, \gamma \in (0,1)$.
- In Proposition 2.3, we have the parameters $\omega, \Theta, m, l, \gamma$. Here ω is the parameter used to describe the remainder $\mathcal{R}_\omega$ and $\omega \in (0, 1/2)$. To apply Proposition 2.3 in our setting, we let
$$\Theta = \nu - 1 + \alpha\beta,\ m = \nu - 2 - \gamma + \beta(2+\alpha),\ l < 1 + 2m,$$
and require that $\beta > \frac{1-\omega}{2}$ such that $m + (1+\omega)\gamma > \Theta$ is guaranteed.

In order to get the desired estimates for the outer problem (2.86), we need
$$\begin{aligned}
&\nu - 1 + \beta(2+\alpha) - \nu_2 > 0, \quad 2\beta - \nu_2 > 0, \quad 0 < \alpha < a < 1,\\
&\beta + \upsilon - \nu_2 > 0, \quad 2\nu_1 - \nu + \beta(2a_1 - \alpha) > 0, \quad \nu_2 < 1,\\
&2\nu - \nu_2 - 1 + 2\alpha\beta > 0, \quad \nu - \beta(\alpha + 2\delta - 2a) > 0.
\end{aligned}$$

In order to get the desired estimates for the inner problems at different modes (2.87)–(2.89), we require
$$\begin{aligned}
&0 < \nu < 1, \quad 1 - \beta(2 + \frac{a}{2}) > 0, \quad 1 + \nu_1 - \nu - \beta(2 + a - a_1) > 0,\\
&1 - 2\beta > 0, \quad \nu - \beta(4 - a) > 0, \quad 2\nu_1 - \nu > 0,\\
&2 - \nu - a\beta > 0, \quad \nu - \beta(a - 2\alpha) > 0, \quad 2 - \nu - \beta(1+a) > 0,\\
&1 - \beta(\delta + 2) > 0, \quad \nu - 2\delta\beta > 0,\\
&0 < \nu_1 < 1, \quad \nu - \nu_1 + \alpha\beta > 0, \quad 2 - \nu_1 - a_1\beta > 0,\\
&2\nu - \nu_1 + 2\alpha\beta - a_1\beta > 0, \quad 1 - \nu_1 - \beta(a_1 - 1) > 0.
\end{aligned}$$

It turns out that suitable choices of the parameters satisfying all the restrictions in this section can be found. Here we give a specific example:
$$\beta \approx \frac{1}{4}\ (\beta > \frac{1}{4}),\ \alpha \approx a \approx a_1 \approx 1,\ \nu \approx \nu_1 \approx 1,\ \nu_2 \approx 0,\ \delta \approx 0.$$

• **Proof of Theorem 2.**

Consider the operator
$$\mathcal{F} = (\mathcal{F}_{\phi^0}, \mathcal{F}_{\phi^1}, \mathcal{F}_{\phi^\perp}, \mathcal{F}_\psi, \mathcal{F}_\kappa, \mathcal{F}_{\lambda_1}, \mathcal{F}_\xi) \tag{2.100}$$
given in (2.99). To prove Theorem 2, our strategy is to show the existence of a fixed point for the operator $\mathcal{F}$ in $\mathcal{B}$ by the Schauder fixed point theorem, where the closed ball $\mathcal{B}$ is defined in (2.98). By collecting the estimates (2.67), (2.72), (2.74), (2.79), (2.94), and using Proposition 2.1, Proposition 2.2, Proposition 2.3, we conclude that for $(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1) \in \mathcal{B}$
$$\begin{cases}
\|\mathcal{F}_{\phi^0}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{*,\nu,a,\delta} \leq CT^\epsilon\\
\|\mathcal{F}_{\phi^1}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{\nu_1,a_1} \leq CT^\epsilon\\
\|\mathcal{F}_{\phi^\perp}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{\nu,a} \leq CT^\epsilon\\
\|\mathcal{F}_\psi(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_* \leq CT^\epsilon\\
\left|\mathcal{F}_\kappa(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1) - \kappa_0\right| \leq C|\log T|^{-1}\\
\|\mathcal{F}_{\lambda_1}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_{*,3-\iota} \leq C|\log T|^{1-\iota}\log^2(|\log T|)\\
\|\mathcal{F}_\xi(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1)\|_G \leq CT^\epsilon
\end{cases} \tag{2.101}$$

where $C > 0$ is a constant independent of T, and $\epsilon > 0$ is a small fixed number. On the other hand, compactness of the operator $\mathcal{F}$ defined in (2.100) can be proved by proper variants of (2.101). Therefore, the existence of the desired blow-up solution for $k = 1$ is concluded from the Schauder fixed point theorem.

The proof of multi-bubble case follows similarly by taking the ansatz (2.39) with nonlocal corrections supported around each concentration zone added. □

3. Harmonic map heat flow and Landau-Lifshitz-Gilbert equation

In this section, we are going to revisit the finite-time singularity formation for the harmonic map heat flow (HMF) and the Landau-Lifshitz-Gilbert equation (LLG). We now briefly introduce the models and background.

Let $\mathcal{M}$ be a m-dimensional Riemannian manifold of the metric g and S^2 be the 2-sphere embedded in $\mathbb{R}^3$. The Landau-Lifshitz-Gilbert equation (LLG) on $\mathcal{M}$ is given by

$$\begin{cases} u_t = -au \wedge (u \wedge \Delta_{\mathcal{M}} u) - bu \wedge \Delta_{\mathcal{M}} u \ \text{ in } \ \mathcal{M} \times (0,T) \\ u(\cdot, 0) = u_0 \in S^2 \ \text{ in } \ \mathcal{M}, \end{cases} \tag{3.1}$$

where $a^2 + b^2 = 1$, $a \geq 0$, $b \in \mathbb{R}$, $\Delta_{\mathcal{M}} u = \frac{1}{\sqrt{g}} \partial_{x_\beta} (g^{\alpha\beta} \sqrt{g} \partial_{x_\alpha} u)$ is the Laplace-Beltrami operator and $u = [u_1, u_2, u_3]^{\mathrm{tr}}$ is a 3-vector with normalized length which is a mapping $u(x,t) : \mathcal{M} \times (0,T) \to S^2$. First formulated by Landau and Lifshitz [64] in 1935, LLG (3.1) is an important system modeling the effects of a magnetic field on ferromagnetic materials in micromagnetics, and it describes the evolution of spin fields in continuum ferromagnetism. See also Gilbert [45]. LLG (3.1) can be viewed as a bridge between the harmonic map flow (HMF) when $a = 1, b = 0$ and the Schrödinger map flow (SMF) when $a = 0, b = -1$.

In the context of HMF, Struwe [89] proved the existence and uniqueness of weak solution with at most finitely many singular points when $\mathcal{M}$ is a Riemann surface. See [37] for further generalizations and [11, 90] for higher dimensional cases. Chang, Ding and Ye [9] first proved the existence of finite time blow-up solutions for HMF from disk into S^2. See also [10, 16, 33, 66, 80, 81, 93, 96] and the references therein. In [94], van den Berg, Hulshof and King used formal analysis to predict the existence of blow-up solutions with quantized rates

$$\lambda_k(t) \sim \frac{|T-t|^k}{|\ln(T-t)|^{\frac{2k}{2k-1}}}, \quad k \in \mathbb{N}^+ \tag{3.2}$$

for the two-dimensional HMF into S^2. For the case $\mathcal{M} = \mathbb{R}^2$ and the target manifold is a revolution surface, using degree 1 harmonic map Q_1 as the building block, Raphaël and Schweyer [83, 84] constructed finite time blow-up solutions with rates (3.2) for all $k \geq 1$ in the equivariant class, where the initial data can be taken arbitrarily close to Q_1 in the energy-critical topology. For the case that $\mathcal{M} \subset \mathbb{R}^2$ is a general bounded domain, Dávila, del Pino and Wei [22] constructed solutions which blow up at finite many points with the type II rate (3.2) for $k = 1$, and they further investigated the stability and reverse bubbling phenomena. The construction in [22] can be generalized to the case $\mathcal{M} = \mathbb{R}^2$.

On the other hand, for SMF with $\mathcal{M} = \mathbb{R}^2$, Merle, Raphaël and Rodnianski [73] constructed the finite time blow-up solution with the rate (3.2) for $k = 1$ in the 1-equivariant class. Analogous to the results in Krieger-Schlag-Tataru [59] for wave maps, Perelman [77] constructed finite time blow-up solutions with continuous rates. See also [5, 6, 7, 8, 56] and the references therein for the global well-posedness results and the dynamics of SMF near ground state.

For LLG, in the case $\mathcal{M} = \mathbb{R}^3, a > 0$, Alouges and Soyeur [1] proved the existence of weak solutions for (3.1) and constructed infinitely many weak solutions. The existence for the weak solution to LLG has been established by Guo and Hong [46, Theorem 4.2] when $\mathcal{M}$ is a closed Riemannian manifold with $m \geq 3$, while for the case that $\mathcal{M}$ is a closed Riemannian surface, the weak solution was shown to be unique and regular except for at most finitely many points [46, Theorem 3.13]. When $\mathcal{M} = \mathbb{R}^2$ and the target manifold is a smooth closed surface embedded in $\mathbb{R}^3$, approximation by discretization was used in [58] to construct a solution of LLG which is smooth away from a two-dimensional locally finite Hausdorff measure. In general, one cannot expect good partial regularity results for weak solutions in the higher dimensional case $m \geq 3$ without further regularity or energy minimizing assumptions. See the famous example by Rivière [85], where weakly harmonic maps from the ball $B^3 \subset \mathbb{R}^3$ into S^2 were constructed for which the singular set Sing u is the closed ball $\overline{B^3}$, and this result can be generalized to higher dimensions. In a similar spirit to the existence results for partially regular solution for HMF in higher dimensions of Chen and Struwe [11], Melcher [71] proved that for $\mathcal{M} = \mathbb{R}^m$ $(m = 3)$

there exists a global weak solution whose singular set has finite 3-dimensional parabolic Hausdorff measure. Later, this result was generalized to $m \le 4$ by Wang [97]. With the additional stability assumption for the weak solution, for $m \le 4$, Moser [76] proved better estimate for the singular set. The partial regularity of LLG (3.1) for $m \ge 5$ still remains open.

For $\mathcal{M} = \mathbb{R}^m$, the global existence, uniqueness and decay properties for the solution of (3.1) were established by Melcher [72] for $m \ge 3$ with initial data u_0 close to a fixed point in S^2 in the L^m norm. Lin, Lai and Wang [67] generalized the result to Morrey space and $m \ge 2$. For u_0 away from a fixed point in S^2 with BMO semi-norm sufficiently small, Gutiérrez and de Laire [50] proved the global existence, uniqueness and regularity results for LLG.

The study of the dynamics for LLG with initial data close to harmonic maps is of special significance and can provide hints on the mechanism of singularity formation. A series of works by Gustafson, Kang, Nakanishi and Tsai [47, 48, 49] are devoted to the behavior of the solutions to LLG with $\mathcal{M} = \mathbb{R}^2$ with initial data u_0 close to the harmonic map in the n-equivariant class. They found, among other things, that there is no finite time blow-up for LLG and HMF with u_0 close to n-equivariant harmonic maps for $n \ge 3$ and $n = 2$, respectively. See [49, Theorem 1.1], [49, Theorem 1.2].

The singularity formation for LLG is an important and challenging topic. For the case that $\mathcal{M}$ is a compact manifold with or without boundary in dimensions $m = 3, 4$, Ding and Wang [32] obtained the existence of a smooth finite time blow-up solution for LLG. For $\mathcal{M} \subset \mathbb{R}^2$, as an analogue of Qing [80] for HMF, Harpes [53] gave descriptions of solutions to LLG (3.1) near the singular points. For the energy critical case that $\mathcal{M}$ is a disk in $\mathbb{R}^2$, in an interesting paper [95], van den Berg and Williams predicted the existence of finite time blow-up by formal asymptotic analysis supported with numerical simulations. For $\mathcal{M} = \mathbb{R}^2$, Xu and Zhao [101] rigorously constructed a finite time blow-up solution to (3.1) in the 1-equivariant class.

One of the versatilities of the systematic gluing method is that it does not rely on any symmetry assumptions, and thus the construction can be done in a general non-radially symmetric setting. Our main theorems for multi-bubble blow-ups for HMF and LLG are given as below.

Theorem 3. *Given points $q = (q_1, \dots, q_k) \in \Omega^k$ and any sufficiently small $T > 0$, there exist u_0 such the solution $u_q(x,t)$ of problem*

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0,T) \tag{3.3}$$

$$u = [0,0,1]^{\mathsf{tr}} \quad \text{on } \partial\Omega \times (0,T) \tag{3.4}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega \tag{3.5}$$

blows-up at exactly those k points as $t \uparrow T$. More precisely, there exist numbers $\kappa_i^* > 0$, α_i^* and a function $u_* \in H^1(\Omega) \cap C(\bar\Omega)$ such that

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} e^{J\alpha_i^*}\left[W\left(\frac{x-q_i}{\lambda_i}\right) - W(\infty)\right] \ \to\ 0 \quad \text{as} \quad t \to T, \tag{3.6}$$

in the H^1 and uniform senses in Ω where

$$\lambda_i(t) = \kappa_i^* \frac{T-t}{|\log(T-t)|^2}\big(1 + o(1)\big) \quad \text{as} \quad t \to T. \tag{3.7}$$

In particular, we have

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 8\pi \sum_{j=1}^{k} \delta_{q_j} \quad \text{as} \quad t \to T$$

as a weak-star convergence of Radon measure.

Remark 3.1. *The same construction works for the Cauchy problem in the entire space $\mathbb{R}^2$ as well.*

We consider the LLG equation with target manifold S^2 and domain $\mathcal{M} = \mathbb{R}^2$, and positive damping parameter $a > 0$

$$\begin{cases} u_t = a(\Delta u + |\nabla u|^2 u) - b u \wedge \Delta u & \text{in } \mathbb{R}^2 \times (0,T), \\ u(\cdot,0) = u_0 \in S^2 & \text{in } \mathbb{R}^2. \end{cases} \tag{3.8}$$

We are interested in the case of *multiple bubbles* to LLG (3.8) in the general *non-radially symmetric setting.* Our construction is based on the following degree 1 profile

$$W(y) = \frac{1}{|y|^2+1}\begin{bmatrix} 2y_1 \\ 2y_2 \\ |y|^2-1 \end{bmatrix}, \quad y=(y_1,y_2)\in\mathbb{R}^2,$$

and clearly

$$Q_\gamma W\left(\frac{x-\xi}{\lambda}\right)$$

solves the stationary equation of LLG (3.8) for any $\xi\in\mathbb{R}^2$, $\lambda>0$, and any γ-rotation matrix around z-axis

$$Q_\gamma := \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us define $U_\infty = [0,0,1]^{\mathsf{tr}}$, and $U_\infty = W(\infty)$. Our main result is stated as follows.

Theorem 4. *For any prescribed N distinct points $q^{[j]}\in\mathbb{R}^2$, $j=1,2,\ldots,N$, $N\in\mathbb{Z}_+$ and T sufficiently small, there exists a smooth initial data u_0 such that the gradient of the solution u to LLG (3.8) with $a>0$ blows up at these N points at finite time $t=T$. More precisely, the solution u takes the sharply scaled degree 1 profile around each point $q^{[j]}$*

$$u(x,t) = -(N-1)U_\infty + \sum_{j=1}^N Q_{\gamma_j} W\left(\frac{x-\xi^{[j]}}{\lambda_j}\right) + \Phi_{\rm per}$$

with

$$\lambda_j(t)=\kappa_j^*\lambda_*(t)(1+o(1)),\quad \lambda_*(t)=\frac{|\ln T|(T-t)}{|\ln(T-t)|^2},\quad \xi^{[j]}(t)=q^{[j]}+o(1),\quad \gamma_j(t)=\gamma_j^*(1+o(1))$$

where $o(1)\to 0$ as $t\to T$; $\kappa_j^*>0$, $\gamma_j^*\in\mathbb{R}$; the leading part of κ_j^* is determined by ∇u_0 and independent of a,b; the leading part of γ_j^* is determined by ∇u_0 and a,b; $\|\Phi_{\rm per}\|_{L^\infty(\mathbb{R}^2\times(0,T))}\ll 1$, and $\lambda_*(t)\|\nabla\Phi_{\rm per}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}\lesssim \lambda_*^\epsilon(t)$ for $\epsilon>0$ small. Moreover,

$$u(x,t)-u_*(x)-\sum_{j=1}^N Q_{\gamma_j}\left[W\left(\frac{x-\xi^{[j]}}{\lambda_j}\right)-W(\infty)\right]\to 0 \quad as \quad t\to T$$

in $H^1_{\rm loc}(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ for some $u_*(x)\in H^1_{\rm loc}(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, and in particular,

$$|\nabla u(\cdot,t)|^2\,dx \rightharpoonup |\nabla u_*|^2\,dx + 8\pi\sum_{j=1}^N \delta_{q^{[j]}} \quad as \quad t\to T$$

as weak-star convergence of Radon measure.

Remark 3.2.

- *The positivity of the damping term (a) plays a crucial role in our construction.*
- *In fact, due to the method employed, the construction works equally well for the case of smooth, bounded domain $\Omega\subset\mathbb{R}^2$ as long as the technical ingredient in [34] can be generalized to general domains with proper boundary conditions.*

3.1. **Harmonic map heat flow.** We sketch the construction of Theorem 7.

• **The 1-corrotational harmonic maps and their linearized operator.**

The harmonic map equation for functions $U:\mathbb{R}^2\to S^2$ is the elliptic problem

$$\Delta U + |\nabla U|^2 U \quad \text{in } \mathbb{R}^2, \quad |U|=1. \tag{3.9}$$

For $\xi\in\mathbb{R}^2$, $\omega\in\mathbb{R}$, $\lambda>0$, we consider the family of solutions of (3.9) given by the following 1-corrotational harmonic maps

$$U_{\lambda,\xi,\omega}(x) := Q_\omega W\Big(\frac{x-\xi}{\lambda}\Big),$$

where W is the canonical least energy harmonic map

$$W(y) = \frac{1}{1+|y|^2}\begin{pmatrix} 2y \\ |y|^2-1 \end{pmatrix}, \quad y \in \mathbb{R}^2,$$

and Q_ω is the ω-rotation matrix

$$Q_\omega := \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The linearized operator for (3.9) around $U = U_{\lambda,\xi,\omega}$ is the elliptic operator

$$L_U[\varphi] = \Delta\varphi + |\nabla U|^2\varphi + 2(\nabla\varphi\cdot\nabla U)U.$$

Differentiating U with respect to each of its parameters we obtain functions that annihilate this operator, namely solutions of $L_U[\varphi]=0$. Setting $y = \frac{x-\xi}{\lambda}$, these functions are

$$\begin{aligned} \partial_\lambda U_{\lambda,\xi,\omega}(x) =& \frac{1}{\lambda} Q_\omega \nabla W(y)\cdot y, \\ \partial_\omega U_{\lambda,\xi,\omega}(x) =& (\partial_\omega Q_\omega) W(y) \\ \partial_{\xi_j} U_{\lambda,\xi,\omega}(x) =& \frac{1}{\lambda} Q_\omega \partial_{y_j} W(y). \end{aligned}$$

We observe that

$$(\partial_\omega Q_\omega) = Q_\omega J_0, \quad J_0 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can represent $W(y)$ in polar coordinates,

$$W(y) = \begin{pmatrix} e^{i\theta}\sin w(\rho) \\ \cos w(\rho) \end{pmatrix}, \quad w(\rho) = \pi - 2\arctan(\rho), \quad y = \rho e^{i\theta}.$$

We notice that

$$w_\rho = -\frac{2}{1+\rho^2}, \quad \sin w = -\rho w_\rho = \frac{2\rho}{1+\rho^2}, \quad \cos w = \frac{\rho^2-1}{1+\rho^2},$$

and derive the alternative expressions

$$\begin{aligned} \partial_\lambda U_{\lambda,\xi,\omega}(x) &= \frac{1}{\lambda} Q_\omega Z_{01}(y), & Z_{01}(y) &= \rho w_\rho(\rho)\, E_1(y) \\ \partial_\omega U_{\lambda,\xi,\omega}(x) &= \quad Q_\omega Z_{02}(y), & Z_{02}(y) &= \rho w_\rho(\rho)\, E_2(y) \\ \partial_{\xi_j} U_{\lambda,\xi,\omega}(x) &= \frac{1}{\lambda} Q_\omega Z_{11}(y), & Z_{11}(y) &= w_\rho(\rho)\,[\cos\theta\, E_1(y) + \sin\theta\, E_2(y)] \\ \partial_{\xi_j} U_{\lambda,\xi,\omega}(x) &= \frac{1}{\lambda} Q_\omega Z_{12}(y), & Z_{12}(y) &= w_\rho(\rho)\,[\sin\theta\, E_1(y) - \cos\theta\, E_2(y)], \end{aligned} \tag{3.10}$$

where

$$E_1(y) = \begin{pmatrix} e^{i\theta}\cos w(\rho) \\ -\sin w(\rho) \end{pmatrix}, \quad E_2(y) = \begin{pmatrix} ie^{i\theta} \\ 0 \end{pmatrix}.$$

The relation $|U_{\lambda,\xi,\omega}| = 1$ implies that all the functions Z_{ij} are pointwise orthogonal to $U_{\lambda,\xi,\omega}$. In fact the vectors $E_1(y)$, $E_2(y)$ constitute an orthonormal basis of the tangent space to S^2 at the point $W(y)$.

We have $L_W[Z_{lj}] = 0$ where for a function $\phi(y)$ we define

$$L_W[\phi] = \Delta_y\phi + |\nabla W(y)|^2\phi + 2(\nabla W(y)\cdot\nabla\phi)W(y).$$

In addition to the elements (3.10) in the kernel of L_W there are also two other relevant functions in the kernel, namely

$$\begin{aligned} Z_{-1,1} &= \rho^2 w_\rho(\rho)(\cos\theta E_1 - \sin\theta E_2) \\ Z_{-1,2} &= \rho^2 w_\rho(\rho)(\sin\theta E_1 + \cos\theta E_2). \end{aligned} \tag{3.11}$$

It is worth noticing the connection between this operator and L_U which is given by

$$L_U[\varphi] = \frac{1}{\lambda^2} Q_\omega L_W[\phi], \quad \varphi(x) = \phi(y), \quad y = \frac{x-\xi}{\lambda}.$$

• **The linearized operator at functions orthogonal to U.** It will be especially significant to compute the action of L_U on functions with values pointwise orthogonal to U. In what remains of this section we will derive various formulas that will be very useful later on.

For an arbitrary function $\Phi(x)$ with values in $\mathbb{R}^3$ we denote

$$\Pi_{U^\perp}\Phi := \Phi - (\Phi\cdot U)U$$

A direct computation shows the validity following:

$$L_U[\Pi_{U^\perp}\Phi] = \Pi_{U^\perp}\Delta\Phi + \tilde L_U[\Phi]$$

where

$$\tilde L_U[\Phi] := |\nabla U|^2\Pi_{U^\perp}\Phi - 2\nabla(\Phi\cdot U)\nabla U,$$

and

$$\nabla(\Phi\cdot U)\nabla U = \partial_{x_j}(\Phi\cdot U)\,\partial_{x_j}U.$$

A very convenient expression for $\tilde L_U[\Phi]$ is obtained if we use polar coordinates. Writing in complex notation

$$\Phi(x) = \Phi(r,\theta), \quad x = \xi + re^{i\theta},$$

we find

$$\tilde L_U[\Phi] = -\frac{2}{\lambda}w_\rho(\rho)\,[(\Phi_r\cdot U)Q_\omega E_1 - \frac{1}{r}(\Phi_\theta\cdot U)Q_\omega E_2], \quad \rho = \frac{r}{\lambda}. \tag{3.12}$$

We single out two consequences of formula (3.12) which will be crucial for later purposes. Let us assume that $\Phi(x)$ is a C^1 function $\Phi:\Omega\to\mathbb{C}\times\mathbb{R}$, which we express in the form

$$\Phi(x) \;=\; \begin{pmatrix}\varphi_1(x)+i\varphi_2(x)\\ \varphi_3(x)\end{pmatrix}. \tag{3.13}$$

We also denote

$$\varphi = \varphi_1 + i\varphi_2, \quad \bar\varphi = \varphi_1 - i\varphi_2$$

and define the operators

$$\operatorname{div}\varphi = \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2, \quad \operatorname{curl}\varphi = \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1.$$

We have the validity of the following formula

$$\tilde L_U[\Phi] = \tilde L_U[\Phi]_0 + \tilde L_U[\Phi]_1 + \tilde L_U[\Phi]_2\ , \tag{3.14}$$

where

$$\begin{cases} \tilde L_U[\Phi]_0 = \lambda^{-1}\rho w_\rho^2\,\big[\,\operatorname{div}(e^{-i\omega}\varphi)\,Q_\omega E_1 + \operatorname{curl}(e^{-i\omega}\varphi)\,Q_\omega E_2\,\big]\\ \tilde L_U[\Phi]_1 = -\,2\lambda^{-1}w_\rho\cos w\,\big[\,(\partial_{x_1}\varphi_3)\cos\theta + (\partial_{x_2}\varphi_3)\sin\theta\,\big]\,Q_\omega E_1\\ \qquad\qquad -\,2\lambda^{-1}w_\rho\cos w\,\big[\,(\partial_{x_1}\varphi_3)\sin\theta - (\partial_{x_2}\varphi_3)\cos\theta\,\big]\,Q_\omega E_2\ ,\\ \tilde L_U[\Phi]_2 = \quad\lambda^{-1}\rho w_\rho^2\,\big[\,\operatorname{div}(e^{i\omega}\bar\varphi)\,\cos2\theta - \operatorname{curl}(e^{i\omega}\bar\varphi)\,\sin2\theta\,\big]\,Q_\omega E_1\\ \qquad\qquad +\,\lambda^{-1}\rho w_\rho^2\,\big[\,\operatorname{div}(e^{i\omega}\bar\varphi)\,\sin2\theta + \operatorname{curl}(e^{i\omega}\bar\varphi)\,\cos2\theta\,\big]\,Q_\omega E_2. \end{cases} \tag{3.15}$$

Another corollary of formula (3.12) that we single out is the following: assume that

$$\Phi(x) = \begin{pmatrix}\phi(r)e^{i\theta}\\ 0\end{pmatrix}, \quad x = \xi + re^{i\theta}, \quad \rho = \frac{r}{\lambda}$$

where $\phi(r)$ is complex valued. Then

$$\tilde L_U[\Phi] = \frac{2}{\lambda}w_\rho(\rho)^2\left[\operatorname{Re}(e^{-i\omega}\partial_r\phi(r))Q_\omega E_1 + \frac{1}{r}\operatorname{Im}(e^{-i\omega}\phi(r))Q_\omega E_2\right]. \tag{3.16}$$

A final result in this section is a computation (in polar coordinates) of the operator L_U acting on a function of the form

$$\Phi(x) = \varphi_1(\rho,\theta)Q_\omega E_1 + \varphi_2(\rho,\theta)Q_\omega E_2, \quad x = \xi + \lambda\rho e^{i\theta}.$$

We have:

$$L_U[\Phi] = \lambda^{-2}\left(\partial_\rho^2\varphi_1 + \frac{\partial_\rho\varphi_1}{\rho} + \frac{\partial_\theta^2\varphi_1}{\rho^2} + (2w_\rho^2 - \frac{1}{\rho^2})\varphi_1 - \frac{2}{\rho^2}\partial_\theta\varphi_2\cos w\right) Q_\omega E_1$$
$$+ \lambda^{-2}\left(\partial_\rho^2\varphi_2 + \frac{\partial_\rho\varphi_2}{\rho} + \frac{\partial_\theta^2\varphi_2}{\rho^2} + (2w_\rho^2 - \frac{1}{\rho^2})\varphi_2 + \frac{2}{\rho^2}\partial_\theta\varphi_1\cos w\right) Q_\omega E_2.$$

• **The ansatz for a blowing-up solution.**

In what follows we shall closely follow notation and computational formulas derived in the previous sections, here applied in a time-dependent framework. Thus we consider the semilinear parabolic equation

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \Omega\times(0,T) \tag{3.17}$$
$$u = u_{\partial\Omega} \quad \text{on } \partial\Omega\times(0,T) \tag{3.18}$$
$$u(\cdot,0) = u_0 \quad \text{in } \Omega \tag{3.19}$$

for a function $u : \bar\Omega\times[0,T)\to S^2$. Here $u_0 : \bar\Omega \to S^2$ is a given smooth map and

$$u_{\partial\Omega} = u_0\big|_{\partial\Omega} \equiv \mathbf{e_3} \quad \text{on } \partial\Omega. \tag{3.20}$$

Here and in what follows we denote

$$\mathbf{e_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{3.21}$$

The constant boundary value $\mathbf{e_3}$ precisely corresponds to $W(\infty)$ where W is the standard degree 1 harmonic map. This choice of $u_{\partial\Omega}$ as a constant is made for convenience, in fact sufficiently small non-constant perturbations of it are also admissible in all arguments below.

In order to keep the notation to a minimum, we shall do this in the case $k=1$ of a single bubbling point. We will later indicate the necessary changes in the general case. Given a fixed point $q\in\Omega$, and any sufficiently small number $T>0$ we look for a solution $u(x,t)$ of problem (3.17)-(3.19) which at main order looks like

$$U(x,t) := U_{\lambda(t),\xi(t),\omega(t)}(x) = Q_{\omega(t)}\, W\Big(\frac{x-\xi(t)}{\lambda(t)}\Big)$$

for certain functions $\xi(t)$, $\lambda(t)$ and $\omega(t)$ of class $C^1([0,T])$ such that

$$\xi(T) = q, \quad \lambda(T) = 0,$$

so that $u(x,t)$ blows-up at time T and the point q. We shall find values for these functions so that for a small remainder $v(x,t)$ we have that $u = U+v$ solves (3.17)-(3.19) for $u_0(x) = U(x,0)+v(x,0)$. Let us denote

$$S(u) := -u_t + \Delta u + |\nabla u|^2 u$$

A useful observation that we make is that as long as the constraint $|u|=1$ is kept at all times and $u=U+v$ with $|v|\le\frac12$ uniformly, then for u to solve equation (3.17) it suffices that

$$S(U+v) = b(x,t)U \tag{3.22}$$

for some scalar function b. Indeed, we observe that since $|u|\equiv 1$ we have

$$b\,(U\cdot u) = S(u)\cdot u = -\frac12\frac{d}{dt}|u|^2 + \frac12\Delta|u|^2 = 0,$$

and since $U\cdot u\ge\frac12$, we find that $b\equiv 0$.

We can parametrize all small functions $v(x,t)$ such that $|U+v|=1$ in the form

$$v = \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U, \tag{3.23}$$

where φ is an arbitrary small function with values into $\mathbb{R}^3$, and

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi\cdot U)U, \quad a(\zeta) := \sqrt{1-|\zeta|^2} - 1.$$

Using that

$$\Delta U + |\nabla U|^2 U = 0$$

we find the following expansion for $S(U+v)$ with v given by (3.23):

$$S(U+\Pi_{U^\perp}\varphi + aU) = -U_t - \partial_t\Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi) + c(\Pi_{U^\perp}\varphi)U$$

where for $\zeta = \Pi_{U^\perp}\varphi$, $a = a(\zeta)$,

$$
\begin{aligned}
L_U(\zeta) &= \Delta\zeta + |\nabla U|^2\zeta + 2(\nabla U\cdot\zeta)U\\
N_U(\zeta) &= \left[2\nabla(aU)\cdot\nabla(U+\zeta) + 2\nabla U\cdot\nabla\zeta + |\nabla\zeta|^2 + |\nabla(aU)|^2\right]\zeta - aU_t\\
&\quad + 2\nabla a\nabla U,\\
c(\zeta) &= \Delta a - a_t + (|\nabla(U+\zeta+aU)|^2 - |\nabla U|^2)(1+a) - 2\nabla U\cdot\nabla\zeta
\end{aligned}
$$

Since we just need to have an equation of the form (3.22) satisfied, we find that

$$
u = U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U
$$

solves (3.17) if and only if φ satisfies

$$
0 = -U_t - \partial_t\Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi) + b(x,t)U, \tag{3.24}
$$

for some scalar function b. The logic of the construction goes like this: We decompose φ into the sum of two functions $\varphi = \varphi^i + \varphi^o$, the "inner" and "outer" solutions and reduce equation (3.24) to solving a system of two equations in (φ^i, φ^o) that we call the inner and outer problems.

The inner function $\varphi^i(x,t)$ will be assumed supported only near $x = \xi(t)$ and better read as a function of the scaled space variable $y = \frac{x-\xi(t)}{\lambda(t)}$ with zero initial condition and such that $\varphi^i\cdot U = 0$, so that $\Pi_{U^\perp}\varphi^i = \varphi^i$. The outer function $\varphi^o(x,t)$ will be made out of several pieces and its role is essentially to satisfy (3.24) far away from the concentration point $x = \xi(t)$.

We write equation (3.24) in the following way:

$$
\begin{aligned}
0 &= -\partial_t\varphi^i + L_U[\varphi^i] + \tilde L_U[\varphi^o] - \Pi_{U^\perp}[\partial_t\varphi^o - \Delta\varphi^o + U_t]\\
&\quad + N_U(\varphi^i + \Pi_{U^\perp}\varphi^o) + (\varphi^o\cdot U)U_t + bU.
\end{aligned}\tag{3.25}
$$

For the outer problem, we consider a function Φ^0 that depends explicitly on the parameter functions chosen in such a way that $\Pi_{U^\perp}[\partial_t\Phi^0 - \Delta\Phi^0 + U_t]$ gets concentrated near $x = \xi(t)$ by elimination of the terms in the first error U_t associated to dilation and rotation. Then we write

$$
\varphi^o(x,t) = \Phi^0(x,t) + \Psi^*(x,t). \tag{3.26}
$$

For the inner solution, we consider a smooth smooth cut-off function $\eta_0(s)$ with $\eta_0(s) = 1$ for $s < 1$ and $= 0$ for $s > \frac{3}{2}$. We also consider a positive, large smooth function $R(t) \to +\infty$ as $t \to T$ that we will later specify. We define

$$
\eta(x,t) := \eta_0\left(R(t)^{-1}|y|\right), \quad y = \frac{x-\xi(t)}{\lambda(t)}
$$

and let

$$
\varphi^i(x,t) = \eta(x,t)Q_\omega\phi(y,t), \quad y = \frac{x-\xi(t)}{\lambda(t)}
$$

for a function $\phi(y,t)$ with initial condition $\phi(\cdot,0) = 0$ that satisfies $\phi(\cdot,t)\cdot W \equiv 0$, defined for $|y| \le 2R(t)$ and that vanishes as $t \to T$. Then we have

$$
\begin{aligned}
Q_{-\omega}L_U[\varphi^i] &= \lambda^{-2}\eta L_W[\phi] + (\Delta_x\eta)\phi + 2\lambda^{-1}\nabla_x\eta\nabla_y\phi\\
Q_{-\omega}\varphi^i_t &= \eta\big(\phi_t - \lambda^{-1}\dot\lambda y\cdot\nabla_y\phi - \lambda^{-1}\dot\xi\cdot\nabla_y\phi + \dot\omega Q_{-\omega}\partial_\omega Q_\omega\phi\big) + \eta_t\phi.
\end{aligned}
$$

Equation (3.25) then becomes

$$
\begin{aligned}
0 &= \lambda^{-2}\eta Q_\omega[-\lambda^2\phi_t + \ L_W[\phi] + \lambda^2 Q_{-\omega}\tilde L_U[\Psi^*]]\\
&\quad + \eta Q_\omega(\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi + \lambda^{-1}\dot\xi\cdot\nabla_y\phi - \dot\omega J\phi)\\
&\quad + \tilde L_U[\Phi^0] + \Pi_{U^\perp}[\partial_t\Phi^0 - \Delta_x\Phi^0 + U_t]\\
&\quad - \partial_t\Psi^* + \Delta\Psi^* + (1-\eta)\tilde L_U[\Psi^*] + Q_\omega[(\Delta_x\eta)\phi + 2\nabla_x\eta\nabla_x\phi - \eta_t\phi]\\
&\quad + N_U(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0 + \Psi^*)) + ((\Psi^* + \Phi^0)\cdot U)U_t + bU.
\end{aligned}\tag{3.27}
$$

Next we will define precisely the operator Φ^0 and estimate the quantity

$$
\tilde L_U[\Phi^0] + \Pi_{U^\perp}[\partial_t\Phi^0 - \Delta_x\Phi^0 + U_t]. \tag{3.28}
$$

The idea is to choose Φ^0 such that $\partial_t\Phi^0 - \Delta_x\Phi^0 + U_t \approx 0$ whenever $|x-\xi| \gg \lambda$, so that in particular the last error term in the outer equation (3.26) is of smaller order.

Invoking formulas (3.10) to compute U_t we get

$$U_t = \dot\lambda\partial_\lambda U_{\lambda,\xi,\omega} + \dot\omega\partial_\omega U_{\lambda,\xi,\omega} + \partial_\xi U_{\lambda,\xi,\omega}\cdot\dot\xi = \mathcal{E}_0 + \mathcal{E}_1,$$

where, setting $y = \frac{x-\xi}{\lambda} = \rho e^{i\theta}$, we have

$$\begin{aligned}\mathcal{E}_0(x,t) &= -Q_\omega[\frac{\dot\lambda}{\lambda}\rho w_\rho(\rho)\,E_1(y) \,+\, \dot\omega\rho w_\rho(\rho)\,E_2(y)\,]\\ \mathcal{E}_1(x,t) &= -\frac{\dot\xi_1}{\lambda}\,w_\rho(\rho)\,Q_\omega[\,\cos\theta\,E_1(y) + \sin\theta\,E_2(y)]\\ &\quad -\frac{\dot\xi_2}{\lambda}\,w_\rho(\rho)\,Q_\omega[\sin\theta\,E_1(y) - \cos\theta\,E_2(y)\,].\end{aligned}$$

Since $\mathcal{E}_1$ has faster space decay in ρ than $\mathcal{E}_0$ we will choose Φ^0 to be an approximate solution of

$$\Phi^0_t - \Delta_x\Phi^0 + \mathcal{E}_0 = 0. \tag{3.29}$$

For $x = \xi + re^{i\theta}$ and $r \gg \lambda$ we have

$$\mathcal{E}_0(x,t) = -\frac{2r}{r^2+\lambda^2}\left[\dot\lambda Q_\omega E_1 + \lambda\dot\omega Q_\omega E_2\right] \approx -\frac{2r}{r^2+\lambda^2}\begin{bmatrix}(\dot\lambda + i\lambda\dot\omega)e^{i(\theta+\omega)}\\ 0\end{bmatrix}.$$

Here and in what follows we let

$$p(t) = \lambda(t)e^{i\omega(t)}.$$

Then

$$-\frac{2r}{r^2+\lambda^2}\begin{bmatrix}(\dot\lambda + i\lambda\dot\omega)e^{i(\theta+\omega)}\\ 0\end{bmatrix} = -\frac{2r}{r^2+\lambda^2}\begin{bmatrix}\dot p(t)e^{i\theta}\\ 0\end{bmatrix} =: \tilde{\mathcal{E}}_0(x,t).$$

With the aid of Duhamel's formula for the standard heat equation, we find that the following function is a good approximate solution of $\Phi^0_t - \Delta_x\Phi^0 + \tilde{\mathcal{E}}_0 = 0$ and hence of (3.29). We define

$$\begin{aligned}\Phi^0[\omega,\lambda,\xi] &:= \begin{bmatrix}\varphi^0(r,t)e^{i\theta}\\ 0\end{bmatrix}\\ \varphi^0(r,t) &= -\int_{-T}^t \dot p(s)rk(z(r),t-s)\,ds\\ z(r) &= \sqrt{r^2+\lambda^2},\quad k(z,t) = 2\frac{1-e^{-\frac{z^2}{4t}}}{z^2},\end{aligned}$$

where for technical reasons that will be made clear later on, $p(t)$ is also assumed to be defined for negative values of t.

A direct computations yields

$$\Phi^0_t + \Delta_x\Phi^0 + \tilde{\mathcal{E}}_0 = \tilde{\mathbb{R}}_0 + \tilde{\mathbb{R}}_1,\quad \tilde{\mathbb{R}}_0 = \begin{pmatrix}\mathbb{R}_0\\ 0\end{pmatrix},\quad \tilde{\mathbb{R}}_1 = \begin{pmatrix}\mathbb{R}_1\\ 0\end{pmatrix}$$

where

$$\mathbb{R}_0 := -re^{i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t \dot p(s)(zk_z - z^2k_{zz})(z(r),t-s)\,ds$$

and

$$\begin{aligned}\mathbb{R}_1 := &-e^{i\theta}\mathrm{Re}\,(e^{-i\theta}\dot\xi(t))\int_{-T}^t \dot p(s)\,k(z(r),t-s)\,ds\\ &+\frac{r}{z^2}e^{i\theta}\,(\lambda\dot\lambda(t) - \mathrm{Re}\,(re^{i\theta}\dot\xi(t)))\int_{-T}^t \dot p(s)\;zk_z(z(r),t-s)\,ds.\end{aligned}$$

We observe that $\mathbb{R}_1$ is actually a term of smaller order. Using formulas (3.14), (3.16) and the facts

$$\frac{\lambda^2 r}{z^4} = \frac{1}{4\lambda}\rho w_\rho^2,\quad \frac{r}{z^2}(1-\cos w) = \frac{1}{2\lambda}\rho w_\rho^2,$$

we derive an expression for the quantity (3.28):

$$
\begin{aligned}
&\tilde{L}_U[\Phi^0] + \Pi_{U^\perp}[-U_t + \Delta\Phi^0 - \Phi^0_t] \\
&= \tilde{L}_U[\Phi^0] - \mathcal{E}_1 + \Pi_{U^\perp}[\tilde{\mathcal{E}}_0] - \mathcal{E}_0 + \Pi_{U^\perp}[\tilde{\mathbb{R}}_0] + \Pi_{U^\perp}[\tilde{\mathbb{R}}_1] \\
&= \mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi] + \Pi_{U^\perp}[\tilde{\mathbb{R}}_1]
\end{aligned}
$$

where

$$
\mathcal{K}_0[p,\xi] = \mathcal{K}_{01}[p,\xi] + \mathcal{K}_{02}[p,\xi]
$$

with

$$
\begin{aligned}
\mathcal{K}_{01}[p,\xi] := -\frac{2}{\lambda}\rho w_\rho^2 \int_{-T}^{t} \Big[\mathrm{Re}\,(\dot{p}(s)e^{-i\omega(t)})Q_\omega E_1 + \mathrm{Im}\,(\dot{p}(s)e^{-i\omega(t)})Q_\omega E_2\Big] \\
\cdot k(z,t-s)\,ds
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
\mathcal{K}_{02}[p,\xi] :=& \frac{1}{\lambda}\rho w_\rho^2 \left[\dot{\lambda} - \int_{-T}^{t} \mathrm{Re}\,(\dot{p}(s)e^{-i\omega(t)}) r k_z(z,t-s) z_r\,ds\right] Q_\omega E_1 \\
&- \frac{1}{4\lambda}\rho w_\rho^2 \cos w \left[\int_{-T}^{t} \mathrm{Re}\,(\dot{p}(s)e^{-i\omega(t)})\,(z k_z - z^2 k_{zz})(z,t-s)\,ds\right] Q_\omega E_1 \\
&- \frac{1}{4\lambda}\rho w_\rho^2 \left[\int_{-T}^{t} \mathrm{Im}\,(\dot{p}(s)e^{-i\omega(t)})\,(z k_z - z^2 k_{zz})(z,t-s)\,ds\right] Q_\omega E_2,
\end{aligned}
\tag{3.31}
$$

$$
\mathcal{K}_1[p,\xi] := \frac{1}{\lambda} w_\rho \left[\Re\big((\dot{\xi}_1 - i\dot{\xi}_2)e^{i\theta}\big)Q_\omega E_1 + \Im\big((\dot{\xi}_1 - i\dot{\xi}_2)e^{i\theta}\big)Q_\omega E_2\right].
\tag{3.32}
$$

We insert this decomposition in equation (3.27) and see that we will have a solution to the equation if the pair (ϕ, Ψ^*) solves the *inner-outer gluing system*

$$
\begin{cases}
\lambda^2 \phi_t = L_W[\phi] + \lambda^2 Q_{-\omega}\Big[\tilde{L}_U[\Psi^*] + \mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]\Big] \quad \text{in } \mathcal{D}_{2R} \\
\phi \cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\
\phi(\cdot,0) = 0 = \phi(\cdot,T),
\end{cases}
\tag{3.33}
$$

$$
\Psi^*_t = \Delta_x \Psi^* + g[p,\xi,\Psi^*,\phi] \quad \text{in } \Omega\times(0,T)
\tag{3.34}
$$

where

$$
\begin{aligned}
g[p,\xi,\Psi^*,\phi] :=& (1-\eta)\tilde{L}_U[\Psi^*] + (\Psi^*\cdot U)U_t \\
&+ Q_\omega\big((\Delta_x\eta)\phi + 2\nabla_x\eta\nabla_x\phi - \eta_t\phi\big) \\
&+ \eta Q_\omega\big(-\dot{\omega}J\phi + \lambda^{-1}\dot{\lambda}y\cdot\nabla_y\phi + \lambda^{-1}\dot{\xi}\cdot\nabla_y\phi\big) \\
&+ (1-\eta)[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] + \Pi_{U^\perp}[\tilde{\mathbb{R}}_1] + (\Phi^0\cdot U)U_t \\
&+ N_U(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0 + \Psi^*)),
\end{aligned}
\tag{3.35}
$$

and we denote

$$
\mathcal{D}_{\gamma R} = \{(y,t)\in\mathbb{R}^2\times(0,T)\ /\ |y| < \gamma R(t)\}.
$$

Indeed if (ϕ,Ψ^*) solves this system, then we have that

$$
u(x,t) = U + \Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega\phi] + a(\Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega\phi])U
\tag{3.36}
$$

solves equation (3.17). The boundary condition (3.20) $u = \mathbf{e}_3$ amounts to

$$
\Pi_{U^\perp}[\Phi^0 + \Psi^*] + a(\Pi_{U^\perp}[U + \Phi^0 + \Psi^*])U = (\mathbf{e}_3 - U)
$$

and then it suffices that we take the boundary condition for (3.34)

$$
\Psi^*\big|_{\partial\Omega} = \mathbf{e}_3 - U - \Phi^0.
\tag{3.37}
$$

Since we want that $u(x,t)$ be a small perturbation of $U(x,t)$ when we stand close to (q,T), it is natural to require that Ψ^* satisfies the final condition

$$
\Psi^*\big(q,T\big) = 0.
$$

This constraint amounts to three Lagrange multipliers when we solve the problem, which we choose to put in the initial condition. Then we assume

$$\Psi^*(x,0) = Z_0^*(x) + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3,$$

where c_1, c_2, c_3 are undetermined constants and $Z_0^*(x)$ is a small function for which specific assumptions will later be made.

• **The reduced equations.** In this section we will informally discuss the procedure to achieve our purpose in particular deriving the order of vanishing of the scaling parameter $\lambda(t)$ as $t \to T$.

The main term that couples equations (3.33) and (3.34) inside the second equation is the linear expression

$$Q_\omega[(\Delta_x\eta)\phi + 2\nabla_x\eta\nabla_x\phi + \eta_t\phi],$$

which is supported in $|y| = O(R)$. This motivates the fact that we want ϕ to exhibit some type of space decay in $|y|$ since in that way Ψ^* will eventually be smaller and in turn that would make the two equations at main order *uncoupled*. Equation (3.33) has the form

$$\begin{aligned} &\lambda^2\phi_t = L_W[\phi] + h[p,\xi,\Psi^*](y,t) \quad \text{in } \mathcal{D}_{2R} \\ &\phi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ &\phi(\cdot,0) = 0 \quad \text{in } B_{2R(0)}, \end{aligned}$$

where, for convenience we assume that $h(y,t)$ is defined for all $y \in \mathbb{R}^2$ extending outside $\mathcal{D}_{2R}$ as

$$h[p,\xi,\Psi^*] = \lambda^2 Q_{-\omega}\tilde{L}_U[\Psi^*]\chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi] + \lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]\chi_{\mathcal{D}_{2R}}, \tag{3.38}$$

where χ_A designates characteristic function of a set A, $\mathcal{K}_0$ is defined in (3.30), (3.31) and $\mathcal{K}_1$ in (3.32). If $\lambda(t)$ has a relatively smooth vanishing as $t \to T$ it seems natural that the term $\lambda^2\phi_t$ be of smaller order and then the equation is approximately represented by the elliptic problem

$$L_W[\phi] + h[p,\xi,\Psi^*] = 0, \quad \phi\cdot W = 0 \quad \text{in } \mathbb{R}^2. \tag{3.39}$$

Let us consider the decaying functions $Z_{lj}(y)$ defined in formula (3.10), which satisfy $L_W[Z_{lj}] = 0$. If $\phi(y,t)$ is a solution of (3.39) with sufficient decay, then necessarily

$$\int_{\mathbb{R}^2} h[p,\xi,\Psi^*](y,t)\cdot Z_{lj}(y)\,dy = 0 \qquad \text{for all} \quad t \in (0,T), \tag{3.40}$$

for $l = 0,1$, $j = 1,2$. These relations amount to an integro-differential system of equations for $p(t)$, $\xi(t)$, which, as a matter of fact, *detemine* the correct values of the parameters so that the solution (ϕ,Ψ^*) with appropriate asymptotics exists.

We derive next useful expressions for relations (3.40). Let us first compute the quantities

$$\mathcal{B}_{0j}[p](t) := \frac{\lambda}{2\pi}\int_{\mathbb{R}^2} Q_{-\omega}[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]]\cdot Z_{0j}(y)\,dy. \tag{3.41}$$

Using (3.30), (3.31) the following expressions for $\mathcal{B}_{01}$, $\mathcal{B}_{02}$ are readily obtained:

$$\mathcal{B}_{01}[p](t) = \int_{-T}^{t} \mathrm{Re}\,(\dot p(s)e^{-i\omega(t)})\,\Gamma_1\left(\frac{\lambda(t)^2}{t-s}\right)\frac{ds}{t-s} - 2\dot\lambda(t)$$

$$\mathcal{B}_{02}[p](t) = \int_{-T}^{t} \mathrm{Im}\,(\dot p(s)e^{-i\omega(t)})\,\Gamma_2\left(\frac{\lambda(t)^2}{t-s}\right)\frac{ds}{t-s}$$

where $\Gamma_j(\tau)$, $j = 1,2$ are the smooth functions defined as follows:

$$\Gamma_1(\tau) = -\int_0^\infty \rho^3 w_\rho^3\left[K(\zeta) + 2\zeta K_\zeta(\zeta)\frac{\rho^2}{1+\rho^2} - 4\cos(w)\zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho$$

$$\Gamma_2(\tau) = -\int_0^\infty \rho^3 w_\rho^3\left[K(\zeta) - \zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho$$

where

$$K(\zeta) = 2\frac{1-e^{-\frac{\zeta}{4}}}{\zeta},$$

and we have used that $\int_0^\infty \rho^3 w_\rho^3 d\rho = -2$. Using these expressions we find that

$$
\begin{aligned}
|\Gamma_l(\tau) - 1| &\le C\tau(1 + |\log\tau|) \qquad \text{for } \tau < 1, \\
|\Gamma_l(\tau)| &\le \frac{C}{\tau} \qquad\qquad\quad \text{for } \tau > 1.
\end{aligned} \tag{3.42}
$$

Let us define

$$
\mathcal{B}_0[p] := \frac{1}{2} e^{i\omega(t)} \left(\mathcal{B}_{01}[p] + i\mathcal{B}_{02}[p] \right) \tag{3.43}
$$

and

$$
\begin{aligned}
a_{0j}[p, \xi, \Psi^*] &:= -\frac{\lambda}{2\pi} \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*] \cdot Z_{0j}(y)\, dy \\
a_0[p, \xi, \Psi^*] &:= \frac{1}{2} e^{i\omega(t)} \left(a_{01}[p, \xi, \Psi^*] + i a_{02}[p, \xi, \Psi^*] \right).
\end{aligned} \tag{3.44}
$$

Similarly, we let

$$
\begin{aligned}
\mathcal{B}_{1j}[\xi](t) &:= \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} Q_{-\omega}[\mathcal{K}_0[p, \xi] + \mathcal{K}_1[p, \xi]] \cdot Z_{1j}(y)\, dy, \\
\mathcal{B}_1[\xi](t) &:= \mathcal{B}_{11}[\xi](t) + i\mathcal{B}_{12}[\xi](t).
\end{aligned}
$$

Using (3.32), (3.10) and the fact that $\int_0^\infty \rho w_\rho^2 d\rho = 2$ we get

$$
\mathcal{B}_1[\xi](t) \;=\; 2[\, \dot{\xi}_1(t) + i\dot{\xi}_2(t) \,].
$$

At last, we set

$$
\begin{aligned}
a_{1j}[p, \xi, \Psi^*] &:= \frac{\lambda}{2\pi} \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*] \cdot Z_{1j}(y)\, dy \\
a_1[p, \xi, \Psi^*] &:= -e^{i\omega(t)} (a_{11}[p, \xi, \Psi^*] + i a_{12}[p, \xi, \Psi^*]).
\end{aligned}
$$

We get that the four conditions (3.40) reduce to the system of two complex equations

$$
\mathcal{B}_0[p] = a_0[p, \xi, \Psi^*], \tag{3.45}
$$

$$
\mathcal{B}_1[\xi] = a_1[p, \xi, \Psi^*]. \tag{3.46}
$$

At this point we will make some preliminary considerations on this system that will allow us to find a first guess of the parameters $p(t)$ and $\xi(t)$. First, we observe that

$$
\mathcal{B}_0[p] = \int_{-T}^{t-\lambda^2} \frac{\dot{p}(s)}{t - s} ds \; + O\big(\|\dot{p}\|_\infty\big).
$$

To get an approximation for a_0, we analyze the operator $\tilde{L}_U$ in a_0. For this let us write

$$
\Psi^* = \begin{bmatrix} \psi^* \\ \psi_3^* \end{bmatrix}, \quad \psi^* = \psi_1^* + i\psi_2^*.
$$

From formula (3.14) we find that

$$
\tilde{L}_U[\Psi^*](y) = [\tilde{L}_U]_0[\Psi^*] + [\tilde{L}_U]_1[\Psi^*] + [\tilde{L}_U]_2[\Psi^*],
$$

where

$$
\begin{aligned}
\lambda Q_{-\omega}[\tilde{L}_U]_0[\Psi^*] =& \quad \rho w_\rho^2 \left[\, \operatorname{div}(e^{-i\omega}\psi^*)\, E_1 + \operatorname{curl}(e^{-i\omega}\psi^*)\, E_2 \,\right] \\
\lambda Q_{-\omega}[\tilde{L}_U]_1[\Psi^*] =& -2w_\rho \cos w \left[\, (\partial_{x_1}\psi_3^*) \cos\theta + (\partial_{x_2}\psi_3^*) \sin\theta \,\right] E_1 \\
& - 2w_\rho \cos w \left[\, (\partial_{x_1}\psi_3^*) \sin\theta - (\partial_{x_2}\psi_3^*) \cos\theta \,\right] E_2 \;, \\
\lambda Q_{-\omega}[\tilde{L}_U]_2[\Psi^*] =& \quad \rho w_\rho^2 \left[\, \operatorname{div}(e^{i\omega}\bar{\psi}^*)\, \cos 2\theta - \operatorname{curl}(e^{i\omega}\bar{\psi}^*)\, \sin 2\theta \,\right] E_1 \\
& + \rho w_\rho^2 \left[\, \operatorname{div}(e^{i\omega}\bar{\psi}^*)\, \sin 2\theta + \operatorname{curl}(e^{i\omega}\bar{\psi}^*)\, \cos 2\theta \,\right] E_2,
\end{aligned}
$$

and the differential operators in Ψ^* on the right hand sides are evaluated at (x, t) with $x = \xi(t) + \lambda(t)y$, $y = \rho e^{i\theta}$ while $E_l = E_l(y)$, $l = 1, 2$.

From the above decomposition, assuming that Ψ^* is of class C^1 in space variable, we find that

$$
a_0[p, \xi, \Psi^*] = [\operatorname{div}\psi^* + i \operatorname{curl}\psi^*](\xi, t) + o(1),
$$

where $o(1) \to 0$ as $t \to T$.

Similarly, we have that

$$a_1(p,\xi) = 2(\partial_{x_1}\psi_3^* + i\partial_{x_2}\psi_3^*)(\xi,t)\int_0^\infty \cos w\, w_\rho^2 \rho\, d\rho + o(1)$$
$$= o(1) \quad \text{as} \quad t \to T,$$

since $\int_0^\infty w_\rho^2 \cos w\rho\, d\rho = 0$.

Let us discuss informally how to handle (3.45)-(3.46). For this we simplify this system in the form

$$\int_{-T}^{t-\lambda^2} \frac{\dot p(s)}{t-s} ds = [\operatorname{div}\psi^* + i\operatorname{curl}\psi^*](\xi(t),t) + o(1) + O(\|\dot p\|_\infty)$$
$$\dot\xi(t) = o(1) \quad \text{as} \quad t \to T. \tag{3.47}$$

We assume for the moment that the function $\Psi^*(x,t)$ is fixed, sufficiently regular, and we regard T as a parameter that will always be taken smaller if necessary. We recall that we want $\xi(T) = q$ where $q \in \Omega$ is given, and $\lambda(T) = 0$. Equation (3.47) immediately suggests us to take $\xi(t) \equiv q$ as a first approximation. Neglecting lower order terms, we arrive at the "clean" equation for $p(t) = \lambda(t)e^{i\omega(t)}$,

$$\int_{-T}^{t-\lambda(t)^2} \frac{\dot p(s)}{t-s} ds = \operatorname{div}\psi^*(q,0) + i\operatorname{curl}\psi^*(q,0) =: a_0^* \tag{3.48}$$

At this point we make the following assumption:

$$\operatorname{div}\psi^*(q,0) < 0. \tag{3.49}$$

This implies that $a_0^* = -|a_0^*|e^{i\omega_0}$ for a unique $\omega_0 \in (-\frac{\pi}{2},\frac{\pi}{2})$. Let us take $\omega(t) \equiv \omega_0$. Then equation (3.48) becomes

$$\int_{-T}^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s} ds = -|a_0^*|. \tag{3.50}$$

We claim that a good approximate solution of (3.50) as $t \to T$ is given by

$$\dot\lambda(t) = -\frac{\kappa}{\log^2(T-t)}$$

for a suitable $\kappa > 0$. In fact, substituting, we have

$$\begin{aligned}\int_{-T}^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s}\, ds \;&=\; \int_{-T}^{t-(T-t)} \frac{\dot\lambda(s)}{t-s}\, ds + \dot\lambda(t)\left[\log(T-t) - 2\log(\lambda(t))\right]\\ &\quad + \int_{t-(T-t)}^{t-\lambda(t)^2} \frac{\dot\lambda(s)-\dot\lambda(t)}{t-s} ds\\ &\approx \int_{-T}^{t} \frac{\dot\lambda(s)}{T-s}\, ds \;-\; \dot\lambda(t)\log(T-t) \;=:\beta(t)\end{aligned} \tag{3.51}$$

as $t \to T$. We see that

$$\log(T-t)\frac{d\beta}{dt}(t) = \frac{d}{dt}(\log^2(T-t)\,\dot\lambda(t)) = 0$$

from the explicit form of $\dot\lambda(t)$. Hence $\beta(t)$ is constant. As a conclusion, equation (3.50) is approximately satisfied if κ is such that

$$\kappa\int_{-T}^{T} \frac{\dot\lambda(s)}{T-s} \;=\; -|a_0^*|.$$

And this finally gives us the approximate expression

$$\dot\lambda(t) = -|\operatorname{div}\psi^*(q,0) + i\operatorname{curl}\psi^*(q,0)|\,\dot\lambda_*(t),$$

where

$$\dot\lambda_*(t) = -\frac{|\log T|}{\log^2(T-t)}.$$

Naturally imposing $\lambda_*(T)=0$ we then have

$$\lambda_*(t)=\frac{|\log T|}{\log^2(T-t)}(T-t)\,(1+o(1))\quad \text{as}\quad t\to T.$$

• **Solving the inner-outer gluing system.**

Our purpose is to determine, for a given $q\in\Omega$ and a sufficiently small $T>0$, a solution (ϕ,Ψ^*) of system (3.33)-(3.34) with a boundary condition of the form (3.37) such that $u(x,t)$ given by (3.36) blows up with $U(x,t)$ as its main order profile. This will only be possible for adequate choices of the parameter functions $\xi(t)$ and $p(t)=\lambda(t)e^{i\omega(t)}$. These functions will eventually be found by fixed point arguments, but a priori we need to make some assumptions regarding their behavior. For some positive numbers a_1,a_2,σ independent of T we will assume that

$$a_1|\dot\lambda_*(t)|\le|\dot p(t)|\le a_2|\dot\lambda_*(t)|\quad\text{for all}\quad t\in(0,T), \tag{3.52}$$

$$|\dot\xi(t)|\le\lambda_*(t)^{\sigma}\qquad\text{for all}\quad t\in(0,T). \tag{3.53}$$

We also take

$$R(t)=\lambda_*(t)^{-\beta}, \tag{3.54}$$

where $\beta\in(0,\frac12)$.

To solve the outer equation (3.34) we will decompose Ψ^* in the form

$$\Psi^*=Z^*+\psi$$

where we let $Z^*:\Omega\times(0,\infty)\to\mathbb{R}^3$ satisfy

$$\begin{cases} Z^*_t=\Delta Z^* \quad \text{in } \Omega\times(0,\infty),\\ Z^*(\cdot,t)=0\quad\text{in } \partial\Omega\times(0,\infty),\\ Z^*(\cdot,0)=Z^*_0\quad\text{in } \Omega,\end{cases} \tag{3.55}$$

with $Z_0^*(x)$ a function satisfying certain conditions to be described below. Since we would like that $u(x,t)$ given by (3.36) has a blow-up behavior given at main order by that of $U(x,t)$, we will require

$$\Psi^*(q,T)=0.$$

This constraint has three parameters. Therefore we need three "Lagrange multipliers" which we include in the initial datum.

• **Assumptions on Z_0^*.**

To describe the assumptions on Z_0^*, let us write

$$Z_0^*(x)=\begin{bmatrix} z_0^*(x)\\ z_{03}^*(x)\end{bmatrix},\quad z_0^*(x)=z_{01}^*(x)+iz_{02}^*(x). \tag{3.56}$$

A first condition that we require, consistent with (3.49), is $\operatorname{div} z_0^*(q)<0$. In addition we require that $Z_0^*(q)\approx 0$ in a non-degenerate way.

We want also Z^* to be sufficiently small, but independently of T, so that the heat equation (3.55) is a good approximation of the linearized harmonic map flow far from the singularity. In order to achieve later the desired stability property, it is convenient split Z_0^* into two parts

$$Z_0^*=Z_0^{*0}+Z_0^{*1},$$

where Z_0^{*0} is sufficiently smooth and Z_0^{*1} allows more irregular perturbations. More precisely, for Z_0^{*0} we assume that for some $\alpha_0>0$ small and some $\alpha_1,\alpha_2>0$, all independent of T, we have

$$\begin{cases} \|Z_0^{*0}\|_{C^3(\overline\Omega)}\le\alpha_0,\\ |Z_0^{*0}(q)|\le 5T,\\ |(Dz_0^{*0}(q))^{-1}|\le\alpha_1,\\ -\alpha_1\le\operatorname{div} z_0^{*0}(q)\le-\alpha_2.\end{cases} \tag{3.57}$$

(The notation here is analogous to (3.56).)

To describe Z_0^{*1} we introduce the following norm

$$\|Z_0^{*1}\|_* = \sup_\Omega |Z_0^{*1}(x)| + \frac{1}{|\log \varepsilon_*|} \sup_\Omega |\nabla_x Z_0^{*1}(x)| + \frac{1}{|\log \varepsilon_*|^{1/2}} \sup_\Omega (|x - q_0| + \varepsilon_*) |D_x^2 Z_0^{*1}(x)|, \tag{3.58}$$

where

$$\varepsilon_* = \lambda_*(0).$$

Then we assume that for some $\sigma > 0$ fixed we have

$$\|Z_0^{*1}\|_* \le T^\sigma. \tag{3.59}$$

In summary, the conditions on Z_0^* are the following:

$$Z_0^* = Z_0^{*0} + Z_0^{*1} \text{ with } Z_0^{*0},\ Z_0^{*1} \text{ satisfying (3.57) and (3.59).} \tag{3.60}$$

• **Linear theory for the inner problem.** The inner problem (3.33) is written as

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h[p, \xi, \Psi^*] & \text{in } \mathcal{D}_{2R} \\ \phi \cdot W = 0 & \text{in } \mathcal{D}_{2R} \\ \phi(\cdot, 0) = 0 & \text{in } B_{2R(0)} \end{cases}$$

where $h[p, \xi, \Psi^*]$ is given by (3.38). To find a good solution to this problem we would like that $h[p, \xi, \Psi^*]$ satisfies the orthogonality conditions (3.40).

We split the right hand side $h[p, \xi, \Psi^*]$ and the inner solution into components with different roles regarding these orthogonality conditions.

Recall that

$$h[p, \xi, \Psi^*] = \lambda^2 Q_{-\omega} \tilde{L}_U[\Psi^*] \chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi] + \lambda^2 Q_{-\omega} \mathcal{K}_1[p, \xi] \chi_{\mathcal{D}_{2R}},$$

the decomposition of $\tilde{L}_U$ given in (3.14):

$$\tilde{L}_U[\Psi^*] = \tilde{L}_U[\Psi^*]_0 + \tilde{L}_U[\Psi^*]_1 + \tilde{L}_U[\Psi^*]_2\ ,$$

with $\tilde{L}_U[\Phi]_j$ defined in (3.15). Using the notation (3.13), we then define

$$\begin{aligned} \tilde{L}_U[\Phi]_1^{(0)} = & -2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1} \varphi_3(\xi(t), t)) \cos\theta + (\partial_{x_2} \varphi_3(\xi(t), t))) \sin\theta \right] Q_\omega E_1 \\ & -2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1} \varphi_3(\xi(t), t))) \sin\theta - (\partial_{x_2} \varphi_3(\xi(t), t))) \cos\theta \right] Q_\omega E_2\ . \end{aligned}$$

We then decompose

$$h = h_1 + h_2 + h_3$$

where

$$\begin{aligned} h_1[p, \xi, \Psi^*] &= \lambda^2 Q_{-\omega} (\tilde{L}_U[\Psi^*]_0 + \tilde{L}_U[\Psi^*]_2) \chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi], \\ h_2[p, \xi, \Psi^*] &= \lambda^2 Q_{-\omega} \tilde{L}_U[\Psi^*]_1^{(0)} \chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega} \mathcal{K}_1[p, \xi] \chi_{\mathcal{D}_{2R}} \\ h_3[p, \xi, \Psi^*] &= \lambda^2 Q_{-\omega} (\tilde{L}_U[\Psi^*]_1 - \tilde{L}_U[\Psi^*]_1^{(0)}) \chi_{\mathcal{D}_{2R}}. \end{aligned}$$

Next we decompose $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$. The function ϕ_1 will solve the inner problem with right hand side $h_1[p, \xi, \Psi^*]$ projected so that it satisfies essentially (3.40). The advantage of doing this is that h_1 has faster spatial decay, which gives better bounds for the solution. For this we let, for any function $h(y, t)$ defined in $\mathbb{R}^2 \times (0, T)$ with sufficient decay,

$$c_{lj}[h](t) := \frac{1}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{lj}|^2} \int_{\mathbb{R}^2} h(y, t) \cdot Z_{lj}(y)\, dy. \tag{3.61}$$

Note that $h[p, \xi, \Psi^*]$ is defined in $\mathbb{R}^2 \times (0, T)$, and for simplicity we will assume that the right hand sides appearing in the different linear equations are always defined in $\mathbb{R}^2 \times (0, T)$.

We would like that ϕ_1 solves

$$\lambda^2 \partial_t \phi_1 = L_W[\phi_1] + h_1[p, \xi, \Psi^*] - \sum_{l=-1}^{1} \sum_{j=1}^{2} c_{lj}[h_1(p, \xi, \Psi^*)] w_\rho^2 Z_{lj} \quad \text{in } \mathcal{D}_{2R},$$

but the estimates for ϕ_1 are better if the projections $c_{0j}[h(p,\xi,\Psi^*)]$ are modified slightly.

Here is the precise result that we will use later. We define the norms

$$\|h\|_{\nu,a} = \sup_{\mathbb{R}^2\times(0,T)} \frac{|h(y,t)|}{\lambda_*^\nu(1+|y|)^{-a}}, \tag{3.62}$$

and

$$\|\phi\|_{*,\nu,a,\delta} = \sup_{\mathcal{D}_{2R}} \frac{|\phi(y,\tau)| + (1+|y|)|\nabla_y\phi(y,\tau)|}{\lambda_*^\nu \max(\frac{R^{\delta(5-a)}}{(1+|y|)^3}, \frac{1}{(1+|y|)^{a-2}})}. \tag{3.63}$$

Proposition 3.1. *Let $a\in(2,3)$, $\delta\in(0,1)$, $\nu>0$. Assume $\|h\|_{\nu,a}<\infty$. Then there is a solution $\phi=\mathcal{T}_{\lambda,1}[h]$, $\tilde c_{0j}[h]$ of*

$$\begin{cases} \lambda^2\partial_t\phi = L_W[\phi] + h - \displaystyle\sum_{j=1,2}\tilde c_{0j}[h]Z_{0j}\chi_{B_1} - \sum_{\substack{l=-1,1\\ j=1,2}} c_{lj}[h]Z_{lj}\chi_{B_1} \quad \text{in } \mathcal{D}_{2R} \\ \phi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi(\cdot,0)=0 \quad \text{in } B_{2R(0)} \end{cases}$$

where c_{lj} is defined in (3.61), which is linear in h, such that

$$\|\phi\|_{*,\nu,a,\delta} \le C\|h\|_{\nu,a}$$

and such that

$$|c_{0j}[h] - \tilde c_{0j}[h]| \le C\lambda_*^\nu R^{-\frac12\delta(a-2)}\|h\|_{\nu,a}.$$

The function ϕ_2 solves the equation with right hand side $h_2[p,\xi,\Psi^*]$, which is in *mode 1*, a notion that we define next (this is basically motivated by the analysis of section 3.1, where we consider the linearized parabolic equation and use a Fourier decomposition of the right hand side and the solution).

Let $h(y,t)\in\mathbb{R}^3$, be defined in $\mathbb{R}^2\times(0,T)$ or $\mathcal{D}_{2R}$ with $h\cdot W=0$. We say that h is un mode $k\in\mathbb{Z}$ if h has the form

$$h(y,t) = \Re(\tilde h_k(|y|,t)e^{ik\theta})E_1 + \Re(\tilde h_k(|y|,t)e^{ik\theta})E_2,$$

for some complex valued function $\tilde h_k(\rho,t)$.

Consider then

$$\begin{cases} \lambda^2\partial_t\phi = L_W[\phi] + h - \displaystyle\sum_{j=1,2} c_{1j}[h]w_\rho^2 Z_{1j} \quad \text{in } \mathcal{D}_{2R} \\ \phi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi(\cdot,0)=0 \quad \text{in } B_{2R(0)} \end{cases} \tag{3.64}$$

Proposition 3.2. *Let $a\in(2,3)$, $\delta\in(0,1)$, $\nu>0$. Assume that h is in mode 1 and $\|h\|_{\nu,a}<\infty$. Then there is a solution $\phi=\mathcal{T}_{\lambda,2}[h]$ of (3.64), which is linear in h, such that*

$$\|\phi\|_{\nu,a-2} \le C\|h\|_{\nu,a}.$$

In the above statemen the norm $\|\phi\|_{\nu,a-2}$ analogous to the one in (3.62), but the supremum is taken in $\mathcal{D}_{2R}$.

Another piece of the inner solution, ϕ_3, will handle $h_3[p,\xi,\Psi^*]$, which does not satisfy orthogonality conditions in mode 0. We will still project it to satisfy the orthogonality condition in mode 1. Let us consider then (3.64) without any orthogonality conditions on h in mode 0. We define

$$\|\phi\|_{**,\nu} = \sup_{\mathcal{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)\,|\nabla_y\phi(y,t)|}{\lambda_*(t)^\nu R(t)^2(1+|y|)^{-1}}. \tag{3.65}$$

Proposition 3.3. *Let $1<a<3$ and $\nu>0$. There exists a $C>0$ such that if $\|h\|_{a,\nu}<+\infty$ there is a solution $\phi=\mathcal{T}_{\lambda,3}[h]$ of (3.64), which is linear in h and satisfies the estimate*

$$\|\phi\|_{**,\nu} \ \le\ C\|h\|_{a,\nu}.$$

Note that we allow a to be less than 2 in the previous proposition.

Next we have a variant of Proposition 3.3 when h is in mode -1.

Proposition 3.4. *Let $2 < a < 3$ and $\nu > 0$. There exists a $C > 0$ such that for any h in mode -1 with $\|h\|_{a,\nu} < +\infty$, there is a solution $\phi = \mathcal{T}_{\lambda,4}[h]$ of problem (3.64), which is linear in h and satisfies the estimate*

$$\|\phi\|_{***,\nu} \le C\|h\|_{a,\nu},$$

where

$$\|\phi\|_{***,\nu} = \sup_{\mathcal{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)\,|\nabla_y \phi(y,t)|}{\lambda_*(t)^\nu \log(R(t))}.$$

All propositions stated here are corollaries of Proposition 3.11 and proved in section 3.1.

• **The equations for $p = \lambda e^{i\omega}$.** We need to choose the free parameters p, ξ so that $c_{lj}[h(p,\xi,\Psi^*)] = 0$ for $l = -1, 0, 1$, $j = 1, 2$. This will be easy to do for $l = 1$ (mode 1), but mode $l = 0$ is more complicated.

To handle c_{0j} we note that by definitions (3.38), (3.41), (3.44)

$$c_{0,j}[h(p,\xi,\Psi^*)] = \frac{2\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} \left(\mathcal{B}_{0j}[p] - a_{0j}[p,\xi,\Psi^*]\right)$$

where B_0, a_0 are defined in (3.43), (3.44) and we recall that $p = \lambda e^{i\omega}$.

So to achieve $c_{0j}[h(p,\xi,\Psi^*)] = 0$ we should solve

$$\mathcal{B}_0[p](t) = a_0[p,\xi,\Psi^*](t), \quad t \in [0,T], \tag{3.66}$$

adjusting the parameters $\lambda(t)$ and $\omega(t)$. This equation is delicate and we will instead impose a modified version of this condition. The modification of (3.66) consists in introducing another term in the equation, essentially modifying the operator $\mathcal{B}_0$.

To make this precise we define the following norms. Let I denote either the interval $[0,T]$ or $[-T,T]$. For $\Theta \in (0,1)$, $l \in \mathbb{R}$ and a continuous function $g : I \to \mathbb{C}$ we let

$$\|g\|_{\Theta,l} = \sup_{t\in I} (T-t)^{-\Theta} |\log(T-t)|^l |g(t)|, \tag{3.67}$$

and for $\gamma \in (0,1)$, $m \in (0,\infty)$, and $l \in \mathbb{R}$ we let

$$[g]_{\gamma,m,l} = \sup\, (T-t)^{-m} |\log(T-t)|^l \frac{|g(t)-g(s)|}{(t-s)^\gamma}, \tag{3.68}$$

where the supremum is taken over $s \le t$ in I such that $t - s \le \frac{1}{10}(T-t)$.

We have then the following result.

Proposition 3.5. *Let $\alpha, \gamma \in (0, \frac{1}{2})$, $l \in \mathbb{R}$, $C_1 > 1$. There is $\alpha_0 > 0$ such that if $\Theta \in (0, \alpha_0)$ and $m \le \Theta - \gamma$, then for $a : [0,T] \to \mathbb{C}$ is such that*

$$\mathrm{Re}(a(T)) < 0, \quad \text{with} \quad \frac{1}{C_1} \le \mathrm{Re}(a(T)) \le C_1$$

and

$$T^\Theta |\log T|^{1+\sigma-l} \|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \le C_1,$$

for some $\sigma > 0$, then, for $T > 0$ small enough there are two operators $\mathcal{P}$ and $\mathcal{R}_0$ so that $p = \mathcal{P}[a] : [-T,T] \to \mathbb{C}$ satisfies

$$\mathcal{B}_0[p](t) = a(t) + \mathcal{R}_0[a](t), \quad t \in [0,T], \tag{3.69}$$

with

$$\begin{aligned} &|\mathcal{R}_0[a](t)| \\ &\le C\Big(T^\sigma + T^\Theta \frac{\log|\log T|}{|\log T|} \|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}\Big) \frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}, \end{aligned} \tag{3.70}$$

for some $\sigma > 0$.

We have additional properties of the solution to this problem.

Proposition 3.6. *Let us make the same assumptions as in Proposition 3.5. Then* $\mathcal{P}[a]$ *can be written as*

$$\mathcal{P}[a] = p_{0,\kappa[a]} + \mathcal{P}_1[a] + \mathcal{P}_2[a]$$

where $p_{0,\kappa}$ *is defined by*

$$p_{0,\kappa}(t) = \kappa|\log T| \int_t^T \frac{1}{|\log(T-s)|^2}\, ds, \quad t \le T, \tag{3.71}$$

and each term

$$\kappa = \kappa[a], \quad p_1 = \mathcal{P}_1[a], \quad p_2 = \mathcal{P}_2[a],$$

has the following bounds:

$$\kappa = |a(T)|(1 + O(\frac{1}{|\log T|}),$$
$$|\dot p_1(t) - \dot p_{0,\kappa}(t)| \le C\frac{|\log T|^{1-\sigma}\log(|\log T|)^2}{|\log(T-t)|^{3-\sigma}},$$
$$|\ddot p_1(t)| \le C\frac{|\log T|}{|\log(T-t)|^3(T-t)},$$

$$\|\dot p_2\|_{\Theta,l} \le C\big(T^{\frac12+\sigma-\Theta} + \|a(\cdot)-a(T)\|_{\Theta,l-1}),$$
$$[\dot p_2]_{\gamma,m,l} \le C(|\log T|^{l-3}T^{\alpha_0-m-\gamma} + T^{\Theta}\frac{\log|\log T|}{|\log T|}\|a(\cdot)-a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}),$$

where $\alpha_0 > 0$ *is some fixed some constant and* $\sigma > 0$ *is arbitrary (with* C *depending on* σ*).*

Roughly speaking, to obtain the modified equation (3.69) we notice that the main term in p in $\mathcal{B}_0[p]$ is the integral operator

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{t-s} ds.$$

Thus we define

$$\tilde{\mathcal{B}}_0[p] = \mathcal{B}_0[p] - \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{t-s} ds.$$

It will be sufficient to solve approximately equations (3.40) replacing in part this integral operator by a "regularized" version of it following the logic of the formal derivation of the rate (3.51). For $\alpha > 0$ let us write

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{t-s} ds = S_\alpha[\dot p] + R_\alpha[\dot p]$$

where

$$S_\alpha[g] := g(t)[-2\log\lambda_*(t) + (1+\alpha)\log(T-t)] + \int_{-T}^{t-(T-t)^{1+\alpha}} \frac{g(s)}{t-s} ds, \tag{3.72}$$

$$R_\alpha[g] := -\int_{t-(T-t)^{1+\alpha}}^{t-\lambda_*^2} \frac{g(t)-g(s)}{t-s} ds. \tag{3.73}$$

Thus equation (3.66) can be written in the form

$$S_\alpha[\dot p] + R_\alpha[\dot p] + \tilde{\mathcal{B}}_0[p] = a(t), \quad \text{in } [0,T],$$

for some function $a(t)$. The modified equation is

$$S_\alpha[\dot p] + \tilde{\mathcal{B}}_0[p] = a(t) \quad \text{in } [0,T],$$

and the remainder $\mathcal{R}_0$ is essentially $R_\alpha[\dot p]$. This is a sketch of how we obtain the modified equation and remainder.

Another modification to equations (3.66) that we introduce is to replace $a_0[p,\xi,\Psi^*]$ by its main term. To do this we write

$$a_0[p,\xi,\Psi] = a_0^{(0)}[p,\xi,\Psi] + a_0^{(1)}[p,\xi,\Psi] + a_0^{(2)}[p,\xi,\Psi]$$

where

$$a_0^{(l)}[p,\xi,\Psi] = -\frac{\lambda}{4\pi}e^{i\omega}\int_{B_{2R}}\left(Q_{-\omega}\tilde{L}_U[\Psi]_l\cdot Z_{01} + iQ_{-\omega}\tilde{L}_U[\Psi]_l\cdot Z_{02}\right)dy$$

for $l = 0,1,2$.

We define

$$c_0^*[p,\xi,\Psi^*](t) := \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2|Z_{01}|^2}e^{-i\omega}\Big(\mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*]\right](t) + a_0^{(1)}[p,\xi,\Psi^*](t) + a_0^{(2)}[p,\xi,\Psi^*](t)\Big) - (c_0[h[p,\xi,\Psi^*]] - \tilde{c}_0[h_1[p,\xi,\Psi^*]]),$$

and

$$c_{01}^* := \mathrm{Re}(c_0^*), \quad c_{02}^* := \mathrm{Im}(c_0^*),$$

where $\mathcal{R}_0$ is the operator given Proposition 3.5 and $\tilde{c}_0 = \tilde{c}_{01} + i\tilde{c}_{02}$ are the operators defined in Proposition 3.1.

• **The system of equations.**

We transform the system (3.33)-(3.34) in the problem of finding functions $\psi(x,t)$, $\phi_1,\ldots,\phi_4$, parameters $p(t) = \lambda(t)e^{i\omega(t)}$, $\xi(t)$ and constants c_1, c_2, c_3 such that the following system is satisfied:

$$\begin{cases} \psi_t = \Delta_x\psi + g(p,\xi,Z^*+\psi,\phi_1+\phi_2+\phi_3+\phi_4) & \text{in } \Omega\times(0,T) \\ \psi = (\mathbf{e}_3 - U) - \Phi^0 & \text{on } \partial\Omega\times(0,T) \\ \psi(\cdot,0) = (c_1\,\mathbf{e_1} + c_2\,\mathbf{e_2} + c_3\,\mathbf{e_3})\chi + (1-\chi)(\mathbf{e}_3 - U - \Phi^0) & \text{in } \Omega \\ \psi(q,T) = -Z^*(q,T) \end{cases} \tag{3.74}$$

$$\begin{cases} \lambda^2\partial_t\phi_1 = L_W[\phi_1] + h_1[p,\xi,\Psi^*] - \sum_{j=1,2}\tilde{c}_{0j}[h_1[p,\xi,\Psi^*]]w_\rho^2 Z_{0j} \\ \qquad - \sum_{\substack{l=-1,1\\ j=1,2}} c_{lj}[h_1[p,\xi,\Psi^*]]w_\rho^2 Z_{lj} \quad \text{in } \mathcal{D}_{2R} \\ \phi_1\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi_1(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{3.75}$$

$$\begin{cases} \lambda^2\partial_t\phi_2 = L_W[\phi_2] + h_2[p,\xi,\Psi^*] - \sum_{j=1,2} c_{1j}[h_2[p,\xi,\Psi^*]]w_\rho^2 Z_{1j} \quad \text{in } \mathcal{D}_{2R} \\ \phi_2\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi_2(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{3.76}$$

$$\begin{cases} \lambda^2\partial_t\phi_3 = L_W[\phi_3] + h_3 - \sum_{j=1,2} c_{1j}[h_3[p,\xi,\Psi^*]]w_\rho^2 Z_{1j} \\ \qquad + \sum_{j=1,2} c_{0j}^*[p,\xi,\Psi^*]w_\rho^2 Z_{0j} \quad \text{in } \mathcal{D}_{2R} \\ \phi_3\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi_3(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{3.77}$$

$$\begin{cases} \lambda^2\partial_t\phi_4 = L_W[\phi_4] + \sum_{j=1,2} c_{-1,j}[h_1[p,\xi,\Psi^*]]w_\rho^2 Z_{-1j} \\ \phi_4\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi_4(\cdot,t) = 0 \quad \text{on } \partial B_{2R(t)} \\ \phi_4(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{3.78}$$

$$c_{0j}[h(p,\xi,\Psi^*)](t) - \tilde{c}_{0j}[p,\xi,\Psi^*](t) = 0 \quad \text{for all} \quad t\in(0,T), \quad j=1,2, \tag{3.79}$$

$$c_{1j}[h(p,\xi,\Psi^*)](t) = 0 \quad \text{for all} \quad t\in(0,T), \quad j=1,2. \tag{3.80}$$

In (3.74) χ is a smooth cut-off function with compact support in Ω which is identically 1 on a fixed neighborhood of q independent of T and the function $g(p,\xi,\Psi^*,\phi)$ is given by (3.35).

We see that if $(\phi_1,\phi_2,\phi_3,\phi_4,\psi,p,\xi)$ satisfies system (3.74)–(3.80) then the functions

$$\phi=\phi_1+\phi_2+\phi_3+\phi_4,\quad \Psi^*=Z^*+\psi$$

solve the outer-inner gluing system (3.33)–(3.34).

The way in which we will proceed to solve the full problem (3.74)–(3.80) is the following. For given functions $\phi_1,\ldots,\phi_4$ and parameters p, ξ in a suitable class, we solve first the outer problem (3.74) in the form of an operator $\psi=\Psi[\phi_1+\phi_2+\phi_3+\phi_4,p,\xi]$ and denote $\Psi^*[\phi_1+\phi_2+\phi_3,p,\xi]=Z^*+\Psi[\phi_1+\phi_2+\phi_3+\phi_4,p,\xi]$. Then we substitute $\Psi^*[\phi_1+\phi_2+\phi_3+\phi_4,p,\xi]$ in (3.75)–(3.78) and solve for ϕ_1, ϕ_2, ϕ_3, ϕ_4 as operators of the pair (p,ξ). Finally, we solve for p and ξ the remaining equations. All this will be done by suitable control on the linear parts of the equation and contraction mapping principle.

• **The outer problem.** Our main result for problem (3.74) is the existence of a small solution for all small T, with certain precise absolute and Lipschitz estimates satisfied. To obtain this result we need a suitable norm that we define next.

Given $\Theta>0$, $\gamma\in(0,\frac12)$ we define

$$\begin{aligned}\|\psi\|_{\sharp,\Theta,\gamma}:=&\ \lambda_*(0)^{-\Theta}\frac{1}{|\log T|\lambda_*(0)R(0)}\|\psi\|_{L^\infty(\Omega\times(0,T))}+\lambda_*(0)^{-\Theta}\|\nabla_x\psi\|_{L^\infty(\Omega\times(0,T))}\\&+\sup_{\Omega\times(0,T)}\lambda_*(t)^{-\Theta-1}R(t)^{-1}\frac{1}{|\log(T-t)|}|\psi(x,t)-\psi(x,T)|\\&+\sup_{\Omega\times(0,T)}\lambda_*(t)^{-\Theta}|\nabla_x\psi(x,t)-\nabla_x\psi(x,T)|\\&+\sup\lambda_*(t)^{-\Theta}(\lambda_*(t)R(t))^{2\gamma}\frac{|\nabla_x\psi(x,t)-\nabla_x\psi(x',t')|}{(|x-x'|^2+|t-t'|)^\gamma},\end{aligned}\tag{3.81}$$

where the last supremum in taken in the region

$$x,x'\in\Omega,\quad t,t'\in(0,T),\quad |x-x'|\le 2\lambda_*R(t),\quad |t-t'|<\frac14(T-t).$$

• **Choice of constants.** We explain here the constants used in the different norms and the values we choose for them.

- $\beta\in(0,\frac12)$ is so that $R(t)=\lambda_*(t)^{-\beta}$.
- $\alpha\in(0,\frac12)$ appears in Proposition 3.5. It is the parameter used to define the remainder $\mathcal{R}_\alpha$ in (3.73).
- We use the norm $\|\ \|_{*,\nu_1,a_1,\delta}$ (3.63) to measure the solution ϕ_1 in (3.75). Here we will ask that $\nu_1\in(0,1)$, $a_1\in(2,3)$, and $\delta>0$ small and fixed.
- We use the norm $\|\ \|_{\nu_2,a_2-2}$ (3.62) to measure the solution ϕ_2 in (3.76), with $\nu_2\in(0,1)$, $a_2\in(2,3)$.
- We use the norm $\|\ \|_{**,\nu_3}$ (3.65) for the solution ϕ_3 of (3.77), with $\nu_3>0$.
- We use the norm $\|\ \|_{***,\nu_4}$ for the solution ϕ_4 of (3.78), with $\nu_4>0$.
- We are going to use the norm $\|\ \|_{\sharp,\Theta,\gamma}$ with a parameters Θ, γ satisfying some restrictions given below.
- We have parameters m, l in Proposition 3.5. We work with m given by $m:=\Theta-2\gamma(1-\beta)$ and l satisfying $l<1+2m$.

We will need a series of additional inequalities satisfied by these parameters ensuring the gluing procedure. See [22].

• **The outer problem.** The next proposition gives a solution to the outer problem (3.74). To state this result we define the spaces

$$\begin{aligned}E_1&=\{\phi_1\in L^\infty(\mathcal{D}_{2R}):\nabla_y\phi_1\in L^\infty(\mathcal{D}_{2R}),\ \|\phi_1\|_{*,\nu_1,a_1,\delta}<\infty\}\\E_2&=\{\phi_2\in L^\infty(\mathcal{D}_{2R}):\nabla_y\phi_2\in L^\infty(\mathcal{D}_{2R}),\ \|\phi_2\|_{\nu_2,a_2}<\infty\}\\E_3&=\{\phi_3\in L^\infty(\mathcal{D}_{2R}):\nabla_y\phi_3\in L^\infty(\mathcal{D}_{2R}),\ \|\phi_3\|_{**,\nu_3}<\infty\}\\E_4&=\{\phi_4\in L^\infty(\mathcal{D}_{2R}):\nabla_y\phi_4\in L^\infty(\mathcal{D}_{2R}),\ \|\phi_4\|_{***,\nu_4}<\infty\}\end{aligned}$$

and use the notation

$$E=E_1\times E_2\times E_3\times E_4,$$

$$\Phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in E$$
$$\|\Phi\|_E = \|\phi_1\|_{*,\nu_1,a_1,\delta} + \|\phi_2\|_{\nu_2,a_2-2} + \|\phi_3\|_{**,\nu_3} + \|\phi_4\|_{***,\nu_4}$$

We define the closed ball
$$\mathcal{B} = \{\Phi \in X : \|\Phi\|_E \le 1\}.$$

Proposition 3.7. *Assume Z_0^* satisfies (3.60). Let $p(t) = \lambda(t)e^{i\omega(t)}$ and $\xi(t)$ satisfy estimates (3.52), (3.53), $\Phi \in \mathcal{B}$. Then there exists $C > 0$ such that if $T > 0$ is sufficiently small then there exists a solution $\psi = \Psi(p, \xi, \Phi, Z_0^*)$ to equation (3.74) such that*
$$\|\Psi(p, \xi, \Phi, Z_0^*)\|_{\sharp,\Theta,\gamma} \le CT^\sigma(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*).$$

The operator $\Psi(p, \xi, \Phi, Z_0^*)$ satisfies Lipschitz properties, which are consequence of its construction.

Corollary 3.1. *Let $\Psi(p, \xi, \Phi, Z_0^*)$ be the solution to the outer problem constructed in Proposition 3.7. Let p_l, ξ satisfy (3.52), (3.53) and $p_l = \lambda e^{i\omega_l}$, $\|\Phi_l\|_E \le 1$, and $\|Z_{0l}^*\|_* < \infty$, $l = 1, 2$. Then*
$$\|\Psi(p_1, \xi, \Phi_1, Z_{01}^*) - \Psi(p_2, \xi, \Phi_2, Z_{02}^*)\|_{\sharp,\Theta,\gamma}$$
$$\le CT^\sigma(\|\Phi_1 - \Phi_2\|_E + \|\lambda_*(\dot\omega_1 - \dot\omega_2)\|_\infty + \|Z_{01}^* - Z_{02}^*\|_*).$$

Corollary 3.1 gives a partial Lipschitz property of the exterior solution $\Psi(p, \xi, \phi)$ with respect to p, namely it only considers variations of $p = \lambda e^{i\omega}$ with respect to ω. We will need Lipschitz estimates for variations of $p = \lambda e^{i\omega}$ in λ and also variations with respect to ξ. These estimates are obtained for $\Psi(p, \xi, \phi)$ when considered as a function of the inner variable $(y, t) \in \mathcal{D}_{2R}$.

For this let us introduce some notation. Suppose that $\psi(x, t)$ is defined in $\Omega \times (0, T)$. We let
$$\tilde\psi(y, t) = \psi(\xi(t) + \lambda(t)y, t), \quad (y, t) \in \mathcal{D}_{2R}.$$
The following expression is $\|\psi\|_{\sharp,\Theta,\gamma}$ expressed in terms of $\tilde\psi$ (and restricted to $\mathcal{D}_{2R}$):
$$\begin{aligned}
\|\tilde\psi\|_{\sharp\sharp,\Theta,\gamma} := {} & \lambda_*(0)^{-\Theta}\frac{1}{|\log T|\lambda_*(0)R(0)}\|\tilde\psi\|_{L^\infty(\mathcal{D}_{2R})} + \lambda_*(0)^{-\Theta-1}\|\nabla_y\psi\|_{L^\infty(\mathcal{D}_{2R})} \\
& + \sup_{\mathcal{D}_{2R}} \lambda_*(t)^{-\Theta-1}R(t)^{-1}\frac{1}{|\log(T-t)|}|\tilde\psi(y,t) - \tilde\psi(y,T)| \\
& + \sup_{(y,t)\in\mathcal{D}_{2R}} \lambda_*(t)^{-\Theta-1}|\nabla_y\tilde\psi(y,t) - \nabla_y\tilde\psi(y,T)| \\
& + \sup_{(y,t),(y',t)\in\mathcal{D}_{2R}} \lambda_*(t)^{-\Theta-1}R(t)^{2\gamma}\frac{|\nabla_y\tilde\psi(y,t) - \nabla_y\tilde\psi(y',t)|}{|y-y'|^{2\gamma}} \\
& + \sup \lambda_*(t)^{-\Theta-1}(\lambda_*(t)R(t))^{2\gamma}\frac{|\nabla_y\tilde\psi(y,t) - \nabla_y\tilde\psi(x',t')|}{|t-t'|^\gamma},
\end{aligned}$$
where the last supremum is taken in the region
$$(y,t), (y,t'), \in \mathcal{D}_{2R}, \quad |t-t'| \le \frac{1}{10}(T-t).$$

Corollary 3.2. *Let $\Psi(p, \xi, \phi)$ be the outer solution in Proposition 3.7. Let $p_l = \lambda_l e^{i\omega}$, ξ_l satisfy (3.52), (3.53) and $\|\phi\|_{*,a,\nu} \le 1$. Then for $\tilde\Theta \in (0, \Theta)$ we have*
$$\|\tilde\Psi(p_1, \xi_1, \phi) - \tilde\Psi(p_2, \xi_2, \phi)\|_{\sharp\sharp,\tilde\Theta,\gamma}$$
$$\le C\Big[\Big\|\frac{\lambda_1 - \lambda_2}{\lambda_*}\Big\|_{L^\infty} + \|\dot\lambda_1 - \dot\lambda_2\|_{L^\infty} + \Big\|\frac{\xi_1 - \xi_2}{\lambda_* R}\Big\|_{L^\infty} + \Big\|\frac{\dot\xi_1 - \dot\xi_2}{R}\Big\|_{L^\infty}\Big].$$

What we do next is to take $\Phi \in E$ with $\|\Phi\|_E \le 1$ and write $\Psi^*(p, \xi, \Phi, Z_0^*) = Z^* + \Psi(p, \xi, \Phi, Z_0^*)$. We can then write the inner problems as the fixed point problem
$$\Phi = \mathcal{F}(\Phi) \tag{3.82}$$
where
$$\mathcal{F}(\Phi) = (\mathcal{F}_1(\Phi), \mathcal{F}_2(\Phi), \mathcal{F}_3(\Phi)), \quad \mathcal{F} : \bar{\mathcal{B}}_1 \subset E \to E$$

is defined as

$$\mathcal{F}(\Phi) = (\mathcal{F}_1(\Phi), \mathcal{F}_2(\Phi), \mathcal{F}_3(\Phi), \mathcal{F}_4(\Phi)),$$

$$\begin{aligned}
\mathcal{F}_1(\Phi) &= \mathcal{T}_{\lambda,1}(h_1[p, \xi, \Psi^*(p, \xi, \Phi, Z_0^*)]) \\
\mathcal{F}_2(\Phi) &= \mathcal{T}_{\lambda,2}(h_2[p, \xi, \Psi^*(p, \xi, \Phi, Z_0^*)]) \\
\mathcal{F}_3(\Phi) &= \mathcal{T}_{\lambda,3}\Big(h_3[p, \xi, \Psi^*(p, \xi, \Phi, Z_0^*)] + \sum_{j=1}^{2} c_{0j}^*[p, \xi, \Psi^*(p, \xi, \Phi, Z_0^*)] w_\rho^2 Z_{0j}\Big) \\
\mathcal{F}_4(\Phi) &= \mathcal{T}_{\lambda,4}\Big(\sum_{j=1}^{2} c_{-1,j}[h_1[p, \xi, \Psi^*(p, \xi, \Phi, Z_0^*)]] w_\rho^2 Z_{-1,j}\Big).
\end{aligned}$$

Although $\mathcal{F}$ also depends on p, ξ, Z_0^* we will omit this dependence from the notation for the moment.

Our next step is to solve problem (3.82).

• **The inner problem.**

Proposition 3.8. *Assume that p and ξ satisfy estimates (3.52) and that Z_0^* satisfies (3.60). Then the system of equations (3.82) for $\Phi = (\phi_1, \phi_2, \phi_3)$ has a solution $\Phi(p, \xi, Z_0^*)$ in $\bar{\mathcal{B}}_1 \subset E$.*

Let $\Phi(p, \xi, Z_0^*)$ be the solution of (3.82) constructed in Proposition 3.8. Next we show that the solution $\Phi(p, \xi, Z_0^*)$ is Lipschitz in the parameters p, ξ, Z_0^*.

Proposition 3.9. *Assume that p_1, p_2 and ξ_1, ξ_2 satisfy estimates (3.52) and that $Z_{0,1}^*$, $Z_{0,2}^*$ have the form*

$$Z_{0,l}^* = Z_0^{*0} + Z_{0,l}^{*1}, \quad l = 1, 2,$$

with Z_0^{*0} satisfying (3.57) and

$$\|Z_{0,l}^{*1}\|_* \le T^\sigma,$$

Let us write $p_j = \lambda_j e^{i\omega_j}$ for $j = 1, 2$. for some $\sigma > 0$. Then

$$\begin{aligned}
\|\Phi(p_1, \xi_1, Z_{0,1}^*) - \Phi(p_2, \xi_2, Z_{0,2}^*)\|_E \le \lambda_*(0)^\sigma \Big[&\|\lambda_*(\dot\omega_1 - \dot\omega_2)\|_\infty + \Big\| \frac{\lambda_1 - \lambda_2}{\lambda_*} \Big\|_{L^\infty} \\
&+ \|\dot\lambda_1 - \dot\lambda_2\|_{L^\infty} + \Big\| \frac{\xi_1 - \xi_2}{\lambda_* R} \Big\|_{L^\infty} + \Big\| \frac{\dot\xi_1 - \dot\xi_2}{R} \Big\|_{L^\infty} \\
&+ \|Z_{0,1}^{*1} - Z_{0,2}^{*1}\|_* \Big],
\end{aligned}$$

for some possibly smaller $\sigma > 0$.

With this we can now state the following result. Let $\Phi(p, \xi, Z_0^*)$ denote the solution of (3.82) constructed in Proposition 3.8.

Proposition 3.10. *Given Z_0^* of the form (3.60) there exists $p = \lambda e^{i\omega}$ and ξ such that (3.79) and (3.80) are satisfied.*

The proposition above yields the existence of a blow-up solution.

• **Linear theory for the inner problem.**

At the very heart of capturing the bubbling structure is the construction of an inverse for the linearized heat operator around the basic harmonic map. We consider the linear equation

$$\begin{aligned}
\lambda^2 \partial_t \phi &= L_W[\phi] + h(y, t) \quad \text{in } \mathcal{D}_{2R} \\
\phi(\cdot, 0) &= 0 \quad \text{in } B_{2R(0)} \\
\phi \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R}
\end{aligned} \tag{3.83}$$

where

$$\mathcal{D}_{2R} = \{(y, t) \ / \ t \in (0, T), \ y \in B_{2R(t)}(0)\}.$$

We assume that $h(y, t)$ is defined for all $(y, t) \in \mathbb{R}^2 \times (0, T)$ and satisfies

$$h \cdot W = 0, \quad |h(y, t)| \le C \frac{\lambda_*^\nu}{(1 + |y|)^a},$$

where $\nu > 0$ and $a \in (2, 3)$ (so that $\|h\|_{a,\nu} < \infty$).

The parameter R is given by (3.54), that is $R(t)=\lambda_*(t)^{-\beta}$, $\beta\in(\frac14,\frac12)$. Also, we assume that the parameter function $\lambda(t)$ satisfies we have that

$$a\lambda_*(t)\le\lambda(t)\le b\lambda_*(t)\quad\text{for all}\quad t\in(0,T)$$

for some positive numbers a,b,c independent of T.

We observe that a priori we are not imposing boundary conditions in problem (3.83). Our purpose is to construct a solution ϕ that defines a linear operator of h and satisfies uniform bounds in terms of suitable norms.

All functions $h(y,t)$ with $h(y,t)\cdot W(y)\equiv0$ can be expressed in polar form as

$$h(y,t)\;=\;h^1(\rho,\theta,t)E_1(y)+h^2(\rho,\theta,t)E_2(y),\quad y=\rho e^{i\theta}. \tag{3.84}$$

We can also expand in Fourier series

$$\tilde h(\rho,\theta,t):=h^1+ih^2\;=\;\sum_{k=-\infty}^{\infty}\tilde h_k(\rho,t)e^{ik\theta},\quad \tilde h_k=\tilde h_{k1}+i\tilde h_{k2} \tag{3.85}$$

so that

$$h(y,t)=\sum_{k=-\infty}^{\infty}h_k(y,t)=:h_0(y,t)+h_1(y,t)+h_{-1}(y,t)+h^{\perp}(y,t), \tag{3.86}$$

where

$$h_k(y,t)=\mathrm{Re}\,(\tilde h_k(\rho,t)e^{ik\theta})\,E_1+\mathrm{Im}\,(\tilde h_k(\rho,t)e^{ik\theta})\,E_2. \tag{3.87}$$

We consider the functions $Z_{kj}(y)$ defined in (3.10) and (3.11) and define for $k=-1,0,1$,

$$\bar h_k(y,t):=\sum_{j=1}^{2}\frac{\chi Z_{kj}(y)}{\int_{\mathbb R^2}\chi|Z_{kj}|^2}\int_{\mathbb R^2}h(x,t)\cdot Z_{kj}(z)\,dz,$$

where

$$\chi(y,t)=\begin{cases}w_\rho^2(|y|) & \text{if } |y|<2R(t),\\ 0 & \text{if } |y|\ge 2R(t).\end{cases}$$

The main result in this section is the following, where we use the norm $\|h\|_{a,\nu}$ defined in (3.62).

Proposition 3.11. *Let $2<a<3$, $\nu>0$ and let h with $\|h\|_{a,\nu}<+\infty$. Let us write $h=h_0+h_1+h_{-1}+h^{\perp}$ with $h^{\perp}=\sum_{k\neq0,\pm1}h_k$. Then there exists a solution $\phi[h]$ of problem (3.83), which defines a linear operator of h, and satisfies the following estimate in $\mathcal D_{2R}$:*

$$\begin{aligned}&(1+|y|)\,|\nabla_y\phi(y,t)|+|\phi(y,t)|\\
&\lesssim\frac{\lambda_*(t)^\nu R(t)^{\frac{5-a}{2}}}{1+|y|}\,\min\{1,R^{\frac{5-a}{2}}|y|^{-2}\}\,\|h_0-\bar h_0\|_{a,\nu}+\frac{\lambda_*(t)^\nu R(t)^2}{1+|y|}\|\bar h_0\|_{a,\nu}\\
&\quad+\frac{\lambda_*(t)^\nu}{1+|y|^{a-2}}\,\big\|h_1-\bar h_1\big\|_{a,\nu}+\frac{\lambda_*(t)^\nu R(t)^4}{1+|y|^2}\,\big\|\bar h_1\big\|_{a,\nu}\\
&\quad+\frac{\lambda_*(t)^\nu R(t)^{\frac{5-a}{2}}}{1+|y|}\,\min\{1,R^{\frac{5-a}{2}}|y|^{-2}\}\,\|h_{-1}-\bar h_{-1}\|_{a,\nu}+\lambda_*(t)^\nu\log R(t)\,\|\bar h_{-1}\|_{a,\nu}\\
&\quad+\frac{\lambda_*(t)^\nu}{1+|y|^{a-2}}\,\|h^{\perp}\|_{a,\nu}.\end{aligned}$$

The construction of the operator $\phi[h]$ as stated in the proposition will be carried out mode by mode in the Fourier series expansion. We shall use the convention that $h(y,t)=0$ for $|y|>2R(t)$. Let us write

$$\phi=\sum_{k=-\infty}^{\infty}\phi_k,\qquad \phi_k(y,t)=\mathrm{Re}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_1+\mathrm{Im}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_2.$$

We shall build a solution of (3.83) by solving separately each of the equations

$$\begin{aligned}\lambda^2\partial_t\phi_k&=L_W[\phi_k]+h_k(y,t)=0\quad\text{in }\mathcal D_{4R},\\
\phi_k(y,0)&=0\quad\text{in }B_{4R(0)}(0),\end{aligned} \tag{3.88}$$

which, are equivalent to the problems

$$\begin{aligned} \lambda^2\partial_t\varphi_k &= \mathcal{L}_k[\varphi_k] + \tilde h_k(\rho,t) \quad \text{in } \tilde D_{4R}, \\ \varphi_k(\rho,0) &= 0 \quad \text{in } (0,4R_0) \end{aligned}$$

with

$$\tilde D_{4R} = \{(\rho,t)\ /\ t\in(0,T),\ \rho\in(0,4R(t))\}$$

and we recall

$$\mathcal{L}_k[\varphi_k] := \partial_\rho^2\varphi_k + \frac{\partial_\rho\varphi_k}{\rho} - (k^2+2k\cos w+\cos(2w))\frac{\varphi_k}{\rho^2}$$

We have the validity of the following result.

Lemma 3.1. *Let $\nu>0$ and $0<a<3$, $a\neq 1,2$. Assume that*

$$\|h_k(y,t)\|_{a,\nu}<+\infty.$$

Then problem (3.88) has a unique bounded solution $\phi_k(y,t)$ of the form

$$\phi_k(y,t) \;=\; \mathrm{Re}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_1 + \mathrm{Im}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_2$$

which in addition satisfies the boundary condition

$$\phi_k(y,t)=0 \quad \textit{for all } t\in(0,T),\quad y\in\partial B_{R(t)}(0). \tag{3.89}$$

These solutions satisfy the estimates

$$|\phi_k(y,t)| \;\le\; C\|h\|_{a,\nu}\,\lambda_*^\nu k^{-2}\left\{\begin{array}{lll} R^{2-a} & \textit{if} & a<2,\\ (1+\rho)^{2-a} & \textit{if} & a>2,\end{array}\right. \quad \textit{if } k\ge 2.$$

$$|\phi_{-1}(y,t)| \;\le\; C\|h\|_{a,\nu}\,\lambda_*^\nu\left\{\begin{array}{lll} R^{2-a} & \textit{if} & a<2,\\ \log R & \textit{if} & a>2,\end{array}\right.$$

$$|\phi_0(y,t)| \;\le\; C\|h\|_{a,\nu}\lambda_*^\nu(1+\rho)^{-1}\left\{\begin{array}{lll} R^{2} & \textit{if} & a>1,\\ R^{3-a} & \textit{if} & a<1,\end{array}\right.$$

$$|\phi_1(y,t)| \;\le\; C\|h\|_{a,\nu}\lambda_*^\nu(1+\rho)^{-2}R^4.$$

with C independent of R and k.

Proof. Standard parabolic theory yields existence of a unique solution to equation (3.88) that satisfies the boundary condition (3.89), for each k. Equivalently, the problem

$$\begin{aligned} \lambda^2\partial_t\varphi_k &= \mathcal{L}_k[\varphi_k] + \tilde h_k(\rho,t) \quad \text{in } \tilde D_{4R}, \\ \varphi_k(t,4R) &= 0 \quad \text{for all} \quad t\in(0,T) \\ \varphi_k(0,\rho) &= 0 \quad \text{in } (0,4R(0)), \end{aligned} \tag{3.90}$$

$$\mathcal{L}_k[\varphi_k] = \partial_\rho^2\varphi_k + \frac{\partial_\rho\varphi_k}{\rho} - (k^2+2k\cos w+\cos(2w))\frac{\varphi_k}{\rho^2}$$

has a unique solution $\varphi_k(\rho,t)$ which is bounded in ρ for each t.

We use barriers to derive the desired estimates. A first observation we make is that for mode $k=-1$ the elliptic equation $\mathcal{L}_{-1}[\varphi]+g(\rho)=0$ in $(0,4R)$ with $\varphi(4R)=0$ has a unique bounded solution given by the variation of parameters formula

$$\varphi(\rho) := Z_{-1}(\rho)\int_\rho^{4R}\frac{dr}{\rho Z_{-1}(r)^2}\int_0^r g(s)Z_{-1}(s)s\,ds, \tag{3.91}$$

$$Z_{-1}(\rho) = -\rho^2 w_\rho = \frac{2\rho^2}{\rho^2+1}.$$

Here we have used that $\mathcal{L}_{-1}[Z_{-1}]=0$. Let us call $\varphi_0(\rho)$ the function in (3.91) with $g(\rho):=2(1+\rho)^{-a}$. We readily estimate

$$|\varphi_0(\rho)| \;\le\; \begin{cases} R^{2-a} & \text{if } a<2,\\ (1+\rho)^{2-a} & \text{if } a>2.\end{cases}.$$

Let us call $\bar\varphi(\rho,t)=\lambda_*(t)^\nu\varphi_0(\rho)$. Then we see that

$$\begin{aligned}-\lambda^2\bar\varphi_t(\rho,t)+\mathcal{L}_{-1}[\bar\varphi(\rho,t)]+\frac{\lambda_*^\nu}{(1+\rho)^a}&\le c\,\lambda_*^{\nu+1}|\dot\lambda_*|\varphi_0(\rho)-\frac{\lambda_*^\nu}{(1+\rho)^a}\\&\le -\lambda_*^\nu(1+\rho)^{-a}\left[1-C\lambda_*R^{2-a}(1+\rho)^a\right]\\&<0\end{aligned}$$

in $\tilde D_{4R}$. Indeed, since $R(t)\ll\lambda_*^{-\frac12}$, the inequality holds provided that T was chosen sufficiently small. Thus for $k=-1$ the barrier $\|h\|_{a,\nu}\,\bar\varphi(\rho,t)$ dominates both, real and imaginary parts of $\varphi_{-1}(\rho,t)$. As a conclusion, we find

$$|\phi_{-1}(y,t)|\ \le\ C\|h\|_{a,\nu}\lambda_*^\nu\begin{cases}R^{2-a}&\text{if } a<2,\\(1+\rho)^{2-a}&\text{if } a>2,\end{cases}\quad\text{in } \mathcal{D}_{4R}.$$

The cases $k=0,1,-2$ can be dealt with in exactly the same manner, by replacing Z_{-1} in Formula (3.91) respectively by the functions

$$Z_0(\rho)=\frac{\rho}{\rho^2+1},\quad Z_1(\rho)=\frac{1}{\rho^2+1},\quad Z_{-2}(\rho)=\frac{\rho^3}{\rho^2+1}.\tag{3.92}$$

The estimates for ϕ_k predicted in the lemma then readily follow for $k=-2,-1,0,1$. Finally, let us now consider k with $|k|\ge2$ and $k\ne-2$ and the function $\bar\varphi(\rho,t)$ as above. Now we find

$$\begin{aligned}-\lambda^2\bar\varphi_t(\rho,t)+\mathcal{L}_k[\bar\varphi(\rho,t)]&\le(\mathcal{L}_k-\mathcal{L}_{-1})[\bar\varphi(\rho,t)]\\&\le -C\lambda_*^\nu(k^2-1+2(k-1))\frac{1}{\rho^2}(1+\rho)^{2-a}\\&< -C(k^2-1+2(k-1))\frac{\lambda_*^\nu}{(1+\rho)^a}\quad\text{in } \tilde{\mathcal{D}}_{4R}.\end{aligned}$$

The latter quantity is negative provided that $|k|\ge2$ and $k\ne-2$ and hence we get the estimate

$$|\phi_k(y,t)|\ \le\ \frac{C}{k^2}\|h\|_{a,\nu}\lambda_*^{-\nu}\begin{cases}R^{2-a}&\text{if } a<2,\\(1+\rho)^{2-a}&\text{if } a>2,\end{cases}\quad\text{in } \mathcal{D}_{4R}.$$

The proof is concluded. ☐

We can get gradient estimates for the solutions built in the above lemma by means of the following result.

Lemma 3.2. *Let ϕ be a solution of the equation*

$$\begin{aligned}&\lambda^2\partial_t\phi=L_W[\phi]+h(y,t)\quad\text{in } \mathcal{D}_{4\gamma R}\\&\phi(\cdot,0)=0\quad\text{in } B_{4\gamma R(0)}.\end{aligned}\tag{3.93}$$

Given numbers a,b,γ, there exists a C such that if for some $M>0$ we have

$$|\phi(y,t)|+(1+|y|)^2|h(y,t)|\le M\,\lambda_*(t)^b(1+|y|)^{-a}\quad\text{in } \mathcal{D}_{4\gamma R},\tag{3.94}$$

then

$$(1+|y|)|\nabla_y\phi(y,t)|\ \le\ C\,M\lambda_*(t)^b(1+|y|)^{-a}\quad\text{in } \mathcal{D}_{3\gamma R}\tag{3.95}$$

and we recall

$$\mathcal{D}_{\gamma R}=\{(y,t)\ /\ |y|<\gamma R(t),\quad t\in(0,T)\}.$$

If in addition we know that ϕ satisfies the boundary condition $\phi(\cdot,t)=0$ on $\partial B_{4\gamma R(t)}$ for all $t\in(0,T)$ then estimate (3.95) holds in the entire region $\mathcal{D}_{4\gamma R}$.

Proof. To prove the gradient estimates, we change the time variable, defining

$$\tau(t)=\int_0^t\frac{ds}{\lambda(s)^2},$$

so that (3.93) becomes in the variables (y,τ)

$$\begin{aligned}&\partial_\tau\phi=L_W[\phi]+h(y,\tau)\quad\text{in } \mathcal{D}_{4\gamma R}\\&\phi(\cdot,0)=0\quad\text{in } B_{4R(0)}\end{aligned}$$

Let $\tau_1 > 0$ and $y_1 \in B_{3\gamma R(\tau_1)}(0)$. Let $\rho = \frac{|y_1|}{5} + 1$ so that $B_\rho(y_1) \subset B_{4\gamma R(\tau_1)}(0)$. Let us define

$$\tilde\phi(z,t) := \phi(y_1 + \rho z, \tau_1 + \rho^2 s), \quad z \in B_1(0), \quad s > -\frac{\tau_1}{\rho^2}.$$

We distinguish two cases. First, when $\tau_1 \geq \rho^2$, we use interior estimates for parabolic equations, while for the case $\tau_1 < \rho^2$, we use estimates for a parabolic equation with initial condition.

Assume $\tau_1 \geq \rho^2$. Then $\tilde\phi(z,s)$ satisfies an equation of the form

$$\tilde\phi_s = \Delta_z \tilde\phi + A\nabla_z\tilde\phi + B\tilde\phi + \tilde h(z,s) \quad \text{in } B_1(0) \times (-1,0]$$

with coefficients $A(z,s)$ and $B(z,s)$ uniformly bounded by $O((1+\rho)^{-2})$ in $B_1(0) \times (-1,0]$ and

$$\tilde h(z,s) = \rho^2 h(y_1 + \rho z, \tau_1 + \rho^2 s).$$

Since $\rho \leq CR(\tau_1)$ and $R(\tau_1)^2 \ll \tau_1$ for τ_1 large we get

$$\lambda_*(\tau_1)^b \lesssim \lambda_*(\tau_1 + \rho^2 s)^b \lesssim \ \lambda_*(\tau_1)^b, \quad s \in (-1,0].$$

Standard parabolic estimates and assumption (3.94) yield

$$\begin{aligned} \|\nabla_z \tilde\phi\|_{L^\infty(B_{\frac14}(0)\times(1,2))} &\lesssim \ \|\tilde\phi\|_{L^\infty B_{\frac12}(0)\times(0,2)} + \|\tilde h\|_{L^\infty(B_{\frac12}(0)\times(0,2))} \\ &\lesssim \ M\,\lambda_*(\tau_1)^b \rho^{2-a}, \end{aligned}$$

so that in particular

$$\rho|\nabla_y \phi(y_1,\tau_1)| \ = \ |\nabla_z \tilde\phi(0,1)| \lesssim M\,\lambda_*(\tau_1)^b\rho^{2-a}.$$

In the case $\tau_1 \geq \rho^2$ the argument is similar, but the equation for $\tilde\phi$ holds in $B_1(0) \times (-\frac{\tau_1}{\rho^2}, 0]$ and has initial condition 0 at $s = -\frac{\tau_1}{\rho^2}$. Finally, for the last assertion we argue in similar way but using boundary rather that interior gradient estimates. □

In addition to estimate (3.95) we have a Hölder gradient estimate which is more natural to express using the variable τ as follows. We denote

$$\mathcal{B}_\ell(y,\tau) = \{(y',\tau') \ / \ |y-y'|^2 + |\tau'-\tau| < \ell^2\}.$$

For a function $g(y,\tau)$, a number $0<\alpha<1$, and a set A we let

$$[g]_{\alpha,A} := \sup\Big\{ \frac{|f(y,\tau) - f(y',\tau')|}{(|y-y'|^2 + |\tau'-\tau|)^{\frac{\alpha}{2}}} \ / \ (y,\tau),\, (y',\tau') \in A \Big\}.$$

Corollary 3.3. *Let ϕ be a solution of the equation (3.93) with $h(y,\tau) = \operatorname{div} H(y,\tau)$. Given $\alpha \in (0,1)$ and constants a, b, γ there is C such that if*

$$|\phi(y,\tau)| + (1+|y|)|H(y,\tau)| + (1+|y|)^{1+\alpha}[H]_{\mathcal{B}_{\ell(y)}(y,\tau)\cap\mathcal{D}_{4\gamma R}} \leq M\,\lambda_*(\tau)^b(1+|y|)^{-a}$$

in $\mathcal{D}_{4\gamma R}$, where $\ell(y) = 1 + \frac{|y|}{4}$, then

$$(1+|y|)|\nabla_y\phi(y,\tau)| + (1+|y|)^{1+\alpha}[\nabla_y\phi]_{\mathcal{B}_{\ell(y)}(y,\tau)\cap\mathcal{D}_{4\gamma R}} \leq C\,M\lambda_*(t)^b(1+|y|)^{-a} \tag{3.96}$$

in $\mathcal{D}_{3\gamma R}$. If in addition we know that ϕ satisfies the boundary condition $\phi(\cdot,t) = 0$ on $\partial B_{4\gamma R(t)}$ for all $t \in (0,T)$ then estimate (3.96) holds in the entire region $\mathcal{D}_{4\gamma R}$.

Our next goal is to construct an inverse for modes $k = -1, 0, 1$ with a better control when subject to a certain solvability condition.

• **Mode $k = 0$.** Let us consider again equation (3.88) for $k=0$ and the functions $Z_{0j}(y)$ defined in (3.10) . We have the following result.

Lemma 3.3. *Let assume that $2 < a < 3$, $k = 0$ and*

$$\int_{\mathbb{R}^2} h_0(y,t)\cdot Z_{0j}(y)\,dy \ = \ 0 \quad \text{for all} \quad t\in[0,T) \tag{3.97}$$

for $j = 1,2$. Then there exist a solution ϕ_0 to equation (3.88) for $k=0$ that defines a linear operator of h_0 and satisfies the estimate in $\mathcal{D}_{3R}$,

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu} R^{\frac{5-a}{2}}\,\lambda_*^\nu (1+|y|)^{-1}\ \min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}\ . \tag{3.98}$$

A central feature of estimate (3.98) is that it matches the size of the solutions obtained in Lemma 3.1 for $k \neq 0, 1$ when $|y| \sim R$.

Proof. We observe that conditions (3.97) can be written as

$$\int_0^{2R} \tilde{h}_0(\rho, t)\, Z_0(\rho)\rho\, d\rho \;=\; 0 \quad \text{for all} \quad \tau \in (0, T). \tag{3.99}$$

Let us consider the complex valued functions

$$\tilde{H}_0(\rho, t) := -Z_0(\rho) \int_\rho^\infty \frac{1}{sZ_0(s)^2} \int_s^\infty \tilde{h}_0(\zeta, t) Z_0(\zeta)\zeta\, d\zeta, \quad k = 0, 1.$$

They are well-defined thanks to (3.99). Then the function

$$H_0(y, t) := \mathrm{Re}\,(\tilde{H}_0(\rho, \tau))E_1(y) + \mathrm{Re}\,(\tilde{H}_0(\rho, t))E_2(y)$$

solves

$$L_W[H_0(y, \tau)] = h_0(y, \tau) \quad \text{in } \mathcal{D}_{4R}$$

and satisfies

$$|H_0(y, t)| \;\lesssim\; \lambda_*(t)^\nu (1 + |y|)^{2-a} \|h_0\|_{a,\nu} \quad \text{in } \mathcal{D}_{4R}.$$

Moreover, elliptic gradient estimates yield

$$|\nabla_y H_0(y, \tau)| \lesssim \lambda_*(t)^\nu (1 + |y|)^{1-a} \|h_0\|_{a,\nu} \quad \text{in } \mathcal{D}_{3R}.$$

Let us consider the problem

$$\begin{aligned} \lambda^2 \Phi_t &= L_W[\Phi] + H_0(y, t) \quad \text{in } \mathcal{D}_{4R}, \\ \Phi(y, 0) &= 0 \quad \text{in } B_{4R}(0) \\ \Phi(y, t) &= 0 \quad \text{for all } t \in (0, T), \quad y \in \partial B_{4R(0)}(0) \end{aligned} \tag{3.100}$$

According to Lemma 3.1, this problem has unique solution $\Phi = \Phi_0$ that satisfies the estimates

$$|\Phi_0(y, t)| \;\leq C \|H_0\|_{a-2,\nu} \lambda_*(\tau)^\nu (1 + |y|)^{-1} \;\; R^{5-a} \quad \text{in } \mathcal{D}_{4R}.$$

Applying Lemma 3.2 we deduce that, also,

$$|\nabla_y \Phi_0(y, t)| \lesssim \|H_0\|_{a-2,\nu} \lambda_*(\tau)^\nu (1 + |y|)^{-2} \;\; R^{5-a} \quad \text{in } \mathcal{D}_{3R}$$

Let us write

$$\Phi_{0j} := \partial_{y_j} \Phi_0, \quad H_{0j} := \partial_{y_j} H_0$$

Then we have

$$\begin{aligned} \lambda^2 \partial_t \Phi_{0j} &= L_W[\Phi_{0j}] + \partial_{y_j} |\nabla W|^2 \Phi_0 + 2\nabla \partial_{y_j} W \nabla \Phi_0 + H_{0j}(y, \tau) \\ &\quad + 2(\nabla \Phi_0 \partial_{y_j} \nabla W)W + 2(\nabla \Phi_0 \nabla W)\partial_{y_j} W \quad \text{in } \mathcal{D}_{3R}, \\ \Phi_{0j}(y, 0) &= 0 \quad \text{for all } y \in B_{3R(0)}(0) \end{aligned}$$

According to Lemma 3.2 and the above estimates we obtain that

$$\begin{aligned} (1 + |y|)|\nabla \Phi_{0j}(y, t)| &\lesssim \|h_0\|_{a,\nu} \lambda_*(t)^\nu (1 + |y|)^{-2} R^{5-a} \\ &\quad + \|h_0\|_{a,\nu} \lambda_*(t)^\nu (1 + |y|)^{4-a} \quad \text{in } \mathcal{D}_{3R}. \end{aligned}$$

Then we define

$$\phi_0 := L_W[\Phi_0]$$

so that $\phi = \phi_0$ solves

$$\begin{aligned} \lambda^2 \phi_t &= L_W[\phi] + h_0(y, t) \quad \text{in } \mathcal{D}_{3R}, \\ \phi(y, 0) &= 0 \quad \text{for all} \quad y \in B_{3R(0)}(0) \end{aligned}$$

and defines a linear operator of the function h_0. Moreover, observing that

$$|L_W[\Phi_0]| \lesssim \left|D_y^2 \Phi_0\right| + O(\rho^{-4})\, |\Phi_0| + O(\rho^{-2})\, |D_y \Phi_0|$$

we then get the estimate

$$|\phi_0(y, t)| \lesssim \|h_0\|_{a,\nu} R^{5-a} \lambda_*(t)^\nu (1 + |y|)^{-3}\,. \tag{3.101}$$

To complete the proof of estimate (3.98), we let φ_0 be the complex valued function defined as

$$\phi_0(y, t) \;=\; \mathrm{Re}\,(\varphi_0(\rho, t))\, E_1 + \mathrm{Im}\,(\varphi_0(\rho, t))\, E_2$$

so that letting $R' = R^{\frac{5-a}{4}} \ll R$, using the notation in (3.90), φ_0 satisfies the equation

$$\lambda^2 \partial_t \varphi_0 = \mathcal{L}_0[\varphi_0] + \tilde{h}_0(\rho, t) \quad \text{in } \tilde{D}_{R'}, \qquad (3.102)$$
$$\varphi_0(0,\rho) = 0 \quad \text{in } (0, R'),$$

and from (3.101), we can find an explicit supersolution for the real and imaginary parts of equation (3.102), which also dominates their boundary values at R', which yields

$$|\varphi_0(y,t)| \lesssim \|h_0\|_{a,\nu} \lambda_*^\nu \, |R'|^2 (1+|y|)^{-1}, \quad |y| < R'.$$

Combining this estimate and (3.101) yields the validity of (3.98). □

We mention next a variant of Lemma 3.3, in which we weaken the hypothesis on the right hand side, allowing it to be a divergence of Hölder continuous function. This will be needed when analyzing estimates of the derivative with respect to λ of operator $\mathcal{T}_{\lambda,2}$ (Proposition 3.3).

Lemma 3.4. *Let assume that $2 < a < 3$, $\nu > 0$, and $k = 0$. Let h_0 have the form*

$$h_0(y,\tau) = \operatorname{div} H_0(y,\tau)$$

such that

$$(1+|y|)|H_0(y,\tau)| + (1+|y|)^{1+\alpha}[H_0]_{\mathcal{B}_{\ell(y)}(y,\tau)\cap\mathcal{D}_{4R}} \le \lambda_*(\tau)^\nu (1+|y|)^{-a},$$

in $\mathcal{D}_{4R}$, where $\alpha \in (0,1)$ and $\ell(y) = 1 + \frac{|y|}{4}$. Assume also that

$$\int_{\mathbb{R}^2} h_0(y,t)\cdot Z_{0j}(y)\, dy \;=\; 0 \quad \text{for all} \quad t \in [0,T)$$

for $j = 1,2$. Then there exist a solution ϕ_0 to equation (3.88) for $k = 0$ that defines a linear operator of h_0 and satisfies

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu} R^{\frac{5-a}{2}} \lambda_0^\nu (1+|y|)^{-1} \min\{1, R^{\frac{5-a}{2}}|y|^{-2}\},$$

in $\mathcal{D}_{3R}$.

• **Mode** $k = -1$**.** Let us consider equation (3.88) for $k = -1$ and the functions $Z_{-1j}(y)$ defined in (3.11) . We have the following result.

Lemma 3.5. *Let assume that $2 < a < 3$, $k = 0$ and*

$$\int_{\mathbb{R}^2} h_{-1}(y,t)\cdot Z_{-1j}(y)\, dy = 0 \quad \text{for all} \quad t \in [0,T)$$

for $j = 1,2$. Then there exist a solution ϕ_{-1} to equation (3.88) for $k = -1$ that defines a linear operator of h_0 and satisfies the estimate in $\mathcal{D}_{3R}$,

$$|\phi_{-1}(y,t)| \lesssim \|h_{-1}\|_{a,\nu} \lambda_*^\nu \min\{\log R, R^{4-a}|y|^{-2}\}.$$

Proof. The proof is essentially the same as that of Lemma 3.3. □

• **Mode** $k = 1$**.** Now we deal with (3.88) for $k = 1$. For convenience we give the result for a right hand side more general than strictly need for the proof of Proposition 3.11. Let us assume that h_1 is defined in entire $\mathbb{R}^2 \times (0,T)$ and that

$$h_1(y,t) = \operatorname{div}_y G(y,t) \qquad (3.103)$$

where

$$|G(y,t)| \le \frac{\lambda_*(t)^\nu}{1+|y|^{a-1}}, \quad y \in \mathbb{R}^2, \ t \in (0,T), \qquad (3.104)$$

for some $\nu > 0$, $a \in (2,3)$. Then the following result holds.

Lemma 3.6. *Let assume that $2 < a < 3$, $k = 1$, h_1 has the form (3.103) so that (3.104) holds and*

$$\int_{\mathbb{R}^2} h_1(y,t)\cdot Z_1^j(y)\, dy \;=\; 0 \quad \text{for all} \quad t \in (0,T)$$

for $j = 1,2$. Then there exist a solution ϕ_1 to equation (3.88) for $k = 1$ that defines a linear operator of h_1 and satisfies the estimate in $\mathcal{D}_{3R}$,

$$|\phi_1(y,t)| \lesssim \lambda_*(t)^\nu (1+|y|)^{2-a}.$$

From this we get directly the next result.

Corollary 3.4. *Let assume that* $2 < a < 3$, $k = 1$ *and*

$$\int_{B_{2R}} h_1(y,t) \cdot Z_1^j(y)\, dy \;=\; 0 \quad \textit{for all} \quad t \in (0,T)$$

for $j = 1, 2$. *Then there exist a solution* ϕ_1 *to equation (3.88) for* $k = 1$ *that defines a linear operator of* h_1 *and satisfies the estimate in* $\mathcal{D}_{3R}$,

$$|\phi_1(y,t)| \;\lesssim\; \|h_1\|_{a,\nu} \lambda_*(t)^{\nu} (1+|y|)^{2-a} \;.$$

Let us do the same change of the time variable so that (3.88) for $k = 1$ in entire $\mathbb{R}^2$ becomes in the variables (y, τ)

$$\begin{aligned} \partial_\tau \phi \;&=\; L_W[\phi] + h \quad \text{in } \mathbb{R}^2 \times (0,\infty), \\ \phi(\cdot, 0) \;&=\; 0 \quad \text{in } \mathbb{R}^2. \end{aligned} \tag{3.105}$$

Thus, we consider a function $h(y, \tau)$ defined in entire $\mathbb{R}^2 \times (0, +\infty)$ of the form

$$h = \operatorname{Re}(\tilde{h} e^{i\theta})\, E_1 + \operatorname{Im}(\tilde{h} e^{i\theta})\, E_2, \tag{3.106}$$

that satisfies the orthogonality conditions for $j = 1, 2$

$$\int_{\mathbb{R}^2} h(\cdot, \tau) \cdot Z_{1j} \;=\; 0 \quad \text{for all} \quad \tau \in (0,\infty) \tag{3.107}$$

and such that $h(y, \tau) = 0$ for $|y| \ge 2R(\tau)$.

By standard parabolic theory, this problem has a unique solution, which is therefore of the form

$$\phi = \operatorname{Re}(\varphi e^{i\theta})\, E_1 + \operatorname{Im}(\varphi e^{i\theta})\, E_2, \tag{3.108}$$

where the complex valued function $\varphi(\rho, \tau)$ solves the initial value problem

$$\begin{aligned} \partial_\tau \varphi \;&= \mathcal{L}_1[\varphi] + \tilde{h}(\rho, \tau) \quad \text{in } (0,\infty) \times (0,\infty), \\ \varphi(\rho, 0) \;&= 0 \quad \text{in } (0,\infty), \end{aligned} \tag{3.109}$$

$$\mathcal{L}_1[\varphi] = \partial_\rho^2 \varphi + \frac{\partial_\rho \varphi}{\rho} - (1 + 2\cos w + \cos(2w)) \frac{\varphi}{\rho^2}.$$

We have the validity of the following result.

Lemma 3.7. *Let* $0 < \sigma < 1$, $\nu > 0$. *Assume that* h *is mode 1, that is, has the form (3.106), satisfies the orthogonality conditions (3.107), and can be written as in* (3.103) *with* g_j *satisfying* (3.104) *where* $b = 1 + \sigma$. *Then there exists a constant* $C > 0$ *such that the solution* ϕ *of problem (3.105) satisfies the estimate*

$$|\phi(y,t)| \le C \frac{\lambda_*(t)^{\nu}}{1 + |y|^{\sigma}}. \tag{3.110}$$

The proof of this result is done by the blow-up argument together with a Liouville theorem, a non-degeneracy of the blow-up profile, and the convolution estimates in order to get a contradiction. See Appendix B.

Proof of Lemma 3.6. We take h to be the extension as zero of the function h_1 as in the statement of the lemma. Then we let ϕ be the unique solution of the initial value problem (3.105), which clearly defines a linear operator of h_1. From Lemma 3.7, expressing the resulting estimate in the variables (y, t), we have that for any $t_1 \in (0, T)$

$$|\phi(y,t)| \le C \lambda_*(t)^{\nu} (1+|y|)^{-\sigma} \|h\|_{2+\sigma, t_1} \quad \text{for all} \quad t \in (0, t_1), \quad y \in \mathbb{R}^2.$$

Then letting $\phi_1 := \phi\big|_{\mathcal{D}_{3R}}$ and letting $t_1 \uparrow T$ the result follows. □

Combining the estimates for all the modes readily yields the validity of Proposition 3.11.

• **Modified theory for mode 0 (re-gluing).**

We will perform another gluing procedure that we refer to as the re-gluing process, for mode 0 to further refine the estimates. The re-gluing is not limited to this specific mode and can be in fact applied to a wider class of operators. For a more general setting, see Appendix D.

Let us consider the problem

$$\begin{cases} \varphi_\tau = L_W\varphi + h(y,\tau) + \sum_{j=1,2} \tilde c_{0j} Z_{0j} w_\rho^2 & \text{in } \mathcal{D}_{2R} \\ \varphi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \varphi = 0 \quad \text{on } \partial B_{2R}\times(\tau_0,\infty) \\ \varphi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R(\tau_0)}, \end{cases} \tag{3.111}$$

in mode 0. We work with the norm $\| \ \|_{\nu,b}$ defined in (3.62).

Proposition 3.12. *Let $\sigma\in(0,1)$, $\delta\in(0,1)$, $\nu>0$. Assume $\|h\|_{\nu,2+\sigma}<\infty$. Then there is a solution ϕ, $\tilde c_{0j}$ of (3.111), which is linear in h, such that*

$$|\varphi(y,\tau)| + (1+|y|)|\nabla_y\varphi(y,\tau)| \le C\tau^{-\nu}\|h\|_{\nu,2+\sigma} \begin{cases} \frac{R^{\delta(3-\sigma)}}{(1+|y|)^3} & |y|\le 2R^\delta \\ \frac{1}{(1+|y|)^\sigma} & 2R^\delta \le |y| \le R, \end{cases}$$

and such that

$$\tilde c_{0j}[h] = -\frac{\int_{B_{\mathbb{R}^2}} h\cdot Z_{0j}}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} - G[h]$$

where G is a linear operator of h satisfying the estimate

$$|G[h]| \le C\tau^{-\nu}R^{-\delta\sigma'}\|h\|_{\nu,2+\sigma}.$$

with $0<\sigma'<\sigma$.

We will construct φ solving (3.111) of the form

$$\varphi = \eta\phi + \psi$$

where

$$\eta(y,\tau) = \eta_1\Big(\frac{|y|}{R_1}\Big)$$

and $\eta_1(r)=1$ for $r\le 1$, $\eta_1(r)=0$ for $r\ge 2$. Here $R_1 = R^\delta$.

We find a solution to (3.111) if we get ϕ, ψ solving the system

$$\begin{cases} \partial_\tau\phi = L_W\phi + B\psi + h(y,\tau) + \sum_{j=1,2} c_{0j} Z_{0j} w_\rho^2 & \text{in } \mathcal{D}_{2R_1} \\ \phi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R_1} \\ \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R_1(\tau_0)}, \end{cases} \tag{3.112}$$

$$\begin{cases} \partial_\tau\psi = \Delta\psi + (1-\eta)B\psi + A\phi + (1-\eta)h(y,\tau) & \text{in } \mathcal{D}_{2R} \\ \psi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \psi = 0 \quad \text{on } \partial B_{2R}\times(\tau_0,\infty) \\ \psi(\cdot,\tau_0) = 0 \quad \text{on } B_{2R(\tau_0)}, \end{cases} \tag{3.113}$$

where

$$B\psi = |\nabla W|^2\psi + 2(\nabla W\cdot\nabla\psi)W$$
$$A\phi = \phi\Delta\eta + 2\nabla\phi\nabla\eta - \phi\eta_t + \phi(\nabla W\nabla\eta)W.$$

Consider

$$\begin{cases} \partial_\tau\psi = \Delta\psi + (1-\eta)B\psi + h(y,\tau) & \text{in } \mathcal{D}_{2R} \\ \psi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \psi = 0 \quad \text{on } \partial B_{2R}\times(\tau_0,\infty), \\ \psi(y,\tau_0) = 0 \quad \forall y\in B_{2R}, \end{cases} \tag{3.114}$$

Let

$$\|\psi\|_{\nu,\sigma}^{(1)} = \sup_{\mathcal{D}_{2R}} \tau^\nu(1+|y|)^\sigma\left[\,|\psi(y,\tau)| + (1+|y|)|\nabla_y\psi(y,\tau)|\,\right]$$

Lemma 3.8. *Let $\sigma \in (0,1)$, $\nu > 0$ and let ψ solve (3.114). If R_1 is sufficiently large, then*

$$\|\psi\|^{(1)}_{\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}. \tag{3.115}$$

If in (3.114) h is replaced by $(1-\eta)h$ we get the additional estimate

$$|\psi(y,t)| + R_1|\nabla\psi(y,t)| \le \frac{1}{\tau^\nu}\frac{1}{R_1^\sigma} \quad |y| \le 2R_1.$$

Proof. To prove this lemma, we first claim that for the equation

$$\begin{cases} \partial_\tau \psi = \Delta\psi + h(y,\tau) \quad \text{in } \mathcal{D}_{2R} \\ \psi = 0 \quad \text{on } \partial B_{2R}\times(\tau_0,\infty), \\ \psi(y,\tau_0) = 0 \quad \forall y \in B_{2R}, \end{cases}$$

in mode 0, we have

$$\|\psi\|^{(1)}_{\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}. \tag{3.116}$$

To find the estimate for the solution of (3.114) we need to estimate $\|(1-\eta)B\psi\|_{\nu,2+\sigma}$. We have that

$$\begin{aligned}(1-\eta)|\nabla W|^2\,|\psi| &\le (1-\eta)\tau^{-\nu}(1+|y|)^{-4-\sigma}\|\psi\|^{(1)}_{\nu,\sigma} \\ &\le R_1^{-2}\tau^{-\nu}(1+|y|)^{-2-\sigma}\|\psi\|^{(1)}_{\nu,\sigma}.\end{aligned}$$

Similarly

$$\begin{aligned}(1-\eta)|\nabla W\cdot\nabla\psi)W| &\le (1-\eta)\tau^{-\nu}(1+|y|)^{-3-\sigma}\|\psi\|^{(1)}_{\nu,\sigma} \\ &\le R_1^{-1}\tau^{-\nu}(1+|y|)^{-2-\sigma}\|\psi\|^{(1)}_{\nu,\sigma}.\end{aligned}$$

Therefore

$$\|(1-\eta)B\psi\|_{\nu,2+\sigma} \le CR_1^{-1}\|\psi\|^{(1)}_{\nu,\sigma}.$$

Then, if ψ satisfies (3.114), using (3.116) we get

$$\|\psi\|^{(1)}_{\nu,\sigma} \le C\|(1-\eta)B\psi + h\|_{\nu,2+\sigma} \le CR_1^{-1}\|\psi\|^{(1)}_{\nu,\sigma} + C\|h\|_{\nu,2+\sigma}.$$

If R_1 is large enough, we obtain (3.115). ☐

Proof of Proposition 3.12. We use Lemma 3.8 to find a solution $\psi[\phi]$ of (3.114) with h replaced by $A\phi$, and a solution $\psi[h]$ of (3.114) with h replaced by $(1-\eta)h$, so that $\psi[\phi]+\psi[h]$ is the solution of (3.113).

Let $\sigma_1 \in (0,1)$. We also get the estimate

$$\|\psi[\phi]\|^{(1)}_{\nu,\sigma_1} \le C\|A\phi\|_{\nu,2+\sigma_1}. \tag{3.117}$$

We take $R_1 = R^\delta$ and construct a solution of the system (3.112), (3.113). For this it suffices to find ϕ such that

$$\begin{cases} \partial_\tau\phi = L_W\phi + B\psi[\phi] + B\psi[h] + h(y,\tau) + \sum\limits_{j=1,2} c_{0j}Z_{0j}\chi_{B_1} \quad \text{in } \mathcal{D}_{2R_1} \\ \phi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R_1} \\ \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R_1(\tau_0)}. \end{cases} \tag{3.118}$$

Let $\mathcal{T}$ denote the linear operator given by Lemma 3.3, Applied in $\mathcal{D}_{2R_1}$. Then to solve (3.118) we consider the fixed point problem

$$\phi = \mathcal{T}[B\psi[\phi] + B\psi[h] + h].$$

Let $\sigma \in (0,1)$. By Lemma 3.3,

$$\|\mathcal{T}[g]\|_{*,\nu,2+\sigma} \le \|g\|_{\nu,2+\sigma}, \tag{3.119}$$

where

$$\|\phi\|_{*,\nu,\sigma} = \sup\frac{\tau^\nu(1+|y|)^3}{R_1^{3-\sigma}}\left[|\phi(y,\tau)| + (1+|y|)|\nabla_y\phi(y,\tau)|\right].$$

We claim that if

$$\sigma_1 < \sigma$$

then

$$\|A\phi\|_{\nu,2+\sigma_1} \le CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}. \tag{3.120}$$

Indeed, we have

$$\begin{aligned}|\phi\Delta\eta| &\le \frac{1}{R_1^2}\tau^{-\nu}\frac{R_1^{3-\sigma}}{(1+|y|)^3}|\Delta\eta_1|\|\phi\|_{*,\nu,\sigma} \le C\tau^{-\nu}\frac{R_1^{\sigma_1-\sigma}}{(1+|y|)^{2+\sigma_1}}\|\phi\|_{*,\nu,\sigma}\\ &\le CR_1(0)^{\sigma_1-\sigma}\tau^{-\nu}\frac{1}{(1+|y|)^{2+\sigma_1}}\|\phi\|_{*,\nu,\sigma}.\end{aligned}$$

Similarly

$$|\nabla\phi\nabla\eta| \le \frac{1}{R_1}\tau^{-\nu}\frac{R_1^{3-\sigma}}{(1+|y|)^4}|\nabla\eta_1|\|\phi\|_{*,\nu,\sigma} \le C\tau^{-\nu}\frac{R_1^{\sigma_1-\sigma}}{(1+|y|)^{2+\sigma_1}}\|\phi\|_{*,\nu,\sigma}.$$

Similar estimates for the remaining terms in A prove (3.120).

¿From (3.117) and (3.120) we find

$$\|\psi[\phi]\|^{(1)}_{\nu,\sigma_1} \le CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}. \tag{3.121}$$

Now we claim that

$$\|B\psi\|_{\nu,2+\sigma} \le C\|\psi\|^{(1)}_{\nu,\sigma_1}. \tag{3.122}$$

Indeed, since $|\nabla W|^2 \le C(1+|y|)^{-4}$, we have

$$|\nabla W|^2\,|\psi| \le C\frac{\tau^{-\nu}}{(1+|y|)^{4+\sigma_1}}\|\psi\|^{(1)}_{\nu,\sigma_1} \le C\frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}\|\psi\|^{(1)}_{\nu,\sigma_1}$$

Also

$$|(\nabla W\cdot\nabla\psi)W| \le C\frac{\tau^{-\nu}}{(1+|y|)^{3+\sigma_1}}\|\psi\|^{(1)}_{\nu,\sigma_1} \le C\frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}\|\psi\|^{(1)}_{\nu,\sigma_1}$$

These two inequalities prove (3.122).

Combining (3.122) and (3.121) we get

$$\|B\psi[\phi\|_{\nu,2+\sigma} \le C\|\psi[\phi]\|^{(1)}_{\nu,\sigma_1} \le CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}.$$

¿From the above inequality and (3.119) we then get

$$\|\mathcal{T}[B\psi[\phi]]\|_{*,\nu,\sigma} \le CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma},$$

which shows that the operator $\phi \mapsto \mathcal{T}[B\psi[\phi]+B\psi[h]+h]$ is a contraction if $R_1(0)$ is sufficiently large, and we find a unique fixed point, which satisfies the estimate

$$\|\phi\|_{*,\nu,\sigma} \le C\|\mathcal{T}[B\psi[h]+h]\|_{*,\nu,\sigma}. \tag{3.123}$$

Next we estimate $\|\mathcal{T}[B\psi[h]+h]\|_{*,\nu,\sigma}$. We have by (3.119)

$$\begin{aligned}\|\mathcal{T}[B\psi[h]+h]\|_{*,\nu,\sigma} &\le C\|B\psi[h]+h\|_{\nu,2+\sigma}\\ &\le C\|\psi[h]\|^{(1)}_{\nu,\sigma} + \|h\|_{\nu,2+\sigma} \le C\|h\|_{\nu,2+\sigma},\end{aligned}$$

and hence

$$\|\phi\|_{*,\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}. \tag{3.124}$$

Similar to (3.121) we have

$$\|\psi[\phi]\|^{(1)}_{\nu,\sigma} \le C\|\phi\|_{*,\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}.$$

and

$$\|\psi[h]\|^{(1)}_{\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}.$$

Recalling that $\varphi = \eta\phi+\psi$ and $R_1 = R^\delta$, we get

$$|\varphi(y,\tau)| + (1+|y|)|\nabla_y\varphi(y,\tau)| \le C\tau^{-\nu}\|h\|_{\nu,2+\sigma}\begin{cases}\frac{R^{\delta(3-\sigma)}}{(1+|y|)^3} & |y|\le 2R^\delta\\ \frac{1}{(1+|y|)^\sigma} & 2R^\delta \le |y| \le R.\end{cases}$$

Finally, thanks to Lemma 3.3, we have that

$$c_{0j}[h] = -\frac{1}{\int_{B_1} |Z_{0j}|^2} \left[\int_{B_{2R_1}} h \cdot Z_{0j} + \int_{B_{2R_1}} (B\psi[\phi] + B\psi[h]) \cdot Z_{0j} \right]$$

The last term is a linear operator of h, which we estimate next. A similar computation as in (3.120) shows that

$$\|A\phi\|_{\nu+\delta(\sigma-\sigma_1),2+\sigma_1} \le C\|\phi\|_{*,\nu,\sigma}.$$

This implies

$$\|\psi[\phi]\|_{\nu+\delta(\sigma-\sigma_1),\sigma_1} \le C\|\phi\|_{*,\nu,\sigma}$$

and therefore

$$\left| \int_{B_{2R_1}} B\psi[\phi] \cdot Z_{0j} \right| \le C\tau^{-\nu} R^{\sigma_1-\sigma} \|\phi\|_{*,\nu,\sigma}$$

and using (3.124)

$$\left| \int_{B_{2R_1}} B\psi[\phi] \cdot Z_{0j} \right| C\tau^{-\nu} R^{\sigma_1-\sigma} \|h\|_{\nu,2+\sigma}.$$

We have for $|y| \le 2R^\delta$

$$|\psi[h](y,t)| + (1+|y|)|\nabla_y \psi[h](y,t)| \le CR_1^{-\sigma} \|h\|_{\nu,2+\sigma}.$$

Then for $|y| \le 2R^\delta$ we have

$$|\nabla W|^2 \, |\psi[h]| \le C\tau^{-\nu}(1+|y|)^{-4} R_1^{-\sigma} \|h\|_{\nu,2+\sigma}$$

and

$$|(\nabla W \cdot \nabla \psi)W| \le C\tau^{-\nu}(1+|y|)^{-4} R_1^{-\sigma} \|h\|_{\nu,2+\sigma}.$$

As a consequence we get

$$|B\psi[h]| \le C\tau^{-\nu}(1+|y|)^{-4} R_1^{-\sigma} \|h\|_{\nu,2+\sigma}.$$

and hence

$$\left| \int_{B_{2R_1}} B\psi[h] \cdot Z_{0j} \right| \le C\tau^{-\nu} R_1^{-\sigma} \|h\|_{\nu,2+\sigma}.$$

□

3.2. **Landau-Lifshitz-Gilbert equation.** The basic ansatz in LLG for the construction is similar to the HMF's. We point out some difference and new aspects as follows.

• **Ansatz of multiple bubbles**

The construction begins with a careful choice of first approximation. Since the target is the 2-sphere, one has to choose some profile for multiple bubbles which is relatively reasonable to analyze. Here we choose

$$U_* = -(N-1)U_\infty + \sum_{j=1}^{N} Q_{\gamma_j} W\left(\frac{x - \xi^{[j]}}{\lambda_j}\right)$$

as the first approximation. Notice that $|U_*| \approx 1$ at any space-times as those bubbles are essentially separated. Based on U_*, we then look for solution to LLG in the form

$$u(x,t) = (1+A)U_* + \Phi - (\Phi \cdot U_*)U_* \tag{3.125}$$

for some perturbation term Φ and scalar A. Here the purpose of the scalar A, depending on Φ, is to preserve the unit-length of the map $u(x,t)$ for any $(x,t) \in \mathbb{R}^2 \times (0,T)$. So here part of the interactions between bubbles get encoded in the scalar A. Let us denote the error of u as

$$S[u] := -u_t + a(\Delta_x u + |\nabla_x u|^2 u) - bu \wedge \Delta_x u.$$

An important observation here is that instead of solving $S[u] = 0$, we only need to solve

$$S[u] = \Xi(x,t)U_*$$

for some scalar function Ξ. Indeed, since $|u|=1$ is kept for all $t\in(0,T)$ and $u=U_*+\tilde w$ where the perturbation $\tilde w$ is uniformly small, then

$$\Xi(U_*\cdot u)=S[u]\cdot u=-\frac{1}{2}\partial_t(|u|^2)+\frac{a}{2}\Delta|u|^2=0.$$

Thus $\Xi\equiv 0$ follows from $U_*\cdot u\geq\delta_0>0$. This provides us the flexibility to adjust the error terms in U_* direction, and we will call this U_**-operation* throughout this note. This operation can simplify analysis especially for the dealing of multiple bubbles.

• **Slow decaying errors and non-local corrections by approximate parabolic system**

The error $S(U_*)$ contains slowing decaying terms

$$\sum_{j=1}^N \mathcal{E}_0^{[j]}\notin L^2(\mathbb{R}^2)$$

which correspond to the re-scaling and rotation around z-axis. To improve the spatial decay of the error at remote region, we add well-designed global/non-local corrections around each bubble. Since the operator

$$-\partial_t+(a-bU^{[j]}\wedge)\Delta_x$$

depends on the blow-up profile U^j as well as the parameters λ_j, γ_j and $\xi^{[j]}$, one cannot expect explicit representation formula apriori without knowing the blow-up dynamics. We consider instead an approximate parabolic operator

$$-\partial_t+(a-bU_\infty\wedge)\Delta_x$$

and add correction $\Phi_0^{*[j]}$ around each bubble $U^{[j]}$ with

$$-\partial_t\Phi_0^{*[j]}+(a-bU_\infty\wedge)\Delta_x\Phi_0^{*[j]}+\mathcal{E}_0^{[j]}\approx 0.$$

Then the new error with corrections is given by those created by $\Phi_0^{*[j]}$ and the remainder $b(U_\infty-U^{[j]})\wedge\Delta_x\Phi_0^{*[j]}$. This is rather important in the analysis of the non-local reduced problems.

• **Formulation of the inner-outer gluing system**

We next look for the perturbation Φ consisting of inner and outer parts with non-local corrections added

$$\Phi(x,t)=\sum_{j=1}^N\left(\eta_R^{[j]}(x,t)Q_{\gamma_j}\Phi_{\rm in}^{[j]}(y^{[j]},t)+\eta_{d_q}^{[j]}(x,t)\Phi_0^{*[j]}\right)+\Phi_{\rm out}(x,t)$$

where $\Phi_{\rm in}^{[j]}$ is on the tangent plane of $W^{[j]}$, $\eta_R^{[j]}$ and $\eta_{d_q}^{[j]}$ are suitable cut-off functions near $q^{[j]}$. Then u solving LLG implies a coupled inner-outer gluing system for $\Phi_{\rm in}^{[j]}$ and $\Phi_{\rm out}$, $j=1,\dots,N$

$$\begin{cases}\lambda_j^2\partial_t\Phi_{\rm in}^{[j]}=\left(a-bW^{[j]}\wedge\right)\left[\Delta_{y^{[j]}}\Phi_{\rm in}^{[j]}+|\nabla_{y^{[j]}}W^{[j]}|^2\Phi_{\rm in}^{[j]}+2\left(\nabla_{y^{[j]}}W^{[j]}\cdot\nabla_{y^{[j]}}\Phi_{\rm in}^{[j]}\right)W^{[j]}\right]\\ \qquad\qquad +\mathcal{H}^{[j]}[\Phi_{\rm in}^{[j]},\Phi_{\rm out},\lambda_j,\gamma_j,\xi^{[j]}]\ \text{ in }\ \mathsf{D}_{2R},\\ \partial_t\Phi_{\rm out}=\mathbf{B}_{\Phi,U_*}\Delta_x\Phi_{\rm out}+\mathcal{G}[\Phi_{\rm in}^{[j]},\Phi_{\rm out},\lambda_j,\gamma_j,\xi^{[j]}]\ \text{ in }\ \mathbb{R}^2\times(0,T),\end{cases}$$

where $W^{[j]}$ is the j-th bubble expressed in the rescaled variable $y^{[j]}=\frac{x-\xi^{[j]}}{\lambda_j}$, the right hand sides $\mathcal{H}^{[j]}$, $\mathcal{G}$ consists of the error terms, couplings and nonlinear terms depending on the parameters λ_j, γ_j, $\xi^{[j]}$, and $\mathbf{B}_{\Phi,U_*}$ is a matrix, that involves the perturbation Φ and the blow-up profile U_*.

For the full system above, finding blow-up of LLG at multiple points now gets reduced to finding well-behaved inner and outer profiles such that gluing procedure can be implemented. In other words, we need to devise appropriate weighted topologies in which the gluing system becomes weakly coupled and thus can be solved by fixed point arguments. For the outer problem, we make use of the sub-Guassian estimate recently proved in [34]. For the inner problem, good solutions with sufficient decay in space and time can only be captured with careful choices of the parameters λ_j, γ_j, $\xi^{[j]}$. We shall develop linear theory for the inner problems with orthogonality conditions, and these orthogonalities in turn determine the blow-up dynamics.

• **Solving the inner problem**

The linear theory for the inner problem is established by analyzing each Fourier mode. Decomposing the complex form in Fourier modes, one obtains the linearized operator at mode k of the form

$$\lambda_j^2\partial_t - (a-ib)\left(\partial_{\rho\rho} + \frac{\partial_\rho}{\rho} - \frac{(k+1)^2\rho^4 + (2k^2-6)\rho^2 + (k-1)^2}{(\rho^2+1)^2}\frac{1}{\rho^2}\right), \tag{3.126}$$

where $\rho_j = |y^{[j]}|$. Then for all the modes $k \in \mathbb{Z}\backslash\{-1\}$, good inner solutions are found by the following

Step 1: we first use energy methods to get a rough pointwise upper bounds for the inner solutions ϕ_k;
Step 2: next we use Duhamel's formula to refine the pointwise bounds and further gain decay estimates;
Step 3: finally we perform re-gluing procedure to obtain better estimates in the innermost region.

See also Appendix D. In Step 1, the following spectral gap for each mode $k \in \mathbb{Z}$ plays a central role:

Lemma 3.9. *Let*

$$\lambda_{R,k} = \inf_{f\in X_0(B_R)\backslash\{0\}} \frac{Q_{R,k}(f,f)}{\|f\|^2_{L^2(B_R)}} \quad \textit{for} \quad k\neq 1, \qquad \lambda_{R,1} = \inf_{f\in H_0^1(B_R)\backslash\{0\}} \frac{Q_{R,1}(f,f)}{\|f\|^2_{L^2(B_R)}} \quad \textit{for} \quad k=1.$$

Here

$$Q_{R,k}(f,f) = 2\pi\int_0^R \left[|\partial_\rho f|^2 + \frac{(k+1)^2\rho^4 + (2k^2-6)\rho^2 + (k-1)^2}{(\rho^2+1)^2}\frac{|f|^2}{\rho^2}\right]\rho d\rho \tag{3.127}$$

denotes the quadratic form of the associated linearized operator. Then $\lambda_{R,k}$ is attained by a real-valued function in $X_0(B_R)$ for $k\neq 1$ and $H_0^1(B_R)$ for $k=1$. When R is large,

$$\lambda_{R,0} \sim (R^2\ln R)^{-1}, \quad \lambda_{R,1} \sim R^{-4}, \quad \lambda_{R,-1} \gtrsim (R^2\ln R)^{-1}, \quad \lambda_{R,k} \gtrsim |k|^2R^{-2} \quad \textit{for} \quad |k|\geq 2.$$

As mentioned earlier, the treatment for mode $k=-1$ is different from the techniques that we employ for all the other modes. The reason is the following: as one can see from (3.126), mode -1 can be roughly viewed as a problem in 2D, which is worse than any other mode as one cannot gain spatial decay in Step 2 above. Motivated by a recent work of Krieger, Miao and Schlag [61] on the stability of blow-up for wave maps beyond the equivariant class as well as a series important works of Krieger-Miao-Schlag-Tataru [59, 60, 62], we try to apply the techniques of distorted Fourier transform (DFT) to this specific mode -1. The general framework and theories of the spectral analysis of half-line Schrödinger operators have been established earlier by Gesztesy-Zinchenko [42]. See also recent applications of DFT in the study of asymptotic stability of solitons and kinks [13, 41, 68, 69].

Fortunately, it turns out that the use of distorted Fourier transform can give us almost the optimal bound. The results and ideas are given in Appendix C.

• **Non-local reduced problems**

The development of the linear theory for the inner problem relies on orthogonalities which are achieved by adjusting modulation parameters λ_j, γ_j, $\xi^{[j]}$. The dynamics for $\xi^{[j]}$ turns out to be governed by an ODE, which is relatively straightforward to solve. However, the non-local feature in the corrections $\Phi_0^{*[j]}$ gets inherited by the mode 0 (λ_j and γ_j) of each bubble as the corrections are essentially for mode 0. Here one might expect the complex system involving both λ_j and γ_j is a rather sophisticated form due to the presence of dispersion. Fortunately, it turns out that the contribution of both $\Phi_0^{*[j]}$ and the remainder $b(U_\infty - U^{[j]})\wedge\Delta_x\Phi_0^{*[j]}$ in the orthogonal equation at mode 0 results in the following well-ordered non-local problem

$$\int_0^{t-\lambda_j^2(t)} \frac{\dot{p}_j(s)}{t-s}ds \sim O(1),$$

where $p_j(t) = \lambda_j(t)e^{i\gamma_j(t)}$. This λ_j-γ_j system was first found and handled in the context of HMF [22]. Surprisingly, this comes with a similar form in LLG with the presence of dispersion.

• **Solving the outer problem**

The linear theory for the outer problem is done by using the sub-Gaussian estimate for the fundamental matrix of the parabolic system in the non-divergence form. Dong-Kim-Lee [34] proved that, for a uniformly parabolic system in the non-divergence form, if the entries belong to $\mathsf{DMO_x}$ (Dini mean oscillation in space), then the fundamental matrix Γ satisfies the sub-Gaussian estimate:

$$|\Gamma(x,t,y,s)| \leq C(t-s)^{-\frac{d}{2}}e^{-c\left(\frac{|x-y|}{\sqrt{t-s}}\right)^{2-\delta}},$$

d is the dimension, c, $C > 0$, and $\delta \in (0,1)$. Here the Gilbert damping $a > 0$ makes sure that the system is uniformly parabolic. Higher regularity estimates can be obtained by the blow-up argument.

In the matrix $\mathbf{B}_{\Phi,U_*}$ appearing in the outer system, the dependence on Φ in the matrix shall be dealt with via Schauder fixed point theorem, and the key thing here is the dependence on the blow-up profile U_*. In fact, to ensure $\mathsf{DMO}_{\mathsf{x}}$-regularity in U_*, the type II blow-up rate

$$\lambda_* \ll \sqrt{T-t}$$

plays a crucial rule.

Roughly speaking, the reason why we choose to work in $\mathsf{DMO}_{\mathsf{x}}$ is that we need weighted C^2 estimates, but C^0 cannot ensure C^2. On the other hand, one cannot expect good C^α bound either since there is a loss of $\lambda_*^{-\alpha}$. So one has to work in some intermediate space between C^0 and C^α that we choose as certain Dini space.

4. Long-time dynamics for 1-equivariant harmonic map flow, critical heat equation in $\mathbb{R}^4$, Keller-Segel system in $\mathbb{R}^2$

In this section, we consider the long-time dynamics for three intimately related problems: the 1-equivariant harmonic map flow, the critical heat equation in $\mathbb{R}^4$, and the Keller-Segel system in $\mathbb{R}^2$. These three models share a common feature that they are all at the borderline separating the regime of slow decay and fast decay.

The harmonic map flow (HMF) from $\mathbb{R}^2$ into S^2 has the form

$$\begin{cases} u_t = \Delta u + |\nabla u|^2 u & \text{in } \ \mathbb{R}^2 \times (0,\infty) \\ u(\cdot,0) = u_0 & \text{in } \ \mathbb{R}^2. \end{cases}$$

A special class of solutions are given by the k-equivariant ansatz

$$u(re^{i\theta},t) = \Big(\cos(k\theta)\sin v(r,t), \sin(k\theta)\sin v(r,t), \cos v(r,t)\Big),$$

and thus HMF gets reduced to a scalar equation for the polar angle

$$\begin{cases} v_t = v_{rr} + \dfrac{1}{r}v_r - \dfrac{k^2\sin(2v)}{2r^2}, & (r,t)\in\mathbb{R}_+\times\mathbb{R}_+ \\ v(r,0) = v_0, & r\in\mathbb{R}_+. \end{cases} \tag{4.1}$$

We consider the 1-equivariant class

$$\begin{cases} v_t = v_{rr} + \dfrac{v_r}{r} - \dfrac{\sin(2v)}{2r^2}, & (r,t)\in\mathbb{R}_+\times(t_0,\infty) \\ v(r,t_0) = v_0(r), & r\in\mathbb{R}_+, \end{cases} \tag{4.2}$$

where $t_0 > 0$ is some large initial time. The aim of this section is to understand the possible long-term behavior of (4.2), and the main result stated below depends precisely on the power decay rate of the initial data v_0.

Define the cut-off function η as $\eta(r) = 1$ for $0 \le r \le 1$, $\eta(r) = 0$ for $r \ge 2$, and $0 \le \eta \le 1$.

Theorem 5. *Given any $\gamma > 1$, for t_0 sufficiently large, there exists an initial data $v_0(r)$ with $v_0(0) = \pi$ and for $r \ge 2\sqrt{t_0}$, $|v_0(r)| \lesssim r^{1-\gamma}$ if $1 < \gamma \le 2$ and $|v_0(r)| \lesssim r^{1-\gamma} + (t_0\ln t_0)^{-1} r e^{-\frac{r^2}{32t_0}}$ if $\gamma > 2$, such that the global solution of (4.2) takes the form*

$$\begin{aligned} v(r,t) &= \eta\left(\frac{r}{\sqrt{t}}\right)\left[\pi - 2\arctan\left(\frac{r}{\mu(t)}\right)\right] + O\left(r\langle r\rangle^{-1} t^{-\frac14\min\{\gamma-1,1\}}\right), \\ v_r(r,t) &= -2\eta\left(\frac{r}{\sqrt{t}}\right)\frac{\mu(t)^{-1}}{1+(\mu(t)^{-1}r)^2} + O\Big(\mu(t)^{-1}t^{-\frac12\min\{\gamma-1,1\}}\Big), \end{aligned}$$

where

$$\mu(t) = \begin{cases} C_\mu(\gamma)t^{1-\frac{\gamma}{2}}(\ln t)^{-1}\left(1+O\left((\ln t)^{-1}\ln\ln t\right)\right), & \text{if } \ 1<\gamma<2 \\ C_\mu(\gamma)\left(1+O\left((\ln t)^{-1}\ln\ln t\right)\right), & \text{if } \ \gamma=2 \\ (\ln t)^{-1}\left(1+O\left((\ln t)^{-1}\ln\ln t\right)\right), & \text{if } \ \gamma>2 \end{cases}$$

with a constant $C_\mu(\gamma) > 0$ depending on γ. In particular, $\|v_r(\cdot,t)\|_{L^\infty([0,\infty))} = 2\mu(t)^{-1}\Big(1+O\Big(t^{-\frac12\min\{\gamma-1,1\}}\Big)\Big)$.

Remark 4.1. *A similar approach in the proof of [100, Lemma A.3] gives the lower bound for* $|v_0(r)|$. *In fact, the initial data of the solutions that we constructed satisfies* $v_0(r) \sim r^{1-\gamma}$ *as* $r \to \infty$.

Due to the natural connection between the k-equivariant harmonic map flow and the critical heat equation, we also built the long time behavior for the critical heat equation in dimension 4

$$\begin{cases} u_t = \Delta u + u^3 & \text{in } \mathbb{R}^4 \times (t_0, \infty), \\ u(x, t_0) = u_0(x) & \text{in } \mathbb{R}^4. \end{cases} \tag{4.3}$$

Theorem 6 ([100])**.** *For* t_0 *sufficiently large, there exists initial value* $u_0 > 0$ *with exponential decay such that the positive solution* $u(x,t)$ *to* (4.3) *blows up at infinite time. More precisely, the solution takes the form of the sharply scaled bubble*

$$u(x,t) = \eta\left(\frac{x-\xi}{\sqrt{t}}\right)\mu^{-1}(t) w\left(\frac{x-\xi(t)}{\mu(t)}\right) + O\big((\ln t)^{-1}\min\{t^{-1}, |x|^{-2}\}\big)$$

where $w(y) = 2^{\frac{3}{2}}\frac{1}{1+|y|^2}$. *The blow-up rate and location are given by*

$$\mu(t) = \frac{1}{\ln t}\Big(1 + O\big(\frac{\ln\ln t}{\ln t}\big)\Big), \quad \xi(t) = O(t^{-1}).$$

The dynamics for establishing Theorem 6 is original from the one in [18]. Consider the classical Keller-Segel problem in $\mathbb{R}^2$,

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) \quad \text{in } \mathbb{R}^2\times(0,\infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1}u := \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2}\log\frac{1}{|x-z|}\,u(z,t)\,dz, \\ u(\cdot,0) = u_0 \geq 0 \quad \text{in } \mathbb{R}^2. \end{cases} \tag{4.4}$$

Theorem 7 ([18])**.** *There exists a nonnegative, radially symmetric function* $u_0^*(x)$ *with critical mass* $\int_{\mathbb{R}^2} u_0^*(x)\,dx = 8\pi$ *and finite second moment* $\int_{\mathbb{R}^2}|x|^2 u_0^*(x)\,dx < +\infty$ *such that for every* $u_1(x)$ *sufficiently close (in suitable sense) to* u_0^* *with* $\int_{\mathbb{R}^2} u_1\,dx = 8\pi$, *we have that the solution* $u(x,t)$ *of system* (4.4) *with initial condition* $u(x,0) = u_1(x)$ *has the form*

$$u(x,t) = \frac{1}{\lambda(t)^2} U\Big(\frac{x-\xi(t)}{\lambda(t)}\Big)(1+o(1)), \quad U(y) = \frac{8}{(1+|y|^2)^2} \tag{4.5}$$

uniformly on bounded sets of $\mathbb{R}^2$, *and*

$$\lambda(t) = \frac{c}{\sqrt{\log t}}(1+o(1)), \quad \xi(t)\to q \quad as \quad t\to+\infty,$$

for some number $c > 0$ *and some* $q \in \mathbb{R}^2$.

Our results are inspired by the formal analysis in [35]. Fila and King [35] conjectured that for

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{N-2}}u, & x\in\mathbb{R}^N, \quad t>0, \\ u(x,0) = u_0(x), & x\in\mathbb{R}^N \end{cases} \tag{4.6}$$

with initial data $\lim_{r\to\infty} r^{\tilde{\gamma}}u_0(r) = A$ for some $A > 0$, the $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}$-norm of threshold solution obeys

	$\frac{N-2}{2} < \tilde{\gamma} < 2$	$\tilde{\gamma} = 2$	$\tilde{\gamma} > 2$
$N=3$	$t^{\frac{\tilde{\gamma}-1}{2}}$	$t^{\frac{1}{2}}(\ln t)^{-1}$	$t^{\frac{1}{2}}$
$N=4$	$t^{-\frac{2-\tilde{\gamma}}{2}}\ln t$	1	$\ln t$
$N=5$	$t^{-\frac{3(2-\tilde{\gamma})}{2}}$	$(\ln t)^{-3}$	1

In particular, the trichotomy constructed in Theorem 5 can be viewed as an analogue of the Fila-King diagram in $\mathbb{R}^4$. Recently, global unbounded solutions for Fujita equation (4.6) in $\mathbb{R}^3$ and $\mathbb{R}^4$ have been rigorously constructed in [26, 100], confirming the existence of upper off-diagonal entries in the above diagram (including a sub-case $1 < \tilde{\gamma} < 2$ when $N = 3$).

The heart of the construction is non-local dynamics, governing the scaling parameter $\mu(t)$ in a unified way:

$$\underbrace{\int_{t/2}^{t-\mu^2(t)} \frac{\dot{\mu}(s)}{t-s} ds}_{:=I_{\mathrm{nl}}} + \underbrace{\frac{\mu(t)}{t}}_{:=I_{\mathrm{ss}}} \sim \underbrace{2C_\gamma v_\gamma(t)}_{:=I_{\mathrm{ic}}}, \quad \forall \gamma > 1. \tag{4.7}$$

Here, I_{nl} is in fact from a non-local correction dealing with the slow spatial decay. Such non-local/global feature usually appears in lower dimensional problems and was first observed in [22, 26]. The second term I_{ss} comes from a self-similar correction improving the error in the intermediate region, and the last term I_{ic} is the contribution from the initial condition v_0 whose expression depends only on γ (cf. (4.12)). The trichotomy in Theorem 5 is captured by approximating the non-local problem by a leading ODE, but the solvability of the full non-local problem is rather involved.

In this part, we will give the proof of Theorem 5 as a prototype for the construction of global solutions.

We use the following notations.

- For $x \in \mathbb{R}^d$, $t_0 \le t$ and admissiable functions $g(x), h(x,t)$, denote

$$\left(T_d \circ g\right)(x,t) := (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} g(y) dy,$$

$$\left(T_d \bullet h\right)(x,t,t_0) := \int_{t_0}^{t} \int_{\mathbb{R}^d} \left[4\pi(t-s)\right]^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} h(y,s) dy ds.$$

- For any $c \in \mathbb{R}$, we use the notation $c-$ (resp. $c+$) to denote a constant less (resp. greater) than c and can be chosen arbitrarily close to c.

4.1. **Approximation and corrections.**

The first approximation is built on the one parameter family of steady states to the equation (4.2)

$$Q_\mu = \pi - 2\arctan\left(\frac{r}{\mu}\right), \quad \mu > 0.$$

Then we have

$$\pi - 2\arctan(\rho) \sim \langle \rho \rangle^{-1}, \quad \sin(2Q_\mu) = \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}, \quad \cos(2Q_\mu) - 1 = -\frac{8\rho^2}{(\rho^2+1)^2} \quad \text{for} \quad \rho := \frac{r}{\mu}. \tag{4.8}$$

We take the first approximate solution of the equation (4.2) to be

$$v_* = \eta\left(\frac{r}{\sqrt{t}}\right) Q_\mu, \quad \mu = \mu(t),$$

and define the error operator as

$$E[v] := -v_t + v_{rr} + \frac{1}{r} v_r - \frac{\sin(2v)}{2r^2}.$$

Let us write

$$z := \frac{r}{\sqrt{t}}.$$

Then we have

$$\begin{aligned} E[v_*] &= \mu^{-1}\dot{\mu}\rho\partial_\rho Q_\mu \eta(z) + \frac{1}{t}\eta''(z) Q_\mu + \frac{2}{\mu\sqrt{t}}\eta'(z)\partial_\rho Q_\mu \\ &\quad + \left(\frac{r}{2t\sqrt{t}} + \frac{1}{r\sqrt{t}}\right)\eta'(z) Q_\mu + \eta(z)\frac{\sin(2Q_\mu)}{2r^2} - \frac{\sin(2\eta(z)Q_\mu)}{2r^2} \\ &= \mathcal{E}_1[\mu] + \mathcal{E}_2[\mu] + \eta(z)\frac{\sin(2Q_\mu)}{2r^2} - \frac{\sin(2\eta(z)Q_\mu)}{2r^2} \end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
\mathcal{E}_2[\mu] &:= \mu^{-1}\dot{\mu}\rho\partial_\rho Q_\mu \eta(z) = r\frac{-2\dot{\mu}}{r^2+\mu^2}\eta(z), \quad \mathcal{E}_1[\mu] := \mathcal{E}_{11}[\mu] + \mathcal{E}_{12}[\mu],\\
\mathcal{E}_{11}[\mu] &:= \frac{2}{t\rho}\eta''(z) - \frac{4}{\mu\sqrt{t}\rho^2}\eta'(z) + \frac{2}{\rho}\left(\frac{r}{2t\sqrt{t}} + \frac{1}{r\sqrt{t}}\right)\eta'(z)\\
&= 2r\mu t^{-2}\left(z^{-2}\eta''(z) + 2^{-1}z^{-1}\eta'(z) - z^{-3}\eta'(z)\right),\\
\mathcal{E}_{12}[\mu] &:= \left[\frac{1}{t}\eta''(z) + \left(\frac{r}{2t\sqrt{t}} + \frac{1}{r\sqrt{t}}\right)\eta'(z)\right]\left(Q_\mu - \frac{2}{\rho}\right) + \frac{2}{\mu\sqrt{t}}\left(\partial_\rho Q_\mu + \frac{2}{\rho^2}\right)\eta'(z).
\end{aligned} \tag{4.10}$$

Throughout this section, we make the ansatz $\mu(t) < \sqrt{t}/9$. Then

$$\mathcal{E}_{12}[\mu] = O\left(rt^{-3}\mu^3 \mathbf{1}_{\{\sqrt{t}\le r\le 2\sqrt{t}\}}\right). \tag{4.11}$$

The initial data plays an important role in determining the leading term of μ. Set

$$\partial_t \Psi_* = \partial_{rr}\Psi_* + \frac{1}{r}\partial_r\Psi_* - \frac{1}{r^2}\Psi_*, \quad \Psi_*(r,0) = r\langle r\rangle^{-\gamma},$$

where Ψ_* is given by

$$\Psi_*(r,t) = r\psi_*(r,t), \quad \psi_*(r,t) = (4\pi t)^{-2}\int_{\mathbb{R}^4} e^{-\frac{|r\mathbf{e}_1 - y|^2}{4t}}\langle y\rangle^{-\gamma}dy$$

with $\mathbf{e}_1 := [1,0,0,0]$. For $t \ge 1$, the leading term of ψ_* is given by

$$\psi_*(0,t) = v_\gamma(t)(C_\gamma + g_\gamma(t)),$$

where

$$v_\gamma(t) = \begin{cases} t^{-\frac{\gamma}{2}}, & \gamma < 4\\ t^{-2}\ln(1+t), & \gamma = 4\\ t^{-2}, & \gamma > 4,\end{cases} \quad C_\gamma = \begin{cases} (4\pi)^{-2}\int_{\mathbb{R}^4} e^{-\frac{|z|^2}{4}}|z|^{-\gamma}dz, & \gamma < 4\\ (4\pi)^{-2}\frac{1}{2}|S^3|, & \gamma = 4\\ (4\pi)^{-2}\int_{\mathbb{R}^4}\langle y\rangle^{-\gamma}dy, & \gamma > 4,\end{cases} \tag{4.12}$$

$$g_\gamma(t) = O\left(\begin{cases} t^{-1}, & \gamma < 2\\ t^{-1}\langle \ln t\rangle, & \gamma = 2\\ t^{\frac{\gamma-4}{2}}, & 2 < \gamma < 4\\ (\ln(1+t))^{-1}, & \gamma = 4\\ t^{\frac{4-\gamma}{2}}, & \gamma < 6\\ t^{-1}\langle \ln t\rangle, & \gamma = 6\\ t^{-1} & \gamma > 6\end{cases}\right).$$

The remainder term is bounded by

$$\begin{aligned}
|\psi_*(r,t) - \psi_*(0,t)| &= \left|(4\pi t)^{-2}\int_{\mathbb{R}^4}\int_0^1 e^{-\frac{|\theta r\mathbf{e}_1 - y|^2}{4t}}\frac{-(\theta r\mathbf{e}_1 - y)\cdot r\mathbf{e}_1}{2t}\langle y\rangle^{-\gamma}d\theta dy\right|\\
&\lesssim rt^{-\frac{5}{2}}\int_{\mathbb{R}^4}\int_0^1 e^{-\frac{|\theta r\mathbf{e}_1 - y|^2}{8t}}\langle y\rangle^{-\gamma}d\theta dy \lesssim rt^{-\frac{1}{2}}v_\gamma(t).
\end{aligned}$$

Combining [100, Lemma A.3], we have

$$\psi_* = \left[v_\gamma(t)\left(C_\gamma + g_\gamma(t)\right) + O\left(rt^{-\frac{1}{2}}v_\gamma(t)\right)\right]\mathbf{1}_{\{r\le 2t^{\frac{1}{2}}\}} + O\left(\begin{cases} r^{-\gamma}, & \gamma < 4\\ r^{-4} + t^{-2}e^{-\frac{r^2}{16t}}\ln(r+2), & \gamma = 4\\ r^{-\gamma} + t^{-2}e^{-\frac{r^2}{16t}}, & \gamma > 4\end{cases}\right)\mathbf{1}_{\{r>2t^{\frac{1}{2}}\}}. \tag{4.13}$$

4.1.1. *Non-local corrections.* We add two corrections Φ_i, $i=1,2$ to transfer the error $\mathcal{E}_i$ of slow spatial decay, where

$$\partial_t\Phi_i = \partial_{rr}\Phi_i + \frac{1}{r}\partial_r\Phi_i - \frac{1}{r^2}\Phi_i + \mathcal{E}_i\,[\mu]\,. \tag{4.14}$$

Set $\Phi_i = r\varphi_i$, $i=1,2$. It suffices to consider

$$\partial_t\varphi_i = \partial_{rr}\varphi_i + \frac{3}{r}\partial_r\varphi_i + r^{-1}\mathcal{E}_i\,[\mu]\,. \tag{4.15}$$

By a similar reason to derive [100, Lemma 2.1], the leading term of φ_1 is obtained from the self-similar solution, while the remaining smaller errors are resolved using Duhamel's formula. More specifically,

$$\varphi_1 = \varphi_1[\mu] := \mu\hat\varphi_1 + T_4 \bullet \left(r^{-1}\mathcal{E}_{12}[\mu] - \dot\mu\hat\varphi_1\right)(r,t,\frac{t_0}{2}), \quad \hat\varphi_1 := 2r^{-2}\left(e^{-\frac{r^2}{4t}} - \eta(\frac{r}{\sqrt{t}})\right) \tag{4.16}$$

and $\hat\varphi_1$ satisfies

$$\begin{aligned} &\hat\varphi_1 = \left(-2^{-1}t^{-1} + O(t^{-2}r^2)\right)\mathbf{1}_{\{|r\le t^{\frac12}\}} + O\left(r^{-2}e^{-\frac{r^2}{4t}}\right)\mathbf{1}_{\{r>t^{\frac12}\}}, \\ &|\partial_r\hat\varphi_1| \lesssim rt^{-2}\mathbf{1}_{\{r\le t^{\frac12}\}} + r^{-1}t^{-1}e^{-\frac{r^2}{4t}}\mathbf{1}_{\{r>t^{\frac12}\}}. \end{aligned} \tag{4.17}$$

By the same argument for deriving [100, Lemma 2.2], φ_2 is given by Duhamel's formula

$$\varphi_2 = \varphi_2[\mu] := T_4 \bullet \left(r^{-1}\mathcal{E}_2[\mu]\right)(r,t,\frac{t_0}{2}). \tag{4.18}$$

Denote $\varphi = \varphi[\mu] := \varphi_1 + \varphi_2$.

Proposition 4.1. *Suppose that $C_1^{-1}\mu(t) \le \mu(s) \le C_1\mu(t)$ for all $s\in[t,2t]$ with a constant $C_1>1$, $\mu(t)\sim\mu_*(t)\le\sqrt{t/2}$; μ_1 satisfies $|\mu_1|\le\mu/2$. Then we have*

$$\begin{aligned} |\varphi[\mu]| \lesssim\ & (\mu t^{-1} + g[\mu])\mathbf{1}_{\{r\le 2t^{\frac12}\}} + \left(\mu r^{-2} + g[\mu]\right)e^{-\frac{r^2}{32t}}\mathbf{1}_{\{r>2t^{\frac12}\}} \\ & + \sup_{t_1\in[t/2,t]}|\dot\mu(t_1)|\left(\langle\ln(\mu^{-1}t^{\frac12})\rangle\mathbf{1}_{\{r\le\mu(t)\}} + \langle\ln(r^{-1}t^{\frac12})\rangle\mathbf{1}_{\{\mu(t)<r\le 2t^{\frac12}\}} + tr^{-2}e^{-\frac{r^2}{32t}}\mathbf{1}_{\{r>2t^{\frac12}\}}\right) \end{aligned} \tag{4.19}$$

where $g[\mu] := t^{-2}\int_{t_0/2}^{t}\left(s^{-1}\mu^3(s) + s|\dot\mu(s)|\right)ds$. For $r\le 2t^{\frac12}$,

$$\begin{aligned} &F[\mu](r,t) := \varphi[\mu] + 2^{-1}\Big(\mu t^{-1} + \int_{t/2}^{t-\mu_*^2}\frac{\dot\mu(s)}{t-s}ds\Big) \\ &= \mu\left(\hat\varphi_1 + 2^{-1}t^{-1}\right) + T_4\bullet\left(r^{-1}\mathcal{E}_{12}[\mu] - \dot\mu\hat\varphi_1\right)(r,t,\frac{t_0}{2}) + T_4\bullet(r^{-1}\mathcal{E}_2[\mu])(r,t,\frac{t_0}{2}) + 2^{-1}\int_{t/2}^{t-\mu_*^2}\frac{\dot\mu(s)}{t-s}ds \\ &= O\Big(\mu t^{-2}r^2 + \sup_{t_1\in[t/2,t]}|\dot\mu(t_1)| + \min\left\{\mu^{-1}r,\ln t\right\}\sup_{t_1\in[t/2,t-\mu_*^2(t)]}|\dot\mu(t_1)| + g[\mu]\Big). \end{aligned} \tag{4.20}$$

$$\begin{aligned} &\left|\varphi[\mu+\mu_1] - \varphi[\mu]\right| \lesssim \left(|\mu_1|t^{-1} + \tilde g[\mu,\mu_1]\right)\mathbf{1}_{\{r\le 2t^{\frac12}\}} + \Big(\sup_{t_1\in[t/2,t]}|\mu_1(t_1)|r^{-2} + \tilde g[\mu,\mu_1]\Big)e^{-\frac{r^2}{32t}}\mathbf{1}_{\{r>2t^{\frac12}\}} \\ &+ \sup_{t_1\in[t/2,t]}\left(|\dot\mu_1| + \mu^{-1}|\dot\mu\mu_1|\right)(t_1)\left(\langle\ln(\mu^{-1}t^{\frac12})\rangle\mathbf{1}_{\{r\le\mu(t)\}} + \langle\ln(r^{-1}t^{\frac12})\rangle\mathbf{1}_{\{\mu(t)<r\le 2t^{\frac12}\}} + tr^{-2}e^{-\frac{r^2}{32t}}\mathbf{1}_{\{r>2t^{\frac12}\}}\right) \end{aligned} \tag{4.21}$$

where

$$\tilde g[\mu,\mu_1] := t^{-2}\mu^2\sup_{t_1\in[t/2,t]}|\mu_1(t_1)| + t^{-2}\int_{t_0/2}^{t/2}\left[s\left(|\dot\mu_1| + \mu^{-1}|\dot\mu\mu_1|\right)(s) + s^{-1}|\mu_1(s)|\mu^2(s)\right]ds. \tag{4.22}$$

For $r \le 2t^{\frac{1}{2}}$,

$$\begin{aligned}
&\tilde{F}[\mu,\mu_1](r,t) := \varphi[\mu+\mu_1]-\varphi[\mu]+2^{-1}\Big(\mu_1 t^{-1}+\int_{t/2}^{t-\mu_*^2}\frac{\dot\mu_1(s)}{t-s}ds\Big)\\
&=\mu_1\left(\hat\varphi_1+2^{-1}t^{-1}\right)+T_4\bullet\left(r^{-1}\left(\mathcal{E}_{12}[\mu+\mu_1]-\mathcal{E}_{12}[\mu]\right)-\dot\mu_1\hat\varphi_1\right)(r,t,\frac{t_0}{2})\\
&\quad+T_4\bullet\left(r^{-1}\left(\mathcal{E}_{2}[\mu+\mu_1]-\mathcal{E}_{2}[\mu]\right)\right)(r,t,\frac{t_0}{2})+2^{-1}\int_{t/2}^{t-\mu_*^2}\frac{\dot\mu_1(s)}{t-s}ds\\
&=O\Big(|\mu_1|t^{-2}r^2+\left(1+\min\left\{\mu^{-1}r,\ln t\right\}\right)\sup_{t_1\in[t/2,t]}\left(|\dot\mu_1|+\mu^{-1}|\dot\mu\mu_1|\right)(t_1)+\tilde g[\mu,\mu_1]\Big).
\end{aligned}\tag{4.23}$$

Remark 4.2. *The precise form (4.20), (4.23) are prepared for solving the leading term and minor terms of the scaling parameter* μ *respectively.*

Proof. The proof is similar to [100, Lemma 2.1, Lemma 2.2]. See also [18, Lemma 7.1, Lemma 7.3]. First, we estimate φ_1 given in (4.16). The first term $\hat\varphi_1$ is deduced in [100, Lemma 2.1], which satisfies

$$\partial_t\hat\varphi_1=\partial_{rr}\hat\varphi_1+\frac{3}{r}\partial_r\hat\varphi_1+2t^{-2}\left(z^{-2}\eta''(z)+2^{-1}z^{-1}\eta'(z)-z^{-3}\eta'(z)\right).$$

By (4.11) and [100, Lemma A.1], we have

$$\left|T_4\bullet\left(r^{-1}\mathcal{E}_{12}[\mu]\right)\right|\lesssim t^{-2}e^{-\frac{r^2}{16t}}\int_{t_0/2}^{t/2}\mu^3(s)s^{-1}ds+\mu^3t^{-2}\mathbf{1}_{\{r\le t^{\frac12}\}}+\mu^3t^{-1}r^{-2}e^{-\frac{r^2}{16t}}\mathbf{1}_{\{r>t^{\frac12}\}}.\tag{4.24}$$

$$\begin{aligned}
&r^{-1}\left(\mathcal{E}_{12}[\mu+\mu_1]-\mathcal{E}_{12}[\mu]\right)=\left[\frac1t\eta''(z)+\left(\frac{r}{2t\sqrt t}+\frac{1}{r\sqrt t}\right)\eta'(z)\right]\int_0^1\frac{-2\mu_1(\mu+\theta\mu_1)^2}{r^2\left[(\mu+\theta\mu_1)^2+r^2\right]}d\theta\\
&+\frac{\eta'(z)}{\sqrt t}\frac{4\mu_1}{r^3}\int_0^1\frac{3r^2(\mu+\theta\mu_1)^2+(\mu+\theta\mu_1)^4}{\left[r^2+(\mu+\theta\mu_1)^2\right]^2}d\theta,
\end{aligned}\tag{4.25}$$

which implies

$$r^{-1}\left|\mathcal{E}_{12}[\mu+\mu_1]-\mathcal{E}_{12}[\mu]\right|\lesssim|\mu_1|\mu^2t^{-3}\mathbf{1}_{\{\sqrt t\le r\le 2\sqrt t\}}.\tag{4.26}$$

So we get

$$\begin{aligned}
&\left|T_4\bullet\left(r^{-1}\left(\mathcal{E}_{12}[\mu+\mu_1]-\mathcal{E}_{12}[\mu]\right)\right)\right|\\
&\lesssim t^{-2}e^{-\frac{r^2}{16t}}\int_{t_0/2}^{t/2}|\mu_1(s)|\mu^2(s)s^{-1}ds+\sup_{t_1\in[t/2,t]}|\mu_1(t_1)|\left(\mu^2t^{-2}\mathbf{1}_{\{r\le t^{\frac12}\}}+\mu^2t^{-1}r^{-2}e^{-\frac{r^2}{16t}}\mathbf{1}_{\{r>t^{\frac12}\}}\right).
\end{aligned}\tag{4.27}$$

Note that $|\dot\mu\hat\varphi_1|\lesssim|\dot\mu|t^{-1}\mathbf{1}_{\{r\le t^{\frac12}\}}+|\dot\mu|r^{-2}e^{-\frac{r^2}{4t}}\mathbf{1}_{\{r>t^{\frac12}\}}$. By Lemma A.2, one has

$$\left|T_4\bullet\left(\dot\mu\hat\varphi_1\right)\right|\lesssim t^{-2}e^{-\frac{r^2}{32t}}\int_{t_0/2}^{t/2}s|\dot\mu(s)|ds+\sup_{t_1\in[t/2,t]}|\dot\mu(t_1)|\left(\mathbf{1}_{\{r\le t^{\frac12}\}}+tr^{-2}e^{-\frac{r^2}{32t}}\mathbf{1}_{\{r>t^{\frac12}\}}\right).\tag{4.28}$$

Similarly,

$$\left|T_4\bullet\left(\dot\mu_1\hat\varphi_1\right)\right|\lesssim t^{-2}e^{-\frac{r^2}{32t}}\int_{t_0/2}^{t/2}s|\dot\mu_1(s)|ds+\sup_{t_1\in[t/2,t]}|\dot\mu_1(t_1)|\left(\mathbf{1}_{\{r\le t^{\frac12}\}}+tr^{-2}e^{-\frac{r^2}{32t}}\mathbf{1}_{\{r>t^{\frac12}\}}\right).\tag{4.29}$$

Next, we consider φ_2. By Lemma A.1,

$$\begin{aligned}
&\left|T_4\bullet\left(r^{-1}\mathcal{E}_2\right)\right|\lesssim t^{-2}e^{-\frac{r^2}{16t}}\int_{t_0/2}^{t/2}s|\dot\mu(s)|ds\\
&\quad+\sup_{t_1\in[t/2,t]}|\dot\mu(t_1)|\left(\langle\ln(\mu^{-1}t^{\frac12})\rangle\mathbf{1}_{\{r\le\mu(t)\}}+\langle\ln(r^{-1}t^{\frac12})\rangle\mathbf{1}_{\{\mu(t)<r\le t^{\frac12}\}}+tr^{-2}e^{-\frac{r^2}{16t}}\mathbf{1}_{\{r>t^{\frac12}\}}\right).
\end{aligned}\tag{4.30}$$

Combining (4.17), (4.24), (4.28), (4.30), we get (4.19). Note that

$$r^{-1}\left(\mathcal{E}_2[\mu+\mu_1]-\mathcal{E}_2[\mu]\right)=-2\left[\frac{\dot\mu_1}{r^2+(\mu+\mu_1)^2}+\frac{-\dot\mu\mu_1\left(2\mu+\mu_1\right)}{\left[r^2+(\mu+\mu_1)^2\right]\left(r^2+\mu^2\right)}\right]\eta(z).\tag{4.31}$$

For μ_1 satisfying $|\mu_1|\le\mu/2$, we have

$$\left|r^{-1}\left(\mathcal{E}_2[\mu+\mu_1]-\mathcal{E}_2[\mu]\right)\right|\lesssim\left(|\dot\mu_1|+\mu^{-1}\left|\dot\mu\mu_1\right|\right)(r+\mu)^{-2}\eta(z).\tag{4.32}$$

By Lemma A.1,

$$\left|T_4 \bullet \left(r^{-1}\left(\mathcal{E}_2[\mu+\mu_1]-\mathcal{E}_2[\mu]\right)\right)\right| \lesssim t^{-2} e^{-\frac{r^2}{16t}} \int_{t_0/2}^{t/2} s\left(|\dot\mu_1|+\mu^{-1}|\dot\mu\mu_1|\right)(s)ds$$
$$+\sup_{t_1\in[t/2,t]}\left|\left(|\dot\mu_1|+\mu^{-1}|\dot\mu\mu_1|\right)(t_1)\right|\left(\langle\ln(\mu^{-1}t^{\frac12})\rangle \mathbf{1}_{\{r\le\mu(t)\}}+\langle\ln(r^{-1}t^{\frac12})\rangle\mathbf{1}_{\{\mu(t)<r\le t^{\frac12}\}}+tr^{-2}e^{-\frac{r^2}{16t}}\mathbf{1}_{\{r>t^{\frac12}\}}\right). \tag{4.33}$$

Combining (4.17), (4.27), (4.29), (4.33), we get (4.21).

In order to extract the dominating part of φ_2 for later purpose of solving the orthogonal equation, we split φ_2 into several parts to estimate.

$$\varphi_2=\left(\int_{t_0/2}^{t/2}+\int_{t/2}^{t-\mu_*^2(t)}+\int_{t-\mu_*^2(t)}^{t}\right)\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|x-y|^2}{4(t-s)}}|y|^{-1}\mathcal{E}_2[\mu](y,s)dyds := I_1[\mu]+I_2[\mu]+I_3[\mu].$$

For I_1, by Lemma A.1 (for the cases $v(s)=|\dot\mu(s)|\mathbf{1}_{\{t_0/2\le s\le t/2\}}$ and $v(s)=\left(|\dot\mu_1|+\mu^{-1}|\dot\mu\mu_1|\right)(s)\mathbf{1}_{\{t_0/2\le s\le t/2\}}$), we have

$$|I_1[\mu]|\lesssim t^{-2}e^{-\frac{r^2}{16t}}\int_{t_0/2}^{t/2}s|\dot\mu(s)|ds, \quad \left|I_1[\mu+\mu_1]-I_1[\mu]\right|\lesssim t^{-2}e^{-\frac{r^2}{16t}}\int_{t_0/2}^{t/2}s\left(|\dot\mu_1|+\mu^{-1}|\dot\mu\mu_1|\right)(s)ds. \tag{4.34}$$

For I_3, given $s\in[t-\mu_*^2(t),t]$, we have

$$\left|r^{-1}\mathcal{E}_2[\mu](r,s)\right|\lesssim \sup_{t_1\in[t-\mu_*^2(t),t]}|\dot\mu(t_1)|\left(r+\mu(t)\right)^{-2}\mathbf{1}_{\{r\le 2t^{\frac12}\}},$$
$$\left|r^{-1}\left(\mathcal{E}_2[\mu+\mu_1]-\mathcal{E}_2[\mu]\right)(r,s)\right|\lesssim \sup_{t_1\in[t-\mu_*^2(t),t]}\left(|\dot\mu_1|+\mu^{-1}|\dot\mu\mu_1|\right)(t_1)\left(r+\mu(t)\right)^{-2}\mathbf{1}_{\{r\le 2t^{\frac12}\}}.$$

Then

$$\begin{aligned}&|I_3[\mu]|\lesssim \sup_{t_1\in[t-\mu_*^2(t),t]}|\dot\mu(t_1)|\int_{t-\mu_*^2(t)}^{t}(t-s)^{-2}\int_0^{2\sqrt t}e^{-\frac{r^2}{4(t-s)}}\left(r+\mu(t)\right)^{-2}r^3drds\lesssim \sup_{t_1\in[t-\mu_*^2(t),t]}|\dot\mu(t_1)|,\\ &|I_3[\mu+\mu_1]-I_3[\mu]|\lesssim \sup_{t_1\in[t-\mu_*^2(t),t]}\left(|\dot\mu_1|+\mu^{-1}|\dot\mu\mu_1|\right)(t_1).\end{aligned} \tag{4.35}$$

For I_2, more delicate calculations are needed to single out the leading term. Set

$$I_2[\mu](0,t):=I_*[\mu]+I_{021}[\mu]+I_{022}[\mu]$$

where

$$\begin{aligned}I_*[\mu]:=&-2\int_{t/2}^{t-\mu_*^2(t)}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|y|^2}{4(t-s)}}\dot\mu(s)|y|^{-2}dyds=-2^{-1}\int_{t/2}^{t-\mu_*^2(t)}\frac{\dot\mu(s)}{t-s}ds,\\ I_{021}[\mu]:=&\,2\int_{t/2}^{t-\mu_*^2(t)}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|y|^2}{4(t-s)}}\dot\mu(s)|y|^{-2}\Big(1-\eta(\frac{|y|}{\sqrt s})\Big)dyds,\\ I_{022}[\mu]:=&\int_{t/2}^{t-\mu_*^2(t)}\int_{\mathbb{R}^4}(4\pi(t-s))^{-2}e^{-\frac{|y|^2}{4(t-s)}}\Big(\frac{-2\dot\mu(s)}{|y|^2+\mu^2(s)}+2\dot\mu(s)|y|^{-2}\Big)\eta(\frac{|y|}{\sqrt s})dyds.\end{aligned}$$

For I_{021}, by the same calculation in [100, p. 10], we get

$$|I_{021}[\mu]|\lesssim \sup_{t_1\in[t/2,t-\mu_*^2(t)]}|\dot\mu(t_1)|, \quad \left|I_{021}[\mu+\mu_1]-I_{021}[\mu]\right|\lesssim \sup_{t_1\in[t/2,t-\mu_*^2(t)]}|\dot\mu_1(t_1)|. \tag{4.36}$$

For I_{022}, since

$$\begin{aligned}&\frac{-2\dot\mu(s)}{|y|^2+\mu^2(s)}+2\dot\mu(s)|y|^{-2}=\frac{2\dot\mu(s)\mu^2(s)}{|y|^2\left(|y|^2+\mu^2(s)\right)},\\ &\frac{2\left(\dot\mu+\dot\mu_1\right)(s)\left(\mu+\mu_1\right)^2(s)}{|y|^2\left[|y|^2+\left(\mu+\mu_1\right)^2(s)\right]}-\frac{2\dot\mu(s)\mu^2(s)}{|y|^2\left(|y|^2+\mu^2(s)\right)}\\ =&\frac{2}{|y|^2}\left\{\frac{\dot\mu_1(s)(\mu+\mu_1)^2(s)}{|y|^2+(\mu+\mu_1)^2(s)}+\frac{\dot\mu(s)\mu_1(s)\left(2\mu+\mu_1\right)(s)|y|^2}{[|y|^2+(\mu+\mu_1)^2(s)]\left(|y|^2+\mu^2(s)\right)}\right\}\\ =&\,O\left(\mu^2\left(|\dot\mu_1|+\mu^{-1}|\dot\mu\mu_1|\right)|y|^{-2}\left(|y|+\mu\right)^{-2}\right),\end{aligned} \tag{4.37}$$

by the same calculation in [100, pp. 10-11], we have

$$|I_{022}[\mu]| \lesssim \sup_{t_1\in[t/2,t-\mu_*^2(t)]} |\dot{\mu}(t_1)|, \quad |I_{022}[\mu+\mu_1]-I_{022}[\mu]| \lesssim \sup_{t_1\in[t/2,t-\mu_*^2(t)]} \left|\left(|\dot{\mu}_1|+\mu^{-1}|\dot{\mu}\mu_1|\right)(t_1)\right|. \tag{4.38}$$

By the same calculation in [100, pp. 11-12], then

$$\begin{aligned} &|I_2[\mu](r,t)-I_2[\mu](0,t)| \lesssim \min\left\{\mu^{-1}r, \ln t\right\} \sup_{t_1\in[t/2,t-\mu_*^2(t)]} |\dot{\mu}(t_1)|,\\ &|(I_2[\mu+\mu_1](r,t)-I_2[\mu+\mu_1](0,t))-(I_2[\mu](r,t)-I_2[\mu](0,t))|\\ \lesssim\ & \min\left\{\mu^{-1}r, \ln t\right\} \sup_{t_1\in[t/2,t-\mu_*^2(t)]} \left(|\dot{\mu}_1|+\mu^{-1}|\dot{\mu}\mu_1|\right)(t_1). \end{aligned} \tag{4.39}$$

Combining (4.17), (4.24), (4.28), (4.34), (4.35), (4.36), (4.38), and (4.39), we get (4.20). Combining (4.17), (4.27), (4.29), (4.34), (4.35), (4.36), (4.38), and (4.39), we get (4.23). The proposition is concluded.

□

4.2. **Elliptic improvement and the leading dynamics of $\mu(t)$.** In order to improve the time decay of the error, we will introduce Φ_e by solving the linearized elliptic equation. Let us first denote

$$v_1(r,t) := \eta(z)Q_\mu + \Phi_1 + \Phi_2 + \Psi_* + \eta(4z)\Phi_e$$

where $\eta(4z)$ is used to restrict the influence of Φ_e within the self-similar region. Then we compute

$$\begin{aligned} E[v_1] =& -\partial_t\left(\eta(4z)\Phi_e\right) + \partial_{rr}\left(\eta(4z)\Phi_e\right) + \frac{1}{r}\partial_r\left(\eta(4z)\Phi_e\right) + \frac{1}{r^2}(\Phi_1+\Phi_2+\Psi_*)\\ & - \frac{\sin[2(\eta(z)Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e)]}{2r^2} + \eta(z)\frac{\sin(2Q_\mu)}{2r^2}\\ =& -\partial_t\left(\eta(4z)\Phi_e\right) + \partial_{rr}\left(\eta(4z)\Phi_e\right) + \frac{1}{r}\partial_r\left(\eta(4z)\Phi_e\right) - \eta(4z)\frac{\cos(2Q_\mu)}{r^2}\Phi_e\\ & - \eta(z)\frac{\cos(2Q_\mu)-1}{r^2}\left(\Phi_1+\Phi_2+\Psi_*\right) + E_e \end{aligned} \tag{4.40}$$

where

$$\begin{aligned} E_e :=& -\eta(z)\frac{1}{2r^2}\Big[\sin[2(Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e)] - \sin(2Q_\mu) - 2\cos(2Q_\mu)\left(\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e\right)\Big]\\ &+ \frac{1}{2r^2}\Big[-\sin[2(\eta(z)Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e)] + \eta(z)\sin[2(Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e)]\\ &+ 2\left(1-\eta(z)\right)\left(\Phi_1+\Phi_2+\Psi_*\right)\Big]. \end{aligned} \tag{4.41}$$

Roughly speaking, we will choose $\Phi_e(\rho,t)$ which solves

$$\partial_{rr}\Phi_e + \frac{1}{r}\partial_r\Phi_e - \frac{\cos(2Q_\mu)}{r^2}\Phi_e \approx \eta(z)\frac{\cos(2Q_\mu)-1}{r^2}(\Phi_1+\Phi_2+\Psi_*).$$

By (4.8), it is equivalent to

$$\partial_{\rho\rho}\Phi_e + \frac{1}{\rho}\partial_\rho\Phi_e - \frac{\rho^4-6\rho^2+1}{\rho^2(\rho^2+1)^2}\Phi_e \approx \eta(\frac{\mu\rho}{\sqrt{t}})\mu\frac{-8\rho}{(\rho^2+1)^2}\left(\varphi[\mu](\mu\rho,t)+\psi_*(\mu\rho,t)\right).$$

The linearly independent kernels $\mathcal{Z}, \tilde{\mathcal{Z}}$ of the homogeneous part satisfying the Wronskian $W[\mathcal{Z},\tilde{\mathcal{Z}}] = \rho^{-1}$ are given as follows:

$$\mathcal{Z}(\rho) = \frac{\rho}{\rho^2+1}, \quad \tilde{\mathcal{Z}}(\rho) = \frac{\rho^4+4\rho^2\ln\rho-1}{2\rho(\rho^2+1)}.$$

Let us write the orthogonality

$$\begin{aligned} \mathcal{M}[\mu] :=& \int_0^\infty \eta(\frac{\mu\rho}{\sqrt{t}})\frac{8\rho}{(\rho^2+1)^2}\left(\varphi[\mu](\mu\rho,t)+\psi_*(\mu\rho,t)\right)\mathcal{Z}(\rho)\rho d\rho\\ =& \int_0^\infty \eta(\frac{\mu\rho}{\sqrt{t}})\frac{8\rho^3}{(\rho^2+1)^3}\left(\varphi[\mu](\mu\rho,t)+\psi_*(\mu\rho,t)\right)d\rho. \end{aligned} \tag{4.42}$$

Written as the leading term of μ, μ_0 will be determined soon. By (4.13) and using (4.20) for the case $\mu_* = \mu_0$, we have

$$
\begin{aligned}
\mathcal{M}[\mu] &= \int_0^\infty \eta(\frac{\mu\rho}{\sqrt{t}})\frac{8\rho^3}{(\rho^2+1)^3}\Big[-2^{-1}\Big(\mu t^{-1} + \int_{t/2}^{t-\mu_0^2}\frac{\dot\mu(s)}{t-s}ds\Big) + O\Big(\mu^3 t^{-2}\rho^2 + \langle\rho\rangle \sup_{t_1\in[t/2,t]}|\dot\mu(t_1)| + g[\mu]\Big) \\
&\quad + v_\gamma(t)(C_\gamma + g_\gamma(t)) + O(\mu\rho t^{-\frac12}v_\gamma(t))\Big]d\rho \\
&= \Big[-\Big(\mu t^{-1} + \int_{t/2}^{t-\mu_0^2}\frac{\dot\mu(s)}{t-s}ds\Big) + 2v_\gamma(t)(C_\gamma + g_\gamma(t))\Big]\Big[1 + \int_0^\infty\Big(\eta(\frac{\mu\rho}{\sqrt{t}})-1\Big)\frac{4\rho^3}{(\rho^2+1)^3}d\rho\Big] \\
&\quad + O\Big(\mu^3 t^{-2}\ln(t^{\frac12}\mu^{-1}) + \sup_{t_1\in[t/2,t]}|\dot\mu(t_1)| + g[\mu]\Big) + O(\mu t^{-\frac12}v_\gamma(t)),
\end{aligned}
\tag{4.43}
$$

where we used $\int_0^\infty \frac{4\rho^3}{(\rho^2+1)^3}d\rho = 1$.

In order to make $\mathcal{M}[\mu]$ close to 0, we single out the leading terms

$$
\mu t^{-1} + \int_{t/2}^{t-\mu_0^2}\frac{\dot\mu(s)}{t-s}ds \approx 2v_\gamma(t)\left(C_\gamma + g_\gamma(t)\right). \tag{4.44}
$$

The leading term μ_0 of the scaling parameter μ will be derived by balancing (4.44). Suppose μ_0 has the form $\mu_0(t) = c_1 t^{1-p_0}(\ln t)^{-1}$ with $1/2 \le p_0 < 1$, then $\dot\mu_0(t) = c_1(1-p_0)t^{-p_0}(\ln t)^{-1}\left[1-(1-p_0)^{-1}(\ln t)^{-1}\right]$. For $2 \le t_1 \le t/2$, one has

$$
\begin{aligned}
&\int_{t_1}^{t-\mu_0^2(t)}\frac{\dot\mu_0(s)}{t-s}ds = \int_{\frac{t_1}{t}}^{1-\frac{\mu_0^2(t)}{t}}\frac{\dot\mu_0(tz)}{1-z}dz \\
&= c_1(1-p_0)t^{-p_0}\int_{\frac{t_1}{t}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}z^{-p_0}(\ln(tz))^{-1}\left[1-(1-p_0)^{-1}(\ln(tz))^{-1}\right]dz \\
&= c_1(1-p_0)(2p_0-1)t^{-p_0} + O(t^{-p_0}(\ln t)^{-1}\ln\ln t),
\end{aligned}
$$

where we have used the following estimates in the last step

$$
\begin{aligned}
&\int_{\frac{t_1}{t}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}z^{-p_0}(\ln(tz))^{-1}dz \\
&= (\ln t)^{-1}\int_{\frac{t_1}{t}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}z^{-p_0}dz + \int_{\frac{t_1}{t}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}z^{-p_0}\left((\ln(tz))^{-1}-(\ln t)^{-1}\right)dz \\
&= (\ln t)^{-1}\int_{\frac{t_1}{t}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}dz + (\ln t)^{-1}\int_{\frac{t_1}{t}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}\left(z^{-p_0}-1\right)dz \\
&\quad + \int_{\frac{t_1}{t}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}z^{-p_0}\frac{-\ln z}{(\ln t+\ln z)\ln t}dz \\
&= (\ln t)^{-1}\left(-\ln(t^{-1}\mu_0^2(t)) + \ln(1-t^{-1}t_1)\right) + O((\ln t)^{-1}) \\
&= (\ln t)^{-1}\left(-\ln(c_1^2) - (1-2p_0)\ln t + 2\ln\ln t + \ln(1-t_1t^{-1})\right) + O((\ln t)^{-1}) \\
&= 2p_0 - 1 + O((\ln t)^{-1}\ln\ln t);
\end{aligned}
$$

and

$$
\int_{\frac{t_1}{t}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}z^{-p_0}(\ln(tz))^{-2}dz = O((\ln t)^{-1})
$$

since

$$\int_{\frac{1}{2}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}z^{-p_0}(\ln(tz))^{-2}dz = O((\ln t)^{-2})\int_{\frac{1}{2}}^{1-\frac{\mu_0^2(t)}{t}}(1-z)^{-1}dz = O((\ln t)^{-1}),$$

$$\int_{\frac{t_1}{t}}^{\frac{1}{2}}(1-z)^{-1}z^{-p_0}(\ln(tz))^{-2}dz \sim \int_{\frac{t_1}{t}}^{\frac{1}{2}} z^{-p_0}(\ln(tz))^{-2}dz = t^{p_0-1}\int_{t_1}^{\frac{t}{2}} a^{-p_0}(\ln a)^{-2}da = O((\ln t)^{-2}).$$

- For $1<\gamma<2$, in order to balance out (4.44),

$$c_1 t^{-p_0}(\ln t)^{-1} + c_1(1-p_0)(2p_0-1)t^{-p_0} + O\left(t^{-p_0}(\ln t)^{-1}\ln\ln t\right) \approx 2v_\gamma(t)\left(C_\gamma + g_\gamma(t)\right),$$

we take $p_0=\frac{\gamma}{2}$, $c_1=(1-\frac{\gamma}{2})^{-1}(\gamma-1)^{-1}2C_\gamma$, which implies $\mu_0(t)=(1-\frac{\gamma}{2})^{-1}(\gamma-1)^{-1}2C_\gamma t^{1-\frac{\gamma}{2}}(\ln t)^{-1}$.
- For $\gamma>2$, by the same argument in [100, pp. 13-14], a good approximation is $\mu_0=(\ln t)^{-1}$.
- For $\gamma=2$, we choose $\mu_0=2C_\gamma+(\ln t)^{-1}$ by a similar argument as in the case $\gamma>2$.

In conclusion, the non-local problem (4.44) has a good approximation of the form

$$\mu_0(t) := \begin{cases} (1-\frac{\gamma}{2})^{-1}(\gamma-1)^{-1}2C_\gamma t^{1-\frac{\gamma}{2}}(\ln t)^{-1}, & 1<\gamma<2\\ 2C_\gamma+(\ln t)^{-1}, & \gamma=2\\ (\ln t)^{-1}, & \gamma>2, \end{cases} \tag{4.45}$$

where the constant C_γ is defined in (4.12). Obviously, $\mu_0 \ll t^{\frac{1}{2}}$ since $\gamma>1$. Moreover, for $\gamma\geq 2$, by the similar calculation for the case $1<\gamma<2$ shown above (see also [100, p. 14], which is related to the case $\gamma>2$), we obtain

$$-\left(\mu_0 t^{-1} + \int_{t/2}^{t-\mu_0^2}\frac{\dot\mu_0(s)}{t-s}ds\right) + 2v_\gamma(t)(C_\gamma+g_\gamma(t)) = O\left(\begin{cases} t^{-\frac{\gamma}{2}}(\ln t)^{-1}\ln\ln t, & 1<\gamma<2\\ t^{-1}(\ln t)^{-2}, & \gamma=2\\ t^{-1}(\ln t)^{-2}\ln\ln t, & \gamma>2 \end{cases}\right) = O\left(\ln\ln t|\dot\mu_0|\right). \tag{4.46}$$

This cancellation is crucial when solving the orthogonal equations.

After several times correction, we achieve that for t_0 sufficiently large, there exists $\bar\mu_0=\bar\mu_0(t)$ such that

$$\begin{aligned} &\bar\mu_0 = \mu_0\left(1+O\left((\ln t)^{-1}\ln\ln t\right)\right), \quad \dot{\bar\mu}_0 = \dot\mu_0\left(1+O\left((\ln t)^{-1}\ln\ln t\right)\right),\\ &8\bar\mu_0/9 \le \mu_0 \le 9\bar\mu_0/8, \quad 8|\dot{\bar\mu}_0|/9 \le |\dot\mu_0| \le 9|\dot{\bar\mu}_0|/8, \qquad \mathcal{M}[\bar\mu_0]=O(t^{-2}). \end{aligned} \tag{4.47}$$

The following corollary is a direct consequence of Proposition 4.1 by our choice of μ_0 in (4.45).

Corollary 4.1. *For $f\sim\mu_0$, $|\dot f|\lesssim|\dot\mu_0|$ and μ_1 satisfying $|\mu_1|\le f/2$, we have*

$$g[f]\lesssim|\dot\mu_0|\sim\begin{cases} t^{-\frac{\gamma}{2}}(\ln t)^{-1}, & 1<\gamma<2\\ t^{-1}(\ln t)^{-2}, & \gamma\ge 2. \end{cases} \tag{4.48}$$

$$|\varphi[f]|\lesssim\begin{cases} t^{-\frac{\gamma}{2}}\mathbf{1}_{\{r\le\mu_0\}} + t^{-\frac{\gamma}{2}}(\ln t)^{-1}\left(\langle\ln(r^{-1}t^{\frac{1}{2}})\rangle\mathbf{1}_{\{\mu_0<r\le t^{\frac{1}{2}}\}} + e^{-\frac{r^2}{32t}}\mathbf{1}_{\{r>t^{\frac{1}{2}}\}}\right), & \text{if } 1<\gamma<2\\ t^{-1}e^{-\frac{r^2}{32t}}, & \text{if } \gamma=2\\ (t\ln t)^{-1}e^{-\frac{r^2}{32t}}, & \text{if } \gamma>2. \end{cases} \tag{4.49}$$

Combining (4.13) and (4.49), we have

$$|\varphi[f]|+|\psi_*|\lesssim\begin{cases} t^{-\frac{\gamma}{2}}\mathbf{1}_{\{r\le 2t^{\frac{1}{2}}\}} + r^{-\gamma}\mathbf{1}_{\{r>2t^{\frac{1}{2}}\}}, & 1<\gamma\le 2\\ (t\ln t)^{-1}\mathbf{1}_{\{r\le 2t^{\frac{1}{2}}\}} + \left(r^{-\gamma}+(t\ln t)^{-1}e^{-\frac{r^2}{32t}}\right)\mathbf{1}_{\{r>2t^{\frac{1}{2}}\}}, & \gamma>2. \end{cases} \tag{4.50}$$

$$\begin{aligned} |\partial_r\varphi[f]|+|\partial_r\psi_*| \lesssim{}& \mathbf{1}_{\{r\le 9\sqrt{t}\}}\mu_0^{-1}\begin{cases} t^{-\frac{\gamma}{2}}(\ln t)^{-\frac{1}{2}}, & 1<\gamma<2\\ t^{-1}(\ln t)^{-1}, & \gamma=2\\ t^{-1}(\ln t)^{-\frac{3}{2}}, & \gamma>2 \end{cases}\\ &+\mathbf{1}_{\{r>9\sqrt{t}\}}t^{-\frac{1}{2}}\begin{cases} r^{-\gamma}, & 1<\gamma\le 2\\ r^{-\gamma}+(t\ln t)^{-1}e^{-\frac{r^2}{64t}}, & \gamma>2. \end{cases} \end{aligned} \tag{4.51}$$

$$|\tilde g[f,\mu_1]|\lesssim t^{-2}\mu_0^2\sup_{t_1\in[t/2,t]}|\mu_1(t_1)| + t^{-2}\int_{t_0/2}^{t/2} s\left(|\dot\mu_1|+\mu_0^{-1}|\dot\mu_0||\mu_1|\right)(s)ds. \tag{4.52}$$

$$|\varphi[f+\mu_1]-\varphi[f]| \lesssim \left(|\mu_1|t^{-1}+\tilde{g}[f,\mu_1]\right)\mathbf{1}_{\{r\le 2t^{\frac12}\}}+\Big(\sup_{t_1\in[t/2,t]}|\mu_1(t_1)|r^{-2}+\tilde{g}[f,\mu_1]\Big)e^{-\frac{r^2}{32t}}\mathbf{1}_{\{r>2t^{\frac12}\}}$$
$$+\sup_{t_1\in[t/2,t]}\left(|\dot\mu_1|+\mu_0(t)^{-1}|\dot\mu_0(t)|\,|\mu_1|\right)(t_1)\left(\ln t\mathbf{1}_{\{r\le f(t)\}}+\langle\ln(r^{-1}t^{\frac12})\rangle\mathbf{1}_{\{f(t)<r\le 2t^{\frac12}\}}+tr^{-2}e^{-\frac{r^2}{32t}}\mathbf{1}_{\{r>2t^{\frac12}\}}\right). \tag{4.53}$$

Proof. We only prove (4.51) and the other terms can be deduced by Proposition 4.1 directly. Recall (4.15), (4.10). Then

$$\partial_t\varphi[f]=\partial_{rr}\varphi[f]+\frac{3}{r}\partial_r\varphi[f]+r^{-1}\mathcal{E}_1[f]+r^{-1}\mathcal{E}_2[f]$$

where

$$\left|r^{-1}\mathcal{E}_1[f]+r^{-1}\mathcal{E}_2[f]\right|\lesssim \mathbf{1}_{\{r\le 2\sqrt{t}\}}\mu_0^{-2}\,|\dot\mu_0|\sim \mathbf{1}_{\{r\le 2\sqrt{t}\}}\mu_0^{-2}\begin{cases}t^{-\frac{\gamma}{2}}(\ln t)^{-1}, & 1<\gamma<2\\ t^{-1}(\ln t)^{-2}, & \gamma\ge 2.\end{cases}$$

Combining (4.50) and using the scaling argument twice, we have (4.51).

□

Set $\mu=\bar\mu_0+\mu_1$, where μ_1 is the next order term with the ansatz

$$|\mu_1|\le\bar\mu_0/9,\quad |\dot\mu_1|\le|\dot{\bar\mu}_0|/9. \tag{4.54}$$

Then

$$\mu_0(t)/2\le\mu(t)\le 2\mu_0(t),\quad |\dot\mu_0(t)|/2\le|\dot\mu(t)|\le 2|\dot\mu_0(t)|. \tag{4.55}$$

Combining (4.13) (4.20) and (4.46), for $r\le 2t^{\frac12}$, we have

$$\begin{aligned}\varphi[\mu]+\psi_*&=-2^{-1}\Big(\mu_1t^{-1}+\int_{t/2}^{t-\mu_0^2}\frac{\dot\mu_1(s)}{t-s}ds\Big)\\&+O\Big(\mu t^{-2}r^2+\big(1+\min\{\mu^{-1}r,\ln t\}\big)\sup_{t_1\in[t/2,t]}|\dot\mu(t_1)|+g[\mu]+\ln\ln t|\dot\mu_0|+rt^{-\frac12}v_\gamma(t)\Big)\end{aligned}\tag{4.56}$$

where we used the cancellation in $r\le 2t^{\frac12}$ due to the choice of $\bar\mu_0$.

Given $\bar\mu_0$ above, we are now able to describe Φ_e rigorously. Set $\bar\rho=\frac{r}{\bar\mu_0}$ and consider $\Phi_e=\Phi_e(\bar\rho,t)$ solving

$$\partial_{\bar\rho\bar\rho}\Phi_e+\frac{1}{\bar\rho}\partial_{\bar\rho}\Phi_e-\frac{\bar\rho^4-6\bar\rho^2+1}{\bar\rho^2(\bar\rho^2+1)^2}\Phi_e=\tilde H(\bar\rho,t)$$

where

$$\tilde H(\bar\rho,t):=\bar\mu_0\eta(\frac{\bar\mu_0\bar\rho}{\sqrt t})\frac{-8\bar\rho}{(\bar\rho^2+1)^2}\left(\varphi[\bar\mu_0](\bar\mu_0\bar\rho,t)+\psi_*(\bar\mu_0\bar\rho,t)\right)+\bar\mu_0\mathcal{M}[\bar\mu_0]\frac{\eta(\bar\rho)\mathcal{Z}(\bar\rho)}{\int_0^3\eta(x)\mathcal{Z}^2(x)xdx}.$$

Φ_e is taken as

$$\Phi_e(\bar\rho,t)=\tilde{\mathcal{Z}}(\bar\rho)\int_0^{\bar\rho}\tilde H(x,t)\mathcal{Z}(x)xdx-\mathcal{Z}(\bar\rho)\int_0^{\bar\rho}\tilde H(x,t)\tilde{\mathcal{Z}}(x)xdx.$$

By the definition of $\mathcal{M}[\bar\mu_0]$, one clearly has

$$\int_0^\infty\tilde H(x,t)\mathcal{Z}(x)xdx=0. \tag{4.57}$$

Using (4.56) for the case $\mu_1\equiv0$ and (4.48), we have

$$\tilde H=\bar\mu_0\eta(\frac{\bar\mu_0\bar\rho}{\sqrt t})\frac{-8\bar\rho}{(\bar\rho^2+1)^2}O\big(\mu_0^3t^{-2}\bar\rho^2+|\dot\mu_0|\min\{\langle\bar\rho\rangle,\ln t\}+\ln\ln t|\dot\mu_0|+\mu_0\bar\rho t^{-\frac12}v_\gamma(t)\big)+\frac{\bar\mu_0O(t^{-2})\eta(\bar\rho)\mathcal{Z}(\bar\rho)}{\int_0^3\eta(x)\mathcal{Z}^2(x)xdx}.$$

Thus

$$|\tilde H|\lesssim\mu_0\eta(\frac{\bar\mu_0\bar\rho}{\sqrt t})\bar\rho\langle\bar\rho\rangle^{-3}\begin{cases}t^{-\frac{\gamma}{2}}(\ln t)^{-1}\ln\ln t, & 1<\gamma<2\\ t^{-1}(\ln t)^{-2}\ln\ln t, & \gamma\ge2.\end{cases}$$

By (4.50), another upper bound of $\tilde H$ with faster spatial decay but slower time decay is given by

$$|\tilde H|\lesssim\mu_0\eta(\frac{\bar\mu_0\bar\rho}{\sqrt t})\bar\rho\langle\bar\rho\rangle^{-4}\begin{cases}t^{-\frac{\gamma}{2}}, & 1<\gamma\le2\\ (t\ln t)^{-1}, & \gamma>2.\end{cases}$$

Combining the above two upper bounds, $\tilde{H}$ is then bounded by

$$|\tilde{H}| \lesssim \mu_0\eta(\frac{\bar{\mu}_0\bar{\rho}}{\sqrt{t}}) \begin{cases} \min\left\{\bar{\rho}\langle\bar{\rho}\rangle^{-4}t^{-\frac{\gamma}{2}}, \bar{\rho}\langle\bar{\rho}\rangle^{-3}t^{-\frac{\gamma}{2}}(\ln t)^{-1}\ln\ln t\right\}, & 1<\gamma<2 \\ \min\left\{\bar{\rho}\langle\bar{\rho}\rangle^{-4}t^{-1}, \bar{\rho}\langle\bar{\rho}\rangle^{-3}t^{-1}(\ln t)^{-2}\ln\ln t\right\}, & \gamma=2 \\ \min\left\{\bar{\rho}\langle\bar{\rho}\rangle^{-4}(t\ln t)^{-1}, \bar{\rho}\langle\bar{\rho}\rangle^{-3}t^{-1}(\ln t)^{-2}\ln\ln t\right\}, & \gamma>2. \end{cases}$$

Using (4.57), we have

$$\langle\bar{\rho}\rangle|\partial_{\bar{\rho}}\Phi_e| + |\Phi_e| \lesssim \mu_0\bar{\rho}^3\langle\bar{\rho}\rangle^{-3} \begin{cases} \min\left\{t^{-\frac{\gamma}{2}}\langle\bar{\rho}\rangle^{-1}\ln(\bar{\rho}+2), t^{-\frac{\gamma}{2}}(\ln t)^{-1}\ln\ln t\right\}, & 1<\gamma<2 \\ \min\left\{t^{-1}\langle\bar{\rho}\rangle^{-1}\ln(\bar{\rho}+2), t^{-1}(\ln t)^{-2}\ln\ln t\right\}, & \gamma=2 \\ \min\left\{(t\ln t)^{-1}\langle\bar{\rho}\rangle^{-1}\ln(\bar{\rho}+2), t^{-1}(\ln t)^{-2}\ln\ln t\right\}, & \gamma>2. \end{cases} \tag{4.58}$$

By the same argument in [100, (2.28)], we also have

$$|\partial_t\Phi_e| \lesssim \mu_0\bar{\rho}^3\langle\bar{\rho}\rangle^{-3} \begin{cases} t^{-1-\frac{\gamma}{2}}\ln t\langle\bar{\rho}\rangle^{-1}\ln(\bar{\rho}+2), & 1<\gamma<2 \\ t^{-2}\ln t\langle\bar{\rho}\rangle^{-1}\ln(\bar{\rho}+2), & \gamma=2 \\ t^{-2}\langle\bar{\rho}\rangle^{-1}\ln(\bar{\rho}+2), & \gamma>2. \end{cases} \tag{4.59}$$

We now use the expression of Φ_e to compute the new error.

$$\begin{aligned} E[v_1] &= \Phi_e\eta'(4z)\frac{2r}{t^{\frac{3}{2}}} - \eta(4z)\partial_t\Phi_e + \frac{16}{t}\eta''(4z)\Phi_e + \frac{8}{\sqrt{t}}\eta'(4z)\partial_r\Phi_e + \frac{1}{r}\frac{4}{\sqrt{t}}\eta'(4z)\Phi_e \\ &+ (\eta(4z)-\eta(z))\frac{-8\bar{\mu}_0^{-1}\bar{\rho}}{(\bar{\rho}^2+1)^2}\left(\varphi[\bar{\mu}_0](r,t)+\psi_*(r,t)\right) \\ &+ \eta(z)\left[\left(\frac{8\mu^{-1}\rho}{(\rho^2+1)^2} - \frac{8\bar{\mu}_0^{-1}\bar{\rho}}{(\bar{\rho}^2+1)^2}\right)\left(\varphi[\bar{\mu}_0](r,t)+\psi_*(r,t)\right) + \frac{8\mu^{-1}\rho}{(\rho^2+1)^2}\left(\varphi[\mu](r,t)-\varphi[\bar{\mu}_0](r,t)\right)\right] \\ &+ \bar{\mu}_0^{-1}\mathcal{M}[\bar{\mu}_0]\frac{\eta(\bar{\rho})\mathcal{Z}(\bar{\rho})}{\int_0^3\eta(x)\mathcal{Z}^2(x)xdx} + \eta(4z)\bar{\mu}_0^{-2}\left(\cos(2Q_{\bar{\mu}_0})-\cos(2Q_\mu)\right)\bar{\rho}^{-2}\Phi_e + E_e. \end{aligned} \tag{4.60}$$

4.3. **Gluing system.** Having improved the spatial decay by non-local corrections and time decay by solving the linearized elliptic equation, we are now ready to formulate the gluing system to deal with the remaining errors. We introduce the correction term

$$\Psi(r,t) + \eta_R(\rho)\Phi(\rho,t), \quad \rho = \mu^{-1}r$$

where $\eta_R(\rho) = \eta(R^{-1}\rho)$ with $R = R(t) = t^\omega$ with $\omega > 0$ to be determined later. In order to restrict the inner problem within the self-similar region, we make the ansatz

$$\mu R \ll \sqrt{t}, \tag{4.61}$$

and by (4.45) and (4.55), it is true provided

$$0 < \omega < \begin{cases} \frac{\gamma-1}{2}, & 1<\gamma<2 \\ \frac{1}{2}, & \gamma\ge 2. \end{cases} \tag{4.62}$$

By direct calculation, one has

$$\begin{aligned} E[v_1+\Psi+\eta_R\Phi] &= -\partial_t\Psi + \partial_{rr}\Psi + \frac{1}{r}\partial_r\Psi - \frac{1}{r^2}\Psi + \Lambda[\Phi] \\ &- \eta_R\partial_t\Phi + \eta_R\mu^{-2}\partial_{\rho\rho}\Phi + \eta_R\mu^{-2}\frac{\partial_\rho\Phi}{\rho} - \mu^{-2}\frac{\rho^4-6\rho^2+1}{\rho^2(\rho^2+1)^2}\eta_R\Phi + \eta_R\rho\partial_\rho\Phi\mu^{-1}\dot{\mu} + \eta_R\frac{8\mu^{-2}}{(\rho^2+1)^2}\Psi \\ &+ (1-\eta_R)\frac{8\mu^{-2}}{(\rho^2+1)^2}\Psi - \frac{\eta(z)-1}{r^2}\cos(2Q_\mu)\Psi + \mathcal{N} + E[v_1], \end{aligned}$$

where $E[v_1]$ is given in (4.60),

$$\Lambda[\Phi] := \eta''(\frac{\rho}{R})(\mu R)^{-2}\Phi + \eta'(\frac{\rho}{R})\mu^{-2}(\rho R)^{-1}\Phi + 2(\mu R)^{-1}\mu^{-1}\eta'(\frac{\rho}{R})\partial_\rho\Phi + \eta'(\frac{\rho}{R})\frac{\rho}{R}\frac{(\mu R)'}{\mu R}\Phi \tag{4.63}$$

and we used (4.61) and

$$\frac{1}{2r^2}\left(\sin\left(2v_1\right)-\sin\left(2\left(v_1+\Psi+\eta_R\Phi\right)\right)\right)=-\frac{\eta(z)}{r^2}\cos(2Q_\mu)\left(\Psi+\eta_R\Phi\right)+\mathcal{N},$$

$$\begin{aligned}\mathcal{N}=\mathcal{N}[\Psi,\Phi,\mu_1]:=&\frac{1}{2r^2}\Bigg[\sin\left(2\left(\eta(z)Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e\right)\right)\\&-\eta(z)\sin\left(2\left(Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e\right)\right)+\eta(z)\sin\left(2\left(Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e+\Psi+\eta_R\Phi\right)\right)\\&-\sin\left(2\left(\eta(z)Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e+\Psi+\eta_R\Phi\right)\right)\Bigg]\\&+\frac{\eta(z)}{2r^2}\Bigg\{\sin\left(2\left(Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e\right)\right)-\sin(2Q_\mu)-2\cos(2Q_\mu)\left(\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e\right)\\&-\Big[\sin\left(2\left(Q_\mu+\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e+\Psi+\eta_R\Phi\right)\right)-\sin(2Q_\mu)\\&-2\cos(2Q_\mu)\left(\Phi_1+\Phi_2+\Psi_*+\eta(4z)\Phi_e+\Psi+\eta_R\Phi\right)\Big]\Bigg\}.\end{aligned}\tag{4.64}$$

In order to make $E\left[v_1+\Psi+\eta_R\Phi\right]=0$, it suffices to solve the following gluing system.

• The outer problem:

$$\begin{cases}\partial_t\Psi=\partial_{rr}\Psi+\frac{1}{r}\partial_r\Psi-\frac{1}{r^2}\Psi+\mathcal{G} & \text{for } (r,t)\in(0,\infty)\times(t_0,\infty)\\ \Psi(\cdot,t_0)=0 & \text{in } (0,\infty)\end{cases}\tag{4.65}$$

where

$$\begin{aligned}\mathcal{G}=\mathcal{G}[\Psi,\Phi,\mu_1]:=&\Lambda[\Phi]+\Phi_e\eta'(4z)\frac{2r}{t^{\frac{3}{2}}}-\eta(4z)(1-\eta_R)\partial_t\Phi_e+\frac{16}{t}\eta''(4z)\Phi_e+\frac{8}{\sqrt{t}}\eta'(4z)\partial_r\Phi_e+\frac{1}{r}\frac{4}{\sqrt{t}}\eta'(4z)\Phi_e\\&+\left(\eta(4z)-\eta(z)\right)\frac{-8\bar\mu_0^{-1}\bar\rho}{(\bar\rho^2+1)^2}\left(\varphi[\bar\mu_0](r,t)+\psi_*(r,t)\right)\\&+(1-\eta_R)\eta(z)\left[\left(\frac{8\mu^{-1}\rho}{\left(\rho^2+1\right)^2}-\frac{8\bar\mu_0^{-1}\bar\rho}{(\bar\rho^2+1)^2}\right)\left(\varphi[\bar\mu_0](r,t)+\psi_*(r,t)\right)+\frac{8\mu^{-1}\rho}{\left(\rho^2+1\right)^2}\left(\varphi[\mu](r,t)-\varphi[\bar\mu_0](r,t)\right)\right]\\&+(1-\eta_R)\eta(4z)\bar\mu_0^{-2}\left(\cos(2Q_{\bar\mu_0})-\cos(2Q_\mu)\right)\bar\rho^{-2}\Phi_e+(1-\eta_R)\left[\frac{8\mu^{-2}}{\left(\rho^2+1\right)^2}\Psi+\tilde{\mathcal{N}}\right],\end{aligned}\tag{4.66}$$

$$\tilde{\mathcal{N}}=\tilde{\mathcal{N}}[\Psi,\Phi,\mu_1]:=\mathcal{N}+E_e-\frac{\eta(z)-1}{r^2}\cos(2Q_\mu)\Psi.\tag{4.67}$$

• The inner problem:

$$\begin{cases}\mu^2\partial_t\Phi=\partial_{\rho\rho}\Phi+\frac{\partial_\rho\Phi}{\rho}-\frac{\rho^4-6\rho^2+1}{\rho^2(\rho^2+1)^2}\Phi+\dot\mu\mu\rho\partial_\rho\Phi+\mathcal{H}_1+\mathcal{H}_2 & \text{for } t\in(t_0,\infty),\rho\in(0,2R(t))\\ \Phi(\cdot,t_0)=0 & \text{in } (0,2R(t_0))\end{cases}\tag{4.68}$$

where

$$\begin{aligned}\mathcal{H}_1=\mathcal{H}_1[\Psi,\mu_1]:=&\frac{8\Psi}{\left(\rho^2+1\right)^2}+\mu^2\Bigg[\left(\frac{8\mu^{-1}\rho}{\left(\rho^2+1\right)^2}-\frac{8\bar\mu_0^{-1}\bar\rho}{(\bar\rho^2+1)^2}\right)\left(\varphi[\bar\mu_0](r,t)+\psi_*(r,t)\right)\\&+\frac{8\mu^{-1}\rho}{\left(\rho^2+1\right)^2}\left(\varphi[\mu](r,t)-\varphi[\bar\mu_0](r,t)\right)+\bar\mu_0^{-2}\left(\cos(2Q_{\bar\mu_0})-\cos(2Q_\mu)\right)\bar\rho^{-2}\Phi_e\Bigg],\end{aligned}\tag{4.69}$$

$$\mathcal{H}_2=\mathcal{H}_2[\Psi,\Phi,\mu_1]:=\mu^2\left(\tilde{\mathcal{N}}-\partial_t\Phi_e+\bar\mu_0^{-1}\mathcal{M}[\bar\mu_0]\frac{\eta(\bar\rho)\mathcal{Z}(\bar\rho)}{\int_0^3\eta(x)\mathcal{Z}^2(x)xdx}\right).\tag{4.70}$$

Set

$$\Psi:=r\psi(r,t),\quad \Phi:=\rho\phi(\rho,t).\tag{4.71}$$

In order to solve (4.65) and (4.68), it suffices to solve the following system

$$\begin{cases}\partial_t\psi = \partial_{rr}\psi + \frac{3}{r}\partial_r\psi + r^{-1}\mathcal{G}[r\psi,\rho\phi,\mu_1] & \text{for } (r,t)\in(0,\infty)\times(t_0,\infty)\\ \psi(\cdot,t_0)=0 & \text{in } (0,\infty),\end{cases} \tag{4.72}$$

$$\begin{cases}\mu^2\partial_t\phi = L_{\text{in}}[\phi] + \rho^{-1}\left(\mathcal{H}_1[\mu\rho\psi,\mu_1]+\mathcal{H}_2[\mu\rho\psi,\rho\phi,\mu_1]\right) & \text{for } t\in(t_0,\infty),\rho\in(0,2R(t))\\ \phi(\cdot,t_0)=0 & \text{in } (0,2R(t_0))\end{cases} \tag{4.73}$$

where

$$L_{\text{in}}[\phi] := \partial_{\rho\rho}\phi + \frac{3}{\rho}\partial_\rho\phi + \frac{8\phi}{(\rho^2+1)^2} + \dot{\mu}\mu\left(\phi+\rho\partial_\rho\phi\right). \tag{4.74}$$

For the dealing of the inner problem, it will be more convenient to use the time variable with

$$\tau = \tau(t) := \int_{t_0}^t \mu^{-2}(s)ds + C_\tau t_0\mu^{-2}(t_0), \quad \tau_0 := \tau(t_0) = C_\tau t_0 \mu^{-2}(t_0) \tag{4.75}$$

where C_τ is a large constant. By (4.55) and (4.45), we have

$$\tau(t)\sim t\mu_0^{-2}\sim\begin{cases} t^{\gamma-1}(\ln t)^2, & 1<\gamma<2\\ t, & \gamma=2\\ t(\ln t)^2, & \gamma>2,\end{cases} \qquad t(\tau)\sim\begin{cases}[\tau(\ln\tau)^{-2}]^{\frac{1}{\gamma-1}}, & 1<\gamma<2\\ \tau, & \gamma=2\\ \tau(\ln\tau)^{-2}, & \gamma>2,\end{cases} \tag{4.76}$$

and thus

$$|\mu(t)\dot{\mu}(t)|\lesssim\tau^{-1}(t). \tag{4.77}$$

Under the parameter restriction (4.62), we have

$$R(t(\tau))\ll\sqrt{\tau}. \tag{4.78}$$

We introduce the norm

$$\|f\|_{i,\kappa,a} := \sup_{\tau\in(\tau_0,\infty),\rho\in(0,2R(t(\tau)))} \tau^\kappa\langle\rho\rangle^a\left(\langle\rho\rangle|\partial_\rho f(\rho,\tau)|+|f(\rho,\tau)|\right) \tag{4.79}$$

for some positive constants $a,\kappa>0$ to be determined later and will solve the inner problem in the space

$$B_{\text{in}} := \Big\{f(\rho,\tau) \text{ is } C^1 \text{ in } \rho \text{ for } \tau\in(\tau_0,\infty),\ \rho\in(0,2R(t(\tau)))\ :\ \|f\|_{i,\kappa,a}\le 1\Big\}. \tag{4.80}$$

The outer problem (4.72) is handled in the following proposition.

Proposition 4.2. *Consider*

$$\begin{cases}\partial_t\psi = \partial_{rr}\psi + \dfrac{3}{r}\partial_r\psi + r^{-1}\mathcal{G}[r\psi,\rho\phi,\mu_1] & \textit{for } (r,t)\in(0,\infty)\times(t_0,\infty)\\ \partial_r\psi(0,t)=0 \ \textit{for } t\in(t_0,\infty), \quad \psi(r,t_0)=0 \ \textit{for } r\in(0,\infty)\end{cases} \tag{4.81}$$

where $\mathcal{G}$ is given in (4.66). Suppose that $\phi\in B_{\text{in}}$, μ_1 satisfies $t|\dot{\mu}_1(t)|,|\mu_1(t)|\le C_1 t(\ln t)^{-\frac{1}{2}}\vartheta(t)$ for a constant $C_1>0$ where

$$\vartheta(t) := \begin{cases} t^{-a\omega+\frac{\gamma}{2}-1-\kappa(\gamma-1)}(\ln t)^{1-2\kappa}, & 1<\gamma<2\\ t^{-a\omega-\kappa}, & \gamma=2\\ t^{-a\omega-\kappa}(\ln t)^{1-2\kappa}, & \gamma>2\end{cases} \sim \tau^{-\kappa}(t)\mu_0^{-1}R^{-a} \tag{4.82}$$

and the parameters satisfy

$$\begin{cases}\gamma-1<\kappa(\gamma-1)+a\omega<2(\gamma-1), \quad a\omega<\kappa(\gamma-1), \quad (2+a)\omega<2\kappa(\gamma-1), & \textit{if } 1<\gamma<2\\ 1<\kappa+a\omega<2, \quad a\omega<\kappa, \quad (2+a)\omega<2\kappa, & \textit{if } \gamma\ge 2,\end{cases} \tag{4.83}$$

then there exists a solution $\psi=\psi[\phi,\mu_1]$ satisfying $\psi = T_4\bullet[r^{-1}\mathcal{G}[r\psi,\rho\phi,\mu_1]](r,t,t_0)$ with the following estimates:

$$|\psi(r,t)|\lesssim\vartheta(t)\left(\mathbf{1}_{\{r\le t^{\frac12}\}}+tr^{-2}\mathbf{1}_{\{r>t^{\frac12}\}}\right), \quad |\partial_r\psi(r,t)|\lesssim\vartheta(t)\left((\mu_0R)^{-1}\mathbf{1}_{\{r\le 9\sqrt{t}\}}+t^{-\frac12}tr^{-2}\mathbf{1}_{\{r>9\sqrt{t}\}}\right), \tag{4.84}$$

$$\sup_{r\in[0,\infty)}\ \sup_{t_1,t_2\in[t-\frac{\lambda_o^2}{4},t]}\frac{|\psi(r,t_1)-\psi(r,t_2)|}{|t_1-t_2|^\alpha}\lesssim\lambda_o^{-2\alpha}\vartheta(t)+\lambda_o^{2-2\alpha}\vartheta(t)(\mu_0R)^{-2} \tag{4.85}$$

for any $\alpha\in(0,1)$, $0<\lambda_o\le t^{\frac12}$.

In order to solve the inner problem (4.73), it suffices to solve the following two equations. The inner problem with orthogonality condition:

$$\begin{cases}\mu^2\partial_t\phi_1 = L_{\rm in}[\phi_1] + \rho^{-1}\left(\mathcal{H}_1[\mu\rho\psi,\mu_1] + \mathcal{H}_*[\mu_1]\right) & \text{for } \ t\in(t_0,\infty), \rho\in(0,2R(t)) \\ \phi_1(\cdot,t_0)=0 & \text{in } \ (0,2R(t_0));\end{cases} \tag{4.86}$$

and the inner problem without orthogonality condition:

$$\begin{cases}\mu^2\partial_t\phi_2 = L_{\rm in}[\phi_2] + \rho^{-1}\left(\mathcal{H}_2[\mu\rho\psi,\rho\phi,\mu_1] - \mathcal{H}_*[\mu_1]\right) & \text{for } \ t\in(t_0,\infty), \rho\in(0,2R(t)) \\ \phi_2(\cdot,t_0)=0 & \text{in } \ (0,2R(t_0))\end{cases} \tag{4.87}$$

where $\phi=\phi_1+\phi_2$ and $\mathcal{H}_*[\mu_1]$ will be given in (4.97) later.

4.4. Solving the orthogonal equation and Hölder estimate about $\dot\mu_1$.

4.4.1. *Solving the orthogonal equation.* Hereafter, for $\vartheta(t)$ given in (4.82), we make the ansatz

$$t|\dot\mu_1(t)|, |\mu_1(t)| \le t(\ln t)^{-\frac12}\vartheta(t). \tag{4.88}$$

Under the assumption (4.83), (4.88) implies the desired ansatz (4.54) with t_0 sufficiently large.

In order to find a well-behaved solution for the orthogonal inner problem (4.86), we will use Proposition D.2 with the conditions $f(\tau)=C_f\tau^{-1}$ with a large constant C_f, $P_V=4$, $d=4$, $Z(\rho)=\frac{1}{\rho^2+1}$, $\beta=-2$,

$$R=(t(\tau))^{\omega}, \quad R_0=(t(\tau))^{\delta_0} \ \text{ with } \ 0<\delta_0<\omega. \tag{4.89}$$

By (4.78), we have $R_0\le R/3 \ll \sqrt{\tau}$. Given the definition (D.9) of $c_{\rm in}$ in Proposition D.2, roughly speaking, we will solve the orthogonal equation

$$c_{\rm in}\left[\rho^{-1}(\mathcal{H}_1[\mu\rho\psi[\phi,\mu_1]] + \mathcal{H}_*[\mu_1])\right](\tau(t)) = 0 \tag{4.90}$$

with a well-chosen function $\mathcal{H}_*[\mu_1]$ to make the orthogonal equation manageable. Note that $c_{\rm in}$ includes a minor term $c_{{\rm in},1}$ defined in Proposition D.2.

The effect from the outer solution is given by

$$\int_0^{R_0}\frac{8\mu\psi[\phi,\mu_1](\mu\rho,t)}{(\rho^2+1)^2}\frac{\rho^3}{\rho^2+1}d\rho = \mu\int_0^{R_0}\frac{8\rho^3\psi[\phi,\mu_1](\mu\rho,t)}{(\rho^2+1)^3}d\rho.$$

Recall $\bar\rho=\frac{r}{\bar\mu_0}$. Notice that for any $a,b,c\in\mathbb{R}$,

$$\mu^c\frac{\rho^a}{(\rho^2+1)^b} - \bar\mu_0^c\frac{\bar\rho^a}{(\bar\rho^2+1)^b} = \mu^{c-1}\mu_1\frac{(2b-a+c)\rho^{a+2}+(c-a)\rho^a}{(\rho^2+1)^{b+1}} + O\Big(\mu^{-1}|\mu_1|\,\mu^{c-1}|\mu_1|\,\rho^a\langle\rho\rangle^{-2b}\Big) \tag{4.91}$$

since for

$$f(\theta):=\mu_\theta^c\frac{\rho_\theta^a}{(\rho_\theta^2+1)^b}, \quad \rho_\theta:=\frac{r}{\mu_\theta}, \quad \mu_\theta:=\theta\mu+(1-\theta)\bar\mu_0=\mu+(\theta-1)\mu_1$$

we have

$$f'(\theta)=\mu_\theta^{c-1}\mu_1\frac{(2b-a+c)\rho_\theta^{a+2}+(c-a)\rho_\theta^a}{(\rho_\theta^2+1)^{b+1}}.$$

In particular,

$$\frac{8\mu^{-1}\rho}{(\rho^2+1)^2} - \frac{8\bar\mu_0^{-1}\bar\rho}{(\bar\rho^2+1)^2} = 8\mu^{-2}\mu_1\frac{2\rho^3-2\rho}{(\rho^2+1)^3} + O\left(\mu^{-1}|\mu_1|\,\mu^{-2}|\mu_1|\,\rho\langle\rho\rangle^{-4}\right). \tag{4.92}$$

By (4.92), (4.56) for the case $\mu_1\equiv 0$, (4.48), and $\gamma>1$, we have

$$\mu^2\int_0^{R_0}\left(\frac{8\mu^{-1}\rho}{(\rho^2+1)^2} - \frac{8\bar\mu_0^{-1}\bar\rho}{(\bar\rho^2+1)^2}\right)(\varphi[\bar\mu_0]+\psi_*)(\mu\rho,t)\frac{\rho^2}{\rho^2+1}d\rho = \mu O\left(|\mu_1|\ln\ln t\mu_0^{-1}|\dot\mu_0|\right) \tag{4.93}$$

where $\bar\rho=\rho\mu(\bar\mu_0)^{-1}$.

By (4.23), we have

$$\begin{aligned}
&\mu^2 \int_0^{R_0} \frac{8\mu^{-1}\rho}{(\rho^2+1)^2} \left(\varphi[\mu]-\varphi[\bar\mu_0]\right)(\mu\rho,t)\frac{\rho^2}{\rho^2+1}d\rho \\
=& -\mu\left(1+C_0(R_0)\right)\Big(\mu_1 t^{-1}+\int_{t/2}^{t-t^{1-\nu}}\frac{\dot\mu_1(s)}{t-s}ds+\dot\mu_1\left[(1-\nu)\ln t-2\ln\mu_0\right]+\mathcal{E}_\nu[\mu_1]\Big)\\
&+\mu\int_0^{R_0}\tilde F[\bar\mu_0,\mu_1](\mu\rho,t)\frac{8\rho^3}{(\rho^2+1)^3}d\rho
\end{aligned}$$

where

$$0<\nu<\min\{\gamma-1,1\} \tag{4.94}$$

so that $\mu_0^2 \ll t^{1-\nu}$;

$$C_0(R_0):=\int_{R_0}^\infty \frac{-4\rho^3}{(\rho^2+1)^3}d\rho = O(R_0^{-2}), \quad \text{and} \quad \mathcal{E}_\nu[\mu_1]=\mathcal{E}_\nu[\mu_1](t):=\int_{t-t^{1-\nu}}^{t-\mu_0^2(t)}\frac{\dot\mu_1(s)-\dot\mu_1(t)}{t-s}ds \tag{4.95}$$

and by (4.23) and (4.52),

$$\begin{aligned}
&\mu\int_0^{R_0}\tilde F[\bar\mu_0,\mu_1](\mu\rho,t)\frac{8\rho^3}{(\rho^2+1)^3}d\rho\\
=&\ \mu O\Big(|\mu_1|t^{-2}\mu^2\ln R_0+\sup_{t_1\in[t/2,t]}\left(|\dot\mu_1|+t^{-1}|\mu_1|\right)(t_1)+t^{-2}\int_{t_0/2}^{t/2}\left(s\,|\dot\mu_1(s)|+|\mu_1(s)|\right)\Big).
\end{aligned} \tag{4.96}$$

We choose

$$\mathcal{H}_*[\mu_1]:=\left(\int_0^2\frac{\eta(x)x^3}{x^2+1}dx\right)^{-1}\mu\left(1+C_0(R_0)\right)\mathcal{E}_\nu[\mu_1]\rho\eta(\rho). \tag{4.97}$$

And we have

$$\mu^2\int_0^{R_0}\bar\mu_0^{-2}\left(\cos(2Q_{\bar\mu_0})-\cos(2Q_\mu)\right)\bar\rho^{-2}\Phi_e(\bar\rho,t)\frac{\rho^2}{\rho^2+1}d\rho=\mu^2 O\Big(|\mu_1|\mu_0^{-2}\begin{cases}t^{-\frac{\gamma}{2}}(\ln t)^{-1}\ln\ln t, & 1<\gamma<2\\ t^{-1}(\ln t)^{-2}\ln\ln t, & \gamma\ge 2\end{cases}\Big). \tag{4.98}$$

Formally, in order to solve (4.90) in (t_0,∞), it suffices to solve the following nonlocal equation

$$\begin{aligned}
\dot\mu_1 =&\ \chi(t)\left[(1-\nu)\ln t-2\ln\mu_0\right]^{-1}\Bigg\{-\mu_1 t^{-1}-\int_{t/2}^{t-t^{1-\nu}}\frac{\dot\mu_1(s)}{t-s}ds+\int_0^{R_0}\frac{8\rho^3\psi[\phi,\mu_1](\mu\rho,t)}{(\rho^2+1)^3}d\rho\\
&+\left(1+C_0(R_0)\right)^{-1}\Bigg[\mu\int_0^{R_0}\left(\frac{8\mu^{-1}\rho}{(\rho^2+1)^2}-\frac{8\bar\mu_0^{-1}\bar\rho}{(\bar\rho^2+1)^2}\right)\left(\varphi[\bar\mu_0]+\psi_*\right)(\mu\rho,t)\frac{\rho^2}{\rho^2+1}d\rho\\
&+\mu\int_0^{R_0}\bar\mu_0^{-2}\left(\cos(2Q_{\bar\mu_0})-\cos(2Q_\mu)\right)\bar\rho^{-2}\Phi_e(\bar\rho,t)\frac{\rho^2}{\rho^2+1}d\rho+\int_0^{R_0}\tilde F[\bar\mu_0,\mu_1](\mu\rho,t)\frac{8\rho^3}{(\rho^2+1)^3}d\rho\\
&+\mu^{-1}c_{\mathrm{in},1}\left[\rho^{-1}(\mathcal{H}_1[\mu\rho\psi[\phi,\mu_1]]+\mathcal{H}_*[\mu_1])\right](\tau(t))\Bigg]\Bigg\}\qquad \text{in } [t_0/4,\infty)
\end{aligned} \tag{4.99}$$

where $\bar\rho=\rho\mu(\bar\mu_0)^{-1}$ and we set $\psi, c_{\mathrm{in},1}=0$ for $t<t_0$ and $\chi(t)$ is a smooth cut-off function such that $\chi(t)=0$ for $t\le\frac{3}{4}t_0$ and $\chi(t)=1$ for $t\ge t_0$.

Recalling $\mathcal{H}_1$ given in (4.69), we have

$$\begin{aligned}
\left|\rho^{-1}\mathcal{H}_1\right|\lesssim&\ \mu\langle\rho\rangle^{-4}\Big[|\psi[\phi,\mu_1](\mu\rho,t)|+|\mu_1|\mu^{-1}\begin{cases}t^{-\frac{\gamma}{2}}, & 1<\gamma\le 2\\ (t\ln t)^{-1}, & \gamma>2\end{cases}\\
&+\ln t\sup_{t_1\in[t/2,t]}\left(|\dot\mu_1|+t^{-1}|\mu_1|\right)(t_1)+t^{-2}\int_{t_0/2}^{t/2}\left(s\,|\dot\mu_1(s)|+|\mu_1(s)|\right)ds\Big].
\end{aligned} \tag{4.100}$$

Also, one has

$$\left|\rho^{-1}\mathcal{H}_*[\mu_1]\right|\lesssim\mu\ln t\sup_{t_1\in[t/2,t]}|\dot\mu_1(t_1)|\eta(\rho). \tag{4.101}$$

We will solve the nonlocal problem (4.99) in the following proposition.

Proposition 4.3. *Given $\phi \in B_{\rm in}$, under the assumptions (4.83) for parameters $a\omega+\kappa(\gamma-1)\neq \frac{\gamma}{2}$ if $1<\gamma<2$; $\nu\in(0,\frac{\min\{\gamma-1,1\}}{2})$;*

$$\ell\in(2,4),\quad 0<\delta_0<\omega,\quad \begin{cases}\omega<\frac{\gamma-1}{2}, & \delta_0<\frac{\gamma-1}{4}, \quad \kappa-2<\frac{\omega(2-\ell-a)}{\gamma-1}, \quad 1<\gamma<2\\ \omega<\frac12, & \delta_0<\frac14, \quad \kappa-2<\omega(2-\ell-a), \quad \gamma\geq 2,\end{cases} \tag{4.102}$$

then for t_0 sufficiently large, there exists a solution $\mu_1=\mu_1[\phi]$ to (4.99) satisfying

$$t|\dot\mu_1(t)|+|\mu_1(t)|\lesssim \vartheta(t)t(\ln t)^{-1} \tag{4.103}$$

where $\vartheta(t)$ is given in (4.82). Moreover, there exists a solution $\phi_1=\phi_1[\phi]$ for (4.86) with the estimate

$$\langle\rho\rangle|\partial_\rho\phi_1|+|\phi_1|\lesssim \tau^{-\kappa}R^{-a}R_0^{6-\ell}\langle\rho\rangle^{2-\ell}.$$

And

$$\begin{aligned}&\left|c_{{\rm in},1}\left[\rho^{-1}(\mathcal H_1[\mu\rho\psi[\phi,\mu_1]]+\mathcal H_*[\mu_1])\right](\tau(t))\right|\lesssim\vartheta(t)\mu_0R_0^{2-\ell},\\ &\left|c_{{\rm in},1}\left[\rho^{-1}(\mathcal H_1[\mu\rho\psi[\phi,\mu_1]]+\mathcal H_*[\mu_1])\right](\tau(t_1))-c_{{\rm in},1}\left[\rho^{-1}(\mathcal H_1[\mu\rho\psi[\phi,\mu_1]]+\mathcal H_*[\mu_1])\right](\tau(t_2))\right|\\ &\lesssim\vartheta(t)\mu_0R_0^{-\ell}t^{-\alpha}\tau(t)|t_1-t_2|^\alpha\end{aligned} \tag{4.104}$$

for $\alpha\in(0,1)$, $t_1,t_2\in[\iota t,t]$ with a constant $\iota\in(0,1)$ close to 1 sufficiently.

Under the additional assumption

$$(6-\ell)\delta_0-\min\{a,\ell-2\}\,\omega<0, \tag{4.105}$$

there exists a constant $\epsilon>0$ small such that

$$\langle\rho\rangle|\partial_\rho\phi_1|+|\phi_1|\lesssim\tau_0^{-\epsilon}\tau^{-\kappa}(t)\langle\rho\rangle^{-a}.$$

Proof. Recall μ_0 in (4.45). For $\nu\neq\gamma-1$,

$$[(1-\nu)\ln t-2\ln\mu_0]^{-1}=C_{\gamma,\nu}(\ln t)^{-1}\left(1+O((\ln t)^{-1}\ln\ln t)\right),\quad C_{\gamma,\nu}:=(\min\{\gamma-1,1\}-\nu)^{-1}. \tag{4.106}$$

Under the ansatz (4.88), $\psi[\phi,\mu_1]$ is given by Proposition 4.2 and there exists $C_1\geq 2$ sufficiently large such that

$$\left|[(1-\nu)\ln t-2\ln\mu_0]^{-1}(1+C_0(R_0))^{-1}\int_0^{R_0}\frac{8\rho^3\psi[\phi,\mu_1](\mu\rho,t)}{(\rho^2+1)^3}d\rho\right|\leq C_1|C_{\gamma,\nu}|t^p(\ln t)^{p_1-1}$$

where $t^p(\ln t)^{p_1}\sim\vartheta(t)$ and

$$p:=\begin{cases}-a\omega+\frac{\gamma}{2}-1-\kappa(\gamma-1), & 1<\gamma<2\\ -a\omega-\kappa, & \gamma\geq 2,\end{cases}\qquad p_1:=\begin{cases}1-2\kappa, & \gamma\in(1,\infty)\backslash\{2\}\\ 0, & \gamma=2.\end{cases} \tag{4.107}$$

For this reason, we introduce the norm

$$\|f\|_*:=\sup_{t\geq t_0/4}\left[|C_{\gamma,\nu}|t^p(\ln t)^{p_1-1}\right]^{-1}|f(t)|, \tag{4.108}$$

and $\dot\mu_1$ will be solved in the space

$$B_{\dot\mu_1}:=\{f\ :\ \|f\|_*\leq C_{\mu_1}C_1\} \tag{4.109}$$

where $C_{\mu_1}\geq 2$ will be determined later and

$$\mu_1(t)=\mu_1[\dot\mu_1](t):=\begin{cases}\int_{t_0}^t\dot\mu_1(s)ds, & p\geq -1\\ -\int_t^\infty\dot\mu_1(s)ds, & p<-1.\end{cases}$$

For any $\dot\mu_1\in B_{\mu_1}$, since $p\neq-1$, we have

$$|\mu_1(t)|\leq|C_{\gamma,\nu}|C(p,p_1)t^{p+1}(\ln t)^{p_1-1}\|\dot\mu_1\|_*,$$

with a constant $C(p,p_1)\geq 1$ depending on p,p_1. For any $\dot\mu_1\in B_{\mu_1}$, under the assumptions

$$\begin{cases}(\kappa-1)(\gamma-1)+a\omega>0, & 1<\gamma<2\\ \kappa+a\omega-1>0, & \gamma\geq 2,\end{cases} \tag{4.110}$$

$\dot\mu_1$, μ_1 satisfies the ansatz (4.88) when t_0 is sufficiently large.

For brevity, we denote $\chi(t)A[\psi,\mu,\dot\mu_1]$ as the right hand side of (4.99). In order to solve (4.99), it suffices to consider the following fixed point problem for $\dot\mu_1$

$$\mathcal{S}[\dot\mu_1](t) := \chi(t)A[\psi[\phi,\mu_1[\dot\mu_1]],\mu_1[\dot\mu_1],\dot\mu_1] \quad \text{in} \quad [t_0/4,\infty). \tag{4.111}$$

We will estimate $A[\psi[\phi,\mu_1[\dot\mu_1]],\mu_1[\dot\mu_1],\dot\mu_1]$ term by term. For simplicity, we write $\mu_1=\mu_1[\dot\mu_1]$.

$$\Big|\int_{t/2}^{t-t^{1-\nu}}\frac{\dot\mu_1(s)}{t-s}ds\Big| \le |C_{\gamma,\nu}|\|\dot\mu_1\|_*\int_{t/2}^{t-t^{1-\nu}}\frac{s^p(\ln s)^{p_1-1}}{t-s}ds$$
$$= |C_{\gamma,\nu}|\|\dot\mu_1\|_* t^p(\ln t)^{p_1-1}\Bigg[\int_{1/2}^{1-t^{-\nu}}\frac{(x^p-1)\left[1+(\ln t)^{-1}\ln x\right]^{p_1-1}}{1-x}dx+\int_{1/2}^{1-t^{-\nu}}\frac{\left[1+(\ln t)^{-1}\ln x\right]^{p_1-1}-1}{1-x}dx$$
$$+\int_{1/2}^{1-t^{-\nu}}\frac{1}{1-x}dx\Bigg] = |C_{\gamma,\nu}|t^p(\ln t)^{p_1}\nu\left(1+O((\ln t)^{-1})\right)\|\dot\mu_1\|_*.$$

By (4.93),

$$\left|\mu\int_0^{R_0}\left(\frac{8\mu^{-1}\rho}{(\rho^2+1)^2}-\frac{8\bar\mu_0^{-1}\bar\rho}{(\bar\rho^2+1)^2}\right)(\varphi[\bar\mu_0]+\psi_*)(\mu\rho,t)\frac{\rho^2}{\rho^2+1}d\rho\right| \lesssim t^p(\ln t)^{p_1-1}\ln\ln t\|\dot\mu_1\|_*.$$

By (4.98),

$$\left|\mu\int_0^{R_0}\bar\mu_0^{-2}\left(\cos(2Q_{\bar\mu_0})-\cos(2Q_\mu)\right)\bar\rho^{-2}\Phi_e(\bar\rho,t)\frac{\rho^2}{\rho^2+1}d\rho\right| \lesssim t^p(\ln t)^{p_1-1}\ln\ln t\|\dot\mu_1\|_*.$$

By (4.96), $\gamma>1$ and the assumptions (4.83) on parameters, we get

$$\left|\int_0^{R_0}\tilde F[\bar\mu_0,\mu_1](\mu\rho,t)\frac{8\rho^3}{(\rho^2+1)^3}d\rho\right| \lesssim t^p(\ln t)^{p_1-1}\|\dot\mu_1\|_*.$$

By (4.100) and (4.101), under the assumptions (4.83), we have

$$\left|\rho^{-1}\mathcal{H}_1[\mu\rho\psi[\phi,\mu_1]]\right| \lesssim \mu_0 t^p(\ln t)^{p_1}\langle\rho\rangle^{-4}\left(1+\|\dot\mu_1\|_*\right),\quad \left|\rho^{-1}\mathcal{H}_*[\mu_1]\right|\lesssim \mu_0 t^p(\ln t)^{p_1}\eta(\rho)\|\dot\mu_1\|_*.$$

Under the assumption (4.102), by Proposition D.2 (for the case $v(\tau)=\tau^{-\kappa}(t(\tau))^{-a\omega}\sim\mu_0t^p(\ln t)^{p_1}$), we see that

$$c_{\mathrm{in},1}\left[\rho^{-1}(\mathcal{H}_1[\mu\rho\psi[\phi,\mu_1]]+\mathcal{H}_*[\mu_1])\right]$$

is well-defined and satisfies

$$\left|\mu^{-1}c_{\mathrm{in},1}\left[\rho^{-1}(\mathcal{H}_1[\mu\rho\psi[\phi,\mu_1]]+\mathcal{H}_*[\mu_1])\right](\tau(t))\right| \lesssim R_0^{2-\ell}t^p(\ln t)^{p_1}\left(1+\|\dot\mu_1\|_*\right)$$

and for $\alpha\in(0,1)$, $\tau_1,\tau_2\in[\frac{7}{8}\tau,\tau]$,

$$\begin{aligned}&\left|c_{\mathrm{in},1}\left[\rho^{-1}(\mathcal{H}_1[\mu\rho\psi[\phi,\mu_1]]+\mathcal{H}_*[\mu_1])\right](\tau_1)-c_{\mathrm{in},1}\left[\rho^{-1}(\mathcal{H}_1[\mu\rho\psi[\phi,\mu_1]]+\mathcal{H}_*[\mu_1])\right](\tau_2)\right|\\&\lesssim \tau^{-\kappa}R^{-a}\Big[\tau^{-1}R_0^{-\ell}|\tau_1-\tau_2|+|\tau_1-\tau_2|^\alpha\left(\tau^{-\alpha}R_0^{2-\ell}+\tau^{1-\alpha}R_0^{-\ell}\right)\Big]\left(1+\|\dot\mu_1\|_*\right)\\&\lesssim \tau^{-\kappa}R^{-a}\tau^{1-\alpha}R_0^{-\ell}|\tau_1-\tau_2|^\alpha\left(1+\|\dot\mu_1\|_*\right)\end{aligned}\tag{4.112}$$

where we have used $R_0\lesssim\sqrt{\tau}$.

Incorporating the above estimate and using $\|\dot\mu_1\|_*\le C_{\mu_1}C_1$, we obtain that there exists a large constant $C_2>1$ which is independent of C_{μ_1} such that for $t\ge\frac{t_0}{4}$,

$$\begin{aligned}|\mathcal{S}[\dot\mu_1]|\le\chi(t)\Bigg\{&C_1|C_{\gamma,\nu}|t^p(\ln t)^{p_1-1}+\nu|C_{\gamma,\nu}|\left(1+O((\ln t)^{-1}\ln\ln t)\right)|C_{\gamma,\nu}|t^p(\ln t)^{p_1-1}C_{\mu_1}C_1\\&+C_2C_{\mu_1}C_1(\ln t)^{-1}\left[t^p(\ln t)^{p_1-1}\ln\ln t+R_0^{2-\ell}t^p(\ln t)^{p_1}\right]\Bigg\}.\end{aligned}$$

The parameters will be determined in the following order. First, we take $\nu|C_{\gamma,\nu}|<1$, which is equivalent to $\nu<\min\left\{\frac{\gamma-1}{2},\frac{1}{2}\right\}$, and then we choose C_{μ_1} sufficiently large such that $\nu|C_{\gamma,\nu}|C_{\mu_1}+1<C_{\mu_1}$. Finally, we take t_0 sufficiently large. We thus conclude that $\mathcal{S}[\dot\mu_1]\in B_{\mu_1}$.

By (4.23), (4.99), and the parabolic regularity theory, $A[\psi[\phi,\mu_1],\mu_1,\dot{\mu}_1]$ enjoys the Hölder continuity in time, which provides compactness for the mapping. Thus, we find a solution $\dot{\mu}_1 \in B_{\dot{\mu}_1}$ by the Schauder fixed-point theorem.

Once we have solved (4.99), i.e., (4.90) holds, then by Proposition D.2, we deduce the existence and the estimates for $\phi_1 = \phi_1[\phi]$.

For $t_1, t_2 \in [\iota t, t]$ and $\iota \in (0,1)$, by (4.75), (4.76), we have $\tau(t_1) \sim \tau(t_2) \sim \tau(t)$ and

$$|\tau(t_1) - \tau(t_2)| \le C|t_1 - t_2|\mu_0^{-2}(t) \le C\,(1-\iota)\min\{\tau(t_1),\tau(t_2)\} \le \frac{1}{9}\min\{\tau(t_1),\tau(t_2)\}$$

for ι arbitrarily close to 1. Then combining (4.112) and (4.76), we get (4.104).

□

4.4.2. *Estimate about the Hölder semi-norm of $\dot{\mu}_1$.* The pointwise estimate (4.101) for $\rho^{-1}\mathcal{H}_*[\mu_1]$ is not sufficient to find a solution ϕ_2 to (4.87) in $B_{\rm in}$. For the purpose of deriving better estimate for $\rho^{-1}\mathcal{H}_*[\mu_1]$, we will give the time decay estimate about the Hölder semi-norm of $\dot{\mu}_1$.

Claim: Let $\dot{\mu}_1$ be the solution to (4.99) given by Proposition 4.3. Under the assumptions in Proposition 4.3, $\dot{\mu}_1$ has the Hölder estimate

$$[\dot{\mu}_1]_{C^\alpha[\iota t,t]} \lesssim \varrho(t) \tag{4.113}$$

with $\alpha \in (0,1)$, and

$$\varrho(t) := \vartheta(t)(\ln t)^{-1} t^{-\alpha}\left(t^{-2\omega} + t^{-\delta_0\ell}\right)\begin{cases} t^{\gamma-1}(\ln t)^2, & 1<\gamma<2 \\ t, & \gamma = 2 \\ t(\ln t)^2, & \gamma>2. \end{cases} \tag{4.114}$$

Thus, recalling (4.97), (4.95), and by (4.113), we have

$$\left|\rho^{-1}\mathcal{H}_*[\mu_1]\right| \lesssim \mu_0\left|\mathcal{E}_\nu[\mu_1]\right|\eta(\rho) \lesssim \eta(\rho)\mu_0 t^{\alpha(1-\nu)}\varrho(t). \tag{4.115}$$

4.5. **Solving the inner problem and the proof of Theorem 5.** In this section, we solve the inner problem and complete the construction of desired blow-up solution.

Recall $\mathcal{H}_2$ given in (4.70). We have

$$\left|\rho^{-1}\mathcal{H}_2\right| \lesssim \Big(\begin{cases} t^{2-2\gamma}(\ln t)^{-1}, & 1<\gamma<2 \\ t^{-2}(\ln t)^2, & \gamma\ge 2 \end{cases} + \tau^{-2\kappa}(t) + \tau^{-3\kappa}(t)R^2\Big)\langle\rho\rangle^{-2}. \tag{4.116}$$

We will use Proposition D.1 to find a solution ϕ_2 for the inner problem (4.87) without orthogonality condition. Recall the upper bound of the right hand side of (4.87) in (4.115), (4.114) and (4.116). The assumption (4.62) gives $\tau^{-1}R^2\ln R + |\partial_\tau R|R\ln R \ll 1$ for t_0 sufficiently large. Then by Proposition D.1, there exists a solution ϕ_2 for (4.87) with the estimate

$$\begin{aligned}\langle\rho\rangle|\partial_\rho\phi_2| + |\phi_2| \lesssim{}& \langle\rho\rangle^{-2}R^2\ln R\tau^{-\kappa}(t)R^{-a}(\ln t)^{-1}t^{-\alpha\nu-\min\{2\omega,\delta_0\ell\}}\begin{cases} t^{\gamma-1}(\ln t)^2, & 1<\gamma<2 \\ t, & \gamma=2 \\ t(\ln t)^2, & \gamma>2 \end{cases} \\ &+ \langle\rho\rangle^{-2}R^2\ln R\Big(\begin{cases} t^{2-2\gamma}(\ln t)^{-1}, & 1<\gamma<2 \\ t^{-2}(\ln t)^2, & \gamma\ge 2 \end{cases} + \tau^{-2\kappa}(t) + \tau^{-3\kappa}(t)R^2\Big).\end{aligned} \tag{4.117}$$

Under the constraints

$$\begin{gathered} 0<a<2, \quad (2-a)\omega - \alpha\nu - \min\{2\omega,\delta_0\ell\} + \begin{cases} \gamma-1, & 1<\gamma<2 \\ 1, & \gamma\ge 2 \end{cases} < 0, \\ \begin{cases} 2\omega+2-2\gamma+\kappa(\gamma-1)<0, & 1<\gamma<2 \\ 2\omega-2+\kappa<0, & \gamma\ge 2, \end{cases} \quad \begin{cases} 2\omega-\kappa(\gamma-1)<0, & 1<\gamma<2 \\ 2\omega-\kappa<0, & \gamma\ge 2, \end{cases} \end{gathered} \tag{4.118}$$

there exists a constant $\epsilon>0$ sufficiently small such that

$$\langle\rho\rangle|\partial_\rho\phi_2| + |\phi_2| \lesssim t^{-\epsilon}\tau^{-\kappa}(t)\langle\rho\rangle^{-a}.$$

Recall ϕ_1 in Proposition 4.3. Combining all the restrictions on constants in Proposition 4.3, we conclude that

$$\phi_1 + \phi_2 = \phi_1[\phi] + \phi_2[\phi] \in B_{\text{in}} \tag{4.119}$$

due to the small quantity provided by $t_0^{-\epsilon}$ with t_0 sufficiently large, where B_{in} is defined in (4.80). The compactness of this mapping is provided by the parabolic regulatiy theory. Thus, by the Schauder fixed-point theorem, we find a solution ϕ of (4.73) in B_{in}.

We now collect all the assumptions on parameters measuring the desired topologies (4.62), (4.83), those in Proposition 4.3, (4.118) and write

$$f_\gamma = \frac{f}{\min\{\gamma-1,1\}} \quad \text{for} \quad f = \omega, \nu, \delta_0.$$

Then the system of parameters can be reformulated as

$$\begin{aligned}
&1 < \kappa + a\omega_\gamma < 2, \quad (2+a)\omega_\gamma < 2\kappa, \quad \alpha \in (0,1), \quad \nu_\gamma \in (0, \frac{1}{2})\\
&a\omega + \kappa(\gamma-1) \neq \frac{\gamma}{2} \ \text{ if } \ 1<\gamma<2,\\
&\ell \in (2,4), \quad 0 < \delta_{0\gamma} < \omega_\gamma < \frac{1}{2}, \quad \delta_{0\gamma} < \frac{1}{4}, \quad \kappa - 2 < \omega_\gamma(2-\ell-a),\\
&(6-\ell)\delta_{0\gamma} - \min\{a, \ell-2\}\,\omega_\gamma < 0, \quad 0 < a < 2,\\
&(2-a)\omega_\gamma - \alpha\nu_\gamma - \min\{2\omega_\gamma, \delta_{0\gamma}\ell\} + 1 < 0, \quad 2\omega_\gamma - 2 + \kappa < 0, \quad 2\omega_\gamma - \kappa < 0.
\end{aligned} \tag{4.120}$$

By the software Mathematica, the solutions are given by

$$\begin{aligned}
&\frac{1}{4} < \omega_\gamma < \frac{1}{3}, \quad \frac{1}{2} < a\omega_\gamma < 2\omega_\gamma, \quad \frac{1}{4}\left(\frac{1}{2} - a\omega_\gamma + 2\omega_\gamma\right) < \delta_{0\gamma} < \frac{1}{4}, \quad 2\omega_\gamma < \kappa < 2 - a\omega_\gamma - 2\omega_\gamma,\\
&\kappa \neq \frac{\gamma}{2(\gamma-1)} - a\omega_\gamma \ \text{ if } \ 1<\gamma<2, \quad 1 - a\omega_\gamma < \alpha\nu_\gamma, \quad 1 - a\omega_\gamma < \nu_\gamma < \frac{1}{2}, \quad \frac{1-a\omega_\gamma}{\nu_\gamma} < \alpha < 1,\\
&\max\left\{2+a, \frac{2\omega_\gamma - a\omega_\gamma - \alpha\nu_\gamma + 1}{\delta_{0\gamma}}, 6 - \frac{a\omega_\gamma}{\delta_{0\gamma}}\right\} < \ell < 4.
\end{aligned} \tag{4.121}$$

One may choose a special solution as $\omega_\gamma = \frac{7}{24}$, $a\omega_\gamma = \frac{13}{24}$, $a = \frac{13}{7}$, $\delta_{0\gamma} = \frac{1}{6}$, $\frac{14}{24} < \kappa < \frac{21}{24}$, $\kappa \neq \frac{\gamma}{2(\gamma-1)} - \frac{13}{24}$ if $1 < \gamma < 2$, $\alpha\nu_\gamma = \frac{23}{48}$, $\nu_\gamma = \frac{47}{96}$, $\alpha = \frac{46}{47}$, $\ell = \frac{55}{14}$.

Now we have constructed the solution v of (4.2) with the form

$$v = \eta(\frac{r}{\sqrt{t}})\left(\pi - 2\arctan(\frac{r}{\mu})\right) + r\left(\varphi + \psi_*\right)(r,t) + \eta(\frac{4r}{\sqrt{t}})\Phi_e(\frac{r}{\bar{\mu}_0}, t) + r\psi(r,t) + \eta(\frac{r}{\mu R})\rho\phi(\rho, t)$$

whose initial data is given by

$$v_0(r) = v(r,t_0) = \eta(\frac{r}{\sqrt{t_0}})\left(\pi - 2\arctan(\frac{r}{\mu(t_0)})\right) + r\left(\varphi + \psi_*\right)(r,t_0) + \eta(\frac{4r}{\sqrt{t_0}})\Phi_e(\frac{r}{\bar{\mu}_0(t_0)}, t_0).$$

Recalling (4.50), (4.58), we have $v_0(0) = \pi$ and for $r \geq 2\sqrt{t_0}$, $|v_0(r)| \lesssim r^{1-\gamma}$ if $1 < \gamma \leq 2$ and $|v_0(r)| \lesssim r^{1-\gamma} + (t_0 \ln t_0)^{-1} r e^{-\frac{r^2}{32 t_0}}$ if $\gamma > 2$.

The estimate of v and v_r are given by direct calculation.

Appendix A. Convolution estimates

Lemma A.1. *Let $n > 2$ be an integer, $t > t_0 \geq 0$, $b \in \mathbb{R}$. Suppose that $v(s) \geq 0$ for $s \in [t_0, t]$; $0 \leq l_1(s) \leq l_2(s) \leq C_* s^{\frac{1}{2}}$ for $s \in [t_0, t]$, $C_l^{-1} l_i(t_1) \leq l_i(t_2) \leq C_l l_i(t_1)$, $i = 1, 2$, for all $t_2 \in [t_1, 2t_1] \cap [t_0, t]$, $t_1 \in [t_0, t]$, where*

$C_* > 0, C_l \geq 1$ are constants, then

$$T_n \bullet \left(v(t)\,(|x| + l_1(t))^{-b}\,\mathbf{1}_{\{|x|\leq l_2(t)\}}\right)(x,t,t_0) \lesssim t^{-\frac{n}{2}} e^{-\frac{|x|^2}{16t}} \int_{t_0}^{\frac{t}{2}} v(s) \begin{cases} l_2^{n-b}(s) & \text{if } b<n \\ \langle \ln(\frac{l_2(s)}{l_1(s)})\rangle & \text{if } b=n\ ds \\ l_1^{n-b}(s) & \text{if } b>n \end{cases}$$

$$+ \sup_{t_1\in[t/2,t]} v(t_1) \begin{cases} \begin{cases} l_2^{2-b}(t) & \text{if } b<2 \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=2 \\ l_1^{2-b}(t) & \text{if } b>2 \end{cases} & \text{for } |x|\leq l_1(t) \\ \begin{cases} l_2^{2-b}(t) & \text{if } b<2 \\ \langle \ln(\frac{l_2(t)}{|x|})\rangle & \text{if } b=2 \\ |x|^{2-b} & \text{if } 2<b<n \\ |x|^{2-n}\langle \ln(\frac{|x|}{l_1(t)})\rangle & \text{if } b=n \\ |x|^{2-n} l_1^{n-b}(t) & \text{if } b>n \end{cases} & \text{for } l_1(t)<|x|\leq l_2(t) \\ |x|^{2-n} e^{-\frac{|x|^2}{16t}} \begin{cases} l_2^{n-b}(t) & \text{if } b<n \\ \langle \ln(\frac{l_2(t)}{l_1(t)})\rangle & \text{if } b=n \\ l_1^{n-b}(t) & \text{if } b>n \end{cases} & \text{for } |x|>l_2(t) \end{cases}$$

where we set $v(s) = 0$ if $s \notin [t_0, t]$.

Lemma A.2. *Let n be a positive integer, $t > t_0 \geq 0$, $b \in \mathbb{R}$, $\ell > 0$, $c_0 > 0$, and suppose that $v(s) \geq 0$ for $s \in [t_0, t]$. Denote*

$$u(x,t) = \int_{t_0}^{t} \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} e^{-c_0 \frac{|x-y|^2}{t-s}} v(s)|y|^{-b} e^{-\ell \frac{|y|^2}{s}} \mathbf{1}_{\{|y|\geq s^{\frac{1}{2}}\}} dyds.$$

Then

$$u(x,t) \lesssim \begin{cases} t^{-\frac{n}{2}} \int_{t_0}^{t/2} v(s) s^{\frac{n-b}{2}} ds + \sup_{t_1\in[\frac{t}{2},t]} v(t_1) t^{1-\frac{b}{2}}, & |x|\leq t^{\frac{1}{2}} \\ \left(t^{-\frac{n}{2}} \int_{t_0}^{t/2} v(s) s^{\frac{n-b}{2}} ds + \sup_{t_1\in[\frac{t}{2},t]} v(t_1) t |x|^{-b}\right) e^{-\frac{\min\{c_0,\ell\}}{8}\frac{|x|^2}{t}}, & |x|>t^{\frac{1}{2}} \end{cases}$$

where we set $v(s) = 0$ if $s \notin [t_0, t]$.

Proof. The strategy is basically a verbatim repetition of the proof of [100, Lemma A.2]. □

Appendix B. Linear theory via blow-up argument

Define

$$\|h\|_{a,\nu} := \sup_{(y,\tau)\in\mathbb{R}^n\times(\tau_0,\infty)} \tau^{\nu}\langle y\rangle^{a}|h(y,\tau)|.$$

The main results are the following.

Proposition B.1. *Consider*

$$\begin{cases} \partial_\tau \phi = \Delta\phi + pU^{p-1}(y)\phi + h(y,\tau) & \text{in } \mathbb{R}^n\times(\tau_0,\infty) \\ \phi(y,\tau_0) = e_0 Z_0(y) & \text{in } \mathbb{R}^n. \end{cases} \tag{B.1}$$

Suppose $2 < a < n-2$, $\nu < 1$, $\|h\|_{2+a,\nu} < \infty$ and

$$\int_{\mathbb{R}^n} h(y,\tau) Z_j(y) dy = 0 \quad \text{for all} \quad \tau\in(\tau_0,\infty),\ j=1,2,\cdots,n+1. \tag{B.2}$$

Then for $\tau_0 \geq 1$, there exists a linear mapping $(\phi, e_0) = (\phi[h], e_0[h])$ satisfying (B.1) and

$$\int_{\mathbb{R}^n} \phi(y,\tau) Z_j(y) dy = 0 \quad \text{for all} \quad \tau\in(\tau_0,\infty),\ j=1,2,\cdots,n+1, \tag{B.3}$$

$$\langle y\rangle|\nabla\phi| + |\phi| \lesssim \tau^{-\nu}\langle y\rangle^{-a}\|h\|_{2+a,\nu}, \quad |e_0[h]| \lesssim \tau_0^{-\nu}\|h\|_{2+a,\nu}. \tag{B.4}$$

Proposition B.1 is in fact a consequence of the following

Proposition B.2. *Suppose $2 < a < n-2$, $\nu < 1$, $\|h\|_{2+a,\nu} < \infty$ and*

$$\int_{\mathbb{R}^n} h(y,\tau)Z_j(y)dy = 0 \quad \text{for all} \quad \tau \in (\tau_0,\infty),\ j = 0,1,\cdots,n+1. \tag{B.5}$$

Then for $\tau_0 \geq 1$, there exists a unique solution ϕ of

$$\begin{cases} \partial_\tau \phi = \Delta\phi + pU^{p-1}(y)\phi + h(y,\tau) & \text{in} \ \ \mathbb{R}^n \times (\tau_0,\infty) \\ \phi(y,\tau_0) = 0 & \text{in} \ \ \mathbb{R}^n \end{cases} \tag{B.6}$$

in $L^\infty(\mathbb{R}^n \times (\tau_0,\tilde\tau))$ for all $\tilde\tau > \tau_0$ and ϕ satisfies

$$\int_{\mathbb{R}^n} \phi(y,\tau)Z_j(y)dy = 0 \quad \text{for all} \quad \tau > \tau_0,\ j = 0,1,\cdots,n+1 \tag{B.7}$$

and the estimate

$$\langle y\rangle|\nabla\phi| + |\phi| \lesssim \tau^{-\nu}\langle y\rangle^{-a}\|h\|_{2+a,\nu}. \tag{B.8}$$

We first use Proposition B.2 to deduce Proposition B.1.

Proof of Proposition B.1. First, set $\phi(y,\tau) = \phi_1(y,\tau) + c(\tau)Z_0(y)$. Then it suffices to consider

$$\begin{cases} \partial_\tau \phi_1 = \Delta\phi_1 + pU^{p-1}(y)\phi_1 + h_1 & \text{in} \ \ \mathbb{R}^n \times (\tau_0,\infty) \\ \phi_1(y,\tau_0) = 0 & \text{in} \ \ \mathbb{R}^n. \end{cases} \tag{B.9}$$

with $h_1 := h + c(\tau)\lambda_0 Z_0 - c'(\tau)Z_0$ where we used $\Delta Z_0 + pU^{p-1}(y)Z_0 = \lambda_0 Z_0$ with $\lambda_0 > 0$. We take

$$c'(\tau) - \lambda_0 c(\tau) = \left(\int_{\mathbb{R}^n} Z_0^2(y)dy\right)^{-1} \int_{\mathbb{R}^n} h(y,\tau)Z_0(y)dy$$

with

$$c(\tau) = -\left(\int_{\mathbb{R}^n} Z_0^2(y)dy\right)^{-1} e^{\lambda_0\tau}\int_\tau^\infty e^{-\lambda_0 s}\int_{\mathbb{R}^n} h(y,s)Z_0(y)dyds$$

so that $\int_{\mathbb{R}^n} h_1(y,\tau)Z_0(y)dy = 0$. Combining this with (B.2) implies (B.5). It is direct to see that

$$|c(\tau)| + |c'(\tau)| \lesssim \|h\|_{2+a,\nu}\tau^{-\nu}, \quad \|h_1\|_{2+a,\nu} \lesssim \|h\|_{2+a,\nu}.$$

By Proposition B.2, (B.9) has a unique solution ϕ_1 satisfying

$$\int_{\mathbb{R}^n} \phi_1(y,\tau)Z_j(y)dy = 0 \ \text{ for all } \ \tau > \tau_0,\ j = 0,1,\cdots,n+1, \quad \langle y\rangle|\nabla\phi_1| + |\phi_1| \lesssim \tau^{-\nu}\langle y\rangle^{-a}\|h\|_{2+a,\nu}.$$

Then $\phi = \phi_1 + c(\tau)Z_0$ satisfies (B.3) and (B.4). Finally, letting $e_0 = c_0(\tau_0)$ completes the proof. □

Next, we use blow-up argument to prove Proposition B.2. Our method does not rely on the use of maximum principle, so we expect that this method is applicable to more general equations/systems in the absence of maximum principle (assuming non-degeneracy of the profile in certain sense). Typical examples are parabolic equations with complex coefficients such as

$$A_0\partial_\tau u = \Delta u + V(u,Du) + h,$$

where the complex constant A_0 satisfies $\mathrm{Re}(A_0) > 0$ (with dissipation). This kind of operator naturally arises, for instance, in the study of Landau-Lifshitz-Gilbert equations.

Proof of Proposition B.2. The existence and uniqueness of (B.6) are given by the classical parabolic theory. Denote

$$\|f\|_{a,\nu,\tau_1} := \sup_{(y,\tau)\in\mathbb{R}^n\times(\tau_0,\tau_1)} \tau^\nu\langle y\rangle^a|f(y,\tau)|.$$

For all $\tau > \tau_0$, by the estimate of parabolic fundamental solution (See [38]) and convolution estimate in [100, Lemma A.1, Lemma A.2], for $0 < a < n-2$, we have

$$\|\phi\|_{a,\nu,\tau} < \infty. \tag{B.10}$$

By scaling argument, we have $\|\nabla\phi\|_{1+a,\nu,\tau} < \infty$.

For $a > 2$, multiplying (B.6) by Z_j, $j = 0,1,\ldots,n+1$ and integrating by parts, we obtain (B.7) by (B.5) and the initial data of (B.6).

In order to prove (B.8), it suffices to prove the following claim.
Claim: For all $\tau_1 > \tau_0$ large enough, there exists C independent of τ_1 such that

$$\|\phi\|_{a,\nu,\tau_1} \le C\|h\|_{2+a,\nu,\tau_1}. \tag{B.11}$$

Indeed, by taking $\tau_1 \to \infty$, (B.11) implies (B.8). To prove (B.11), we argue by contradiction. Suppose that there exist sequences $\tau_1^k \to \infty$ and ϕ_k, h_k satisfying

$$\begin{cases} \partial_\tau \phi_k = \Delta\phi_k + pU^{p-1}(y)\phi_k + h_k \quad \text{ in } \mathbb{R}^n \times (\tau_0, \infty) \\ \int_{\mathbb{R}^n} \phi_k(y,\tau) Z_j(y) dy = 0 \quad \text{ for all } \tau \in (\tau_0, \tau_1^k), \quad j = 0, 1, \cdots, n+1 \\ \phi_k(y, \tau_0) = 0 \quad \text{ in } \mathbb{R}^n \end{cases} \tag{B.12}$$

and

$$\|\phi_k\|_{a,\nu,\tau_1^k} = 1, \quad \|h_k\|_{2s+a,\nu,\tau_1^k} = o(1) \ \text{ where } \ o(1) \to 0 \ \text{ as } \ k \to \infty. \tag{B.13}$$

First, we claim that for any compact subset Ω in $\mathbb{R}^n$,

$$\sup_{\tau_0 < \tau < \tau_1^k} \tau^\nu |\phi_k(y,\tau)| \to 0 \ \text{ uniformly in } \ \Omega. \tag{B.14}$$

Assume this is not true. Then there exist a constant $M > 0$ such that $|y_k| \le M$ and $\tau_0 < \tau_2^k < \tau_1^k$,

$$(\tau_2^k)^\nu |\phi_k(y_k, \tau_2^k)| \ge \delta_0 > 0, \tag{B.15}$$

for a constant $\delta_0 > 0$. Since $\|h_k\|_{2+a,\nu,\tau_1^k} = o(1)$, we have $\tau_2^k \to \infty$ by the same reason for getting (B.10). Without loss of generality, we assume $\tau_2^k \ge 9\tau_0$. Set

$$\tilde{\phi}_k(y,t) = (\tau_2^k)^\nu \phi_k(y, \tau_2^k + t), \quad \tilde{h}_k(y,t) = (\tau_2^k)^\nu h_k(y, \tau_2^k + t).$$

Then by (B.12), one has

$$\partial_t \tilde{\phi}_k = \Delta\tilde{\phi}_k + pU^{p-1}(y)\tilde{\phi}_k + \tilde{h}_k \ \text{ in } \ \mathbb{R}^n \times (\tau_0 - \frac{\tau_2^k}{2}, 0] \tag{B.16}$$

with

$$|\tilde{\phi}_k(y,\tau)| \le C(\nu)\langle y\rangle^{-a}, \quad |\tilde{h}_k(y,\tau)| \le o(1)C(\nu)\langle y\rangle^{-2-a} \ \text{ in } \ \mathbb{R}^n \times (\tau_0 - \frac{\tau_2^k}{2}, 0], \tag{B.17}$$

where $C(\nu)$ is a constant only depending on ν.

By the parabolic regularity theorem, up to a subsequence, we have $\tilde{\phi}_k \to \tilde{\phi}$ in C^1_{loc}, that is, $\tilde{\phi}_k \to \tilde{\phi}$ in C^1 topology on any compact subsets of $\mathbb{R}^n \times (-\infty, 0]$. Combining this with (B.15) yields

$$|\tilde{\phi}(y,\tau)| \le C(\nu)\langle y\rangle^{-a}, \quad \tilde{\phi} \ne 0. \tag{B.18}$$

By (B.16), we have

$$\begin{aligned} \tilde{\phi}_k(y,t) = & \int_{\mathbb{R}^n} [4\pi(t - \tau_0 + \frac{\tau_2^k}{2})]^{-\frac{n}{2}} \exp\left(-\frac{|y-z|^2}{4(t - \tau_0 + \frac{\tau_2^k}{2})}\right) \tilde{\phi}_k(z, \tau_0 - \frac{\tau_2^k}{2}) dz \\ & + \int_{\tau_0 - \frac{\tau_2^k}{2}}^t \int_{\mathbb{R}^n} [4\pi(t-s)]^{-\frac{n}{2}} \exp\left(-\frac{|y-z|^2}{4(t-s)}\right) \left(pU^{p-1}(z)\tilde{\phi}_k(z,s) + \tilde{h}_k(z,s)\right) dz ds. \end{aligned}$$

Then for any fixed $(y,t) \in \mathbb{R}^n \times (-\infty, 0]$, by $a > 0$, (B.17), (B.18), [91, Corollary B.4, Lemma B.5] (used for time integral $\int_{t_2}^{t_1} \cdots$ for some $t_1 \le -1$), [100, Lemma A.3] (used for Cauchy integral) and $\tilde{\phi}_k \to \tilde{\phi}$ in C^1_{loc}, we have

$$\tilde{\phi}(y,t) = \int_{-\infty}^t \int_{\mathbb{R}^n} [4\pi(t-s)]^{-\frac{n}{2}} e^{-\frac{|y-z|^2}{4(t-s)}} pU^{p-1}(z)\tilde{\phi}(z,s) dz ds. \tag{B.19}$$

Then the limiting equation reads

$$\begin{cases} \partial_\tau \tilde{\phi} = \Delta\tilde{\phi} + pU^{p-1}(y)\tilde{\phi} \quad \text{ in } \quad \mathbb{R}^n \times (-\infty, 0] \\ \int_{\mathbb{R}^n} \tilde{\phi}(y,\tau) Z_j(y) dy = 0 \ \text{ for all } \ \tau \in (-\infty, 0], \ j = 0, 1, \cdots n+1 \\ |\tilde{\phi}(y,\tau)| \le C(\nu)\langle y\rangle^{-a} \ \text{ in } \ \mathbb{R}^n \times (-\infty, 0] \end{cases} \tag{B.20}$$

where we have used $a > 2$ in the orthogonality by dominated convergence theorem.

Using (B.19), [91, Corollary B.4, Lemma B.5] finitely many times for $\tau \in (-\infty, -M_0)$ with M_0 large and then applying [100, Lemmas A.1, A.2, A.3] in $[M_0, 0]$, we have

$$|\tilde{\phi}| \lesssim \langle y \rangle^{2-n},$$

and $\tilde{\phi}$ is smooth by the parabolic regularity theory. By scaling argument, one has

$$\langle y \rangle^{-1} |D\tilde{\phi}| + |\tilde{\phi}_\tau| + |D^2 \tilde{\phi}| \lesssim \langle y \rangle^{-n}.$$

Differentiating (B.20), we get

$$\partial_\tau \tilde{\phi}_\tau = \Delta \tilde{\phi}_\tau + pU^{p-1}(y)\tilde{\phi}_\tau, \tag{B.21}$$

and then scaling argument gives

$$\langle y \rangle^{-1} |D\tilde{\phi}_\tau| + |\tilde{\phi}_{\tau\tau}| + |D^2 \tilde{\phi}_\tau| \lesssim \langle y \rangle^{-n-2}.$$

Moreover, multiplying (B.21) by $\tilde{\phi}_\tau$ and integrating by parts, we get

$$\frac{1}{2}\partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy + B(\tilde{\phi}_\tau, \tilde{\phi}_\tau) = 0,$$

where

$$B(f, f) := \int_{\mathbb{R}^n} \left(|\nabla f|^2 - pU^{p-1}(y)|f|^2 \right) dy.$$

By orthogonality in (B.20), we have $\int_{\mathbb{R}^n} \partial_\tau \tilde{\phi}(y, \tau) Z_j(y) dy = 0$ for all $\tau \in (-\infty, 0]$, $j = 0, 1, \cdots, n+1$. Then $B(\tilde{\phi}_\tau, \tilde{\phi}_\tau) \geq 0$ by $\int_{\mathbb{R}^n} \partial_\tau \tilde{\phi}(y, \tau) Z_0(y) dy = 0$ since Z_0 is the only eigenfucntion corresponding to the positive eigenvalue. Thus, $\partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy \leq 0$.

Multiplying (B.20) by $\tilde{\phi}_\tau$ and integrating by parts, we have

$$\int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy = -\frac{1}{2}\partial_\tau B(\tilde{\phi}, \tilde{\phi}).$$

From these relations, one has

$$\partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy \leq 0, \quad \int_{-\infty}^0 d\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 dy < \infty.$$

Hence $\tilde{\phi}_\tau = 0$. So $\tilde{\phi}$ is independent of τ and $\Delta\tilde{\phi} + pU^{p-1}(y)\tilde{\phi} = 0$. By the nondegeneracy of $\Delta + pU^{p-1}(y)$ (see [2, Lemma 5.2]), $\tilde{\phi}$ is a linear combination of Z_j, $j = 1, \cdots, n+1$. Due to the orthogonal conditions in (B.20), we must have $\tilde{\phi} \equiv 0$, which contradicts (B.18). Thus (B.14) holds.

By (B.13) and (B.14), there exists a sequence y_k with $|y_k| \to \infty$ such that

$$(\tau_2^k)^\nu \langle y_k \rangle^a |\phi_k(y_k, \tau_2^k)| \geq \frac{1}{2}.$$

Set

$$\tilde{\phi}_k(z, t) := (\tau_2^k)^\nu \langle y_k \rangle^a \phi_k(y_k + |y_k| z, |y_k|^2 t + \tau_2^k).$$

Then

$$|\tilde{\phi}_k(0, 0)| \geq \frac{1}{2}. \tag{B.22}$$

We reformulate (B.12) as

$$\begin{cases} \partial_t \tilde{\phi}_k = \Delta_z \tilde{\phi}_k + p|y_k|^2 U^{p-1}(y_k + |y_k| z)\tilde{\phi}_k + \tilde{h}_k(z, t) \ \text{ in } \ \mathbb{R}^n \times (\frac{\tau_0 - \tau_2^k}{|y_k|^2}, \infty) \\ \tilde{\phi}_k(\cdot, \frac{\tau_0 - \tau_2^k}{|y_k|^2}) = 0 \ \text{ in } \ \mathbb{R}^n \end{cases} \tag{B.23}$$

where

$$\tilde{h}_k(z, t) = (\tau_2^k)^\nu \langle y_k \rangle^a |y_k|^2 h_k(y_k + |y_k| z, |y_k|^2 t + \tau_2^k).$$

By (B.13), one has

$$\begin{aligned} &|\tilde{h}_k(z, t)| \lesssim o(1)(\tau_2^k)^\nu \langle y_k \rangle^a |y_k|^2 \langle y_k + |y_k| z \rangle^{-2-a} \left(|y_k|^2 t + \tau_2^k \right)^{-\nu} \\ &\sim o(1) \left(|y_k|^{-1} + |\hat{y}_k + z| \right)^{-2-a} \left((\tau_2^k)^{-1} |y_k|^2 t + 1 \right)^{-\nu}, \quad \text{for} \quad (z, t) \in \mathbb{R}^n \times (\frac{\tau_0 - \tau_2^k}{|y_k|^2}, \frac{\tau_1^k - \tau_2^k}{|y_k|^2}), \end{aligned}$$

$$\left||y_k|^2 U^{p-1}(y_k+|y_k|z)\tilde{\phi}_k\right| \lesssim |y_k|^2\langle y_k+|y_k|z\rangle^{-4}(\tau_2^k)^{\nu}\langle y_k\rangle^a\langle y_k+|y_k|z\rangle^{-a}\left(|y_k|^2t+\tau_2^k\right)^{-\nu}$$
$$\sim |y_k|^{-2}\left(|y_k|^{-1}+|\hat{y}_k+z|\right)^{-4-a}\left((\tau_2^k)^{-1}|y_k|^2t+1\right)^{-\nu}, \quad \text{for} \quad (z,t)\in\mathbb{R}^n\times(\frac{\tau_0-\tau_2^k}{|y_k|^2},\frac{\tau_1^k-\tau_2^k}{|y_k|^2})$$

where $\hat{y}_k=y_k|y_k|^{-1}$. By (B.23), we have

$$\tilde{\phi}_k(z,t)=\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^{t}\int_{\mathbb{R}^n}[4\pi(t-s)]^{-\frac{n}{2}}e^{-\frac{|z-w|^2}{4(t-s)}}\left(p|y_k|^2U^{p-1}(y_k+|y_k|w)\tilde{\phi}_k(w,s)+\tilde{h}_k(w,s)\right)dwds.$$

Then

$$\left|\tilde{\phi}_k(0,0)\right| \lesssim \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^{0}\int_{\mathbb{R}^n}(-s)^{-\frac{n}{2}}e^{\frac{|w|^2}{4s}}\Big[|y_k|^{-2}\left(|y_k|^{-1}+|\hat{y}_k+w|\right)^{-4-a}\left((\tau_2^k)^{-1}|y_k|^2s+1\right)^{-\nu}$$
$$+o(1)\left(|y_k|^{-1}+|\hat{y}_k+w|\right)^{-2-a}\left((\tau_2^k)^{-1}|y_k|^2s+1\right)^{-\nu}\Big]dwds.$$

Claim: Suppose that $\tau_2^k\ge 2\tau_0$, $|y_k|\ge 2$, $m>2, n>2, \nu<1$, then

$$\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^{0}\int_{\mathbb{R}^n}(-s)^{-\frac{n}{2}}e^{\frac{|w|^2}{4s}}\left(|y_k|^{-1}+|\hat{y}_k+w|\right)^{-m}\left((\tau_2^k)^{-1}|y_k|^2s+1\right)^{-\nu}dwds \lesssim \begin{cases}1, & \text{if} \quad m<n\\ \langle \ln|y_k|\rangle, & \text{if} \quad m=n\\ |y_k|^{m-n}, & \text{if} \quad m>n.\end{cases} \tag{B.24}$$

Assuming (B.24), for $0<a<n-2$, one then has

$$\left|\tilde{\phi}_k(0,0)\right| \lesssim o(1)+\begin{cases}|y_k|^{-2}, & \text{if} \quad 4+a<n\\ |y_k|^{-2}\langle\ln|y_k|\rangle, & \text{if} \quad 4+a=n \to 0 \quad \text{as} \quad k\to\infty\\ |y_k|^{2+a-n}, & \text{if} \quad 4+a>n\end{cases}$$

which contradicts (B.22).

Finally, we prove (B.24).

Proof of (B.24):

$$\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^{0}\int_{\mathbb{R}^n}(-s)^{-\frac{n}{2}}e^{\frac{|w|^2}{4s}}\left(|y_k|^{-1}+|\hat{y}_k+w|\right)^{-m}\left((\tau_2^k)^{-1}|y_k|^2s+1\right)^{-\nu}dwds$$
$$=\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^{0}\left((\tau_2^k)^{-1}|y_k|^2s+1\right)^{-\nu}(-s)^{-\frac{m}{2}}\int_{\mathbb{R}^n}e^{-\frac{|x|^2}{4}}\left((-s)^{-\frac{1}{2}}|y_k|^{-1}+|(-s)^{-\frac{1}{2}}\hat{y}_k+x|\right)^{-m}dxds.$$

Notice for $0 < 2c_0 \le |\vec{v}|$, we estimate the spatial integral as

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4}} (c_0 + |\vec{v}+x|)^{-m}\, dx = \left(\int_{|x|\le \frac{|\vec{v}|}{2}} + \int_{\frac{|\vec{v}|}{2} < |x| \le 2|\vec{v}|} + \int_{|x|>2|\vec{v}|} \right) e^{-\frac{|x|^2}{4}} (c_0 + |\vec{v}+x|)^{-m}\, dx$$

$$\sim \int_{|x|\le \frac{|\vec{v}|}{2}} e^{-\frac{|x|^2}{4}} (c_0 + |\vec{v}|)^{-m}\, dx + \int_{\frac{|\vec{v}|}{2} < |x| \le 2|\vec{v}|} e^{-\frac{|x|^2}{4}} (c_0 + |\vec{v}+x|)^{-m}\, dx + \int_{|x|>2|\vec{v}|} e^{-\frac{|x|^2}{4}} (c_0 + |x|)^{-m}\, dx$$

$$\lesssim |\vec{v}|^{-m} \int_{|x|\le \frac{|\vec{v}|}{2}} e^{-\frac{|x|^2}{4}} dx + e^{-\frac{|\vec{v}|^2}{16}} \int_{|x+\vec{v}|\le 3|\vec{v}|} (c_0 + |\vec{v}+x|)^{-m}\, dx + \int_{|x|>2|\vec{v}|} e^{-\frac{|x|^2}{4}} |x|^{-m} dx$$

$$\lesssim \mathbf{1}_{\{|\vec{v}|\le 1\}} |\vec{v}|^{n-m} + \mathbf{1}_{\{|\vec{v}|>1\}} |\vec{v}|^{-m} + e^{-\frac{|\vec{v}|^2}{16}} \left(\int_0^{c_0} + \int_{c_0}^{3|\vec{v}|} \right) (c_0 + r)^{-m} r^{n-1} dr$$

$$+ \mathbf{1}_{\{|\vec{v}|\le 1\}} \begin{cases} 1, & \text{if } m<n \\ \langle \ln|\vec{v}| \rangle, & \text{if } m=n \\ |\vec{v}|^{n-m}, & \text{if } m>n \end{cases} + \mathbf{1}_{\{|\vec{v}|>1\}} e^{-\frac{|\vec{v}|^2}{2}}$$

$$\lesssim e^{-\frac{|\vec{v}|^2}{16}} \begin{cases} |\vec{v}|^{n-m}, & \text{if } m<n \\ \langle \ln(\frac{|\vec{v}|}{c_0}) \rangle, & \text{if } m=n \\ c_0^{n-m}, & \text{if } m>n \end{cases} + \mathbf{1}_{\{|\vec{v}|\le 1\}} \begin{cases} 1, & \text{if } m<n \\ \langle \ln|\vec{v}| \rangle, & \text{if } m=n \\ |\vec{v}|^{n-m}, & \text{if } m>n \end{cases} + \mathbf{1}_{\{|\vec{v}|>1\}} |\vec{v}|^{-m}$$

$$\sim \mathbf{1}_{\{|\vec{v}|\le 1\}} \begin{cases} 1, & \text{if } m<n \\ \langle \ln|\vec{v}| \rangle + \langle \ln(\frac{|\vec{v}|}{c_0}) \rangle, & \text{if } m=n \\ c_0^{n-m}, & \text{if } m>n \end{cases} + \mathbf{1}_{\{|\vec{v}|>1\}} \begin{cases} |\vec{v}|^{-m}, & \text{if } m<n \\ |\vec{v}|^{-m} + e^{-\frac{|\vec{v}|^2}{16}} \langle \ln(\frac{|\vec{v}|}{c_0}) \rangle, & \text{if } m=n \\ |\vec{v}|^{-m} + e^{-\frac{|\vec{v}|^2}{16}} c_0^{n-m}, & \text{if } m>n. \end{cases}$$

Thus, we obtain

$$\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1} |y_k|^2 s + 1 \right)^{-\nu} (-s)^{-\frac{m}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4}} \left((-s)^{-\frac12} |y_k|^{-1} + |(-s)^{-\frac12} \hat{y}_k + x| \right)^{-m} dx ds$$

$$\lesssim \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1} |y_k|^2 s + 1 \right)^{-\nu} (-s)^{-\frac{m}{2}} \Bigg(\mathbf{1}_{\{s\le -1\}} \begin{cases} 1, & \text{if } m<n \\ \langle \ln(-s) \rangle + \langle \ln|y_k| \rangle, & \text{if } m=n \\ \left((-s)^{-\frac12} |y_k|^{-1} \right)^{n-m}, & \text{if } m>n \end{cases}$$

$$+ \mathbf{1}_{\{s>-1\}} \begin{cases} (-s)^{\frac{m}{2}}, & \text{if } m<n \\ (-s)^{\frac{m}{2}} + e^{\frac{1}{16s}} \langle \ln|y_k| \rangle, & \text{if } m=n \\ (-s)^{\frac{m}{2}} + e^{\frac{1}{16s}} \left((-s)^{-\frac12} |y_k|^{-1} \right)^{n-m}, & \text{if } m>n \end{cases} \Bigg) ds$$

$$= \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1} |y_k|^2 s + 1 \right)^{-\nu} \Bigg(\mathbf{1}_{\{s\le -1\}} \begin{cases} (-s)^{-\frac{m}{2}}, & \text{if } m<n \\ (-s)^{-\frac{n}{2}} \left(\langle \ln(-s) \rangle + \langle \ln|y_k| \rangle \right), & \text{if } m=n \\ (-s)^{-\frac{n}{2}} |y_k|^{m-n}, & \text{if } m>n \end{cases}$$

$$+ \mathbf{1}_{\{s>-1\}} \begin{cases} 1, & \text{if } m<n \\ 1 + (-s)^{-\frac{n}{2}} e^{\frac{1}{16s}} \langle \ln|y_k| \rangle, & \text{if } m=n \\ 1 + (-s)^{-\frac{n}{2}} e^{\frac{1}{16s}} |y_k|^{m-n}, & \text{if } m>n \end{cases} \Bigg) ds := \mathbf{A}.$$

If $\frac{\tau_0-\tau_2^k}{|y_k|^2} \ge -2$, for $|y_k| \ge 2$, we estimate

$$\mathbf{A} \lesssim \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1} |y_k|^2 s + 1 \right)^{-\nu} \begin{cases} 1, & \text{if } m<n \\ \langle \ln|y_k| \rangle, & \text{if } m=n \; ds \\ |y_k|^{m-n}, & \text{if } m>n \end{cases} \lesssim \begin{cases} 1, & \text{if } m<n \\ \langle \ln|y_k| \rangle, & \text{if } m=n \\ |y_k|^{m-n}, & \text{if } m>n \end{cases}$$

where we have used

$$\int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^0 \left((\tau_2^k)^{-1} |y_k|^2 s + 1 \right)^{-\nu} ds = \tau_2^k |y_k|^{-2} (1-\nu)^{-1} \left[1 - (\tau_0 (\tau_2^k)^{-1})^{1-\nu} \right] \lesssim 1$$

for $\nu < 1$.

If $\frac{\tau_0-\tau_2^k}{|y_k|^2} < -2$, we have

$$\mathbf{A} = \int_{-1}^{0} \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} \begin{cases} 1, & \text{if } m < n \\ 1 + (-s)^{-\frac{n}{2}} e^{\frac{1}{16s}} \langle \ln |y_k|\rangle, & \text{if } m = n\ ds \\ 1 + (-s)^{-\frac{n}{2}} e^{\frac{1}{16s}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$+ \left(\int_{\frac{\tau_0-\tau_2^k}{2|y_k|^2}}^{-1} + \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^{\frac{\tau_0-\tau_2^k}{2|y_k|^2}}\right) \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} \begin{cases} (-s)^{-\frac{m}{2}}, & \text{if } m < n \\ (-s)^{-\frac{n}{2}} \left(\langle \ln(-s)\rangle + \langle \ln |y_k|\rangle\right), & \text{if } m = n\ ds \\ (-s)^{-\frac{n}{2}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$\lesssim (1-\nu)^{-1}\tau_2^k |y_k|^{-2} \left[1 - \left(1 - (\tau_2^k)^{-1}|y_k|^2\right)^{1-\nu}\right] \begin{cases} 1, & \text{if } m < n \\ \langle \ln |y_k|\rangle, & \text{if } m = n\ ds \\ |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$+ \int_{\frac{\tau_0-\tau_2^k}{2|y_k|^2}}^{-1} \begin{cases} (-s)^{-\frac{m}{2}}, & \text{if } m < n \\ (-s)^{-\frac{n}{2}} \left(\langle \ln(-s)\rangle + \langle \ln |y_k|\rangle\right), & \text{if } m = n\ ds \\ (-s)^{-\frac{n}{2}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$+ \int_{\frac{\tau_0-\tau_2^k}{|y_k|^2}}^{\frac{\tau_0-\tau_2^k}{2|y_k|^2}} \left((\tau_2^k)^{-1}|y_k|^2 s + 1\right)^{-\nu} \begin{cases} (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{m}{2}}, & \text{if } m < n \\ (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{n}{2}} \left(\langle \ln(\frac{\tau_2^k-\tau_0}{|y_k|^2})\rangle + \langle \ln |y_k|\rangle\right), & \text{if } m = n\ ds \\ (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{n}{2}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$\lesssim \begin{cases} 1, & \text{if } m < n \\ \langle \ln |y_k|\rangle, & \text{if } m = n \\ |y_k|^{m-n}, & \text{if } m > n \end{cases} + \tau_2^k |y_k|^{-2}(1-\nu)^{-1} \left[\left(\frac{\tau_0(\tau_2^k)^{-1}}{2} + \frac{1}{2}\right)^{1-\nu} - (\tau_0(\tau_2^k)^{-1})^{1-\nu}\right]$$

$$\times \begin{cases} (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{m}{2}}, & \text{if } m < n \\ (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{n}{2}} \left(\langle \ln(\frac{\tau_2^k-\tau_0}{|y_k|^2})\rangle + \langle \ln |y_k|\rangle\right), & \text{if } m = n \\ (\frac{\tau_2^k-\tau_0}{|y_k|^2})^{-\frac{n}{2}} |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

$$\lesssim \begin{cases} 1, & \text{if } m < n \\ \langle \ln |y_k|\rangle, & \text{if } m = n \\ |y_k|^{m-n}, & \text{if } m > n \end{cases}$$

where we have used $\tau_2^k \ge 2\tau_0$, $|y_k| \ge 2$, $m > 2$, $n > 2$, $\nu < 1$. Therefore, we conclude the validity of (B.24). □

Appendix C. Linear theory via distorted Fourier transform

Consider the LLG equation in critical dimension

$$\begin{cases} u_t = a(\Delta u + |\nabla u|^2 u) - bu \wedge \Delta u & \text{in } \mathbb{R}^2 \times (0,T), \\ u(\cdot,0) = u_0 \in S^2 & \text{in } \mathbb{R}^2, \end{cases}$$

where $a^2 + b^2 = 1$, $a > 0$, $b \in \mathbb{R}$. The inner linearization around degree 1 harmonic map, in self-similar variables, looks like

$$(a+ib)\partial_\tau - \left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho - \frac{(n+1)^2\rho^4 + (2n^2-6)\rho^2 + (n-1)^2}{(\rho^2+1)^2}\frac{1}{\rho^2}\right).$$

We utilize distorted Fourier transform at mode $n = -1$ in [99, Section 9.6]. The use of aforementioned techniques does imply a solution. However, such solution is not sufficient for the gluing to work as it loses too many R's and makes the nonlinear terms non-controllable. So we use distorted Fourier transform (DFT) instead to get desired estimates. The reason behind this is that formally the worst mode is -1 as it corresponds to 2-dimensional heat operator, and usually estimates in 2D come with a logarithmic loss.

The derivation of desired estimates for mode -1 is done by first deducing the Duhamel's representation via DFT and then estimating pointwisely. For $\ell \in \mathbb{R}$ and $v(\tau) > 0$ and vectorial complex-valued function f, the weighted topology are defined by

$$\|f\|_{v,\ell} := \sup_{(y,\tau)\in\mathbb{R}^2\times(\tau_0,\infty)} v^{-1}(\tau)\langle y\rangle^{\ell}|f(y,\tau)|.$$

Consider

$$\begin{cases}(a+ib)\partial_\tau\phi_n(\rho,\tau) = \mathcal{L}_n\phi_n(\rho,\tau),\\ \phi_n(\rho,\tau_0) = g(\rho),\end{cases}$$

where $\tau_0 \geq 1$,

$$\mathcal{L}_n = \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho - \frac{(n-1)^2}{\rho^2} - \frac{4n}{\rho^2+1} + \frac{8}{(\rho^2+1)^2}.$$

Assume g is a Schwartz function. Set $\phi_n(\rho,\tau) = \rho^{-\frac{1}{2}}A_n(\rho,\tau)$, then

$$\begin{cases}(a+ib)\partial_\tau A_n(\rho,\tau) = \tilde{\mathcal{L}}_n A_n(\rho,\tau),\\ A_n(\rho,\tau_0) = \rho^{\frac{1}{2}}g(\rho).\end{cases}$$

where $\tilde{\mathcal{L}}_n = \partial_{\rho\rho} + \frac{1}{4}\rho^{-2} - \frac{(n-1)^2}{\rho^2} - \frac{4n}{\rho^2+1} + \frac{8}{(\rho^2+1)^2}$.

Recall the generalized eigenfunctions $\Phi^n(\rho,\xi)$ with respect to $-\tilde{\mathcal{L}}_n$ is given by

$$-\tilde{\mathcal{L}}_n\Phi^n(\rho,\xi) = \xi\Phi^n(\rho,\xi).$$

We multiply it by $\Phi^n(\rho,\xi)$ and integrate by parts and get

$$\begin{cases}(a+ib)\partial_\tau\hat{A}_n(\xi,\tau) = -\xi\hat{A}_n(\xi,\tau),\\ \hat{A}_n(\xi,\tau_0) = \int_0^\infty \rho^{\frac{1}{2}}g(\rho)\Phi^n(\rho,\xi)d\rho,\end{cases}$$

where $\hat{A}_n(\xi,\tau) = \int_0^\infty A_{-1}(\rho,\tau)\Phi^n(\rho,\xi)d\rho$. Thus

$$\hat{A}_n(\xi,\tau) = e^{-(a-ib)\xi\tau}\hat{A}_n(\xi,\tau_0).$$

Taking inverse DFT, one has

$$\begin{aligned}A_n(\rho,\tau) &= \int_0^\infty \hat{A}_n(\xi,\tau)\Phi^n(\rho,\xi)\rho_n(d\xi) = \int_0^\infty e^{-(a-ib)\xi\tau}\hat{A}_n(\xi,0)\Phi^n(\rho,\xi)\rho_n(d\xi)\\ &= \int_0^\infty e^{-(a-ib)\xi\tau}\Phi^n(\rho,\xi)\int_0^\infty x^{\frac{1}{2}}g(x)\Phi^n(x,\xi)dx\rho_n(d\xi)\\ &= \int_0^\infty\int_0^\infty e^{-(a-ib)\xi\tau}\Phi^n(\rho,\xi)\Phi^n(x,\xi)\rho_n(d\xi)x^{\frac{1}{2}}g(x)dx.\end{aligned}$$

By Duhamel's principle, it holds that

$$\phi_n(\rho,\tau) = \int_{\tau_0}^\tau\int_0^\infty\int_0^\infty e^{-(a-ib)\xi(\tau-s)}\rho^{-\frac{1}{2}}\Phi^n(\rho,\xi)\Phi^n(x,\xi)x^{\frac{1}{2}}h_n(x,s)\rho_n(d\xi)dxds \tag{C.1}$$

gives a solution to the non-homogeneous equation with RHS h_n and zero initial data.

To estimate solution in above formulation, one needs precise estimates of generalized eigenfunctions and density of spectral measure. For $n = -1$, we summarize the results in [60, Section 4.3.2] as follows.

Proposition C.1 ([60]). *For all $\rho \geq 0$, $\xi \geq 0$, we have*

$$\left|\Phi^{-1}(\rho,\xi)\right| \lesssim \begin{cases}\rho^{\frac{5}{2}}\langle\rho\rangle^{-2} & \text{if } \rho^2\xi \leq 1\\ \xi^{-\frac{1}{4}}\langle\xi\rangle^{-1} & \text{if } \rho^2\xi > 1\end{cases}.$$

$\Phi^{-1}(\rho,\xi)$ has the following expansion:

$$\Phi^{-1}(\rho,\xi) = \Phi_0^{-1}(\rho) + \rho^{\frac{1}{2}}\sum_{j=1}^\infty(-\rho^2\xi)^j\Phi_j(\rho^2),$$

which converges absolutely, where $\Phi_0^{-1}(\rho) = \frac{\rho^{\frac{5}{2}}}{1+\rho^2}$. *It converges uniformly if* $\rho\xi^{\frac{1}{2}}$ *remains bounded. Here* $\Phi_j(u) \ge 0$ *are smooth functions of* $u \ge 0$ *satisfying*

$$\Phi_j(u) \le \frac{1}{j!}\frac{u}{1+u}, \quad \text{for all } \ u \ge 0, j \ge 1,$$

and $\Phi_1(u) \ge c_1 \frac{u}{1+u}$ *for all* $u \ge 0$ *with some absolute constant* $c_1 > 0$.

The spectrum measure $\rho_{-1}(d\xi)$ *of* $-\tilde{\mathcal{L}}_{-1}$ *is absolutely continuous on* $\xi \ge 0$ *with density*

$$\frac{d\rho_{-1}(\xi)}{d\xi} \sim \langle\xi\rangle^2.$$

Our linear theory for LLG mode -1 without orthogonality condition is stated as follows.

Proposition C.2. ([99, Proposition 9.8]) *Consider*

$$\begin{cases} (a+ib)\partial_\tau \phi_{-1}(\rho,\tau) = \mathcal{L}_{-1}\phi_{-1}(\rho,\tau) + h(\rho,\tau) \quad \text{in } \ (0,\infty)\times(\tau_0,\infty), \\ \phi_{-1}(\rho,\tau_0) = 0 \quad \text{in } \ (0,\infty). \end{cases}$$

where $\tau_0 \ge 2$, $\|h\|_{v,\ell} < \infty$, *where* $v(\tau) \ge 0$, $\ell > \frac{3}{2}$. *Then the solution* $\phi_{-1} = \mathcal{T}_{-1}[h]$, *where* $\mathcal{T}_{-1}[h]$ *is given by the linear mapping* (C.1) *with* $k = -1$, *satisfies the following estimate*

$$|\phi_{-1}(\rho,\tau)| \lesssim \|h\|_{v,\ell}\mathbf{1}_{\{\rho \le \tau^{\frac{1}{2}}\}} \begin{cases} \tau^{1-\frac{\ell}{2}} \sup\limits_{s\in[\tau/2,\tau]} v(s) + \tau^{-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell < 2 \\ (\ln\tau)^2 \sup\limits_{s\in[\tau/2,\tau]} v(s) + \tau^{-1}\ln\tau \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell = 2 \\ \ln\tau \sup\limits_{s\in[\tau/2,\tau]} v(s) + \tau^{-1} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell > 2 \end{cases}$$

$$+ \|h\|_{v,\ell}\mathbf{1}_{\{\rho > \tau^{\frac{1}{2}}\}}\rho^{-\frac{1}{2}} \begin{cases} \tau^{\frac{5}{4}-\frac{\ell}{2}} \sup\limits_{s\in[\tau/2,\tau]} v(s) + \tau^{\frac{1}{4}-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell < 2 \\ \tau^{\frac{1}{4}}\langle\ln\tau\rangle \sup\limits_{s\in[\tau/2,\tau]} v(s) + \tau^{-\frac{3}{4}}\langle\ln\tau\rangle \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell = 2 \\ \tau^{\frac{1}{4}} \sup\limits_{s\in[\tau/2,\tau]} v(s) + \tau^{-\frac{3}{4}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell > 2. \end{cases}$$

Inside the self-similar region, better estimates with orthogonality condition imposed can be obtained, see [99, Section 9.6]. Indeed, we have:

Proposition C.3. ([99, Proposition 9.8]) *Under the assumptions in Proposition C.2, assuming* $2 < \ell < \frac{5}{2}$ *and the orthogonality condition*

$$\int_{\mathbb{R}^2} h(y,\tau)\mathcal{Z}_{-1,1}(y)dy = 0 \quad \text{for all } \ \tau > \tau_0, \tag{C.2}$$

we have the following estimate

$$|\phi_{-1}(\rho,\tau)| \lesssim \|h\|_{v,\ell}^{\infty} \begin{cases} \langle\rho\rangle^{2-\ell} \sup\limits_{s\in[\tau/2,\tau]} v(s) + \tau^{-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \rho \le \tau^{\frac{1}{2}} \\ \rho^{-\frac{1}{2}}\Big(\tau^{\frac{5}{4}-\frac{\ell}{2}} \sup\limits_{s\in[\tau/2,\tau]} v(s) + \tau^{\frac{1}{4}-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds\Big) & \text{if } \ \rho > \tau^{\frac{1}{2}}. \end{cases}$$

The proof of Proposition C.2 and Proposition C.3 is done by directly estimating the Duhamel's formula.

Appendix D. Linear theory via re-gluing

General linear theory, which can be applied to the inner problem, will be developed in this section. Technical ingredients include energy estimates, orthogonality conditions, maximum principle, spectral gap, and a re-gluing process.

This section is pretty independent of the other parts in this note. Some symbols which appear in other sections are abused, but there is no relationship between them. Consider

$$\begin{cases} \partial_\tau\phi = L\phi + A(y,\tau)\cdot\nabla\phi + B(y,\tau)\phi + h, \quad \text{in } \ \mathcal{D}_R, \\ \phi(\cdot,\tau_0) = 0, \quad \text{in } \ B_{R(\tau_0)} \end{cases} \tag{D.1}$$

where $A = (A_1, A_2, \cdots, A_d)$ is a vector-valued function and B is a scalar function,

$$L := \Delta + V(y), \quad B_{R(\tau)} := \left\{y \in \mathbb{R}^d \mid |y| < R(\tau)\right\}, \quad \mathcal{D}_R = \left\{(y,\tau) \;\middle|\; y \in B_{R(\tau)},\ \tau \in (\tau_0, \infty)\right\}.$$

We always assume $V(y) = V(|y|)$ is radial in space satisfying $|V(y)| \lesssim \langle y\rangle^{-2}$; L has a radial positive kernel $Z(\rho)$ with the asymptotic behavior

$$Z(\rho) \sim \langle\rho\rangle^{\beta}, \qquad |\partial_\rho Z(\rho)| \lesssim \langle\rho\rangle^{\beta-1}.$$

We shall write $v = v(\tau),\ R = R(\tau)$ for simplicity.

Our aim is to find well-behaved ϕ for right hand side h in the weighted space with the norm

$$\|h\|_{v,\ell} := \sup_{(y,\tau)\in\mathcal{D}_R} (v(\tau))^{-1} \langle y\rangle^{\ell} |h(y,\tau)|$$

for $\ell \in \mathbb{R}$, $v(\tau) \geq 0$. The non-orthogonal linear theory is given as follows.

Proposition D.1. *Consider*

$$\begin{cases} \partial_\tau \phi = L\phi + A(y,\tau)\cdot\nabla\phi + B(y,\tau)\phi + h & \text{in } \mathcal{D}_R, \\ \phi = 0 \quad \text{on } \partial\mathcal{D}_R, \quad \phi(\cdot,\tau_0) = 0 & \text{in } B_{R(\tau_0)}. \end{cases}$$

Suppose $\|h\|_{v,\ell} < \infty$, $R \geq 2$, $v, R \in C^1(\tau_0,\infty)$, $\langle y\rangle^{-1}|A(y,\tau)| + |B(y,\tau)| \leq f(\tau)$, $\left(f(\tau) + |v'|v^{-1}\right)\theta_{R,\ell,1} + \theta_{R,\ell,2} \ll 1$, *where*

$$\theta_{R,\ell,1} := \begin{cases} \begin{cases} R^{2+\max\{0,-\beta-\ell\}}, & \ell < \beta + d \\ R^{2+\max\{0,-d-2\beta\}} \ln R, & \ell = \beta + d \\ R^{2+\max\{0+,-d-2\beta\}}, & \ell > \beta + d \end{cases} & \text{if } \beta < 1 - \frac{d}{2} \\ \begin{cases} R^{2+\max\{0,-\beta-\ell\}}, & \ell < \beta + d \\ R^2 (\ln R)^2, & \ell = \beta + d \\ R^{2+} \ln R, & \ell > \beta + d \end{cases} & \text{if } \beta = 1 - \frac{d}{2} \\ \begin{cases} R^{2+\max\{0,-\beta-\ell\}}, & \ell < 2 - \beta \\ R^2 \ln R, & \ell = 2 - \beta \\ R^2, & 2 - \beta < \ell < \beta + d \\ R^2 \ln R, & \ell = \beta + d \\ R^{2+}, & \ell > \beta + d \end{cases} & \text{if } \beta > 1 - \frac{d}{2}, \end{cases}$$

$$\theta_{R,\ell,2} := \begin{cases} |R'| R^{1+\max\{0,-\beta-\ell\}}, & \ell < \beta + d \\ |R'| R^{1+\max\{0,-d-2\beta\}} \ln R, & \ell = \beta + d \\ |R'| R^{1+\max\{0+,-d-2\beta\}}, & \ell > \beta + d, \end{cases}$$

then there exists a unique solution with the pointwise upper bound

$$\langle y\rangle |\nabla\phi(y,\tau)| + |\phi(y,\tau)| \lesssim v(\tau) \langle y\rangle^{\beta} \Theta_{R,\ell}(|y|) \|h\|_{v,\ell}$$

where

$$\Theta_{R,\ell}(\rho) := \begin{cases} \begin{cases} R^{2-\beta-\ell}, & \ell < \beta + d \\ R^{2-d-2\beta} \ln R, & \ell = \beta + d \\ R^{2-d-2\beta}, & \ell > \beta + d \end{cases} & \text{if } \beta < 1 - \frac{d}{2} \\ \begin{cases} R^{2-\beta-\ell}, & \ell < \beta + d \\ (\ln R)^2, & \ell = \beta + d \\ \ln R, & \ell > \beta + d \end{cases} & \text{if } \beta = 1 - \frac{d}{2} \\ \begin{cases} R^{2-\beta-\ell}, & \ell < 2 - \beta \\ \ln R, & \ell = 2 - \beta \\ \langle\rho\rangle^{2-\beta-\ell}, & 2 - \beta < \ell < \beta + d \\ \langle\rho\rangle^{2-d-2\beta} \ln(\rho + 2), & \ell = \beta + d \\ \langle\rho\rangle^{2-d-2\beta}, & \ell > \beta + d \end{cases} & \text{if } \beta > 1 - \frac{d}{2}. \end{cases} \tag{D.2}$$

Proof. The existence and uniqueness are given by the classical parabolic theory and we will give pointwise estimate by the comparison theorem. Set $\rho = |y|$ and a barrier function with the form $\bar{\phi}(\rho,\tau) = Cvg(\rho, R)$, where

$$Lg(\rho, R) = -\langle \rho\rangle^{-\bar{\ell}}, \quad g(\rho, R) = Z(\rho)\int_\rho^R \frac{dx}{Z^2(x)x^{d-1}}\int_0^x Z(s)s^{d-1}\langle s\rangle^{-\bar{\ell}}ds$$

and $C, \bar{\ell}$ will be determined later. It is easy to get

$$\partial_\rho g(\rho, R) = Z'(\rho)\int_\rho^R \frac{dx}{Z^2(x)x^{d-1}}\int_0^x Z(s)s^{d-1}\langle s\rangle^{-\bar{\ell}}ds - \frac{1}{Z(\rho)\rho^{d-1}}\int_0^\rho Z(s)s^{d-1}\langle s\rangle^{-\bar{\ell}}ds.$$

Direct calculation yields

$$\int_0^x Z(s)s^{d-1}\langle s\rangle^{-\bar{\ell}}ds \sim \begin{cases} x^d, & x \le 1 \\ x^{\beta+d-\bar{\ell}}, & x > 1, \bar{\ell} < \beta + d \\ 1+\ln x, & x > 1, \bar{\ell} = \beta+d \\ 1, & x > 1, \bar{\ell} > \beta + d, \end{cases} \qquad \frac{1}{Z^2(x)x^{d-1}} \sim \begin{cases} x^{1-d}, & x \le 1 \\ x^{1-d-2\beta}, & x > 1, \end{cases}$$

$$\frac{1}{Z^2(x)x^{d-1}}\int_0^x Z(s)s^{d-1}\langle s\rangle^{-\bar{\ell}}ds \sim \begin{cases} x, & x \le 1 \\ x^{1-\beta-\bar{\ell}}, & x > 1, \bar{\ell} < \beta+d \\ x^{1-d-2\beta}(1+\ln x), & x > 1, \bar{\ell} = \beta + d \\ x^{1-d-2\beta}, & x > 1, \bar{\ell} > \beta + d. \end{cases}$$

Considering three cases $1 - d - 2\beta > (=, <) - 1$, namely, $\beta < (=, >) 1 - \frac{d}{2}$, we have

$$\int_\rho^R \frac{dx}{Z^2(x)x^{d-1}}\int_0^x Z(s)s^{d-1}\langle s\rangle^{-\bar{\ell}}ds \lesssim \Theta_{R,\bar{\ell}}(\rho).$$

Then

$$g \lesssim \langle\rho\rangle^\beta \Theta_{R,\bar{\ell}}(\rho).$$

For $\partial_\rho g$, since

$$\frac{1}{Z(\rho)\rho^{d-1}}\int_0^\rho Z(s)s^{d-1}\langle s\rangle^{-\bar{\ell}}ds \sim \begin{cases} \rho, & \rho \le 1 \\ \rho^{1-\bar{\ell}}, & \rho > 1, \bar{\ell} < \beta + d \\ \rho^{1-d-\beta}(1+\ln\rho), & \rho > 1, \bar{\ell} = \beta+d \\ \rho^{1-d-\beta}, & \rho > 1, \bar{\ell} > \beta + d, \end{cases}$$

then

$$|\partial_\rho g| \lesssim \langle\rho\rangle^{\beta-1}\Theta_{R,\bar{\ell}}(\rho).$$

A closer look at $\Theta_{R,\bar{\ell}}(\rho)$, when $\bar{\ell} > \beta + d$, tells that the increase in $\bar{\ell}$ will not improve the estiamte of g. Thus, we take

$$\bar{\ell} := \begin{cases} \ell, & \ell \le \beta + d \\ (\beta+d)+ < \ell, & \ell > \beta + d. \end{cases}$$

Then $\Theta_{R,\bar{\ell}}(\rho) = \Theta_{R,\ell}(\rho)$, which implies

$$g + \langle\rho\rangle|\partial_\rho g| \lesssim \langle\rho\rangle^\beta \Theta_{R,\ell}(\rho).$$

From the choice of g, we have

$$\begin{aligned} & P[\bar{\phi}] := L(Cvg) + h + Cv\left(A\cdot\nabla g + Bg\right) - \partial_\tau(Cvg) \\ = & -Cv\langle y\rangle^{-\bar{\ell}} + h + Cv\left(A\cdot\nabla g + Bg\right) - Cv'g - \frac{CvR'Z(|y|)}{Z^2(R)R^{d-1}}\int_0^R Z(s)s^{d-1}\langle s\rangle^{-\bar{\ell}}ds \\ \le & \, Cv\langle y\rangle^{-\bar{\ell}}\Big[-1 + \langle y\rangle^{\bar{\ell}}\left(A\cdot\nabla g + Bg\right) - v'v^{-1}\langle y\rangle^{\bar{\ell}}g - \frac{R'\langle y\rangle^{\bar{\ell}}Z(|y|)}{Z^2(R)R^{d-1}}\int_0^R Z(s)s^{d-1}\langle s\rangle^{-\bar{\ell}}ds\Big] + v\langle y\rangle^{-\ell}\|h\|_{v,\ell} \\ \le & -\frac{3}{4}Cv\langle y\rangle^{-\bar{\ell}} + v\langle y\rangle^{-\ell}\|h\|_{v,\ell} \end{aligned}$$

where we have used

$$\left|\langle y\rangle^{\bar{\ell}}\left(A\cdot\nabla g + Bg\right)\right| + \left|v'v^{-1}\langle y\rangle^{\bar{\ell}}g\right| \lesssim \left(f(\tau) + |v'|v^{-1}\right)\langle y\rangle^{\bar{\ell}+\beta}\Theta_{R,\ell}(|y|) \lesssim \left(f(\tau)+|v'|v^{-1}\right)\theta_{R,\ell,1} \ll 1$$

$$\Big|\frac{R'\langle y\rangle^{\bar\ell}Z(|y|)}{Z^2(R)R^{d-1}}\int_0^R Z(s)s^{d-1}\langle s\rangle^{-\bar\ell}ds\Big| \lesssim \langle y\rangle^{\bar\ell+\beta}\begin{cases}|R'|R^{1-\beta-\bar\ell}, & \bar\ell<\beta+d\\ |R'|R^{1-d-2\beta}\ln R, & \bar\ell=\beta+d\\ |R'|R^{1-d-2\beta}, & \bar\ell>\beta+d\end{cases} \lesssim \theta_{R,\ell,2}\ll 1.$$

Set $C=2\|h\|_{v,\ell}$, then $P[\bar\phi]\le 0$, which implies $|\phi(y,\tau)|\lesssim v(\tau)\langle y\rangle^{\beta}\Theta_{R,\ell}(|y|)$. Since $\theta_{R,\ell,1}\ge R^2$, then $f(\tau)\ll(\theta_{R,\ell,1})^{-1}\le R^{-2}$. Additionally, from the presumption, $|V(y)|\lesssim\langle y\rangle^{-2}$. Then using scaling argument, we get the gradient estimate.

□

Lemma D.1. *Consider*

$$\begin{cases}\partial_\tau\phi=L\phi+A(y,\tau)\cdot\nabla\phi+B(y,\tau)\phi+h & \text{in } \mathcal{D}_R,\\ \phi(\cdot,\tau_0)=0 & \text{in } B_{R(\tau_0)}.\end{cases}$$

Suppose $\|h\|_{v,\ell}<\infty$, h is radial in space and if $\ell>\beta+d$, h satisfies the orthogonal condition

$$\int_0^R h(u,\tau)Z(u)u^{d-1}du=0 \quad \text{for all } \tau>\tau_0; \tag{D.3}$$

and $\langle y\rangle^{-1}|A(y,\tau)|+|B(y,\tau)|\le f(\tau)$,

$$\begin{aligned}&R\ge 2, v,R\in C^1(\tau_0,\infty), \left(f(\tau)+|v'||v^{-1}\right)\theta_{R,\bar\ell-2,1}+\theta_{R,\bar\ell-2,2}\ll 1,\\ &f(\tau)v(\tau)\theta_{R,\ell,3}\in C^1(\tau_0,\infty), \left[f(\tau)+|(f(\tau)v(\tau)\theta_{R,\ell,3})'|(f(\tau)v(\tau)\theta_{R,\ell,3})^{-1}\right]\theta_{R,p_\ell,1}+\theta_{R,p_\ell,2}\ll 1,\end{aligned} \tag{D.4}$$

where $\theta_{R,c,1}$, $\theta_{R,c,2}$ for $c\in\mathbb{R}$ are defined in Propostion D.1,

$$\bar\ell:=\begin{cases}\ell, & \ell\ne\beta+d,2-\beta\\ \ell-, & \ell=\beta+d \text{ or } \ell=2-\beta,\end{cases}\qquad p_\ell:=\begin{cases}2-\beta & \text{if } \beta\le 1-\frac d2\\ \begin{cases}2-\beta, & \bar\ell\le 4-\beta\\ \bar\ell-2, & 4-\beta<\bar\ell<\beta+d+2\\ (d+\beta)-, & \bar\ell=\beta+d+2\\ d+\beta, & \bar\ell>\beta+d+2\end{cases} & \text{if } \beta>1-\frac d2,\end{cases} \tag{D.5}$$

$$\theta_{R,\ell,3}:=\begin{cases}\begin{cases}R^{4-\beta-\bar\ell}, & \bar\ell<\beta+d+2\\ R^{2-d-2\beta}\ln R, & \bar\ell=\beta+d+2\\ R^{2-d-2\beta}, & \bar\ell>\beta+d+2\end{cases} & \text{if } \beta<1-\frac d2\\ \begin{cases}R^{4-\beta-\bar\ell}, & \bar\ell<\beta+d+2\\ (\ln R)^2, & \bar\ell=\beta+d+2\\ \ln R, & \bar\ell>\beta+d+2\end{cases} & \text{if } \beta=1-\frac d2\\ \begin{cases}R^{4-\beta-\bar\ell}, & \bar\ell<4-\beta\\ \ln R, & \bar\ell=4-\beta\\ 1, & \bar\ell>4-\beta\end{cases} & \text{if } \beta>1-\frac d2.\end{cases}$$

Then there exists a solution $\phi=\phi[h]$ as a linear mapping about h with the estimate

$$\langle y\rangle|\nabla\phi|+|\phi|\lesssim v(\tau)\theta_{R,\ell,3}\left(\langle y\rangle^{-p_\ell}+f(\tau)\langle y\rangle^{\beta}\Theta_{R,p_\ell}(|y|)\right)\|h\|_{v,\ell}$$

where

$$\theta_{R,\ell,3}\langle y\rangle^{-p_\ell}=\begin{cases}\langle y\rangle^{\beta-2}\begin{cases}R^{4-\beta-\bar\ell}, & \bar\ell<\beta+d+2\\ R^{2-d-2\beta}\ln R, & \bar\ell=\beta+d+2\\ R^{2-d-2\beta}, & \bar\ell>\beta+d+2\end{cases} & \text{if } \beta<1-\frac{d}{2}\\ \langle y\rangle^{\beta-2}\begin{cases}R^{4-\beta-\bar\ell}, & \bar\ell<\beta+d+2\\ (\ln R)^2, & \bar\ell=\beta+d+2\\ \ln R, & \bar\ell>\beta+d+2\end{cases} & \text{if } \beta=1-\frac{d}{2}\\ \begin{cases}\langle y\rangle^{\beta-2}R^{4-\beta-\bar\ell}, & \bar\ell<4-\beta\\ \langle y\rangle^{\beta-2}\ln R, & \bar\ell=4-\beta\\ \langle y\rangle^{2-\bar\ell}, & 4-\beta<\bar\ell<\beta+d+2\\ \langle y\rangle^{-[(d+\beta)-]}, & \bar\ell=\beta+d+2\\ \langle y\rangle^{-d-\beta}, & \bar\ell>\beta+d+2\end{cases} & \text{if } \beta>1-\frac{d}{2}.\end{cases}\tag{D.6}$$

Remark D.1. *By direct calculation, for $\ell\le\beta+d$, we have $\langle y\rangle^{\beta}\Theta_{R,\ell}(|y|)\lesssim\langle y\rangle^{-p_\ell}\theta_{R,\ell,3}$. Thus for $\ell\le\beta+d$, the estimate in Proposition D.1 is better than Lemma D.1.*

Proof. Let us first consider an elliptic problem $LH=\tilde h$ where $\tilde h$ is the extension of h as zero outside $\mathcal{D}_R$. H is given by

$$H(\rho,\tau)=\begin{cases}Z(\rho)\displaystyle\int_0^\rho\frac{1}{Z^2(s)s^{d-1}}\int_0^s Z(u)u^{d-1}\tilde h(u,\tau)duds, & \text{if }\ \ell=\beta+d,\beta\le1-\frac{d}{2}\text{ or }\ell\ne\beta+d,\ell\le2-\beta\\ -Z(\rho)\displaystyle\int_\rho^\infty\frac{1}{Z^2(s)s^{d-1}}\int_0^s Z(u)u^{d-1}\tilde h(u,\tau)duds, & \text{if }\ \ell=\beta+d,\beta>1-\frac{d}{2}\text{ or }\ell\ne\beta+d,\ell>2-\beta.\end{cases}$$

The intuition behind the choice of H is to make H behave like integration twice in space about h and keep the information of ℓ. By direct calculation, H has the following estimate

$$\|H\|_{v,\bar\ell-2}\lesssim\|h\|_{v,\ell}\tag{D.7}$$

for $\bar\ell$ given in (D.5). Indeed,

$$\left|\int_0^s Z(u)u^{d-1}\tilde h(u,\tau)du\right|\lesssim\|h\|_{v,\ell}v\begin{cases}s^d, & s\le1\\ s^{\beta+d-\ell}, & s>1,\ell<\beta+d\\ 1+\ln s, & s>1,\ell=\beta+d\\ s^{\beta+d-\ell}, & s>1,\ell>\beta+d,\end{cases}$$

where we require the orthogonal condition (D.3) when $s>1$, $\ell>\beta+d$.

$$\frac{1}{Z^2(s)s^{d-1}}\sim\begin{cases}s^{1-d}, & s\le1\\ s^{1-d-2\beta}, & s>1.\end{cases}$$

Then

$$\left|\frac{1}{Z^2(s)s^{d-1}}\int_0^s Z(u)u^{d-1}\tilde h(u,\tau)du\right|\lesssim\|h\|_{v,\ell}v\begin{cases}s, & s\le1\\ s^{1-d-2\beta}\left(1+\ln s\right), & s>1,\ell=\beta+d\\ s^{1-\beta-\ell}, & s>1,\ell\ne\beta+d.\end{cases}$$

When $\ell=\beta+d,\beta\le1-\frac{d}{2}$ or $\ell\ne\beta+d,\ell\le2-\beta$,

$$\left|\int_0^\rho\frac{1}{Z^2(s)s^{d-1}}\int_0^s Z(u)u^{d-1}\tilde h(u,\tau)duds\right|\lesssim\|h\|_{v,\ell}v\begin{cases}\rho^2, & \rho\le1\\ \rho^{2-d-2\beta}\left(1+\ln\rho\right), & \rho>1,\ell=\beta+d,\beta<1-\frac{d}{2}\\ 1+\left(\ln\rho\right)^2, & \rho>1,\ell=\beta+d,\beta=1-\frac{d}{2}\\ \rho^{2-\beta-\ell}, & \rho>1,\ell\ne\beta+d,\ell<2-\beta\\ 1+\ln\rho, & \rho>1,\ell\ne\beta+d,\ell=2-\beta.\end{cases}$$

For the cases $\ell=\beta+d, \beta>1-\frac{d}{2}$ or $\ell\neq\beta+d, \ell>2-\beta$, this formula cannot recover the information of ℓ. Then

$$|H|\lesssim \|h\|_{v,\ell} v \begin{cases} \rho^2, & \rho\le 1\\ \rho^{2-\ell}\langle \ln\rho\rangle, & \rho>1, \ell=\beta+d, \beta<1-\frac{d}{2}\\ \rho^{2-\ell}\langle \ln\rho\rangle^2, & \rho>1, \ell=\beta+d, \beta=1-\frac{d}{2}\\ \rho^{2-\ell}, & \rho>1, \ell\neq\beta+d, \ell<2-\beta\\ \rho^{2-\ell}\langle \ln\rho\rangle, & \rho>1, \ell\neq\beta+d, \ell=2-\beta. \end{cases}$$

When $\ell=\beta+d, \beta>1-\frac{d}{2}$ or $\ell\neq\beta+d, \ell>2-\beta$,

$$\left|\int_\rho^\infty \frac{1}{Z^2(s)s^{d-1}}\int_0^s Z(u)u^{d-1}\tilde{h}(u,\tau)duds\right| \lesssim \|h\|_{v,\ell} v \begin{cases} \langle\rho\rangle^{2-d-2\beta}\ln(\rho+2), & \ell=\beta+d, \beta>1-\frac{d}{2}\\ \langle\rho\rangle^{2-\beta-\ell}, & \ell\neq\beta+d, \ell>2-\beta \end{cases}$$

which implies

$$|H|\lesssim \|h\|_{v,\ell} v \begin{cases} \langle\rho\rangle^{2-\ell}\ln(\rho+2), & \ell=\beta+d, \beta>1-\frac{d}{2}\\ \langle\rho\rangle^{2-\ell}, & \ell\neq\beta+d, \ell>2-\beta. \end{cases}$$

Thus we complete the proof of (D.7). Next, consider

$$\begin{cases} \partial_\tau\Phi = L\Phi + H \quad \text{in} \quad \mathcal{D}_{2R},\\ \Phi=0 \quad \text{on} \quad \partial\mathcal{D}_{2R} \quad \Phi(\cdot,\tau_0)=0 \quad \text{in} \quad B_{2R(\tau_0)}. \end{cases}$$

Under the assumption $R\ge 2$, $v, R\in C^1(\tau_0,\infty)$, $\left(f(\tau)+|v'|v^{-1}\right)\theta_{R,\bar{\ell}-2,1}+\theta_{R,\bar{\ell}-2,2}\ll 1$, by Proposition D.1, we have a unique solution Φ with the estimate

$$|\Phi|\lesssim v(\tau)\langle y\rangle^\beta \Theta_{R,\bar{\ell}-2}(|y|)\|h\|_{v,\ell}.$$

Set $\phi_1=L\Phi$. By $|V(|y|)|\lesssim\langle y\rangle^{-2}$, scaling argument and the defintion of $\Theta_{R,\bar{\ell}-2}$ in Proposition D.1, we have

$$\begin{aligned} &\langle y\rangle|\nabla\phi_1|+|\phi_1|\lesssim v(\tau)\langle y\rangle^{\beta-2}\Theta_{R,\bar{\ell}-2}(|y|)\|h\|_{v,\ell}\\ &\lesssim v(\tau)\|h\|_{v,\ell}\begin{cases} \langle y\rangle^{\beta-2}\begin{cases} R^{4-\beta-\bar{\ell}}, & \bar{\ell}<\beta+d+2\\ R^{2-d-2\beta}\ln R, & \bar{\ell}=\beta+d+2\\ R^{2-d-2\beta}, & \bar{\ell}>\beta+d+2\end{cases} & \text{if} \quad \beta<1-\frac{d}{2}\\ \langle y\rangle^{\beta-2}\begin{cases} R^{4-\beta-\bar{\ell}}, & \bar{\ell}<\beta+d+2\\ (\ln R)^2, & \bar{\ell}=\beta+d+2\\ \ln R, & \bar{\ell}>\beta+d+2\end{cases} & \text{if} \quad \beta=1-\frac{d}{2}\\ \begin{cases} \langle y\rangle^{\beta-2}R^{4-\beta-\bar{\ell}}, & \bar{\ell}<4-\beta\\ \langle y\rangle^{\beta-2}\ln R, & \bar{\ell}=4-\beta\\ \langle y\rangle^{2-\bar{\ell}}, & 4-\beta<\bar{\ell}<\beta+d+2\\ \langle y\rangle^{-d-\beta}\ln(|y|+2), & \bar{\ell}=\beta+d+2\\ \langle y\rangle^{-d-\beta}, & \bar{\ell}>\beta+d+2\end{cases} & \text{if} \quad \beta>1-\frac{d}{2} \end{cases}\\ &\lesssim v(\tau)\theta_{R,\ell,3}\langle y\rangle^{-p_\ell}\|h\|_{v,\ell}. \end{aligned}$$

In order to retrieve the term $A\cdot\nabla\phi+B\phi$, we consider

$$\begin{cases} \partial_\tau\phi_2 = L\phi_2 + A\cdot\nabla\phi_2 + B\phi_2 + A\cdot\nabla\phi_1 + B\phi_1 \quad \text{in} \quad \mathcal{D}_R,\\ \phi_2=0 \quad \text{on} \quad \partial\mathcal{D}_R, \quad \phi_2(\rho,\tau_0)=0 \quad \text{in} \quad B_{R(\tau_0)} \end{cases}$$

where

$$|A\cdot\nabla\phi_1+B\phi_1|\lesssim f(\tau)\left(\langle y\rangle|\nabla\phi_1|+|\phi_1|\right)\lesssim f(\tau)v(\tau)\theta_{R,\ell,3}\langle y\rangle^{-p_\ell}\|h\|_{v,\ell}.$$

Under the assumption $R\ge 2$, $f(\tau)v(\tau)\theta_{R,\ell,3}, R\in C^1(\tau_0,\infty)$, $\left[f(\tau)+|(f(\tau)v(\tau)\theta_{R,\ell,3})'|(f(\tau)v(\tau)\theta_{R,\ell,3})^{-1}\right]\theta_{R,p_\ell,1}+\theta_{R,p_\ell,2}\ll 1$, by Proposition D.1, we get ϕ_2 with the estimate

$$\langle y\rangle|\nabla\phi_2|+|\phi_2|\lesssim f(\tau)v(\tau)\theta_{R,\ell,3}\langle y\rangle^\beta\Theta_{R,p_\ell}(|y|)\|h\|_{v,\ell}.$$

Set $\phi[h]=\phi_1+\phi_2$. We complete the proof.

□

Next we perform the re-gluing procedure to further improve the linear theory. In the process of re-gluing, since the outer solution will enter the right hand side of the inner problem, in order to make the term involving the outer solution radial in space, we take $A(y,\tau) = \tilde{A}(\rho,\tau)y$, $B(y,\tau) = B(\rho,\tau)$ where $\rho = |y|$ and $\tilde{A}(\rho,\tau)$ is a scalar function.

Proposition D.2. *Consider*

$$\begin{cases} \partial_\tau \phi = L\phi + \tilde{A}(\rho,\tau)\rho\partial_\rho\phi + B(\rho,\tau)\phi + h + c_{\rm in}(\tau)\eta(\rho)Z(\rho) & \text{in } \mathcal{D}_R, \\ \phi(\cdot,\tau_0) = 0 & \text{in } B_{R(\tau_0)} \end{cases}$$

where $|\tilde{A}(\rho,\tau)| + |B(\rho,\tau)| \le f(\tau)$, $\|h\|_{v,\ell} < \infty$. *Suppose that* $R_0 \le R/3 \lesssim \sqrt{\tau}$, $|V(|y|)| \lesssim \langle y\rangle^{-P_V}$, $P_V > 2$, $d > 2$, $-d < \beta < 0$, $\ell \in (\beta+d, \beta+d+2) \cap (2,d)$, $f(\tau)R_0^{\max\{4-d-2\beta, 2+\}} \lesssim 1$; *there exists a constant* $C_* > 1$ *such that* $C_*^{-1}F(\tau) \le F(s) \le C_* F(\tau)$ *with* $F = v, R, R_0$ *for all* $\tau \le s \le 2\tau$, $\tau^{-\frac{d}{2}} \int_{\tau_0}^{\tau} v(s) R^{d-\bar{\ell}_*}(s) ds \lesssim v R^{2-\bar{\ell}_*}$, *where*

$$\bar{\ell}_* := \begin{cases} \bar{\ell}-, & \beta > 1 - \frac{d}{2} \text{ and } \bar{\ell} = 4-\beta \\ \bar{\ell}, & \text{otherwise} \end{cases}$$

for $\bar{\ell}$ *given in* (D.5)*;*

$$v, R_0 \in C^1(\tau_0,\infty), \left(f(\tau) + |v'|v^{-1}\right)\theta_{R_0,\bar{\ell}-2,1} + |R_0'|R_0^{1+\max\left\{0,2-\beta-\bar{\ell}\right\}} \ll 1,$$
$$f(\tau)v(\tau)\theta_{R_0,\ell,3} \in C^1(\tau_0,\infty), \left[f(\tau) + |(f(\tau)v(\tau)\theta_{R_0,\ell,3})'|(f(\tau)v(\tau)\theta_{R_0,\ell,3})^{-1}\right]\theta_{R_0,p_\ell,1} + \theta_{R_0,p_\ell,2} \ll 1,$$

where

$$\theta_{R_0,\bar{\ell}-2,1} = \begin{cases} R_0^{2+\max\left\{0,2-\beta-\bar{\ell}\right\}} & \text{if } \beta \le 1-\frac{d}{2} \\ \begin{cases} R_0^{2+\max\left\{0,2-\beta-\bar{\ell}\right\}}, & \bar{\ell} < 4-\beta \\ R_0^2 \ln R_0, & \bar{\ell} = 4-\beta \\ R_0^2, & 4-\beta < \bar{\ell} < \beta+d+2 \end{cases} & \text{if } \beta > 1-\frac{d}{2}, \end{cases}$$

$$\theta_{R_0,\ell,3} = \begin{cases} R_0^{4-\beta-\bar{\ell}} & \text{if } \beta \le 1-\frac{d}{2} \\ \begin{cases} R_0^{4-\beta-\bar{\ell}}, & \bar{\ell} < 4-\beta \\ \ln R_0, & \bar{\ell} = 4-\beta \\ 1, & \bar{\ell} > 4-\beta \end{cases} & \text{if } \beta > 1-\frac{d}{2}, \end{cases}$$

$$\theta_{R_0,p_\ell,1} = \begin{cases} R_0^{2+\max\{0+,-d-2\beta\}}, & \text{if } \beta < 1-\frac{d}{2} \\ R_0^2(\ln R_0)^2, & \text{if } \beta = 1-\frac{d}{2} \\ \begin{cases} R_0^2 \ln R_0, & \bar{\ell} \le 4-\beta \\ R_0^2, & 4-\beta < \bar{\ell} < \beta+d+2 \end{cases} & \text{if } \beta > 1-\frac{d}{2}, \end{cases}$$

$$\theta_{R_0,p_\ell,2} = \begin{cases} |R_0'|R_0^{1+\max\{0+,-d-2\beta\}}, & \text{if } \beta < 1-\frac{d}{2} \\ |R_0'|R_0 \ln R_0, & \text{if } \beta = 1-\frac{d}{2} \\ |R_0'|R_0, & \text{if } \beta > 1-\frac{d}{2}, \end{cases}$$

for $\inf_{\tau\ge\tau_0} R_0(\tau)$ *sufficiently large and* $\sup_{\tau\ge\tau_0} f(\tau)R^2(\tau)$ *sufficiently small. Then there exists a solution* $(\phi, c_{\rm in}) = (\phi[h], c_{\rm in}[h])$ *which is a linear mapping about* h *with the properties*

$$\langle\rho\rangle|\partial_\rho\phi| + |\phi| \lesssim v\|h\|_{v,\ell} \begin{cases} \langle\rho\rangle^{2-\bar{\ell}} R_0^{4-\beta-\bar{\ell}} & \text{if } \beta \le 1-\frac{d}{2} \\ \begin{cases} \langle\rho\rangle^{2-\bar{\ell}} R_0^{4-\beta-\bar{\ell}}, & \bar{\ell} < 4-\beta \\ \langle\rho\rangle^{(2-\bar{\ell})+} \ln R_0, & \bar{\ell} = 4-\beta \\ \langle\rho\rangle^{2-\bar{\ell}}, & 4-\beta < \bar{\ell} < \beta+d+2 \end{cases} & \text{if } \beta > 1-\frac{d}{2}, \end{cases} \tag{D.8}$$

$$c_{\rm in}(\tau) = c_{\rm in}[h](\tau) = -\Big(\int_0^2 \eta(x)Z^2(x)x^{d-1}dx\Big)^{-1} \left(\int_0^{2R_0} h(x,\tau)Z(x)x^{d-1}dx + c_{{\rm in},1}[h](\tau)\right), \tag{D.9}$$

and $c_{\rm in,1}[h]$ *depends linearly on* h *with the following estimate.*

$$|c_{\rm in,1}[h](\tau)| \lesssim v\|h\|_{v,\ell} R_0^{2-\bar{\ell}_*} \begin{cases} 1, & \beta+d-P_V<0 \\ \ln R_0, & \beta+d-P_V=0 \\ R_0^{\beta+d-P_V}, & \beta+d-P_V>0; \end{cases} \tag{D.10}$$

for $\alpha\in(0,1)$, $\tau_1,\tau_2\in[\tau-\lambda_1^2,\tau]$ *with* $0<\lambda_1\le\sqrt{\tau/8}$,

$$\begin{aligned} |c_{\rm in,1}[h](\tau_1)-c_{\rm in,1}[h](\tau_2)| \lesssim v\|h\|_{v,\ell} \Bigg[& R_0^{1-\bar{\ell}_*+\beta+d-P_V}|R_0(\tau_1)-R_0(\tau_2)| \\ & + |\tau_1-\tau_2|^\alpha \left(\lambda_1^{-2\alpha}R_0^{2-\bar{\ell}_*}+\lambda_1^{2-2\alpha}R_0^{-\bar{\ell}_*}\right) \begin{cases} 1, & \beta+d-P_V<0 \\ \ln R_0, & \beta+d-P_V=0 \\ R_0^{\beta+d-P_V}, & \beta+d-P_V>0 \end{cases} \Bigg]. \end{aligned} \tag{D.11}$$

Proof of Proposition D.2. Set $\phi(\rho,\tau)=\eta_{R_0}(\rho)\phi_i(\rho,\tau)+\phi_o(\rho,\tau)$, where $\eta_{R_0}(\rho)=\eta(\frac{\rho}{R_0})$. In order to find a solution ϕ, it suffices to solve the following inner–outer gluing system for (ϕ_i,ϕ_o).
The outer problem:

$$\begin{cases} \partial_\tau\phi_o=\Delta\phi_o+J[\phi_o,\phi_i]\eta_{2R}(\rho) \ \text{ in } \ \mathbb{R}^d\times(\tau_0,\infty), \\ \phi_o(\cdot,\tau_0)=0 \ \text{ in } \ \mathbb{R}^d, \end{cases} \tag{D.12}$$

the inner problem:

$$\begin{cases} \partial_\tau\phi_i=L\phi_i+\tilde{A}\rho\partial_\rho\phi_i+B\phi_i+V\phi_o+h+c_{\rm in}(\tau)\eta(\rho)Z(\rho) & \text{in } \ \mathcal{D}_{2R_0}, \\ \phi_i(\cdot,\tau_0)=0 & \text{in } \ B_{2R(\tau_0)}, \end{cases} \tag{D.13}$$

where $\eta_{2R}(\rho)=\eta(\frac{\rho}{2R})$,

$$J[\phi_o,\phi_i]:=V\phi_o(1-\eta_{R_0})+\tilde{A}\rho\partial_\rho\phi_o+B\phi_o+\Lambda[\phi_i]+h(1-\eta_{R_0}),$$
$$\Lambda[\phi_i]:=\phi_i\Delta\eta_{R_0}+2\partial_\rho\eta_{R_0}\partial_\rho\phi_i+\tilde{A}\phi_i\rho\partial_\rho\eta_{R_0}-\phi_i\partial_\tau\eta_{R_0}.$$

Here $c_{\rm in}(\tau)$ is given by

$$c_{\rm in}(\tau)=c_{\rm in}[\phi_o](\tau)=C_1\int_0^{2R_0}\left(V(x,\tau)\phi_o(x,\tau)+h(x,\tau)\right)Z(x)x^{d-1}dx, \quad C_1=-\left(\int_0^2\eta(x)Z^2(x)x^{d-1}dx\right)^{-1}$$

to achieve the orthogonal condition

$$\int_0^{2R_0}\left(V(x)\phi_o(x,\tau)+h(x,\tau)+c_{\rm in}(\tau)\eta(x)Z(x)\right)Z(x)x^{d-1}dx=0.$$

We reformulate (D.12) and (D.13) into the following form

$$\phi_o(y,\tau)=\mathcal{T}_d\bullet[J[\phi_o,\phi_i]\eta_{2R}](y,\tau,\tau_0), \quad \phi_i(y,\tau)=\mathcal{T}_i\left[V\phi_o+h+c_{\rm in}\eta(\rho)Z(\rho)\right] \tag{D.14}$$

where $\mathcal{T}_i$ is the linear mappings given by Lemma D.1. We will solve the system (D.14) by the contraction mapping theorem.

Denote the leading term of the right hand side of (D.13) as $H_1:=h+C_1\eta(\rho)Z(\rho)\int_0^{2R_0}h(x,\tau)Z(x)x^{d-1}dx$. It is easy to check $\|H_1\|_{v,\ell}\lesssim\|h\|_{v,\ell}$ under the assumption $\beta+d<\ell$. If H_1 satisfies the orthogonal condition in $\mathcal{D}_{2R_0}$, under the assumption

$$R_0\ge 2, v, R_0\in C^1(\tau_0,\infty), \left(f(\tau)+|v'||v^{-1}\right)\theta_{R_0,\bar{\ell}-2,1}+\theta_{R_0,\bar{\ell}-2,2}\ll 1,$$
$$f(\tau)v(\tau)\theta_{R_0,\ell,3}\in C^1(\tau_0,\infty), \left[f(\tau)+|(f(\tau)v(\tau)\theta_{R_0,\ell,3})'|(f(\tau)v(\tau)\theta_{R_0,\ell,3})^{-1}\right]\theta_{R_0,p_\ell,1}+\theta_{R_0,p_\ell,2}\ll 1,$$

Lemma D.1 gives the following a priori estimate

$$\langle\rho\rangle|\partial_\rho\mathcal{T}_i[H_1]|+|\mathcal{T}_i[H_1]|\le D_i w_i(\rho,\tau)$$

where $D_i\ge 1$ is a constant and

$$w_i(\rho,\tau):=v(\tau)\theta_{R_0,\ell,3}\langle\rho\rangle^{-p_\ell}\|h\|_{v,\ell}$$

where we have used $f(\tau)\langle\rho\rangle^{\beta}\Theta_{R_0,p_\ell}(\rho) \lesssim \langle\rho\rangle^{-p_\ell}$ under the assumption $f(\tau)R_0^{\max\{4-d-2\beta,2+\}} \lesssim 1$. Indeed, by the definition of p_ℓ in (D.5) and Θ_{R_0,p_ℓ} in (D.2),

$$f(\tau)\langle\rho\rangle^{\beta+p_\ell}\Theta_{R_0,p_\ell}(\rho) \lesssim 1 \Longleftrightarrow f(\tau)\begin{cases}\langle\rho\rangle^2 R_0^{2-d-2\beta}, & \text{if } \beta<1-\frac{d}{2}\\ \langle\rho\rangle^2(\ln R_0)^2 & \text{if } \beta=1-\frac{d}{2}\\ \begin{cases}\langle\rho\rangle^2\ln R_0, & p_\ell=2-\beta\\ \langle\rho\rangle^2, & 2-\beta<p_\ell<\beta+d\\ \langle\rho\rangle^2\ln(\rho+2), & p_\ell=\beta+d\end{cases} & \text{if } \beta>1-\frac{d}{2}\end{cases} \lesssim 1.$$

For this reason, we will solve the inner problem (D.13) in the space

$$\mathcal{B}_i = \{g(\rho,\tau) \;:\; \langle\rho\rangle|\partial_\rho g(\rho,\tau)| + |g(\rho,\tau)| \le 2D_i w_i(\rho,\tau)\}.$$

For any fixed $\tilde{\phi}_i \in \mathcal{B}_i$, we will find a solution $\phi_o = \phi_o[\tilde{\phi}_i]$ of the outer problem (D.12) by the contraction mapping theorem. Let us first estimate $J[0,\tilde{\phi}_i]$ term by term. Under the assumption $|R_0'|R_0^{-1} + f(\tau) \lesssim R_0^{-2}$, using (D.6), we have

$$\begin{aligned}|\Lambda[\tilde{\phi}_i]| &\lesssim \left(R_0^{-2} + |R_0'|R_0^{-1} + f(\tau)\right)\mathbf{1}_{\{R_0\le\rho\le 2R_0\}} D_i w_i(R_0,\tau) \sim R_0^{-2}\mathbf{1}_{\{R_0\le\rho\le 2R_0\}} D_i w_i(R_0,\tau)\\
&= D_i R_0^{-2}\mathbf{1}_{\{R_0\le\rho\le 2R_0\}} v\|h\|_{v,\ell}\theta_{R_0,\ell,3}\langle R_0\rangle^{-p_\ell}\\
&\sim D_i\mathbf{1}_{\{R_0\le\rho\le 2R_0\}} v\|h\|_{v,\ell}\begin{cases}\begin{cases}R_0^{-\bar\ell}, & \bar\ell<\beta+d+2\\ R_0^{-\bar\ell}\ln R_0, & \bar\ell=\beta+d+2\\ R_0^{-2-d-\beta}, & \bar\ell>\beta+d+2\end{cases} & \text{if } \beta<1-\frac{d}{2}\\ \begin{cases}R_0^{-\bar\ell}, & \bar\ell<\beta+d+2\\ R_0^{-\bar\ell}(\ln R_0)^2, & \bar\ell=\beta+d+2\\ R_0^{-2-d-\beta}\ln R_0, & \bar\ell>\beta+d+2\end{cases} & \text{if } \beta=1-\frac{d}{2}\\ \begin{cases}R_0^{-\bar\ell}, & \bar\ell<4-\beta\\ R_0^{-\bar\ell}\ln R_0, & \bar\ell=4-\beta\\ R_0^{-\bar\ell}, & 4-\beta<\bar\ell<\beta+d+2\\ R_0^{-2-[(d+\beta)-]}, & \bar\ell=\beta+d+2\\ R_0^{-2-d-\beta}, & \bar\ell>\beta+d+2\end{cases} & \text{if } \beta>1-\frac{d}{2}.\end{cases}\end{aligned}$$

Since the information of ℓ is lost in $|\Lambda[\tilde{\phi}_i]|$ when $\bar\ell > \beta+d+2$, we restrict it to the case $\ell < \beta+d+2$. Additionally,

$$|h(1-\eta_{R_0})| \lesssim \mathbf{1}_{\{\rho\ge R_0\}} v\langle\rho\rangle^{-\ell}\|h\|_{v,\ell}.$$

Thus,

$$\begin{aligned}|J[0,\tilde{\phi}_i]|\eta_{2R} &\lesssim D_i\mathbf{1}_{\{R_0\le\rho\le 4R\}} v\|h\|_{v,\ell}\begin{cases}\langle\rho\rangle^{-\bar\ell}\ln(\rho+2), & \beta>1-\frac{d}{2} \text{ and } \bar\ell=4-\beta\\ \langle\rho\rangle^{-\bar\ell}, & \text{otherwise}\end{cases}\\
&\lesssim D_i v\|h\|_{v,\ell}\left(\mathbf{1}_{\{\rho<R_0\}}R_0^{-\bar\ell_*} + \mathbf{1}_{\{R_0\le\rho\le 4R\}}\rho^{-\bar\ell_*}\right)\end{aligned} \tag{D.15}$$

where

$$\bar\ell_* := \begin{cases}\bar\ell-, & \beta>1-\frac{d}{2} \text{ and } \bar\ell=4-\beta\\ \bar\ell, & \text{otherwise.}\end{cases}$$

Consider (D.12) with the right hand side $J[0,\tilde{\phi}_i]\eta_{2R}$. In order to retrieve the information of ℓ and avoid losing time decay, we restrict it to the case $d>2$, $2<\ell<d$.

By [100, Lemma A.1] and the scaling argument, under the assumption that $C_*^{-1}F(\tau)\le F(s)\le C_*F(\tau)$ with $F=v,R,R_0$ and a costant $C_*>1$ for all $\tau\le s\le 2\tau$, $\tau^{-\frac{d}{2}}\int_{\tau_0}^{\tau}v(s)R^{d-\bar{\ell}_*}(s)ds\lesssim vR^{2-\bar{\ell}_*}$, $R\lesssim\sqrt{\tau}$, we have

$$\langle\rho\rangle\left|\partial_\rho\mathcal{T}_d\bullet[J[0,\tilde{\phi}_i]\eta_{2R}]\right|+\left|\mathcal{T}_d\bullet[J[0,\tilde{\phi}_i]\eta_{2R}]\right|\le CD_i\|h\|_{v,\ell}\left|\mathcal{T}_d\bullet\left[v\big(\mathbf{1}_{\{\rho<R_0\}}R_0^{-\bar{\ell}_*}+\mathbf{1}_{\{R_0\le\rho\le 4R\}}\rho^{-\bar{\ell}_*}\big)\right]\right|$$
$$\le CD_i\|h\|_{v,\ell}\left(\tau^{-\frac{d}{2}}e^{-\frac{\rho^2}{16\tau}}\int_{\tau_0}^{\tau}v(s)R^{d-\bar{\ell}_*}(s)ds+v(\tau)\begin{cases}R_0^{2-\bar{\ell}_*}, & \rho\le R_0\\ \rho^{2-\bar{\ell}_*}, & R_0<\rho\le R\\ \rho^{2-d}e^{-\frac{\rho^2}{16\tau}}R^{d-\bar{\ell}_*}, & \rho>R\end{cases}\right)$$
$$\le w_o(\rho,\tau):=D_oD_i\|h\|_{v,\ell}\left(\mathbf{1}_{\{\rho>8R\}}\tau^{-\frac{d}{2}}e^{-\frac{\rho^2}{16\tau}}\int_{\tau_0}^{\tau}v(s)R^{d-\bar{\ell}_*}(s)ds+v(\tau)\begin{cases}R_0^{2-\bar{\ell}_*}, & \rho\le R_0\\ \rho^{2-\bar{\ell}_*}, & \rho>R_0\end{cases}\right)$$

with a large constant $D_o\ge 1$. This suggests that we solve ϕ_o in the following space:

$$\mathcal{B}_o=\left\{g(\rho,\tau)\ :\ \langle\rho\rangle|\partial_\rho g(\rho,\tau)|+|g(\rho,\tau)|\le 2w_o(\rho,\tau)\right\}. \tag{D.16}$$

For any $\tilde{\phi}_o\in\mathcal{B}_o$, since $|V(\rho)|\lesssim\langle\rho\rangle^{-P_V}$ with $P_V>2$, we have

$$\begin{aligned}|V\tilde{\phi}_o(1-\eta_{R_0})|\eta_{2R}&\lesssim\big(\inf_{\tau\ge\tau_0}R_0(\tau)\big)^{2-P_V}\mathbf{1}_{\{R_0\le\rho\le 4R\}}D_oD_i\|h\|_{v,\ell}v\rho^{-\bar{\ell}_*},\\ \left|\tilde{A}\rho\partial_\rho\phi_o+B\phi_o\right|\eta_{2R}&\lesssim f(\tau)D_oD_i\|h\|_{v,\ell}v\big(R_0^{2-\bar{\ell}_*}\mathbf{1}_{\{\rho\le R_0\}}+\rho^{2-\bar{\ell}_*}\mathbf{1}_{\{R_0<\rho\le 4R\}}\big)\\ &\lesssim\big(\sup_{\tau\ge\tau_0}f(\tau)R^2(\tau)\big)D_oD_i\|h\|_{v,\ell}v\big(R_0^{-\bar{\ell}_*}\mathbf{1}_{\{\rho\le R_0\}}+\rho^{-\bar{\ell}_*}\mathbf{1}_{\{R_0<\rho\le 4R\}}\big).\end{aligned} \tag{D.17}$$

Taking $\inf_{\tau\ge\tau_0}R_0(\tau)$ sufficiently large and $\sup_{\tau\ge\tau_0}f(\tau)R^2(\tau)$ sufficiently small, we have

$$\mathcal{T}_o[J[\tilde{\phi}_o,\tilde{\phi}_i]\eta_{2R}]\in\mathcal{B}_o.$$

The contraction mapping property can be deduced in the same way. Indeed, denote

$$\|g\|_o:=\sup_{\mathbb{R}^d\times(\tau_0,\infty)}\left[(w_o(\rho,\tau))^{-1}|g(\rho,\tau)|\right].$$

Then for any $\tilde{\phi}_o^{(1)},\tilde{\phi}_o^{(2)}\in\mathcal{B}_o$,

$$\begin{aligned}&\left|V(\tilde{\phi}_o^{(1)}-\tilde{\phi}_o^{(2)})(1-\eta_{R_0})\right|\eta_{2R}+\left|\tilde{A}\rho\partial_\rho(\tilde{\phi}_o^{(1)}-\tilde{\phi}_o^{(2)})+B(\tilde{\phi}_o^{(1)}-\tilde{\phi}_o^{(2)})\right|\eta_{2R}\\ \lesssim&\Big(\big(\inf_{\tau\ge\tau_0}R_0(\tau)\big)^{2-P_V}+\sup_{\tau\ge\tau_0}f(\tau)R^2(\tau)\Big)D_oD_i\|h\|_{v,\ell}v\big(R_0^{-\bar{\ell}_*}\mathbf{1}_{\{\rho\le R_0\}}+\rho^{-\bar{\ell}_*}\mathbf{1}_{\{R_0<\rho\le 4R\}}\big)\|\tilde{\phi}_o^{(1)}-\tilde{\phi}_o^{(2)}\|_o\end{aligned}$$

where $\big(\inf_{\tau\ge\tau_0}R_0(\tau)\big)^{2-P_V}+\sup_{\tau\ge\tau_0}f(\tau)R^2(\tau)$ provides small quantity for the contraction mapping.

Now we have found a solution $\phi_o=\phi_o[\tilde{\phi}_i]\in\mathcal{B}_o$. It follows that in the domain $\rho\le 2R_0$,

$$\begin{aligned}&\left|V\phi_o+\eta(\rho)Z(\rho)C_1\int_0^{2R_0}V(x,\tau)\phi_o(x,\tau)Z(x)x^{d-1}dx\right|\\ \lesssim&\ D_oD_i\|h\|_{v,\ell}vR_0^{2-\bar{\ell}_*}\left(\langle\rho\rangle^{-P_V}+\eta(\rho)\begin{cases}1, & \beta+d-P_V<0\\ \ln R_0, & \beta+d-P_V=0\\ R_0^{\beta+d-P_V}, & \beta+d-P_V>0\end{cases}\right)\lesssim\big(\inf_{\tau\ge\tau_0}R_0(\tau)\big)^{-\epsilon}D_oD_i\|h\|_{v,\ell}v\langle\rho\rangle^{-\ell}\end{aligned}$$

for a small constant $\epsilon>0$ provided $P_V>\max\{2,2-\bar{\ell}_*+\beta+d\}$.

Due to the choice of $c_{\rm in}(\tau)$, $H_2:=V\phi_o[\tilde{\phi}_i]+h+c_{\rm in}[\phi_o[\tilde{\phi}_i]](\tau)\eta(\rho)Z(\rho)$ satisfies the orthogonal condition in $\mathcal{D}_{2R_0}$. By Lemma D.1, since $\big(\inf_{\tau\ge\tau_0}R_0(\tau)\big)^{-\epsilon}$ provides smallness, we have

$$\mathcal{T}_i[H_2]\in\mathcal{B}_i.$$

The contraction property can be deduced in the same way. Thus we find a solution

$$\phi_i=\phi_i[h]\in\mathcal{B}_i. \tag{D.18}$$

Finally we obtain a solution (ϕ_o,ϕ_i) for (D.12) and (D.13).

By the same argument in the proof of [100, Proposition 7.1], $\phi=\phi[h]$ is a linear mapping about h.

We will regard D_o, D_i as general constants hereafter. Set

$$c_{\mathrm{in},1}[h](\tau) := \int_0^{2R_0} V(x)\phi_o[h](x,\tau)Z(x)x^{d-1}dx.$$

Since $\phi_o[h] \in \mathcal{B}_o$, then the estimate (D.10) follows.

Combining (D.18), (D.16), (D.6) and then using the scaling argument, we get (D.8).

Combining (D.15) and (D.17), we have $|J[\phi_o,\phi_i]\eta_{2R}| \lesssim v\|h\|_{v,\ell}R_0^{-\bar{\ell}_*}$. By (D.16), then $|\phi_o| \lesssim v\|h\|_{v,\ell}R_0^{2-\bar{\ell}_*}$. Applying again the scaling argument to (D.12), we have

$$\sup_{y\in\mathbb{R}^d}\ \sup_{\tau_1,\tau_2\in[\tau-\lambda_1^2,\tau]} \frac{|\phi_o(y,\tau_1)-\phi_o(y,\tau_2)|}{|\tau_1-\tau_2|^\alpha} \lesssim v\|h\|_{v,\ell}\left(\lambda_1^{-2\alpha}R_0^{2-\bar{\ell}_*} + \lambda_1^{2-2\alpha}R_0^{-\bar{\ell}_*}\right)$$

for $\alpha \in (0,1)$, $0 < \lambda_1 \le \sqrt{\tau/8}$, which yields (D.11).

□

Remark D.2. *The reason why we consider (D.12) in $\mathbb{R}^d$ is to make the scaling argument applicable for a wider range of λ_1.*

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