On a transcendental equation involving quotients of Gamma functions *

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Abstract

The note is aimed at giving a complete characterization of the following equation in p:

$$p\frac{\Gamma(\frac{n}{2}-\frac{s}{p-1})\Gamma(s+\frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2}-\frac{s}{p-1})}=\Big(\frac{\Gamma(\frac{n+2s}{4})}{\Gamma(\frac{n-2s}{4})}\Big)^2.$$

The method is based on some key transformation and the properties of the Gamma function. Applications to fractional nonlinear Lane-Emden equations will be given.

1 Introduction and main results

In this note we consider the following equation for p

$$p\frac{\Gamma(\frac{n}{2}-\frac{s}{p-1})\Gamma(s+\frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2}-\frac{s}{p-1})}=\Big(\frac{\Gamma(\frac{n+2s}{4})}{\Gamma(\frac{n-2s}{4})}\Big)^2, \tag{1.1}$$

where $p>\frac{n}{n-2s}$ and (s,n) satisfies

$$0<s<\frac{n}{2},\ n\in\mathbb{N}^+. \tag{1.2}$$

Equation (1.1) appears frequently in the study of fractional Lane-Emden equation (see [2, 5]), the fractional Yamabe equation with singularities (see [4, 7]) and also some high-order equations (see [3, 8], where $s=2$). For example, consider the singular solutions for the fractional supercritical Lane-Emden equation,

$$(-\Delta)^s u=|u|^{p-1}u,\quad p>\frac{n}{n-2s}. \tag{1.3}$$

By Lemma 1.1 of [5], the singular radial solution of (1.3) u_s is given by

$$u_s(x)=A|x|^{-\frac{2s}{p-1}},\ \text{where } A^{p-1}=\frac{\Gamma(\frac{n}{2}-\frac{s}{p-1})\Gamma(s+\frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2}-\frac{s}{p-1})}. \tag{1.4}$$

By Herbst's generalized Hardy's inequality ([9])

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(\phi(x)-\phi(y))^2}{|x-y|^{n+2s}}dxdy\geq\frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}\int_{\mathbb{R}^n}|x|^{-2s}\phi^2dx,$$

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u_s is stable if and only if the following inequality holds

$$p\frac{\Gamma(\frac{n}{2}-\frac{s}{p-1})\Gamma(s+\frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2}-\frac{s}{p-1})} \leq \Big(\frac{\Gamma(\frac{n+2s}{4})}{\Gamma(\frac{n-2s}{4})}\Big)^2, \tag{1.5}$$

while it is unstable if

$$p\frac{\Gamma(\frac{n}{2}-\frac{s}{p-1})\Gamma(s+\frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2}-\frac{s}{p-1})} > \Big(\frac{\Gamma(\frac{n+2s}{4})}{\Gamma(\frac{n-2s}{4})}\Big)^2. \tag{1.6}$$

In [5], it is proved that for $1 < s < 2$, if $p > 1, p \neq \frac{n+2s}{n-2s}$ and p satisfies (1.6) then all stable and finite Morse index solutions to (1.3) must be trivial. An open question is the classification of the range of p for which (1.5) or (1.6) holds. In this paper we shall give an affirmative answer to this question.

Our first result concerns the classification of the roots of (1.1).

Theorem 1.1. *Assume that $n > 2s, s > 0, p > \frac{n}{n-2s}$. There exists $n_0(s) \in \mathbb{N}^+$ such that*

(1) if $n \leq n_0(s)$, then (1.1) only has one real root p and

$$p = \frac{n+2s-2+2a_{n,s}\sqrt{n}}{n-2s-2+2a_{n,s}\sqrt{n}},$$

where $a_{n,s}$ satisfies $\frac{1}{\sqrt{n}} < a_{n,s} < \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}$ and is the unique positive root of the function $f_{n,s}(a)$ defined in (2.3) of Section 2.

(2) if $n > n_0(s)$, then (1.1) has exactly two real roots p_1 and p_2, where

$$p_1 := \frac{n+2s-2+2a_{n,s}\sqrt{n}}{n-2s-2+2a_{n,s}\sqrt{n}}, \quad p_2 := \frac{n+2s-2-2a_{n,s}\sqrt{n}}{n-2s-2-2a_{n,s}\sqrt{n}}.$$

Moreover, there holds

$$\frac{n}{n-2s} < p_1 < \frac{n+2s}{n-2s} < p_2 < +\infty. \tag{1.7}$$

Remark 1.1. *The integer $n_0(s)$ is the largest integer such that*

$$n-2s-2-2a_{n,s}\sqrt{n} \leq 0. \tag{1.8}$$

Hence when $n > n_0(s), n-2s-2-2a_{n,s}\sqrt{n} > 0$ and thus p_2 is well-defined.

As for the inequalities (1.5)-(1.6), we have the following necessary and sufficient conditions.

Theorem 1.2. *Assume that $n > 2s, s > 0, p > \frac{n}{n-2s}$. Then there exists $n_0(s) \in \mathbb{N}^+$ such that for the inequality (1.6), we have*

(1) if $n \leq n_0(s)$, then the inequality (1.6) holds if and only if

$$p > p_1 := \frac{n+2s-2+2a_{n,s}\sqrt{n}}{n-2s-2+2a_{n,s}\sqrt{n}};$$

(2) if $n > n_0(s)$, then the inequality (1.6) holds if and only if

$$p_1 := \frac{n+2s-2+2a_{n,s}\sqrt{n}}{n-2s-2+2a_{n,s}\sqrt{n}} < p < p_2 := \frac{n+2s-2-2a_{n,s}\sqrt{n}}{n-2s-2-2a_{n,s}\sqrt{n}},$$

where

$$\frac{n}{n-2s} < p_1 < \frac{n+2s}{n-2s} < \frac{n+2s-4}{n-2s-4} < p_2 < +\infty.$$

Remark 1.2. *The root p_1 appears in the Lane-Emden equation with singularities and also fractional Yamabe equation with singularities, while the root p_2 is essential in the study of stability of solutions to the fractional Lane-Emden equation. In the literature, when $s = 1$, the root p_2 is usually called Joseph-Lundgren exponent (See Joseph and Lundgren [10] and Farina [6]). When $s = 1$, the root p_1 plays an important role in constructing singular solutions for Lane-Emden equation with subcritical exponent (Chen-Lin [1]).*

The following corollary gives a complete classification on the stability of the singular radial solutions to (1.3).

Corollary 1.1. *Assume that $n > 2s, s > 0, p > \frac{n}{n-2s}$. Let u_s be given by (1.4). Then there exists $n_0(s) \in \mathbb{N}^+$ such that*

(1) if $n \le n_0(n, s)$, then u_s is stable if and only if $p \ge p_1$;

(2) if $n > n_0(n, s)$, then u_s is stable if and only if

$$p \ge p_2 := \frac{n + 2s - 2 - 2a_{n,s}\sqrt{n}}{n - 2s - 2 - 2a_{n,s}\sqrt{n}}.$$

In all the results above, we have the two numbers $n_0(s)$ and $a_{n,s}$ which are to be implicitly determined. By (1.8), $n_0(s)$ is the largest integer satisfying the following inequality

$$n \le \left(a_{n,s} + \sqrt{a_{n,s}^2 + 2s + 2}\,\right)^2. \tag{1.9}$$

The range of $a_{n,s}$ is important in applications. The bound $\frac{1}{\sqrt{n}} < a_{n,s} < \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}$ is too rough. Next, we give more refined and quantitative estimates on $a_{n,s}$. These results show that $a_{n,s}$ is very close to the constant 1 when n is large.

Theorem 1.3. *Assume that $n > 2s, s > 0$.*

(1) For any $\varepsilon_1 > 0$, there exists $n_1(s, \varepsilon_1)$ such that $a_{n,s} < 1 + \varepsilon_1$ whenever $n > \overline{n}_1(s, \varepsilon_1) := \max\{(1 + \varepsilon_1 + \sqrt{\max\{(1 + \varepsilon_1)^2 + 2s - 2, 0\}})^2, n_1(s, \varepsilon_1)\}$, where $n_1(s, \varepsilon_1)$ is the largest real root of

$$\begin{aligned}\Big((&-\varepsilon_1^2 - 2\varepsilon_1)n^4 + \big[-27 + (18s + 48)(1 + \varepsilon_1)^2\big]n^3 \\ &+ \big[(-36s^2 - 96s - 144)(1 + \varepsilon_1)^2 - 24s^2 - 30s + 88\big]n^2 \\ &+ \big[(24s^3 + 192s^2 + 288s + 192)(1 + \varepsilon_1)^2 + 60s^2 + 64s - 144\big]n \\ &+ 48s^4 + 216s^3 + 352s^2 + 288s + 192\Big) = 0.\end{aligned}$$

(2) For any $\varepsilon_2 > 0$, there exists $n_2(s, \varepsilon_2)$ such that $a_{n,s} > 1 - \varepsilon_2$ whenever $n > \overline{n}_2(s, \varepsilon_2) := \max\{(1 - \varepsilon_2 + \sqrt{\max\{(1 - \varepsilon_2)^2 + 2s - 2, 0\}})^2, n_2(s, \varepsilon_2)\}$, where $n_2(s, \varepsilon_2)$ is the square of the largest real root of the following equation (about variable t)

$$\begin{aligned}\Big(&(\varepsilon_2^2 + 2\varepsilon_2)t^6 - 2(1 - \varepsilon_2)^3 t^5 + \big[18(1 - \varepsilon_2)^2 - 18s - 39\big]t^4 \\ &+ \big[-4(1 - \varepsilon_2)^3 s - 6(1 - \varepsilon_2)\big]t^3 \\ &+ \big[(12s^2 + 36s)(1 - \varepsilon_2)^2 + 36s^2 + 144s + 158\big]t^2 \\ &- 12(1 - \varepsilon_2)st - 24s^3 - 132s^2 - 260s - 192\Big) = 0.\end{aligned}$$

(3) $\lim_{n\to+\infty} a_{n,s} = 1$ for any fixed $s > 0$.

Remark 1.3. *Theorem 1.3 gives precise thresholds for $a_{n,s}$. In fact, for a fixed range of s, say $s \in (2, 3)$, we have $0.7 < a_{n,s} < 1.5$ as long as $n \ge 44$. Moreover, from the Table 1, we have a quantitative estimate of the constants $\overline{n}_1(s, \varepsilon_1)$ and $\overline{n}_2(s, \varepsilon_2)$ (See Theorem 1.3).*

Table 1: **The location of $a_{n,s}$. For simplicity, we set** $a_{n,s} := \mathcal{A}$.

s & $\mathcal{A}$	$\mathcal{A} < 1.5$	$\mathcal{A} > 0.6$	$\mathcal{A} < 1.2$	$\mathcal{A} > 0.8$	$\mathcal{A} < 1.1$	$\mathcal{A} > 0.9$
$s \in (0, 1]$	$n \geq 28$	$n \geq 20$	$n \geq 46$	$n \geq 37$	$n \geq 79$	$n \geq 71$
$s \in (1, 2]$	$n \geq 36$	$n \geq 26$	$n \geq 63$	$n \geq 51$	$n \geq 110$	$n \geq 100$
$s \in (2, 3]$	$n \geq 44$	$n \geq 33$	$n \geq 79$	$n \geq 65$	$n \geq 141$	$n \geq 128$
$s \in (3, 4]$	$n \geq 52$	$n \geq 39$	$n \geq 96$	$n \geq 79$	$n \geq 172$	$n \geq 157$
$s \in (4, 5]$	$n \geq 59$	$n \geq 46$	$n \geq 112$	$n \geq 93$	$n \geq 204$	$n \geq 186$
...	...	...	...	...	...	...

Remark 1.4. *Using the estimates for $a_{n,s}$ we can have some estimates on the critical dimension $n_0(s)$. More precisely, for any $\varepsilon_1 > 0$ we have*

$$n_0(s) \leq \max\{n_1(s, \varepsilon_1), (1 + \varepsilon_1 + \sqrt{(1 + \varepsilon_1)^2 + 2s + 2})^2\}. \tag{1.10}$$

If we select that $\varepsilon_1 = 1$, we get the Table 2 below.

Table 2: **The estimate of $n_0(s)$ for various s.**

$s, n_1(s, \varepsilon_1), n_0(n, s), n_0(s)$	$n_1(s, \varepsilon_1) <$	$n_0(n, s) \leq$	$n_0(s) \leq$
$s \in (0, 1]$	22	24	24
$s \in (1, 2]$	28	27	27
$s \in (2, 3]$	33	30	33
$s \in (3, 4]$	39	33	39
$s \in (4, 5]$	44	36	44
...	...	...	...

Remark 1.5. *Although the explicit formula of $a_{n,s}$ may be very complicated for general $s > 0$, we have seen by Theorem 1.3-(3) that $a_{n,s}$ lies around the constant 1. Therefore, the roots p_1, p_2 obtained in Theorem 1.1 and 1.2 have the following asymptotic formulas:*

$$p_1 := \frac{n+2s-2+2a_{n,s}\sqrt{n}}{n-2s-2+2a_{n,s}\sqrt{n}} \approx \frac{n+2s-2+\sqrt{n}}{n-2s-2+\sqrt{n}},$$
$$p_2 := \frac{n+2s-2-2a_{n,s}\sqrt{n}}{n-2s-2-2a_{n,s}\sqrt{n}} \approx \frac{n+2s-2-\sqrt{n}}{n-2s-2-\sqrt{n}}.$$

On the other hand, to get more precise estimates on the roots p_1, p_2, we just need to select suitable $\varepsilon_1, \varepsilon_2$ in Theorem 1.3.

Remark 1.6. *Recall that when $s = 1$, the Joseph-Lundgren exponent is given by the following formula (see Joseph and Lundgren [10], see also Farina [6]):*

$$p_{JL}(n, s=1) := \begin{cases} \infty & \text{if } n \le 10, \\ \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \ge 11. \end{cases}$$

For the bi-harmonic case, i.e., $s = 2$, Joseph-Lundgren exponent is given by (see Gazzola and Grunau [8], see also Davila, Dupaigne, Wang and Wei [3]):

$$p_{JL}(n, s=2) = \begin{cases} \infty & \text{if } n \le 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \ge 13. \end{cases}$$

In our setting, we obtain the universal Joseph-Lundgren exponent for any $s > 0$, that is,

$$p_{JL}(n, s) = p_2 := \frac{n+2s-2-2a_{n,s}\sqrt{n}}{n-2s-2-2a_{n,s}\sqrt{n}}.$$

In particular, when $s = 1$,

$$p_{JL}(n, s=1) := \begin{cases} \infty & \text{if } n \le 10, \\ \frac{n-2a_{n,1}\sqrt{n}}{n-4-2a_{n,1}\sqrt{n}} & \text{if } n \ge 11, \end{cases} \tag{1.11}$$

where

$$a_{n,1} = \sqrt{\frac{n-1}{n}}.$$

In this case (i.e., $s = 1$), the root p_1 was obtained in Chen-Lin [1] for Lane-Emden equation with subcritical exponent, where

$$p_1 = \frac{n+2a_{n,1}\sqrt{n}}{n-4+2a_{n,1}\sqrt{n}} = \frac{n+2\sqrt{n-1}}{n-4+2\sqrt{n-1}}.$$

See Remark 1.2.

When $s = 2$,

$$p_{JL}(n, s=2) = \begin{cases} \infty & \text{if } n \le 12, \\ \frac{n+2-2a_{n,2}\sqrt{n}}{n-6-2a_{n,2}\sqrt{n}} & \text{if } n \ge 13, \end{cases} \tag{1.12}$$

where

$$a_{n,2} = \sqrt{\frac{2(n-1)(n^2-2n-2)}{n(n^2+4+n\sqrt{(n-4)^2+4})}}.$$

Here we notice that

$$\lim_{n\to+\infty} a_{n,1} = \lim_{n\to+\infty} a_{n,2} = 1.$$

Remark 1.7. *We may also consider equation (1.1) when n is any positive number satisfying $n > 2s$. In fact from the proof below we have used n as a continuous real variable. Theorems 1.1-1.2 hold for general real number $n > 2s$.*

2 Key transformations and analysis

At the first glance, equation (1.1) looks complicated. In this section we introduce a key transformation which puts it in more symmetric form.

First we let

$$k := \frac{2s}{p-1}.$$

Since $\Gamma(s+1) = s\Gamma(s)$, (1.1) becomes

$$\frac{\Gamma(\frac{n-k}{2})\Gamma(s+\frac{k}{2}+1)}{\Gamma(\frac{k}{2}+1)\Gamma(\frac{n-k-2s}{2})} = \Big(\frac{\Gamma(\frac{n+2s}{4})}{\Gamma(\frac{n-2s}{4})}\Big)^2. \tag{2.1}$$

Here we notice that, the sum of the variables of the Gamma function in both the numerator and the denominator on the left hand side of the above equation (2.1) is equal to $\frac{n+2s}{2}+1$ and $\frac{n-2s}{2}+1$, respectively. To make sure that all the variables in the Gamma function in (2.1) have the term $\frac{1}{4}n+\frac{1}{2}s$ or the term $\frac{1}{4}n-\frac{1}{2}s$, we introduce a new parameter $a \in \mathbb{R}$ satisfying

$$k := \frac{n-(2s+2)}{2} + a\sqrt{n}. \tag{2.2}$$

For the reason why the term $\sqrt{n}$ appears, see Remark 3.1. This is a key point. Now (2.1) reads as

$$\frac{\Gamma(\frac{1}{4}n+\frac{1}{2}s+\frac{1}{2}+\frac{1}{2}a\sqrt{n})\Gamma(\frac{1}{4}n+\frac{1}{2}s+\frac{1}{2}-\frac{1}{2}a\sqrt{n})}{\Gamma(\frac{1}{4}n-\frac{1}{2}s+\frac{1}{2}+\frac{1}{2}a\sqrt{n})\Gamma(\frac{1}{4}n-\frac{1}{2}s+\frac{1}{2}-\frac{1}{2}a\sqrt{n})} = \Big(\frac{\Gamma(\frac{1}{4}n+\frac{1}{2}s)}{\Gamma(\frac{1}{4}n-\frac{1}{2}s)}\Big)^2.$$

Now we focus on the new variable $a \in \mathbb{R}$. Taking the logarithm on both sides above we see that (2.1) becomes

$$\begin{aligned}
&\underbrace{\ln\Gamma(\frac{1}{4}n+\frac{1}{2}s+\frac{1}{2}+\frac{1}{2}a\sqrt{n}) - \ln\Gamma(\frac{1}{4}n+\frac{1}{2}s)}\\
&+\underbrace{\ln\Gamma(\frac{1}{4}n+\frac{1}{2}s+\frac{1}{2}-\frac{1}{2}a\sqrt{n}) - \ln\Gamma(\frac{1}{4}n+\frac{1}{2}s))}\\
&-\underbrace{\Big(\ln\Gamma(\frac{1}{4}n-\frac{1}{2}s+\frac{1}{2}+\frac{1}{2}a\sqrt{n}) - \ln\Gamma(\frac{1}{4}n-\frac{1}{2}s\Big)}\\
&-\underbrace{\Big(\ln\Gamma(\frac{1}{4}n-\frac{1}{2}s+\frac{1}{2}-\frac{1}{2}a\sqrt{n}) - \ln\Gamma(\frac{1}{4}n-\frac{1}{2}s)\Big)} = 0.
\end{aligned}$$

Correspondingly, we denote the left hand side (LHS, for short) of the above equation by the following

$$\begin{aligned}
LHS :=& g_1(n,s,a) + g_2(n,s,a) - g_3(n,s,a) - g_4(n,s,a) = 0;\\
LHS :=& \underbrace{g_1(n,s,a) + g_2(n,s,a)} - \underbrace{(g_3(n,s,a) + g_4(n,s,a))} = 0;\\
LHS :=& f_1(n,s,a) - f_2(n,s,a) = 0
\end{aligned}$$

which can be written as

$$f_{n,s}(a) := f_1(n,s,a) - f_2(n,s,a) = 0. \tag{2.3}$$

We note that, to make sure that all the expressions in the Gamma function above are meaningful, we need that $-2 < k < n-2s$, equivalently,

$$-\frac{1}{2}\frac{n-2s}{\sqrt{n}} - \frac{1}{\sqrt{n}} < a < \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}. \tag{2.4}$$

Using the above notations, we first observe that $f_{n,s}$ is an even function.

Lemma 2.1.

$$f_{n,s}(-a) = f_{n,s}(a).$$

Proof. It can be checked directly. □

By evenness we can only discuss the function $f_{n,s}(a)$ for positive variable $a \in [0, \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}})$ only.

To obtain further properties of $f_{n,s}(a)$, we introduce the following function

$$\Psi(x) = \frac{d}{dx}\Big(\ln(\Gamma(x))\Big) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

It is known that

$$\Psi^{(m)}(x) = (-1)^{m+1} m! \sum_{i=0}^{\infty} \frac{1}{(x+i)^{m+1}}, \quad m = 1, 2, ...$$

For $x > 1$, we note that

$$\begin{aligned}\frac{1}{mx^m} &= \int_0^{+\infty} \frac{1}{(x+y)^{m+1}} dy \le \sum_{i=0}^{\infty} \frac{1}{(x+i)^{m+1}} \\ &\le \int_0^{+\infty} \frac{1}{(x+y-1)^{m+1}} dy = \frac{1}{m(x-1)^m}.\end{aligned}$$

Therefore, by letting $m = 2k$ and $m = 2k+1$ respectively, we have the following estimates on the derivatives of $\Psi(x)$:

$$\begin{cases} -\frac{(2k-1)!}{(x-1)^{2k}} \le \Psi^{(2k)}(x) \le -\frac{(2k-1)!}{x^{2k}}, \quad m = 2k, \\ \frac{(2k)!}{x^{2k+1}} \le \Psi^{(2k+1)}(x) \le \frac{(2k)!}{(x-1)^{2k+1}}, \quad m = 2k+1. \end{cases} \tag{2.5}$$

Lemma 2.2. *If $n > 2s$, $s > 0$, then $f_{n,s}(0) > 0$.*

Proof. Consider the function $\ln\Gamma(\frac{1}{2}n + \frac{1}{4}s + x) - \ln\Gamma(\frac{1}{2}n - \frac{1}{4}s + x)$ for $x \ge 0$. Since $s > 0$, we have

$$\begin{aligned}&\frac{d}{dx}\Big(\ln\Gamma(\frac{1}{2}n + \frac{1}{4}s + x) - \ln\Gamma(\frac{1}{2}n - \frac{1}{4}s + x)\Big) \\ &= \Psi(\frac{1}{2}n + \frac{1}{4}s + x) - \Psi(\frac{1}{2}n - \frac{1}{4}s + x) > 0.\end{aligned}$$

Hence we obtain that

$$\begin{aligned}f_{n,s}(0) =& 2\Big(\ln\Gamma(\frac{1}{2}n + \frac{1}{4}s + \frac{1}{2}) - \ln\Gamma(\frac{1}{2}n + \frac{1}{4}s)\Big) - \\ & 2\Big(\ln\Gamma(\frac{1}{2}n - \frac{1}{4}s + \frac{1}{2}) - \ln\Gamma(\frac{1}{2}n - \frac{1}{4}s)\Big) \\ =& 2\Big(\ln\Gamma(\frac{1}{2}n + \frac{1}{4}s + x) - \ln\Gamma(\frac{1}{2}n - \frac{1}{4}s + x)\Big)\ |_{x=\frac{1}{2}} \\ & - 2\Big(\ln\Gamma(\frac{1}{2}n + \frac{1}{4}s + x) - \ln\Gamma(\frac{1}{2}n - \frac{1}{4}s + x)\Big)\ |_{x=0} \\ >& 0.\end{aligned}$$

□

We further prove that

Lemma 2.3. *If $n > 2s$ and $s > 0$, then $f_{n,s}(\frac{1}{\sqrt{n}}) > 0$.*

Proof. By definition we have that

$$\begin{aligned}f_{n,s}(\frac{1}{\sqrt{n}}) = \ln\Gamma(\frac{1}{4}n + \frac{1}{2}s + 1) - \ln\Gamma(\frac{1}{4}n + \frac{1}{2}s) \\ - \Big(\ln\Gamma(\frac{1}{4}n - \frac{1}{2}s + 1) - \ln\Gamma(\frac{1}{4}n - \frac{1}{2}s)\Big).\end{aligned}$$

We divide into two different cases.

Case 1: $s \geq 1$. Then we have

$$\begin{aligned} f_{n,s}(\frac{1}{\sqrt{n}}) &= \ln\Gamma(\frac{1}{4}n + \frac{1}{2}s + 1) - \ln\Gamma(\frac{1}{4}n + \frac{1}{2}s) \\ &\quad - \Big(\ln\Gamma(\frac{1}{4}n - \frac{1}{2}s + 1) - \ln\Gamma(\frac{1}{4}n - \frac{1}{2}s)\Big) \\ &= \Psi(\frac{1}{4}n + \frac{1}{2}s + \theta_1) - \Psi(\frac{1}{4}n - \frac{1}{2}s + \theta_2), \end{aligned}$$

where $\theta_1, \theta_2 \in (0,1)$ by the mean value theorem. Since $(\frac{1}{4}n + \frac{1}{2}s + \theta_1) - (\frac{1}{4}n - \frac{1}{2}s + \theta_2) = s + \theta_1 - \theta_2 > 0$, and $x > 0$, $\Psi'(x) > 0$, we obtain the conclusion.

Case 2: $0 < s < 1$. Then we have

$$\begin{aligned} f_{n,s}(\frac{1}{\sqrt{n}}) &= \ln\Gamma(\frac{1}{4}n + \frac{1}{2}s + 1) - \ln\Gamma(\frac{1}{4}n - \frac{1}{2}s + 1) \\ &\quad - \Big(\ln\Gamma(\frac{1}{4}n + \frac{1}{2}s) - \ln\Gamma(\frac{1}{4}n - \frac{1}{2}s)\Big) \\ &= s\Big(\Psi(\frac{1}{4}n + \frac{\alpha_1}{2}s + 1) - \Psi(\frac{1}{4}n + \frac{\alpha_2}{2}s)\Big), \end{aligned}$$

where $\alpha_1, \alpha_2 \in (-1,1)$ by mean value theorem. Since $(\frac{1}{4}n + \frac{\alpha_1}{2}s + 1) - (\frac{1}{4}n + \frac{\alpha_2}{2}s) = 1 + \frac{\alpha_1 - \alpha_2}{2}s > 1 - s > 0$ and $x > 0$, $\Psi'(x) > 0$, we get the conclusion. □

Lemma 2.4. *Let $a \geq 0$. Then we have $f'_{n,s}(0) = 0$ and $f'_{n,s}(a) < 0$ if $a > 0$.*

Proof. By direct computation we have that

$$\begin{aligned} f'_{n,s}(a) =& \frac{1}{2}\sqrt{n}\Big(\big(\Psi(\frac{1}{4}n + \frac{1}{2}s + \frac{1}{2}a\sqrt{n}) - \Psi(\frac{1}{4}n - \frac{1}{2}s + \frac{1}{2}a\sqrt{n})\big) \\ & - \big(\Psi(\frac{1}{4}n + \frac{1}{2}s - \frac{1}{2}a\sqrt{n}) - \Psi(\frac{1}{4}n - \frac{1}{2}s - \frac{1}{2}a\sqrt{n})\big)\Big). \end{aligned}$$

Since $f_{n,s}(a)$ is an even function, it follows that $f'_{n,s}(0) = 0$.

For $a > 0$, let us consider the function $\Psi(\frac{1}{4}n + x) + \Psi(\frac{1}{4}n - x)$ for $\frac{1}{4}n > x > 0$. By (2.5) we infer that

$$\frac{d}{dx}\Big(\Psi(\frac{1}{4}n + x) + \Psi(\frac{1}{4}n - x)\Big) = \Psi'(\frac{1}{4}n + x) - \Psi'(\frac{1}{4}n - x) < 0, \text{for } x > 0.$$

As a consequence we deduce that

$$\begin{aligned} \frac{d}{da}f_{n,s}(a) = \frac{1}{2}\sqrt{n}\Big(&\big(\Psi(\frac{1}{4}n + x) + \Psi(\frac{1}{4}n - x)\big)\mid_{x = \frac{1}{2}s + a\sqrt{n}} \\ &-\big(\Psi(\frac{1}{4}n + x) + \Psi(\frac{1}{4}n - x)\big)\mid_{x = \frac{1}{2}s - a\sqrt{n}}\Big) < 0. \end{aligned}$$

□

Lemma 2.5. *If $n > 2s$, then*

$$f_{n,s}(a)\mid_{a = \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}} = -\infty.$$

Proof. If $a = \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}$, by a direct calculation, we have that $\frac{1}{4}n - \frac{1}{2}s + \frac{1}{2} - \frac{1}{2}a\sqrt{n} = 0$. Thus $\ln\Gamma(\frac{1}{4}n - \frac{1}{2}s + \frac{1}{2} - \frac{1}{2}a\sqrt{n}) = +\infty$. Note that

$$\begin{aligned} &\ln\Gamma(\frac{1}{4}n - \frac{1}{2}s + \frac{1}{2} + \frac{1}{2}a\sqrt{n}) = \ln\Gamma(\frac{1}{2}(n - 2s) + 1), \\ &\ln\Gamma(\frac{1}{4}n + \frac{1}{2}s + \frac{1}{2} - \frac{1}{2}a\sqrt{n}) = \ln\Gamma(s), \\ &\ln\Gamma(\frac{1}{4}n + \frac{1}{2}s + \frac{1}{2} + \frac{1}{2}a\sqrt{n}) = \ln\Gamma(\frac{1}{2}n + 1). \end{aligned}$$

Therefore, $f(n,s,a)\mid_{a = \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}} = -\infty$. □

Lemmata 2.1-2.5 yield the following result.

Theorem 2.1. *Assume that $n > 2s, s > 0$ and a satisfies* (2.4).

(1) The equation $f_{n,s}(a) = 0$ admits precisely two real roots which are opposite numbers, we denote them as $\pm a_{n,s}$. Moreover, $\frac{1}{\sqrt{n}} < a_{n,s} < \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}$.

(2) The inequality $f_{n,s}(a) > 0$ holds if and only if $-a_{n,s} < a < a_{n,s}$.

Now we return to the variable k. Recalling that $k = \frac{n-(2s+2)}{2} + a\sqrt{n}$, by Theorem 2.1 we immediately have

Theorem 2.2. *Assume that $n > 2s, s > 0$ and a satisfies* (2.4).

(1) The equation (2.1) *of variable k has and only has two real roots, we denote as k_1, k_2. Moreover,*

$$k_1 := \frac{n-(2s+2)}{2} - a_{n,s}\sqrt{n}, \quad k_2 := \frac{n-(2s+2)}{2} + a_{n,s}\sqrt{n},$$

where $\frac{1}{\sqrt{n}} < a_{n,s} < \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}$.

(2) The inequality (1.6) *holds if and only if $k_1 < k < k_2$.*

Now we turn to the original equation (1.1) and the corresponding inequality (1.6).

Proofs of Theorems 1.1-1.2. Applying Theorem 2.2 above, we get k_1, k_2, where $k_1 = \frac{n-(2s+2)}{2} - a_{n,s}\sqrt{n}$, $k_2 = \frac{n-(2s+2)}{2} + a_{n,s}\sqrt{n}$. The only difference between Theorem 1.1-1.2 with Theorem 2.2 is that in Theorem 1.1-1.2 $k > 0$. Since $p > 1$ in Theorem 1.1-1.2, recalling that $\frac{2s}{p-1} = k$, we have $k > 0$. However in Theorem 2.2, the region of k, that is $-2 < k < n - 2s$, is natural from the fact that the Gamma function is positive. It can be easily checked that $-2 < k_1 < n - 2s$, $\frac{n-2s}{2} < k < n - 2s$ since $\frac{1}{\sqrt{n}} < a_{n,s} < \frac{1}{2}\frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}$. Therefore the solution k_1 may be non-positive. So we need to divide into several cases, the borderline determined by the following equation

$$k_1 := \frac{n-(2s+2)}{2} - a_{n,s}\sqrt{n} = 0.$$

Solving this, we have either

$$\sqrt{n} = a_{n,s} - \sqrt{a_{n,s}^2 + 2s + 2} \;\text{ or }\; \sqrt{n} = a_{n,s} + \sqrt{a_{n,s}^2 + 2s + 2}.$$

Since $a_{n,s} - \sqrt{a_{n,s}^2 + 2s + 2} < 0$, we have that $k_1 > 0$ if and only if $n > (a_{n,s} + \sqrt{a_{n,s}^2 + 2s + 2})^2$. The rest of the proofs follow from Theorem 2.2. □

3 The location of $a_{n,s}$ and further discussion

In the section we focus on the constant $a_{n,s}$, which is crucial in our discussion above, i.e., the critical dimension $n_0(s)$ and the roots of p_1 and p_2 of (1.1). In the following, we shall give both lower and upper bounds of the function $f_{n,s}(a)$. Using these bounds, we can have better estimates for $a_{n,s}$ which yields that

$$\lim_{n\to+\infty} a_{n,s} = 1 \;\text{ for any fixed }\; s > 0.$$

Lemma 3.1. *For $n > 2s+4, n > (a+\sqrt{\max\{a^2+2s-2,0\}})^2$, we have the following upper bound for $f_{n,s}(a)$*

$$\begin{aligned}
&f_{n,s}(a)\\
&\le s\Big(\frac{1}{\frac{1}{4}n-\frac{s}{2}-1}-\frac{\frac{1}{4}+\frac{1}{4}a^2n}{(\frac{1}{4}n+\frac{s}{2})^2}+\frac{1}{(\frac{1}{4}n-\frac{s}{2}-1)^3}\frac{(\frac{1}{2}a\sqrt{n}+\frac{1}{2})^3}{3}\Big)\\
&=\frac{4s}{3(n-2s-4)^3(n+2s)^2}\Big\{(-3a^2+3)n^4+2a^3n^{\frac{7}{2}}+\big[-27+(18s+42)a^2\big]n^3\\
&+(8a^3s+6a)n^{\frac{5}{2}}+\big[(-36s^2-120s-144)a^2-24s^2-30s+86\big]n^2\\
&+(8a^3s^2+24as)n^{\frac{3}{2}}+\big[(24s^3+168s^2+288s+192)a^2+60s^2+56s-144\big]n\\
&+24as^2n^{\frac{1}{2}}+48s^4+216s^3+344s^2+288s+192\Big\}
\end{aligned} \tag{3.1}$$

and the following lower bound

$$\begin{aligned}
&f_{n,s}(a)\\
&\ge s\Big(\frac{1}{\frac{1}{4}n+\frac{s}{2}}-\frac{\frac{1}{4}+\frac{1}{4}a^2n}{(\frac{1}{4}n-\frac{s}{2}-1)^2}-\frac{1}{(\frac{1}{4}n-\frac{s}{2}-1)^3}\frac{(\frac{1}{2}a\sqrt{n}-\frac{1}{2})^3}{3}\Big)\\
&=\frac{4s}{3(n+2s)(n-2s-4)^3}\Big((-3a^2+3)n^3-2a^3n^{\frac{5}{2}}+(18a^2-18s-39)n^2\\
&\quad+(-4a^3s-6a)n^{\frac{3}{2}}+\big[(12s^2+36s)a^2+36s^2+144s+158\big]n\\
&\quad-12asn^{\frac{1}{2}}-24s^3-132s^2-260s-192\Big).
\end{aligned} \tag{3.2}$$

Remark 3.1. *Here we obtain better estimates through the transform $k=\frac{n-(2s+2)}{2}+a\sqrt{n}$. The term $\frac{n-(2s+2)}{2}$ seems natural which guarantees that all the variables in the Gamma function of the equation* (1.1) *have the part $\frac{1}{4}n+\frac{1}{2}s$ or $\frac{1}{4}n-\frac{1}{2}s$.*

Proof. If $n > (a+\sqrt{\max\{a^2+2s-2,0\}})^2$ then all the expression in the Gamma function of the function $f_{n,s}(a)$ are positive.

We take the Taylor's expansion of the function $g_j(n,s,a)$:

$$\begin{aligned}
g_1(n,s,a)=&\Psi(\frac{1}{4}n+\frac{1}{2}s)(\frac{1}{2}+\frac{1}{2}a\sqrt{n})+\Psi'(\frac{1}{4}n+\frac{1}{2}s)\frac{(\frac{1}{2}+\frac{1}{2}a\sqrt{n})^2}{2!}\\
&+\Psi''(\frac{1}{4}n+\frac{\theta_{11}}{2}s)\frac{(\frac{1}{2}+\frac{1}{2}a\sqrt{n})^3}{3!};\\
g_2(n,s,a)=&\Psi(\frac{1}{4}n+\frac{1}{2}s)(\frac{1}{2}-\frac{1}{2}a\sqrt{n})+\Psi'(\frac{1}{4}n+\frac{1}{2}s)\frac{(\frac{1}{2}-\frac{1}{2}a\sqrt{n})^2}{2!}\\
&+\Psi''(\frac{1}{4}n+\frac{\theta_{12}}{2}s)\frac{(\frac{1}{2}-\frac{1}{2}a\sqrt{n})^3}{3!};\\
g_3(n,s,a)=&\Psi(\frac{1}{4}n-\frac{1}{2}s)(\frac{1}{2}+\frac{1}{2}a\sqrt{n})+\Psi'(\frac{1}{4}n-\frac{1}{2}s)\frac{(\frac{1}{2}+\frac{1}{2}a\sqrt{n})^2}{2!}\\
&+\Psi''(\frac{1}{4}n-\frac{\theta_{21}}{2}s)\frac{(\frac{1}{2}+\frac{1}{2}a\sqrt{n})^3}{3!};\\
g_4(n,s,a)=&\Psi(\frac{1}{4}n-\frac{1}{2}s)(\frac{1}{2}-\frac{1}{2}a\sqrt{n})+\Psi'(\frac{1}{4}n-\frac{1}{2}s)\frac{(\frac{1}{2}-\frac{1}{2}a\sqrt{n})^2}{2!}\\
&+\Psi''(\frac{1}{4}n-\frac{\theta_{22}}{2}s)\frac{(\frac{1}{2}-\frac{1}{2}a\sqrt{n})^3}{3!}.
\end{aligned}$$

Adding these up and applying the mean value theorem, we have

$$\begin{aligned}
f_{n,s}(a) &= g_1(n,s,a) + g_2(n,s,a) - g_3(n,s,a) - g_4(n,s,a) \\
&= \Psi(\frac{1}{4}n + \frac{1}{2}s) - \Psi(\frac{1}{4}n - \frac{1}{2}s) \\
&\quad + \Big(\Psi'(\frac{1}{4}n + \frac{1}{2}s) - \Psi'(\frac{1}{4}n - \frac{1}{2}s)\Big)(\frac{1}{4} + \frac{1}{4}a^2 n) \\
&\quad + \Big(\Psi''(\frac{1}{4}n + \frac{\theta_{11}}{2}s) - \Psi''(\frac{1}{4}n - \frac{\theta_{21}}{2}s)\Big)\frac{(\frac{1}{2} + \frac{1}{2}a\sqrt{n})^3}{3!} \\
&\quad + \Big(\Psi''(\frac{1}{4}n + \frac{\theta_{12}}{2}s) - \Psi''(\frac{1}{4}n - \frac{\theta_{22}}{2}s)\Big)\frac{(\frac{1}{2} - \frac{1}{2}a\sqrt{n})^3}{3!} \\
&= s\Big(\Psi'(\frac{1}{4}n + \frac{\alpha_1}{2}s) + \Psi''(\frac{1}{4}n + \frac{\alpha_2}{2}s)(\frac{1}{4} + \frac{1}{4}a^2 n) \\
&\quad + \frac{\theta_{11}+\theta_{21}}{2}\Psi'''(\frac{1}{4}n + \frac{\alpha_3}{2}s)\frac{(\frac{1}{2} + \frac{1}{2}a\sqrt{n})^3}{3!} \\
&\quad + \frac{\theta_{21}+\theta_{22}}{2}\Psi'''(\frac{1}{4}n + \frac{\alpha_4}{2}s)\frac{(\frac{1}{2} - \frac{1}{2}a\sqrt{n})^3}{3!}\Big),
\end{aligned}$$

where $\theta_{ij} \in (0,1), \alpha_k \in (-1,1)$. Now in view of the derivative estimates of $\Psi(x)$ in (2.5), we get the upper and lower bounds of $f_{n,s}(a)$. Notice that

$$\begin{aligned}
f_{n,s}(a) &\le c_1(n,s,a)\Big((-3a^2+3)n^4 + \text{lower-order-term}\Big), \\
f(n,s,a) &\ge c_2(n,s,a)\Big((-3a^2+3)n^3 + \text{lower-order-term}\Big).
\end{aligned}$$

Let $n = t^2$. Then all the lower order terms are lower order polynomials, that is,

$$\begin{aligned}
f_{n,s}(a)\mid_{n=t^2} &\le c_1(n,s,a)\Big((-3a^2+3)t^8 + \text{lower-order-term}\Big), \\
f_{n,s}(a)\mid_{n=t^2} &\ge c_2(n,s,a)\Big((-3a^2+3)t^6 + \text{lower-order-term}\Big).
\end{aligned}$$

Let $a^2 > 1$, then there exists $t_1 = t_1(a) > 0$, such that for any $t > t_1$, there holds $(-3a^2+3)t^8 +$ (lower-order-term) < 0. Hence for any $a > 1$, there exists $t_1 = t_1(a) > 0$ such that $f_{n,s}(a) < 0$ when $n > t_1^2$. By Lemma 2.2, we see $f_{n,s}(0) > 0$. Thus, there is a point $a_0 \in (0,a)$ such that $f_{n,s}(a_0) = 0$. Furthermore, since $f_{n,s}(a)$ is non-increasing, then a_0 is the only real root on interval $[0,+\infty)$.

On the other hand, for any $a > 0$ such that $a^2 < 1$, then there exists $t_2 > 0$, such that for any $t > t_2$, there holds $(-3a^2+3)n^3 + l.o.t > 0$. Therefore, for $\varepsilon_1, \varepsilon_2 > 0$, there exist $t_1 = t_1(\varepsilon_1), t_2 = t_1(\varepsilon_2) > 0$, such that when $n > \max\{t_1^2, t_2^2\}$, there holds

$$\begin{aligned}
f_{n,s}(1+\varepsilon_1)\mid_{n=t^2} &\le c_1(n,s,\varepsilon_1)\Big(-3(\varepsilon_1^2+2\varepsilon_1)t^8 + \text{lower-order-term}\Big) < 0, \\
f_{n,s}(1-\varepsilon_2)\mid_{n=t^2} &\ge c_2(n,s,\varepsilon_2)\Big(3\varepsilon_2(2-\varepsilon_2)t^6 + \text{lower-order-term}\Big) > 0.
\end{aligned}$$

Thus, there is an $a \in (1-\varepsilon_2, 1+\varepsilon_1)$ such that $f_{n,s}(a) = 0$ for $n > \max\{t_1^2, t_2^2\}$. Besides, since ε is arbitrary small, we get that $a \to 1$ as $n \to +\infty$. □

By the inverse transformation of $k = \frac{2s}{p-1}$ and $k = \frac{n-(2s+2)}{2} + a\sqrt{n}$, a direct consequence of the above lemma is the following corollary, which complements Theorem 1.1.

Corollary 3.1. *For any $\varepsilon_1, \varepsilon_2 > 0$, there exist $a \in (1-\varepsilon_2, 1+\varepsilon_1)$ and $n_0 = n_0(\varepsilon_1, \varepsilon_2)$, for $n \ge n_0$, the equation* (1.1) *has and only has two real roots p_1 and p_2:*

$$p_1 := \frac{n+2s-2+2a\sqrt{n}}{n-2s-2+2a\sqrt{n}}, \quad p_2 := \frac{n+2s-2-2a\sqrt{n}}{n-2s-2-2a\sqrt{n}}$$

and $a = a_{n,s} \to 1$ as $n \to +\infty$.

Remark 3.2. *Generally speaking, when $\varepsilon_1, \varepsilon_2 \to 0+$, n_0 will be larger and larger; however, when $\varepsilon_1, \varepsilon_2$ are far away from 0, n_0 will be smaller. That is, to make sure the existence of such roots, we need to choose the parameters $\varepsilon_1, \varepsilon_2$ suitably away from 0. On the other hand, to get more accurate estimates on the roots, we need to select $\varepsilon_1, \varepsilon_2 \to 0+$ properly, which requires that n must be large.*

Now we turn to the inequality (1.6). By the transformation above, the inequality (1.6) is equivalent to $f_{n,s}(a) > 0$. Then we have the following

Corollary 3.2. *Assume the inequality (1.6) holds. Then for any $\varepsilon_1, \varepsilon_2 > 0$, there exists an $a = a_{n,s} \in (1-\varepsilon_2, 1+\varepsilon_1)$ and $n_0 = n_0(\varepsilon_1, \varepsilon_2)$, such that for all $n \geq n_0$, we have the following*

$$\frac{n+2s-2+2a\sqrt{n}}{n-2s-2+2a\sqrt{n}} < p < \frac{n+2s-2-2a\sqrt{n}}{n-2s-2-2a\sqrt{n}}$$

and $a = a_{n,s} \to 1$ as $n \to +\infty$.

Remark 3.3. *Generally speaking, when $\varepsilon_1, \varepsilon_2 \to +0$, n_0 becomes larger while n_0 becomes smaller when $\varepsilon_1, \varepsilon_2$ are far away from 0. Therefore, to find the optimal bound, we need to choose both parameters in an optimal way.*

To obtain the optimal $n_0(\varepsilon_1, \varepsilon_2)$ and also optimal upper and lower bound about p in (1.6), we need to generalize Lemma 3.1. As before, we take the Taylor's expansion of the functions $g_i(n,s,a)$ to m order.

$$\begin{aligned}
g_1(n,s,a) &= \sum_{j=0}^{m} \Psi^{(j)}(\frac{1}{4}n + \frac{1}{2}s)\frac{(\frac{1}{2}+\frac{1}{2}a\sqrt{n})^{j+1}}{(j+1)!} \\
&\quad + \Psi^{(m+1)}(\frac{1}{4}n + \frac{\theta_{11}}{2}s)\frac{(\frac{1}{2}+\frac{1}{2}a\sqrt{n})^{m+2}}{(m+2)!}; \\
g_2(n,s,a) &= \sum_{j=0}^{m} \Psi^{(j)}(\frac{1}{4}n + \frac{1}{2}s)\frac{(\frac{1}{2}-\frac{1}{2}a\sqrt{n})^{j+1}}{(j+1)!} \\
&\quad + \Psi^{(m+1)}(\frac{1}{4}n + \frac{\theta_{12}}{2}s)\frac{(\frac{1}{2}-\frac{1}{2}a\sqrt{n})^{m+2}}{(m+2)!}; \\
g_3(n,s,a) &= \sum_{j=0}^{m} \Psi^{(j)}(\frac{1}{4}n - \frac{1}{2}s)\frac{(\frac{1}{2}+\frac{1}{2}a\sqrt{n})^{j+1}}{(j+1)!} \\
&\quad + \Psi^{(m+1)}(\frac{1}{4}n - \frac{\theta_{21}}{2}s)\frac{(\frac{1}{2}+\frac{1}{2}a\sqrt{n})^{m+2}}{(m+2)!}; \\
g_4(n,s,a) &= \sum_{j=0}^{m} \Psi^{(j)}(\frac{1}{4}n - \frac{1}{2}s)\frac{(\frac{1}{2}-\frac{1}{2}a\sqrt{n})^{j+1}}{(j+1)!} \\
&\quad + \Psi^{(m+1)}(\frac{1}{4}n - \frac{\theta_{22}}{2}s)\frac{(\frac{1}{2}-\frac{1}{2}a\sqrt{n})^{m+2}}{(m+2)!}.
\end{aligned}$$

Here $\Psi^{(0)} = \Psi$. Adding these up, we have that

$$
\begin{aligned}
& f(n,s,a) \\
&= g_1(n,s,a) + g_2(n,s,a) - g_3(n,s,a) - g_4(n,s,a) \\
&= \sum_{j=0}^{m} \Big(\Psi^{(j)}(\frac{1}{4}n + \frac{1}{2}s) - \Psi^{(j)}(\frac{1}{4}n - \frac{1}{2}s) \Big) \frac{(\frac{1}{2} + \frac{1}{2}a\sqrt{n})^{j+1} + (\frac{1}{2} - \frac{1}{2}a\sqrt{n})^{j+1}}{(j+1)!} \\
&\quad + \Big(\Psi^{(m+1)}(\frac{1}{4}n + \frac{\theta_{11}}{2}s) - \Psi^{(m+1)}(\frac{1}{4}n - \frac{\theta_{21}}{2}s) \Big) \frac{(\frac{1}{2} + \frac{1}{2}a\sqrt{n})^{m+2}}{(m+2)!} \\
&\quad + \Big(\Psi^{(m+1)}(\frac{1}{4}n + \frac{\theta_{12}}{2}s) - \Psi^{(m+1)}(\frac{1}{4}n - \frac{\theta_{22}}{2}s) \Big) \frac{(\frac{1}{2} - \frac{1}{2}a\sqrt{n})^{m+2}}{(m+2)!} \\
&= s\Big(\sum_{j=0}^{m} \Psi^{(j+1)}(\frac{1}{4}n + \frac{\alpha_j}{2}s) \frac{(\frac{1}{2} + \frac{1}{2}a\sqrt{n})^{j+1} + (\frac{1}{2} - \frac{1}{2}a\sqrt{n})^{j+1}}{(j+1)!} \\
&\quad + \frac{\theta_{11}+\theta_{21}}{2} \Psi^{(m+2)}(\frac{1}{4}n + \frac{\alpha_{m+1}}{2}s) \frac{(\frac{1}{2} + \frac{1}{2}a\sqrt{n})^{m+2}}{(m+2)!} \\
&\quad + \frac{\theta_{12}+\theta_{22}}{2} \Psi^{(m+2)}(\frac{1}{4}n + \frac{\alpha_{m+2}}{2}s) \frac{(\frac{1}{2} - \frac{1}{2}a\sqrt{n})^{m+2}}{(m+2)!} \Big),
\end{aligned} \tag{3.3}
$$

where $\theta_{ij} \in (0,1), \alpha_k \in (-1,1)$. When $m = 2$, we have the following

Lemma 3.2. *Assume that $n > 2s+4, n > (a + \sqrt{\max\{a^2+2s-2, 0\}})^2$. Then we have*

$$
\begin{aligned}
f_{n,s}(a) &\le s\Big(\frac{1}{\frac{1}{4}n - \frac{s}{2} - 1} - \frac{\frac{1}{4} + \frac{1}{4}a^2 n}{(\frac{1}{4}n + \frac{s}{2})^2} + \frac{2!}{(\frac{1}{4}n - \frac{s}{2} - 1)^3}(\frac{1}{24} + \frac{1}{8}a^2 n) \Big) \\
&= \frac{4s}{3(n-2s-4)^3(n+2s)^2} \Big((-3a^2+3)n^4 + \big[-27 + (18s+48)a^2 \big] n^3 \\
&\quad + \big[(-36s^2 - 96s - 144)a^2 - 24s^2 - 30s + 88 \big] n^2 \\
&\quad + \big[(24s^3 + 192s^2 + 288s + 192)a^2 + 60s^2 + 64s - 144 \big] n \\
&\quad + 48s^4 + 216s^3 + 352s^2 + 288s + 192 \Big),
\end{aligned}
$$

and

$$
\begin{aligned}
f_{n,s}(a) &\ge s\Big(\frac{1}{\frac{1}{4}n + \frac{s}{2}} - \frac{\frac{1}{4} + \frac{1}{4}a^2 n}{(\frac{1}{4}n - \frac{s}{2} - 1)^2} + \frac{2!}{(\frac{1}{4}n + \frac{s}{2})^3}(\frac{1}{24} + \frac{1}{8}a^2 n) \\
&\quad - \frac{3!}{(\frac{1}{4}n - \frac{s}{2} - 1)^4}(\frac{1}{192} + \frac{1}{192}a^4 n^2 + \frac{1}{32}a^2 n) \Big) \\
&= \frac{4s}{3(t^2+2s)^3(t^2-2s-4)^4} \Big((-3a^2+3)n^6 + \big[-6a^4 + (-6s+36)a^2 \\
&\quad - 12s - 51 \big] n^5 + \big[-36sa^4 + (24s^2 - 276)a^2 - 12s^2 + 90s + 316 \big] n^4 \\
&\quad + \big[-72s^2a^4 + (48s^3 + 288s^2 + 648s + 1152)a^2 + 96s^3 + 408s^2 + \\
&\quad\ 64s - 886 \big] n^3 \\
&\quad + \big[-48s^3a^4 + (-48s^4 - 768s^3 - 3312s^2 - 4608s - 3072)a^2 - 48s^4 \\
&\quad - 720s^3 - 2208s^2 - 1476s + 1152 \big] n^2 + \big[(-96s^5 - 192s^4 + 864s^3 \\
&\quad + 4608s^2 + 6144s + 3072)a^2 - 192s^5 - 816s^4 - 512s^3 \\
&\quad + 1656s^2 + 1536s - 1024 \big] n + 192s^6 + 1440s^5 + 4288s^4 \\
&\quad + 6224s^3 + 4608s^2 + 2048s + 1024 \Big).
\end{aligned}
$$

Remark 3.4. *Combining with the Taylor's expansion of function $f_{n,s}(a)$ in* (3.3) *and the derivative estimates of Ψ in* (2.5)*, we can obtain the formula with higher order expansions. By this way, we can reduce the bounds $n_1(s, \varepsilon_1)$ and $n_2(s, \varepsilon_2)$.*

Proof of Theorems 1.3. This is a straightforward consequence of Lemma 3.1 and Lemma 3.2. □

4 Example: an application to $s \in (2,3)$.

Let p, n, s satisfy (1.6). From Lemma 3.1, we get that

Lemma 4.1. *Assume that* (1.6) *holds. Then if* $n \geq 59$*, we get that* $0.7 < a < 1.5$ *and hence we obtain the following estimate for the second stability exponent*

$$\frac{n+2s-2+3\sqrt{n}}{n-2s-2+3\sqrt{n}} < p_2 < \frac{n+2s-2-3\sqrt{n}}{n-2s-2-3\sqrt{n}}. \tag{4.1}$$

However, if we apply Lemma 3.2 with higher order Taylor's expansion, we may improve the bound 59 and still get (4.1). Precisely, we have

Lemma 4.2. *Assume that* (1.6) *holds. Then if* $n \geq 44$*, we get that* $0.7 < a < 1.5$ *and that*

$$\frac{n+2s-2+3\sqrt{n}}{n-2s-2+3\sqrt{n}} < p_2 < \frac{n+2s-2-3\sqrt{n}}{n-2s-2-3\sqrt{n}}.$$

These estimates are important in deriving monotonicity formula and classifying stable solutions for the Lane-Emden equation (1.3) in the range $s \in (2,3)$. See [11].

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