On smooth solutions to one phase free boundary problem in $\mathbb{R}^n$

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Abstract

We construct a smooth axially symmetric solution to the classical one phase free boundary problem in $\mathbb{R}^n$, $n \geq 3$. Its free boundary is of "catenoid" type. This is a higher dimensional analogy of the Hauswirth-Helein-Pacard solution [18] in $\mathbb{R}^2$. The existence of such solution is conjectured in [18, Remark 2.4]. This is the first nontrivial smooth solution to the one phase free boundary problem in higher dimensions.

1 Introduction and main results

Free boundary problems arise as mathematical models in many different contexts, e.g., heat conduction, interface dynamics, evolution of ecological systems. In this paper, we are interested in constructing new smooth solutions for the following classical one phase free boundary problem in the whole space:

$$\begin{cases} \Delta u = 0 \text{ in } \Omega := \{u > 0\} \subset \mathbb{R}^n, \\ |\nabla u| = 1 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

Here $\partial\Omega$ is the free boundary. The regularity theory of (1) has been studied for a long time, see for instances [2, 5, 6, 7, 9, 12, 21]. In the literature, the domain

Ω in the one phase problem is called *exceptional domain* and the function u is called *roof function*.

The simplest solution to (1) is the one-dimensional solution x_n^+. This solution is unbounded, which constitutes a major difficulty for the construction of other solutions using this one dimensional profile. Another class of solutions of (1) is the cone type solutions (homogeneous functions of degree one). Consider the Alt-Caffarelli cone in $\mathbb{R}^n$ given by

$$|x_n| < \alpha_n \sqrt{x_1^2 + ... + x_n^2}. \tag{2}$$

It is known that there exists a unique dimensional constant α_n (see [1, 8]) such that there is a solution to (1) whose free boundary is exactly this cone. It has been proved by De Silva and Jerison [11] that in dimension $n = 7$ (actually also for $n = 9, 11, 13, 15$), the solution to (1) corresponding to the cone (2) is a minimizer for the energy functional

$$J_0(u) := \int \left[|\nabla u|^2 + \chi_{(0,+\infty)}(u) \right]. \tag{3}$$

For a discussion on the existence and stability of more general cones other than (2), we refer to [19, 22]. We remark that for a cone type solution which is also a minimizer of the energy functional, it is expected that there should be a family of smooth solutions to (1) whose free boundary is smooth and asymptotic to the cone.

We notice that the cone solution has a singularity at the origin. So far the only nontrivial smooth solution with simply connected phase we know of is the so-called Hauswirth-Helein-Pacard solution [18] in the plane (also called hairpin solution [20]). To describe this solution, we use Φ to denote the map

$$(x, y) \to (x + \cos y \sinh x, y + \sin y \cosh x).$$

Let Ω be the image of the region $\left\{(x, y) : |y| < \frac{\pi}{2}\right\}$ under this map. One checks directly that

$$\Omega = \left\{(x, y) : |y| < \frac{\pi}{2} + \cosh x\right\}.$$

Let $u(x, y) = \cos y \cosh x$. Then the function

$$U(x, y) = u \circ \Phi^{-1}(x, y) \tag{4}$$

is a solution to (1). It turns out that the Hauswirth-Helein-Pacard solution plays an important role in the analysis of other solutions of the one phase free boundary problem in the unit disk with simply connected phase [20], similar to the role played by the catenoids in the minimal surface theory [10].

Using complex function theory, Traizet [35] (see also [36] for related results) established a one-to-one correspondence between solutions to (1) and a special class of minimal surfaces in $\mathbb{R}^3$. Under this correspondence, U is transformed to the catenoid, a classical minimal surface. It also has been proved there that U is the unique (up to a scaling and the trivial one dimensional solution) solution in $\mathbb{R}^2$ with simply connected phase (see also [26, 32]). Unfortunately the correspondence between one phase problem and minimal surface is not available in dimensions $n \geq 3$. However, it is conjectured in [18, Remark 2.4] that the Hauswirth-Helein-Pacard solution should still have higher dimensional analogy. In this paper, we confirm this conjecture.

To state our result, we use $(x_1, ..., x_{n-1}, z)$ to denote the coordinate of $\mathbb{R}^n$ and set $r = \sqrt{x_1^2 + ... + x_{n-1}^2}$.

Theorem 1 *There exists a solution u to* (1) *satisfying the following properties:*
(I) u depends only on r and $|z|$.
(II) The positive phase $\Omega := \{(x_1, ..., x_{n-1}, z) \in \mathbb{R}^n : u(x_1, ..., x_{n-1}, z) > 0\}$ can be described by

$$\mathbb{R}^n \setminus \{(x_1, ..., x_{n-1}, z) : |z| < g(r)\},$$

for a function g with $g(1) = 0$ and

$$\lim_{r \to +\infty} \left(g'(r) r^{n-2}\right) \in [0, +\infty). \tag{5}$$

(III) In Ω, $\partial_z u > 0$ for $z > 0$, and $\partial_r u < 0$ for $r > 0$.

Remark 2 *Due to the scaling invariance of the problem, actually we have a family of solutions $\frac{u(\rho X)}{\rho}$ with $\rho > 0$ being a parameter. It is to be expected that there should exist another two families of axially symmetric solutions whose free boundaries are asymptotic to the Alt-Caffarelli cone. The positive phases should have the form $\{(x_1, ..., x_{n-1}, z) : |z| < h(r)\}$, where h is a positive monotone function defined on $[0, +\infty)$ for the first family of solutions, while h is monotone and defined on $[1, +\infty)$ with $h(1) = 0$ for one of the solutions in the second family.*

Now let us describe the main difficulties and steps of the proof of Theorem 1. A solution of the one phase free boundary problem is *formally* a critical point of the energy functional J_0. Given suitable boundary conditions, while it is relatively easy to use minimizing arguments to obtain *minimizers* (see [1]), variational methods in general are not directly applicable for unstable critical points of J_0, due to the fact that J_0 is **not** differentiable in usual functional spaces. Furthermore for the solutions we are interested in this paper, they are indeed not minimizers. Actually, for $\mathbb{R}^n, n \leq 4$, all stable solutions to the one phase free boundary problem are trivial. For dimension $n = 3$ this result was obtained by Caffarelli, Jerison and Kenig in [8], and they conjectured that it remains true up to dimension $n \leq 6$. Jerison and Savin [22] established the same result for $n = 4$. Another difficulty we are facing is that usually the solutions are unbounded and the energy is actually equal to infinity.

To overcome these difficulties, we proceed the proofs in two steps. We discuss the case of $n = 3$ only and the other cases are similar. In the first step, for each fixed $k > 0$, we construct two-end solutions to the following two-component free boundary problem

$$\begin{cases} \Delta u = 0 \text{ in } \{|u| < 1\}, \\ |\nabla u| = 1 \text{ on } \partial \{|u| < 1\}, \end{cases} \tag{6}$$

where the nodal set $\{u = 0\}$ behaves like $\{|z| = k \log r + b\}$. The solutions to problem (6) are bounded and relatively easier to deal with, though we still need to overcome the problem of nonsmooth profiles and regularity issues since the solutions are not minimizers and are of mountain-pass type in terms of the new energy functional

$$J_1(u) := \int \left[|\nabla u|^2 + \chi_{(-1,+1)}(u) \right].$$

In the second step, we show that the solutions to (6), after some rescaling, as $k \to 0^+$, approach to a nontrivial solution of (1).

The paper is organized as follows. From Section 2 to Section 5, we prove Theorem 1 in the case $n = 3$. Then in Section 6 we indicate the necessary modifications needed for general $n \geq 3$. In Section 2, we consider a family of regularized problems and use variational arguments to show the existence of mountain pass type solutions $U_{\varepsilon,a}$ in bounded domain Ω_a. In Section 3, we prove that as ε tends to zero, these mountain pass solutions converge to a solution V_a of (6) in Ω_a. We also show the regularity of the free boundary of V_a. In Section 4, we enlarge the domain by sending a to infinity, and get a solution W_k for (6) with prescribed asymptotic behavior at infinity (nodal set looks like $k \ln r + b$). In Section 5, we analyze the precise asymptotic behavior of W_k. Then by sending k to zero, we show that suitable blow up sequence of W_k near the origin converges to a solution of (1). In the last section, we consider the general case of $n > 3$.

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2 Mountain pass solutions for a family of regularized problems

As we already mentioned in Section 1, there are three main difficulties in dealing with problem (1): firstly the energy functional is not smooth, secondly the solution we are interested in is not a minimizer and thirdly the solution is unbounded.

To overcome the above mentioned difficulties, we shall regularize the functional J_1 and consider a family of smooth potentials F_ε which approximates the characteristic function $\chi_{(-1,1)}(\cdot)$ of the interval $(-1,1)$. F_ε is defined in the following way. Let $\bar{F}$ be a smooth monotone increasing function in $[0,+\infty)$ such that

$$\bar{F}(s) = \begin{cases} s^2, s \in \left[0, \frac{1}{2}\right], \\ 1 - e^{-s}, s \in (1, +\infty). \end{cases}$$

We may also assume that $\bar{F}'' < 0$ in $\left(\frac{3}{4}, +\infty\right)$. It is worth pointing out that the idea of regularizing the potentials has been explored in some other related contexts, for instances [3, 29].

Let $\rho \geq 0$ be a cutoff function satisfying $\rho(s) + \rho(-s) = 1$ and

$$\rho(s) = \begin{cases} 1, s < -\frac{1}{2}, \\ 0, s > \frac{1}{2}. \end{cases}$$

For each $\varepsilon > 0$ small, we define a smooth even potential F_ε on the interval $[-1,1]$, monotone increasing in $[-1,0]$, to be

$$F_\varepsilon(s) = \rho(s)\bar{F}\left(\frac{s+1}{\varepsilon}\right) + (1-\rho(s))\bar{F}\left(\frac{-s+1}{\varepsilon}\right).$$

With this definition, $F_\varepsilon \leq 1$, and on any compact subinterval of $(-1,1)$, $F_\varepsilon \to 1$, as $\varepsilon \to 0$. We also have

$$F_\varepsilon''(\pm 1) = \frac{2}{\varepsilon^2} \to +\infty, \quad \text{as } \varepsilon \to 0.$$

Then instead of J_1, we shall consider the regularized functional

$$\int \left[|\nabla u|^2 + F_\varepsilon(u)\right].$$

Note that F_ε is a double well type potential and a critical point of this functional solves the equation

$$-\Delta u + \frac{1}{2}F_\varepsilon'(u) = 0. \tag{7}$$

Let H_ε be the heteroclinic solution of the ODE

$$H_\varepsilon'' = \frac{1}{2}F_\varepsilon'(H_\varepsilon),$$

with

$$H_\varepsilon(0) = 0, \quad H_\varepsilon(\pm\infty) = \pm 1.$$

Heuristically, as $\varepsilon \to 0$, H_ε converges to the function

$$\mathcal{H}(x) = \begin{cases} x, & \text{for } -1 < x < 1, \\ 1, & \text{for } x \geq 1, \\ -1, & \text{for } x \leq -1. \end{cases}$$

This is a one-dimensional solution of the following free boundary problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega := \{|u| < 1\} \subset \mathbb{R}^n, \\ |\nabla u| = 1 \text{ on } \partial\Omega. \end{cases} \tag{8}$$

The existence and classification of solutions to this problem has been studied in [24, 31, 37].

We would like to construct mountain pass type solutions for (7) on bounded domains with suitable boundary data, using similar ideas as that of [17], where Morse index one solutions to the Allen-Cahn equation are constructed.

To describe the boundary data, we need to know the asymptotic behavior of H_ε as ε goes to zero.

Lemma 3 *Let $\varepsilon > 0$ be small. For any $x \in [0, 1 + \varepsilon \ln \varepsilon]$, there holds*

$$(1-\varepsilon)\, x \leq H_\varepsilon(x) \leq x.$$

Proof. H_ε satisfies

$$H_\varepsilon'^2 = F_\varepsilon(H_\varepsilon) \leq 1. \tag{9}$$

Hence $H_\varepsilon(x) \leq x$, for $x > 0$. In particular, in the interval $[0, 1 + \varepsilon \ln \varepsilon]$,

$$H(x) \leq 1 + \varepsilon \ln \varepsilon. \tag{10}$$

By the definition of F_ε,

$$1 - F_\varepsilon(s) = 1 - \rho(s)\, \bar{F}\left(\frac{s+1}{\varepsilon}\right) - (1-\rho(s))\, \bar{F}\left(\frac{-s+1}{\varepsilon}\right).$$

If $0 \leq s < 1 + \varepsilon \ln \varepsilon$, then $\bar{F}\left(\frac{s+1}{\varepsilon}\right) = 1 - e^{-\frac{s+1}{\varepsilon}}, \bar{F}\left(\frac{-s+1}{\varepsilon}\right) = 1 - e^{-\frac{-s+1}{\varepsilon}}$. Hence

$$\begin{aligned} 1 - F_\varepsilon(s) &= 1 - \rho(s)\left(1 - e^{-\frac{s+1}{\varepsilon}}\right) - (1-\rho(s))\left(1 - e^{-\frac{-s+1}{\varepsilon}}\right) \\ &= \rho(s)\, e^{-\frac{s+1}{\varepsilon}} + (1-\rho(s))\, e^{-\frac{-s+1}{\varepsilon}}. \end{aligned}$$

Consequently,

$$1 - F_\varepsilon(s) \leq \varepsilon.$$

This together with (10) implies that in the interval $[0, 1 + \varepsilon \ln \varepsilon]$,

$$H_\varepsilon' = \sqrt{F_\varepsilon(H_\varepsilon)} \geq 1 - \varepsilon,$$

provided that ε is small. The conclusion of the lemma then follows from $H_\varepsilon(0) = 0$. ■

We use t_ε to denote the point where

$$H_\varepsilon(t_\varepsilon) = 1 - \frac{\varepsilon}{2}. \tag{11}$$

By (9), $H_\varepsilon'(t_\varepsilon) = \frac{1}{2}$ and $H_\varepsilon''(t_\varepsilon) = \frac{1}{2\varepsilon}$. Additionally, $t_\varepsilon \in [1 + \varepsilon \ln \varepsilon, 1]$.

Lemma 4 *For $t \in [t_\varepsilon, +\infty)$,*

$$H_\varepsilon(t) = 1 - \frac{\varepsilon}{2} e^{\frac{t_\varepsilon}{\varepsilon}} e^{-\frac{t}{\varepsilon}}.$$

Proof. Let $t \geq t_\varepsilon$. Since H_ε is monotone increasing, using Lemma 3, we find that $H_\varepsilon(t) \geq 1 - \frac{\varepsilon}{2}$. It follows that

$$F_\varepsilon(H_\varepsilon) = \frac{(1 - H_\varepsilon)^2}{\varepsilon^2}.$$

Hence by (9),

$$H'_\varepsilon = \frac{1 - H_\varepsilon}{\varepsilon}.$$

Consequently, $H_\varepsilon(t) = 1 - c_\varepsilon e^{-\frac{t}{\varepsilon}}$. It then follows from (11) that

$$1 - c_\varepsilon e^{-\frac{t_\varepsilon}{\varepsilon}} = 1 - \frac{\varepsilon}{2}.$$

This yields $c_\varepsilon = \frac{\varepsilon}{2} e^{\frac{t_\varepsilon}{\varepsilon}}$. The proof is completed. ■

Lemma 3 and Lemma 4, together with the fact that $|H'_\varepsilon| \leq 1$, imply that $H_\varepsilon(s) - s \to 0$ in $C^{0,\alpha}([-1,1])$.

Let $l > 2$ be a large constant and $\delta_\varepsilon = O\left(\varepsilon^{\frac{4}{3}}\right)$ be the constant satisfying

$$H_\varepsilon(2) + H'_\varepsilon(2)\,\delta_\varepsilon + \frac{1}{2} H''_\varepsilon(2)\,\delta_\varepsilon^2 + \frac{1 - H_\varepsilon(2)}{\varepsilon^4}\delta_\varepsilon^3 = 1. \tag{12}$$

Note that for ε small, we have

$$H'_\varepsilon(2) = \frac{1 - H_\varepsilon(2)}{\varepsilon} = \varepsilon H''_\varepsilon(2).$$

We define a family of C^2 functions

$$w_{\varepsilon,l}(x) := \begin{cases} H_\varepsilon(x), x \in [-l, 2], \\ H_\varepsilon(2) + H'_\varepsilon(2)(x-2) + \frac{1}{2} H''_\varepsilon(2)(x-2)^2 + \frac{1 - H_\varepsilon(2)}{\varepsilon^4}(x-2)^3, x \in [2, 2 + \delta_\varepsilon], \\ -H_\varepsilon(l) + H'_\varepsilon(l)(x+l) - \frac{1}{2} H''_\varepsilon(l)(x+l)^2, x \in [-l - \varepsilon, -l]. \end{cases}$$

Observe that for ε small enough, $H'_\varepsilon(l) = -\varepsilon H''_\varepsilon(l)$. Hence $w'_{\varepsilon,l}(-l - \varepsilon) = 0$. Moreover, we have $w'_{\varepsilon,l}(x) \geq 0$.

Lemma 5 *$w_{\varepsilon,l}$ is a subsolution:*

$$-w''_{\varepsilon,l} + \frac{1}{2} F'_\varepsilon(w_{\varepsilon,l}) \leq 0, \text{ for } x \in [-l - \varepsilon, 2 + \delta_\varepsilon].$$

Proof. We first prove this in the interval $[2, 2 + \delta_\varepsilon]$. For $s \in \left[1 - \frac{\varepsilon}{2}, 1\right]$,

$$F_\varepsilon(s) = \varepsilon^{-2}(1 - s)^2,\ F'_\varepsilon(s) = 2\varepsilon^{-2}(s - 1).$$

It follows that

$$
\begin{aligned}
\frac{1}{2}F_{\varepsilon}'\left(w_{\varepsilon,l}\left(x\right)\right) &= \varepsilon^{-2}\left(w_{\varepsilon,l}\left(x\right)-1\right) \\
&= \varepsilon^{-2}\left[H_{\varepsilon}\left(2\right)-1+H_{\varepsilon}'\left(2\right)\left(x-2\right)+\frac{1}{2}H_{\varepsilon}''\left(2\right)\left(x-2\right)^{2}+a\left(x-2\right)^{3}\right].
\end{aligned}
$$

On the other hand, we compute

$$
w_{\varepsilon,l}''\left(x\right)=H_{\varepsilon}''\left(2\right)+\frac{1-H_{\varepsilon}\left(2\right)}{\varepsilon^{4}}6\left(x-2\right).
$$

Then using the fact that $\delta_{\varepsilon}=O\left(\varepsilon^{\frac{4}{3}}\right)$, we find that for $x\in\left[2,2+\delta_{\varepsilon}\right]$,

$$
\begin{aligned}
&-w_{\varepsilon,l}''+\frac{1}{2}F_{\varepsilon}'\left(w_{\varepsilon,l}\right) \\
&=-H_{\varepsilon}''\left(2\right)-\frac{1-H_{\varepsilon}\left(2\right)}{\varepsilon^{4}}6\left(x-2\right) \\
&+\varepsilon^{-2}\left[H_{\varepsilon}\left(2\right)-1+H_{\varepsilon}'\left(2\right)\left(x-2\right)+\frac{1}{2}H_{\varepsilon}''\left(2\right)\left(x-2\right)^{2}+\frac{1-H_{\varepsilon}\left(2\right)}{\varepsilon^{4}}\left(x-2\right)^{3}\right] \\
&=\left(x-2\right)\left[-6\frac{1-H_{\varepsilon}\left(2\right)}{\varepsilon^{4}}+\varepsilon^{-2}\left(H_{\varepsilon}'\left(2\right)+\frac{1}{2}H_{\varepsilon}''\left(2\right)\left(x-2\right)+\frac{1-H_{\varepsilon}\left(2\right)}{\varepsilon^{4}}\left(x-2\right)^{2}\right)\right] \\
&\leq 0,
\end{aligned}
$$

provided that ε is small enough.

Next we consider the case of $x\in\left[-l-\varepsilon,-l\right]$. In this case, we have

$$
\begin{aligned}
\frac{1}{2}F_{\varepsilon}'\left(w_{\varepsilon,l}\left(x\right)\right) &= \varepsilon^{-2}\left(w_{\varepsilon,l}\left(x\right)+1\right) \\
&= \varepsilon^{-2}\left[1-H_{\varepsilon}\left(l\right)+H_{\varepsilon}'\left(l\right)\left(x+l\right)-\frac{1}{2}H_{\varepsilon}''\left(l\right)\left(x+l\right)^{2}\right].
\end{aligned}
$$

Moreover, $w_{\varepsilon,l}''\left(x\right)=-H_{\varepsilon}''\left(l\right)$. Then using the fact that $H''\left(l\right)=\varepsilon^{-2}\left(H_{\varepsilon}\left(l\right)-1\right)$, we get

$$
\begin{aligned}
-w_{\varepsilon,l}''\left(x\right)+\frac{1}{2}F_{\varepsilon}'\left(w_{\varepsilon,l}\left(x\right)\right) &= \varepsilon^{-2}\left[H_{\varepsilon}'\left(l\right)\left(x+l\right)-\frac{1}{2}H_{\varepsilon}''\left(l\right)\left(x+l\right)^{2}\right] \\
&= \varepsilon^{-2}H_{\varepsilon}'\left(l\right)\left(x+l\right)\left(1+\frac{x+l}{2\varepsilon}\right)\leq 0.
\end{aligned}
$$

The proof is finished. ■

As we mentioned before, from Section 2 to Section 5, we will deal with the case of dimension $n=3$. Recall that in the coordinate $\left(r,z\right)$, where $r=\sqrt{x_1^2+x_2^2}$, the catenoids are given by $\epsilon r=\cosh\left(\epsilon z\right)$, with $\epsilon>0$ being a parameter. They are classical minimal surfaces, and can also be described by $\epsilon z=\operatorname{arccosh}\left(\epsilon r\right)$.

Let $k > 0$ be a parameter. For each a large, let

$$\Omega_a := \{(r, z) : r \in [0, a], z \in [0, b_\varepsilon]\},$$

where

$$b_\varepsilon = k \operatorname{arccosh}\left(k^{-1}a\right) + 2 + \delta_\varepsilon. \tag{13}$$

Set $L_a := L_{1,a} \cup L_{2,a}$, where

$$L_{1,a} := \{(a, z) : z \in [0, b_\varepsilon]\}, \quad \text{and} \quad L_{2,a} := \{(r, b_\varepsilon) : r \in [0, a]\}.$$

For fixed k, we then define a function $\omega = \omega(r, z)$, depending on the parameter ε and a, to be

$$\omega(r, z) = w_{\varepsilon, k \operatorname{arccosh}(k^{-1}a) - \varepsilon}\left(z - k \operatorname{arccosh}\left(k^{-1}a\right)\right).$$

Although eventually we are interested in solutions of the free boundary problem in the whole space $\mathbb{R}^3$, it will be crucial to study solutions $u = u(r, z)$ of the following regularized problem in the bounded cylindrical domain Ω_a, with mixed boundary condition:

$$\begin{cases} -\partial_r^2 u - \frac{1}{r}\partial_r u - \partial_z^2 u + \frac{1}{2}F_\varepsilon'(u) = 0 \text{ in } \Omega_a, \\ \partial_r u(0, z) = 0, \partial_z u(r, 0) = 0, \\ u = \omega, \text{ on } L_a. \end{cases} \tag{14}$$

2.1 Solutions of the regularized problems in Ω_a with relatively small energy

For each a large, we would like to construct a mountain pass type solution for the regularized problem (14). We will first of all look for two solutions u_1, u_2 with relatively small energy. Minimaxing in suitable class of paths of functions connecting u_1 and u_2, we then obtain a mountain pass solution. Intuitively, u_1 will have nodal set almost parallel to the horizontal x_1-x_2 plane, while the nodal set of u_2 will be close to a vertical cylinder. Similar construction has been carried out in [17] for the Allen-Cahn equation in the two dimensional case.

2.1.1 A solution with almost horizontal nodal set

For fixed ε, a, consider the following initial value problem for the function $u = u(t; r, z)$:

$$\begin{cases} \partial_t u = \Delta u - \frac{1}{2}F_\varepsilon'(u) \text{ in } \Omega_a \times (0, T), \\ \partial_r u(t; 0, z) = 0, \partial_z u(t; r, 0) = 0, \\ u|_{L_a} = \omega, \\ u(0; r, z) = \omega(r, z). \end{cases} \tag{15}$$

Since the constant function ± 1 solves the equation

$$\partial_t u = \Delta u - \frac{1}{2}F_\varepsilon'(u_\varepsilon),$$

we infer that the solution u of (15) satisfying $-1<u<1$. Hence the L^{∞} norm of the solution does not blow up in finite time and by parabolic regularity, the solution can be extended to the whole time interval $(0,+\infty)$.

Let us set

$$E(u):=\int_{\Omega_a}\left[|\nabla u|^2+F_{\varepsilon}(u)\right]\geq 0.$$

Lemma 6 *There exists a sequence $t_n\to+\infty$, such that $u(t_n;\cdot)$ converges to a solution u_1 of the problem*

$$\begin{cases}\Delta u-\frac{1}{2}F_{\varepsilon}'(u)=0,\\ \partial_r u(t;0,z)=0,\partial_z u(t;r,0)=0,\\ u|_{L_a}=\omega.\end{cases}\tag{16}$$

Proof. Direct computation yields

$$E'(u(t))=-2\int_{\Omega_a}|\partial_t u|^2\leq 0.$$

Hence $E(u(t))$ is decreasing and uniformly bounded. It also follows that

$$\int_0^{+\infty}\int_{\Omega_a}|\partial_t u|^2<+\infty.$$

Hence there exists a sequence $t_n\to+\infty$ such that

$$\int_{\Omega_a}|\partial_t u(t_n;\cdot)|^2\to 0.$$

We then get a P.S. sequence(in the natural functional space $H^{0,1}$, see [17] for related discussion) $\{u(t_n;\cdot)\}$ for the functional E(i.e., $E(u(t_n;\cdot))\leq C$, $dE[u(t_n;\cdot)]\to 0$). Since E satisfies the P.S. condition, using standard variational arguments, we may extract a subsequence converging to a solution u_1 of (16). ■

Lemma 7 *u_1 is monotone in the following sense:*

$$\partial_z u_1>0 \text{ and } \partial_r u_1<0,\ \text{ in } \Omega_a.$$

Proof. The fact that $\partial_z u_1>0$ follows from the moving plane argument. It remains to prove $\partial_r u_1<0$.

By Lemma 5, we know that ω is a subsolution:

$$-\omega''+\frac{1}{2}F_{\varepsilon}'(\omega)\leq 0.$$

In particular,

$$\partial_t\omega-\Delta\omega+\frac{1}{2}F_{\varepsilon}'(\omega)\leq 0.$$

Since $u(0;r,z)=\omega(r,z)$, parabolic comparison principle (cf. [33, Proposition 52.6]) tells us that $u(t;\cdot)\geq\omega(\cdot)$ in Ω_a, for all $t\geq 0$. This then implies that $\partial_r u<0$ on $L_{1,a}$ for any t. Now the function $\phi:=\partial_x u$ satisfies

$$\partial_t\phi-\Delta\phi+\frac{1}{2}F_\varepsilon''(u)\,\phi=0$$

and $\phi(0;\cdot)=0$ and

$$\phi(t;\cdot)\leq 0 \text{ on } \partial\{\Omega_a\cap\{x>0\}\}.$$

Hence by the parabolic maximum principle (cf. [33, Proposition 52.4]), $\phi(t;\cdot)\leq 0$ on Ω_a. This proves the monotonicity of u_1 in r. ■

2.1.2 A solution with almost vertical nodal set

We shall construct a second solution u_2 whose nodal set is close to a vertical cylinder. The energy of u_2 will be less than that of u_1. To show the existence of u_2, we still use the parabolic flow.

Let $U_2>u_1$ be a function such that $\partial_r U_2\leq 0$, $\partial_z U_2\geq 0$, and

$$E(U_2)\leq 10ka\ln a.$$

Roughly speaking, we can construct U_2 whose nodal sets are almost vertical, and locally in the direction transverse to the nodal set, it looks like the one dimensional solution H_ε. Now consider the solution u of the problem

$$\begin{cases}\partial_t u=\Delta u-\frac{1}{2}F_\varepsilon'(u)\,,t\in(0,+\infty)\,,\\ \partial_r u(t;0,z)=0,\partial_z u(t;r,0)=0,\\ u|_{L_a}=\omega,\\ u(0;r,z)=U_2.\end{cases}$$

Similarly as before, we can show that there is a sequence $t_n\to+\infty$, such that $u(t_n,\cdot)$ converges to a solution u_2 of (14). Since $U_2>\omega$, by the comparison principle, we have $u_2>u_1$. We also have

$$\partial_z u_2>0 \text{ and } \partial_r u_2<0 \text{ in } \Omega_a.$$

2.2 Mountain pass type solutions

We have so far obtained two solutions u_1, u_2, with $u_1<u_2$. Now we would like to construct a mountain pass type solution using u_1 and u_2. Let $\mathcal{E}$ be the set of C^1 functions ϕ satisfying the following properties:
(I) $u_1<\phi<u_2$ in Ω_a,
(II) $\partial_z\phi>0;\partial_r\phi<0$, in Ω_a,
(III) $\phi|_{L_a}=\omega$,
(IV)$\partial_r\phi(0,z)=0,\partial_z\phi(r,0)=0.$

To proceed, we define

$$e_\varepsilon = \int_{\mathbb{R}} \left[H_\varepsilon'^2 + F_\varepsilon \left(H_\varepsilon \right) \right] = 2 \int_{-1}^{1} \sqrt{F_\varepsilon \left(s \right)} ds.$$

Note that $e_\varepsilon \to 4$, as $\varepsilon \to 0$. Let ε_0 be a fixed small positive constant. For each $\varepsilon \in (0, \varepsilon_0)$, we can construct a family of C^1 functions $\eta_\varepsilon^* \left(s; r, z \right)$ depending continuously on s, such that $\eta_\varepsilon^* \left(s; \cdot \right) \in \mathcal{E}$ for any $s \in [0, 1]$ and

$$\eta_\varepsilon^* \left(0; \cdot \right) = u_1, \eta_\varepsilon^* \left(1; \cdot \right) = u_2.$$

Moreover, we require $\partial_s \eta_\varepsilon^* \left(s; r, z \right) \geq 0$, and

$$\max_{s \in [0,1]} E \left(\eta_\varepsilon^* \left(s \right) \right) \leq \frac{a^2 e_\varepsilon}{2} + \frac{e_\varepsilon}{2} k^2 \ln a + C. \tag{17}$$

We may also assume that $\left| \nabla_{(r,z)} \eta_\varepsilon^* \left(s \right) \right|$ is uniformly bounded for $s \in [0, 1]$ and $\varepsilon \in (0, \varepsilon_0)$. The existence of this family of solutions essentially follow from geometric properties of catenoids.

Now we shall consider the solution $u = u^* \left(t; s; r, z \right)$ of the initial value problem

$$\begin{cases} \partial_t u^* = \Delta u^* - \frac{1}{2} F_\varepsilon' \left(u^* \right), t \in (0, +\infty), \\ \partial_r u^* \left(t; s; 0, z \right) = 0, \partial_z u^* \left(t; s; r, 0 \right) = 0, \\ u^* \left(t; s; \cdot \right) |_{L_a} = \omega, \\ u^* \left(0; s; \cdot \right) = \eta_\varepsilon^* \left(s; \cdot \right). \end{cases}$$

Using the order preserving property of the parabolic flow, we know that for each $t \geq 0$ and $s \in [0, 1]$, $u^* \left(t; s; \cdot \right) \in \mathcal{E}$. Moreover, $\partial_s u^* \left(t; s; \cdot \right) \geq 0$.

Let

$$P = \left\{ u^* \left(t; \cdot \right) : t \in [0, +\infty) \right\}.$$

We define

$$c^* = \min_{\eta \in P} \max_{s \in [0,1]} E \left(\eta \left(s; \cdot \right) \right).$$

The following lemma gives us the upper bound on c^*.

Lemma 8 *There exists a constant C independent of a and ε, such that*

$$c^* \leq \frac{a^2 e_\varepsilon}{2} + \frac{e_\varepsilon}{2} k^2 \ln a + C.$$

Proof. This follows directly from the property (17) of η_ε^* and the fact that the energy E is decreasing along the parabolic flow. ∎

To prove the existence of mountain pass solution, we need to get a lower bound for c^*. It turn out that the estimate of the lower bound is much more delicate.

Lemma 9 *Suppose* $r_0 \in [k, a]$. *Let* ξ *be a* C^1 *function defined on* $[r_0, a]$ *such that* $\xi(r_0) = 0$ *and* $\xi(a) = k \operatorname{arccosh}\left(k^{-1}a\right)$. *Then*

$$\int_{r_0}^{a} \sqrt{1 + \xi'^2(r)}\, r dr \geq \frac{1}{2}a^2 - \frac{1}{2}r_0^2 + \frac{k^2}{2}\ln a - C_k,$$

where C_k *is independent of* r_0 *and* a.

Proof. Define a new function

$$\bar{\xi}(r) := \begin{cases} \xi(r), r \in [r_0, a], \\ 0, r \in [k, r_0]. \end{cases}$$

Then using the fact that the function $g(r) := k \operatorname{arccosh}\left(k^{-1}r\right)$ represents a minimal surface (a catenoid) and hence it has minimizing area, we get

$$\begin{aligned} \int_{k}^{a} \sqrt{1 + \bar{\xi}'^2(r)}\, r dr &\geq \int_{k}^{a} \sqrt{1 + g'^2(r)}\, r dr \\ &= \int_{0}^{k \operatorname{arccosh}\left(k^{-1}a\right)} \sqrt{1 + \sinh^2(k^{-1}z)}\, k \cosh\left(k^{-1}z\right) dz \\ &= \frac{k^2}{2}\operatorname{arccosh}\left(k^{-1}a\right) + \frac{a^2}{2}\sqrt{1 - k^2 a^{-2}}. \end{aligned}$$

Since $\int_k^{r_0} \sqrt{1 + \bar{\xi}'^2(r)}\, r dr = \frac{1}{2}r_0^2 - \frac{1}{2}k^2$, we then get

$$\int_{r_0}^{a} \sqrt{1 + \bar{\xi}'^2(r)}\, r dr \geq \frac{a^2}{2} - \frac{r_0^2}{2} + \frac{k^2}{2}\ln a - C_k.$$

This is the desired estimate. ■

Proposition 10 *For* ε *small enough, there exists a constant* C *independent of* a, ε, *such that*

$$c^* \geq \frac{1}{2}a^2 e_\varepsilon + \frac{k^2}{20}\ln a - C.$$

Proof. Let $\eta \in P$. Since η is a continuous family of C^1 functions from u_1 to u_2, we know that there is a $s_0 \in (0, 1)$, such that the function $u(\cdot) := \eta(s_0; \cdot)$ is equal to 0 at the point $\left(k, \frac{k}{10}\ln a\right)$. We introduce the notation

$$\Omega_a^- = \{X \in \Omega_a : u(X) < 0\},$$

$$\Omega_a^+ = \{X \in \Omega_a : u(X) > 0\}.$$

By the coarea formula, we have

$$\begin{aligned} \int_{\Omega_a^+} \left[|\nabla u|^2 + F_\varepsilon(u)\right] &\geq 2\int_{\Omega_a^+} \left[|\nabla u| \sqrt{F_\varepsilon(u)}\right] \\ &= 2\int_0^1 A(s)\sqrt{F_\varepsilon(s)}\, ds, \end{aligned}$$

where

$$A(s) = \text{Area of } \{X : u(X) = s\}.$$

Since u is monotone in r and z, we deduce that for $s \in (0,1)$,

$$A(s) \geq \frac{1}{2}a^2 - C,$$

where C does not depend on a and ε. Hence

$$\int_{\Omega_a^+} \left[|\nabla u|^2 + F_\varepsilon(u) \right] \geq \frac{1}{4} e_\varepsilon a^2 - C. \tag{18}$$

Next we estimate the energy in the region Ω_a^-, which is more involved. For $r \geq 0$, we define $s = u(r,0)$. It is a function of r. Applying Lemma 9, we infer that for $s \leq \min\{0, u(0,0)\}$,

$$A(s) \geq \frac{1}{2}a^2 - \frac{1}{2}r^2 + \frac{k^2}{2}\ln a - C.$$

Using this estimate, we find that

$$\begin{aligned}
&\int_{\Omega_a^-} \left[|\nabla u|^2 + F_\varepsilon(u) \right] \geq \int_{-1}^{0} A(s)\sqrt{F_\varepsilon(s)}ds \\
&\geq \int_{\min\{0,u(0,0)\}}^{0} A(s)\sqrt{F_\varepsilon(s)}ds \\
&+ \left(\frac{1}{2}a^2 + \frac{k^2}{2}\ln a \right) \int_{-1}^{\min\{0,u(0,0)\}} \sqrt{F_\varepsilon(s)}ds \\
&- \frac{1}{2}\int_{-1}^{\min\{0,u(0,0)\}} r^2\sqrt{F_\varepsilon(s)}ds - C.
\end{aligned} \tag{19}$$

We would like to estimate the last integral. For this purpose, define a new function $\phi(r) := F_\varepsilon(u(r,0)) = F_\varepsilon(s)$. We distinguish two possibilities.

Case 1.

$$\int_0^a \phi(r)\, r dr > \frac{k^2}{10}\ln a. \tag{20}$$

In this case, we have

$$\begin{aligned}
E(u) &= \int_{\Omega_a \cap \{z>1\}} \left[|\nabla u|^2 + F_\varepsilon(u) \right] \\
&+ \int_{\Omega_a \cap \{0<z<1\}} \left[|\nabla u|^2 + F_\varepsilon(u) \right] \\
&\geq \frac{a^2}{2}e_\varepsilon + \int_{\Omega_a \cap \{0<z<1\}} F_\varepsilon(u) - C.
\end{aligned}$$

Due to the monotonicity of u in the r and z direction, we have

$$\int_{\Omega_a \cap \{0<z<1\}} F_\varepsilon(u) \geq \int_{\Omega_a \cap \{0<z<1\}} F_\varepsilon(u(r,0))$$

$$= \int_0^a \phi(r)\, r dr.$$

It then follows from (20) that

$$E(u) \geq \frac{a^2}{2} e_\varepsilon + \frac{k^2}{10} \ln a - C.$$

This is the desired estimate.

Case 2.

$$\int_0^a \phi(r)\, r dr \leq \frac{k^2}{10} \ln a. \tag{21}$$

In this case, we write

$$\int_{-1}^{\min\{0,u(0,0)\}} r^2 \sqrt{F_\varepsilon(s)} ds = \int_{-1}^{-1+\frac{\varepsilon}{2}} r^2 \sqrt{F_\varepsilon(s)} ds + \int_{-1+\frac{\varepsilon}{2}}^{\min\{0,u(0,0)\}} r^2 \sqrt{F_\varepsilon(s)} ds.$$

Let us estimate these two integrals separately.

Recall that when $s \in \left[-1, -1+\frac{\varepsilon}{2}\right]$, $\phi(r) = F_\varepsilon(s) = \varepsilon^{-2}(s+1)^2$. Let $\bar{t}$ be the point where $u(\bar{t}, 0) = -1 + \frac{\varepsilon}{2}$. Then

$$\int_{-1+\frac{\varepsilon}{2}}^{\min\{0,u(0,0)\}} r^2 \sqrt{F_\varepsilon(s)} ds \leq \bar{t}^2. \tag{22}$$

On the other hand, using the monotonicity of ϕ and (21), we get

$$\phi(r) r^2 \leq 2 \int_0^r \phi(t)\, t dt \leq \frac{k^2}{5} \ln a, \text{ for any } t \in [0, a]. \tag{23}$$

This together with $\phi(\bar{t}) = \frac{1}{2}$ tells us that $\bar{t}^2 \leq \frac{2k^2}{5} \ln a$. Hence in view of (22), we find that

$$\int_{-1+\frac{\varepsilon}{2}}^{\min\{0,u(0,0)\}} r^2 \sqrt{F_\varepsilon(s)} ds \leq \bar{t}^2 \leq \frac{2k^2}{5} \ln a. \tag{24}$$

Next, we compute

$$\begin{aligned}\int_{-1}^{-1+\frac{\varepsilon}{2}} r^2 \sqrt{F_\varepsilon(s)} ds &= -\frac{\varepsilon}{2} \int_{\bar{t}}^a r^2 \phi'(r)\, dr \\ &= -\frac{\varepsilon}{2}\left(\phi(a) a^2 - \phi(\bar{t}) \bar{t}^2\right) + \varepsilon \int_{\bar{t}}^a \phi(t)\, t dt.\end{aligned}$$

Applying (21) and (23), we get

$$\int_{-1}^{-1+\frac{\varepsilon}{2}} r^2 \sqrt{F_\varepsilon(s)} ds \leq \frac{2k^2\varepsilon}{5} \ln a. \tag{25}$$

Combining (18), (19), (24), (25), we obtain

$$E(u) \geq \int_{\Omega_a^-} \left[|\nabla u|^2 + F_\varepsilon(u)\right] + \int_{\Omega_a^+} \left[|\nabla u|^2 + F_\varepsilon(u)\right]$$

$$
\begin{aligned}
&\geq \frac{1}{2}a^2 e_\varepsilon + \frac{k^2}{2} e_\varepsilon \ln a - \frac{1}{2}\int_{-1}^{\min\{0,u(0,0)\}} r^2 \sqrt{F_\varepsilon (s)} ds - C \\
&\geq \frac{1}{2}a^2 e_\varepsilon + \frac{k^2}{2} e_\varepsilon \ln a - \frac{k^2}{5}\ln a - \frac{k^2 \varepsilon}{5}\ln a - C \\
&\geq \frac{1}{2}a^2 e_\varepsilon + \frac{k^2}{20}\ln a - C,
\end{aligned}
$$

provided that ε is small enough. This finishes the proof. ■

Remark 11 *For the purpose of obtaining a mountain pass solution, one only need to prove the estimate*

$$
c^* \geq \frac{1}{2}a^2 e_\varepsilon + \delta,
$$

for some universal constant δ. See Lemma 33 for the corresponding estimate in the higher dimensional case.

Proposition 12 *Let a be large enough. Then there exists a mountain pass solution $U_\varepsilon = U_{\varepsilon,a}$ to* (14) *. Moreover, $\partial_r U_\varepsilon > 0, \partial_z U_\varepsilon < 0$ in Ω_a.*

Proof. By Proposition 10,

$$
c^* \geq \frac{1}{2}a^2 e_\varepsilon + \frac{k^2}{20}\ln a - C > \max_{i=1,2} E(u_i), \tag{26}
$$

provided that a is sufficiently large. Standard arguments in variational methods yield the existence of a solution $U_{\varepsilon,a}$ whose energy is equal to c^*. ■

3 Asymptotic analysis of $\{U_\varepsilon\}$ and regularity of the free boundary of the limiting solution

For each fixed large constant a, we have obtained a family of solutions U_ε to the regularized problem. Using arguments of Section 1.2 of Caffarelli-Salsa[9], we can show that $|\nabla U_{\varepsilon,a}| \leq C$. Therefore, $U_{\varepsilon,a}$ converges in $C^{0,\alpha}(\Omega_a)$ to a function V_a. Since F_ε converges on any compact subinterval of $(-1,1)$ to 1, V_a is a harmonic function in the region $\Xi_a := \{|V_a| < 1\} \cap \Omega_a$. Recall that $U_{\varepsilon,a}$ is monotone, hence

$$
\partial_r V_a < 0 \text{ and } \partial_z V_a > 0 \text{ in } \Xi_a.
$$

In this section, we show that V_a satisfies the free boundary condition $|\nabla V_a| = 1$ on $\partial\{|V_a| < 1\} \cap \Omega_a$ in the classical sense. Let us introduce the notation

$$
\digamma_a := \partial \Xi_a \cap \Omega_a.
$$

We also define

$$
\digamma_a^+ = \digamma_a \cap \{V_a = 1\}, \ \digamma_a^- = \digamma_a \cap \{V_a = -1\}. \tag{27}
$$

To investigate the regularity property of the free boundary $\digamma_a$, the first step is to show that the free boundary is nondegenerated in the sense of [1]. We use $B_\rho(X)$ to denote the ball of radius ρ with center X in $\mathbb{R}^3$.

Lemma 13 *Let $x_0 = (r_0, z_0) \in F_a^+$ with $z_0 > 0$. Let $\rho < \frac{1}{2}$. For any ball $B_\rho \subset B_{\frac{z_0}{2}}(x_0)$, then*

$$\rho^{-3}\int_{\partial B_\rho} V_a \geq C > 0.$$

Proof. Checking the details of the proof of Lemma 3.4 in [1], we find that to prove this nondegeneracy property, we need to show the local minimizing property of V_a, i.e. compare the energy of V_a with another carefully chosen test function larger than V_a. To do this, we shall use suitable minimizing property of the function U_ε and sending ε to 0.

Let B_ρ be a ball of radius ρ in $B_{\frac{z_0}{2}}(x_0)$. For each fixed small ε, consider the smooth family of functions $U_\varepsilon(r, z-k)$, with

$$0 \leq k < b_\varepsilon - z_0 - \rho,$$

where b_ε is the constant appeared in (13). Since U_ε is monotone in z, we have

$$U_\varepsilon(r, z-k_1) < U_\varepsilon(r, z-k_2), \text{ if } k_1 < k_2.$$

Using this monotone family of functions, we can construct a calibration, using the theory developed in [4]. The arguments of Theorem 4.5 in [4] then tell us that

$$\int_{B_\rho}\left[|\nabla U_\varepsilon|^2 + F_\varepsilon(U_\varepsilon)\right] \leq \int_{B_\rho}\left[|\nabla \eta|^2 + F_\varepsilon(\eta)\right], \tag{28}$$

for any smooth function η satisfying $\eta = U_\varepsilon$ on ∂B_ρ, and

$$U_\varepsilon \leq \eta \leq U_\varepsilon(r, z - b_\varepsilon + z_0 + \rho).$$

We observe that due to monotonicity,

$$U_\varepsilon(r, z - b_\varepsilon + z_0 + \rho) \geq 1 - \varepsilon^2. \tag{29}$$

Following Alt-Caffarelli ([1]), we define

$$g_\beta(X) = \beta(\ln|X| - \ln\beta),$$

$$w_\varepsilon(X) = \min\left\{c_0 g_{\frac{\rho}{4}}(X - x_0), 1 - \varepsilon^2\right\},$$

and let $W_\varepsilon = \max\{U_\varepsilon, w_\varepsilon\}$. Here c_0 is the maximum constant choose such that $w_\varepsilon \leq U_\varepsilon$ on ∂B_ρ.

Since $U_\varepsilon \leq W_\varepsilon$ and $U_\varepsilon = W_\varepsilon$ on ∂B_ρ, by (28),

$$\int_{B_\rho}\left[|\nabla U_\varepsilon|^2 + F_\varepsilon(U_\varepsilon)\right] \leq \int_{B_\rho}\left[|\nabla W_\varepsilon|^2 + F_\varepsilon(W_\varepsilon)\right].$$

On the other hand, for any subdomain $\Omega \subset B_\rho$,

$$\int_\Omega\left[|\nabla V_a|^2 + \chi_{(-1,1)}(V_a)\right] \leq \liminf_{\varepsilon\to 0}\int_\Omega\left[|\nabla U_\varepsilon|^2 + F_\varepsilon(U_\varepsilon)\right]. \tag{30}$$

Letting $\varepsilon \to 0$, using (30) in the region where $w_\varepsilon \geq U_\varepsilon$, we obtain

$$\int_{B_\rho} \left[|\nabla V_a|^2 + \chi_{(-1,1)}(V_a) \right] \leq \int_{B_\rho} \left[|\nabla W|^2 + \chi_{(-1,1)}(W) \right]. \tag{31}$$

Once (31) is proved, we may proceed as Lemma 3.4 of [1] to conclude the proof.
■

Next we study the nondegeneracy around Γ_a^-.

Lemma 14 *Let $x_0 = (r_0, z_0) \in \Gamma_a^-$. Suppose there exists $\delta > 0$ such that*

$$\Gamma_a^- \cap \{(r,z) : r \in [r_0 - \delta, r_0 + \delta]\} \subset \{(r,z) : z > 2\delta\}.$$

Then for any ball $B_\rho \subset B_\delta(x_0)$, if V_a is not identically zero in B_ρ, we have

$$\rho^{-3} \int_{\partial B_\rho} V_a \geq C > 0.$$

Proof. Let B_ρ be the ball of radius ρ in $B_\delta(x_0)$ with center (r_*, z_*). Consider the family of functions $U_\varepsilon(r, z-k)$, with $-(z_* - \rho) < k \leq 0$. Due to monotonicity,

$$U_\varepsilon(r, z-k_1) \leq U_\varepsilon(r, z-k_2), \text{ if } k_1 < k_2.$$

The same arguments as Lemma 14 yield

$$\int_{B_\rho} \left[|\nabla U_\varepsilon|^2 + F_\varepsilon(U_\varepsilon) \right] \leq \int_{B_\rho} \left[|\nabla \eta|^2 + F_\varepsilon(\eta) \right],$$

for any smooth function η satisfying $\eta = U_\varepsilon$ on ∂B_ρ, and

$$U_\varepsilon(r, z + z_* - \rho) \leq \eta \leq U_\varepsilon \text{ in } B_\rho.$$

While in Lemma 14 we know from (29) that the function $U_\varepsilon(r, z - b_\varepsilon + z_0 + \rho)$ is close enough to 1, we do not have similar estimate for $U_\varepsilon(r, z + z_* - \rho)$ up to now. Nevertheless, we would like to show

$$F_\varepsilon[U_\varepsilon(r, z + z_* - \rho)] \to 0 \text{ in } B_\rho, \text{ as } \varepsilon \to 0. \tag{32}$$

Once this is proved, the rest of the proof is same as Lemma 14.

For each $r \in [r_0 - \delta, r_0 + \delta]$, we define

$$d(r) = \inf\left\{ z : (r,z) \in \Gamma^- \right\},$$

and

$$\Lambda = \{(r,z) : r \in [r_0 - \delta, r_0 + \delta], z < d(r)\}.$$

The measure of a set S will be denoted by $|S|$.

We claim that for each fixed constant $K > 0$

$$\lim_{\varepsilon \to 0} |\Lambda \cap \{|F'_\varepsilon(U_\varepsilon)| > K\}| = 0. \tag{33}$$

Suppose this were not true. Then we could find a subsequence $\{\varepsilon_n\}$ tending to 0, and r_1, r_2, z_1, z_2, depending on ε_n, such that

$$F'_{\varepsilon_n}(U(r,z)) > K, \text{ for } (r,z) \in D := (r_1, r_2) \times (z_1, z_2) \subset \Lambda. \tag{34}$$

Moreover, we could assume $|z_2 - z_1| = |r_2 - r_1| = \delta > 0$, where δ is independent of ε. Then in the region D, $\Delta U_{\varepsilon_n} = F'_{\varepsilon_n}(U_{\varepsilon_n}) \geq K$. Let ϕ be a function satisfying

$$\Delta \phi = K \text{ in } D, \ \phi = U_{\varepsilon_n} \text{ on } \partial D.$$

Then

$$-\Delta (U_{\varepsilon_n} - \phi) \leq 0 \text{ in } D, \ U_{\varepsilon_n} - \phi \leq 0 \text{ on } \partial D.$$

Hence $U_{\varepsilon_n} \leq \phi$ in D. In view of the fact that $U_{\varepsilon_n} \to 1$ in D as $\varepsilon_n \to 0$, we get $\phi < -1$ at the center of D. This contradicts with the fact that $U_\varepsilon \geq -1$.

To prove (32) , we first show that for each fixed $(r^*, z^*) \in \Lambda$,

$$\lim_{\varepsilon \to 0} F_\varepsilon (U_\varepsilon (r^*, z^*)) = 0. \tag{35}$$

Assume to the contrary that

$$\limsup_{\varepsilon \to 0} F_\varepsilon (U_\varepsilon (r^*, z^*)) = \xi > 0.$$

Then using (33) , we could infer that in the region $\Lambda^* := \{(r,z) : (r,z) \in \Lambda, r < r^*, z > z^*\}$, ΔU_ε converges pointwise to 0. Hence

$$\Delta V_a = 0 \text{ in } \Lambda^*.$$

This contradicts with the maximum principle. Hence we get (32) .

We remark that once (35) is proven, we can show exponentially decay to -1 in Λ, away from the free boundary points. ■

Having obtained sufficiently fast decay to ± 1 away from the free boundary, we prove that V_a is a variational solution (see [40] on a discussion on this topic).

Lemma 15 *V_a is a variational solution in the sense that*

$$\int_\Omega \left\{ \left(|\nabla V_a|^2 + \chi_{(-1,1)}(V_a) \right) \operatorname{div} \phi - 2 \nabla V_a D\phi (\nabla V_a)^T \right\} = 0. \tag{36}$$

for any $\phi \in C_c^\infty \left(\Omega, \mathbb{R}^3\right)$.

Proof. Since U_ε is C^1 and

$$\operatorname{div} \left[\left(|\nabla U_\varepsilon|^2 + F_\varepsilon (U_\varepsilon) \right) \phi \right] = \left(|\nabla U_\varepsilon|^2 + F_\varepsilon (U_\varepsilon) \right) \operatorname{div} \phi - 2 \nabla U_\varepsilon D\phi (\nabla U_\varepsilon)^T ,$$

we have

$$\int_{\Omega_a} \left\{ \left(|\nabla U_\varepsilon|^2 + F_\varepsilon (U_\varepsilon) \right) \operatorname{div} \phi - 2 \nabla U_\varepsilon D\phi (\nabla U_\varepsilon)^T \right\} = 0.$$

Letting $\varepsilon \to 0$, using the fact that ∇U_ε is uniformly bounded with respect to ε and the exponential decay to 0 away from the free boundary, we get the desired result. ■

Lemma 16 *$\digamma_a^{\pm}$ is a smooth curve away from the origin, and V_a is a solution of the free boundary problem*

$$\begin{cases} \Delta V_a = 0 \text{ in } \Xi_a, \\ |\nabla V_a| = 1 \text{ on } \digamma_a^{\pm}. \end{cases}$$

Moreover, the energy of V_a has the following lower bound estimate:

$$J(V_a) = \int_{\Omega_a} \left[|\nabla V_a|^2 + \chi_{(-1,1)}(V_a) \right] \geq 2a^2 + \frac{k^2}{20} \ln a - C. \tag{37}$$

Proof. Let $X \in \digamma_a^{\pm}$. Suppose first of all that X is not on the z axis. Since V_a is nondegenerated and a variational solution, the Weiss monotonicity formula([38, 39]) and a standard blow up analysis tell us that the blow up limit around X is a cone. Due to rotational symmetry around the z axis, this is a two dimensional cone. Hence it must be trivial. Then the usual regularity theory([5, 6, 7]) of free boundary tells us that around X the free boundary is analytic. Now suppose X is on the z axis and is not the origin. The blow up limit around X will be the cone (2), this contradicts with the monotonicity of V_a in the z direction.

In view of the exponential decay of $\nabla U_{\varepsilon,a}$ to 0 in $\Omega_a \backslash \{|V_a| \leq 1\}$ away from the free boundary, we know that $\nabla U_{\varepsilon,a}$ converges almost everywhere to ∇V_a. Dominated converges theorem then yields

$$\int_{\Omega_a} \left[|\nabla V_a|^2 + \chi_{(-1,1)}(V_a) \right] = \lim_{\varepsilon \to 0} \int_{\Omega_a} \left[|\nabla U_\varepsilon|^2 + F_\varepsilon(U_\varepsilon) \right] \geq 2a^2 + \frac{k^2}{20} \ln a - C.$$

This is (37). ■

4 Asymptotic analysis of $\{V_a\}$

In this section, we show that as $a \to +\infty$, up to a subsequence, V_a converges to a solution W_k of the free boundary problem (6).

We will also have some information of the asymptotic behavior of W_k as r tends to infinity. We will need the following

Lemma 17 *Let u be a solution to (6), with smooth free boundary. Then the mean curvature of the surface $\partial\{|u| < 1\}$ is nonnegative, with respect to the unit normal pointing outwards of $\{|u| < 1\}$.*

Proof. By [37, Proposition 2.1], $|\nabla u| \leq 1$ in $\{|u| < 1\}$. Hence the maximum of $|\nabla u|$ is achieved at the free boundary, then the assertion of this lemma follows from [8, Remark 2]. ■

Lemma 18 *Suppose u is a solution to (6) depending only on r and $|z|$. Assume $\partial_r u < 0$ and $\partial_z u > 0$ in $\Omega = \{(r,z) : z > 0 \text{ and } |u(r,z)| < 1\}$. Let r_0 be a large constant. Suppose that in the region where $r > r_0$,*

$$\partial\Omega \cap \{(r,z) : u(r,z) = 1\} = \{(r,z) : z = f_1(r)\},$$

$$\partial\Omega \cap \{(r,z) : u(r,z) = -1\} = \{(r,z) : z = f_2(r)\}.$$

Then there exist $k > 0$ and $b \in \mathbb{R}$, such that

$$\begin{aligned} f_1(r) - k\ln r - b &\to 0, \\ f_2(r) - k\ln r - b + 2 &\to 0, \end{aligned}$$

as $r \to +\infty$.

Proof. We write

$$\frac{rf_1'(r)}{\sqrt{1+f_1'^2}} = \int_{r_0}^{r} \left[\frac{rf_1'(r)}{\sqrt{1+f_1'^2}}\right]' ds + \frac{r_0 f_1'(r_0)}{\sqrt{1+f_1'^2(r_0)}} := a_1(r). \tag{38}$$

Applying Lemma 17, we get

$$\left[\frac{rf_1'(r)}{\sqrt{1+f_1'^2}}\right]' \geq 0.$$

Hence $a_1(\cdot)$ is positive and monotone increasing. Similarly, applying Lemma 17, we have

$$\frac{rf_2'(r)}{\sqrt{1+f_2'^2}} = \int_{r_0}^{r} \left[\frac{rf_2'(r)}{\sqrt{1+f_2'^2}}\right]' ds + \frac{r_0 f_2'(r_0)}{\sqrt{1+f_2'^2(r_0)}} := a_2(r), \tag{39}$$

where a_2 is monotone decreasing. On the other hand, using the monotonicity of u, we can show that as r tends to infinity, u behaves locally like suitable vertical translation of the one dimensional profile $\mathcal{H}$. (By the De Giorgi type classification result, see [37]). This together with (38), (39) imply that

$$\lim_{r\to+\infty} a_1(r) = \lim_{r\to+\infty} a_2(r) = k \in (0,+\infty).$$

Now we can write

$$\begin{aligned} f_1'(r) &= \frac{a_1}{r}\frac{1}{\sqrt{1-r^{-2}a_1^2}} := \frac{a_1(r)}{r} + \eta_1(r), \\ f_2'(r) &= \frac{a_2}{r}\frac{1}{\sqrt{1-r^{-2}a_2^2}} := \frac{a_2(r)}{r} + \eta_2(r), \end{aligned}$$

where $\eta_i(r) = O\left(r^{-3}\right)$ as $r \to +\infty$. Therefore

$$\begin{aligned} f_1(r) - f_2(r) = &\int_{r_0}^{r} \frac{a_1(s) - a_2(s)}{s} ds + \int_{r_0}^{r} (\eta_1(s) - \eta_2(s))\, ds \\ &+ f_1(r_0) - f_2(r_0). \end{aligned} \tag{40}$$

Observe that $\lim_{r\to+\infty}(f_1(r)-f_2(r))=2$. Then (40) together with the fact that $a_1(r)\le k$ and $a_2(r)\ge k$ tell us that

$$\int_{r_0}^{+\infty}\frac{k-a_1(r)}{s}<+\infty,\ \int_{r_0}^{+\infty}\frac{a_2(r)-k}{s}<+\infty.$$

This in turn implies the existence of b such that

$$f_1(r)-k\ln r-b\to 0,\text{ as } r\to+\infty.$$

The proof is thus completed. ■

Our next task is to show that the distance of the free boundary of V_a to the origin O is uniformly bounded.

Lemma 19 *Let $\Gamma_a^{\pm}$ be defined by* (27) *. There exists a constant C independent of a, such that*

$$\operatorname{dist}\left(O,\Gamma_a^{\pm}\right)\le C.$$

Proof. Assume to the contrary that the conclusion of the lemma were not true. There are three possibilities.

Case 1. $\Gamma_a^-\cap\{(r,z):z=0\}=\varnothing.$

In this case, moving plane argument tells us that V_a is the trivial one dimensional (only depends on z variable) solution. To be more precise, let us consider the family of trivial solutions $\mathcal{H}(z-\beta)$ where β is a parameter. We start with $\beta<0$(sufficiently small) and increase β continuously until their free boundaries touch at some point. Monotonicity of the solution implies that the free boundary of $\mathcal{H}$ and V_a must touch inside Ω_a. Maximum principle then tells us that V_a is the trivial one dimensional solution. But this contradicts with the energy estimate (37) . We remark that actually if the free boundary Γ_a^- intersects with $L_{1,a}$ at a point (a,z_0) with $z_0<k\operatorname{arccosh}\left(k^{-1}a\right)-1$, then at this intersection point, they must touch tangentially (see [25, 27]).

Case 2. $\Gamma_a^+\cap\{(r,z):r=0\}=\varnothing.$

Subcase 1. There exists a universal constant C such that

$$\Gamma_a^+\subset\{(r,z):a-C<r<a\}.$$

Then using the fact that $|\nabla V_a|$ is uniformly bounded in a, we estimate

$$\begin{aligned}J(V_a)&=\int\left[|\nabla V_a|^2+\chi_{(-1,1)}(V_a)\right]\\&\le Ca\ln a,\end{aligned}$$

which contradicts with the energy estimate (37) .

Subcase 2. There is a sequence $\{a_i\}$ tending to infinity and a sequence of points $P_i\in\Gamma_{a_i}^+\cap\{(r,z):z=0\}$, with $|P_i|$ also tending to infinity, such that $\operatorname{dist}(P_i,(a_i,0))\to+\infty$.

In this case, from the construction in [30], we know that there is a family of solutions $\bar{u}_\lambda$ to the free boundary problem (6) whose nodal set is close to the family of rescaled catenoids $z = \lambda \operatorname{arccosh}\left(\lambda^{-1} r\right)$, where λ is a (large) parameter. Moving plane type arguments based on $\bar{u}_\lambda$ then tell us that we can touch V_{a_i} inside Ω_{a_i} with some $\bar{u}_\lambda$. This contradicts with the maximum principle.

Case 3. $\digamma_a^+ \cap \{(r,z) : z = 0\} = \varnothing$ and $\digamma_a^- \cap \{(r,z) : r = 0\} = \varnothing$.

Subcase 1. $\operatorname{dist}\left(O, \digamma_{a_i}^+\right) \to +\infty$, for a sequence $\{a_i\}$.

Let P_a be the intersection of $\digamma_a^+$ with the z axis. Then the sequence of functions $h_{a_i}(\cdot) = V_{a_i}(\cdot - P_{a_i})$ converges in $C^{0,\alpha}$ to a function h_∞. h_∞ is a variational solution in the sense of (36). Each V_{a_i} is nondegenerated, hence the free boundary point of h_∞ is also nondegenerated. Blow up analysis then tells us that the free boundary is regular. From De Giorgi type results, we infer that h_∞ is a one dimensional solution. That is, $h_\infty(r,z) = \mathcal{H}(z+1)$. This contradicts with the monotonicity of V_{a_i} in the z direction.

Subcase 2. $\operatorname{dist}\left(O, \digamma_a^-\right) \to +\infty$.

In this case, we can proceed similarly as Subcase 1. We omit the details. ■

With Lemma 19 understood, we state the following

Proposition 20 *For each $k \in (0, +\infty)$, there exists a solution W_k to the free boundary problem (6) whose nodal set $\{(r,z) : z = f(r)\}$ has the following asymptotic behavior: There exists a constant b_k such that*

$$f(r) - k \ln r - b_k \to 0, \ \text{as } r \to +\infty. \tag{41}$$

Before starting the proof, let us establish the following

Lemma 21 *Fix a constant $\bar{k} > 0$ with $\bar{k} \neq k$. Suppose b and a/b is large. Let ξ be a C^1 function satisfying $\xi(b) = \bar{k} \ln b$ and $\xi(a) = k \ln a$. Then*

$$\int_b^a \sqrt{1+\xi'^2}\, r\,dr \geq \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}\frac{\left(k \ln a - \bar{k}\ln b\right)^2}{\ln a - \ln b} - C,$$

where C does not depend on a, b.

Proof. The points $\left(b, \bar{k}\ln b\right)$ and $(a, k \ln a)$ are on the catenoid $z = \sigma \operatorname{arccosh}\left(\sigma^{-1} r\right) + d := \eta(r)$, where σ, d satisfies

$$\begin{cases} \sigma \operatorname{arccosh}\left(\sigma^{-1} b\right) + d = \bar{k} \ln b, \\ \sigma \operatorname{arccosh}\left(\sigma^{-1} a\right) + d = k \ln a. \end{cases}$$

The existence of σ is guaranteed by the assumption that b and a/b is large. σ has the estimate

$$\sigma = \frac{k \ln a - \bar{k} \ln b}{\ln a - \ln b} + O\left(\frac{1}{(\ln a - \ln b)\, b^2}\right).$$

Using this, we then compute

$$\int_b^a \sqrt{1 + \eta'^2(r)}\, r\,dr = \sigma \int_{\bar{k}\ln b - d}^{k \ln a - d} \cosh^2\left(\sigma^{-1} z\right) dz$$

$$\geq \frac{1}{2}\left(a^2 - b^2\right) + \frac{\sigma}{2}\left(\bar{k}\ln b - k\ln a\right) - C$$
$$\geq \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}\frac{\left(k\ln a - \bar{k}\ln b\right)^2}{\ln a - \ln b} - C,$$

provided that b and a/b is large. The desired estimate of this lemma then follows from the fact that the catenoid is a(parametric in this case) minimal surface. ■

Proof of Proposition 20. We would like to get a uniform estimate for the sequence of solutions V_a independent on a. Once we have this estimate, we can let $a \to +\infty$ and get a solution W_k with desired asymptotic behavior at infinity.

By Lemma 19, a subsequence of $\{V_a\}$ converges in $C^{0,\alpha}_{loc}\left(\mathbb{R}^3\right)$ to a solution W of (6) . Since V_a is monotone, W is also monotone(in both r and z direction). By Lemma 18, there exists $\bar{k} > 0$ and $b_k \in \mathbb{R}$ such that

$$f(r) - \bar{k}\ln r - b_k \to 0, \text{ as } r \to +\infty.$$

It suffices to prove that $\bar{k} = k$.

We argue by contradiction and assume $\bar{k} \neq k$. We would like to show that for a sufficiently large, the energy of V_a satisfies

$$J(V_a) - \limsup_{\varepsilon \to 0} c^* > 0.$$

Fix a large constant A_1 such that in the region $\mathbb{R}^3 \backslash B_{A_1}$, the nodal set of W_k is close to $z = \bar{k}\ln r + b_k$. Then we can estimate

$$\int_{B_{A_1}} \left[|\nabla V_a|^2 + \chi_{(-1,1)}(V_a)\right] = 2A_1^2 + 2\bar{k}^2 \ln A_1 + O(1).$$

On the other hand, the energy outside the ball B_{A_1} satisfies

$$\int_{\Omega_a \backslash B_{A_1}} \left[|\nabla V_a|^2 + \chi_{(-1,1)}(V_a)\right] \geq 2\int_{\Omega_a \backslash B_{A_1}} |\nabla V_a| = 2\int_{-1}^{1} |\{V_a = s\} \cap (\Omega_a \backslash B_{A_1})|\, ds.$$

Using Lemma 21,

$$|\{V_a = s\} \cap (\Omega_a \backslash B_{A_1})| \geq \frac{1}{2}a^2 - \frac{1}{2}A_1^2 + \frac{1}{2}\frac{\left(k\ln a - \bar{k}\ln A_1\right)^2}{\ln a - \ln A_1} - C.$$

Therefore, recalling the upper bound estimate of c^*(Lemma 8), we get

$$J(V_a) - \limsup_{\varepsilon \to 0} c^* \geq 2\frac{\left(k\ln a - \bar{k}\ln A_1\right)^2}{\ln a - \ln A_1} + 2\bar{k}^2 \ln A_1 - 2k^2 \ln a - C$$
$$= 2\frac{\left(k - \bar{k}\right)^2 \ln a \ln A_1}{\ln a - \ln A_1} - C$$
$$\geq 2\left(k - \bar{k}\right)^2 \ln A_1 - C > 0,$$

provided that A_1 is large enough. This is a contradiction ■

5 Asymptotic analysis of $\{W_k\}$

In this section, we first show that as $k \to 0$, W_k converges to the function $|z|-1$ in the region $\{(r,z) : |z| < 2\}$. Then we can perform a rescaling on W_k and prove that the resulted sequence of functions converges to the desired solution of the one phase free boundary problem. Some computations in this section are similar to those in [31]. A main step in the argument is the analysis of the asymptotic behavior of W_k.

We set

$$\digamma = \partial \{|W_k| < 1\}, \digamma^{\pm} = \digamma \cap \{W_k = \pm 1\}.$$

Due to monotonicity of the solution, $\digamma^-$ is represented by the graph of a function p_k :

$$\digamma^- = \{(r,z) : z = p_k(r)\}.$$

Proposition 22 *For $k \in (0,1)$, there exist b_k, such that $|b_k| \leq C$ and*

$$|p_k(r) - k \ln r - b_k| \leq C r^{-1}, \text{ for all } r > r_0,$$

where r_0, C are certain constants independent of k.

For notational convenience, we will not write the subscript k if no confusion will arise. The main difficulty in the proof of Proposition 22 is that although $p''(r) \to 0$ as $r \to +\infty$, which follows from the regularity theory of Kinderlehrer-Nirenberg[28], a priori we do not have any decay information on p''. We use (l,s) to denote the Fermi coordinate around the curve $\digamma^-$. Explicitly, for a given point, the relation between its (l,s) and (r,z) coordinate is given by

$$\begin{cases} r = l - \frac{p'}{\sqrt{1+p'^2}} s, \\ z = p + \frac{1}{\sqrt{1+p'^2}} s, \end{cases}$$

where p, p' is evaluated at l. Since p'' is small, this Fermi coordinate is well defined in a large (depending on p'') tubular neighbourhood of $\digamma^-$. Set

$$\Gamma_h := \left\{X + h\nu(X) : X \in \digamma^-\right\},$$

where ν is a unit normal of $\digamma^-$, pointing upwards. Then $\Gamma_0 = \digamma^-$. We also know that $\digamma^+$ can be written as Γ_h, for a function h close to 2.

Now we define an approximate solution $\bar{W}$ in terms of the Fermi coordinate as

$$\bar{W}(l,s) = \frac{s}{1+f(l)} - 1,$$

where $f = \frac{h}{2} - 1$. Then $\bar{W} = -1$ on $\digamma^-$, and $\bar{W} = 1$ on $\digamma^+$. We write W as $\bar{W} + \phi$ and want to estimate ϕ.

It will be important to estimate the error of the approximate solution $\bar{W}$. We use H_M to denote the mean curvature of a surface M. Then we compute the Laplacian of $\bar{W}$ in the Fermi coordinate

$$\Delta \bar{W} = \Delta_{\Gamma_s} \bar{W} + \partial_s^2 \bar{W} - H_{\Gamma_s} \partial_s \bar{W}$$

$$= \Delta_{\Gamma_s} \bar{W} - \frac{H_{\Gamma_s}}{1+f}.$$

Let us use $k_i, i = 1, 2$, to denote the principle curvatures of $\digamma^-$:

$$k_1 = \frac{p''}{\left(p'^2+1\right)^{\frac{3}{2}}}, \quad k_2 = \frac{p'}{r\sqrt{1+p'^2}}.$$

Then the mean curvature of $\digamma^-$ at the point (r, z) is

$$H = \frac{1}{r}\left(\frac{rp'(r)}{\sqrt{1+p'^2(r)}}\right)'.$$

We will set $|A|^2 = k_1^2 + k_2^2$ and

$$t = \frac{s}{1+f(l)} - 1.$$

Lemma 23 *The Laplacian operator on* Γ_0 *acting on* $\bar{W}$ *satisfies*

$$\Delta_{\Gamma_0} \bar{W} = -t\Delta_{\Gamma_0} f + I_1,$$

where

$$I_1 = -tf\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}\left(\frac{sf^2}{1+f}\right).$$

Proof. We can write

$$\begin{aligned} \bar{W} &= \frac{s}{1+f(l)} - 1 = s\left(1 - f + \frac{f^2}{1+f}\right) - 1 \\ &= s - sf + \frac{sf^2}{1+f} - 1. \end{aligned}$$

We then compute

$$\Delta_{\Gamma_0} \bar{W} = -s\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}\left(\frac{sf^2}{1+f}\right).$$

Inserting the relation $s = t(1+f)$ into the left hand side, we get

$$\begin{aligned} \Delta_{\Gamma_0} \bar{W} &= -t(1+f)\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}\left(\frac{sf^2}{1+f}\right) \\ &= -t\Delta_{\Gamma_0} f - tf\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}\left(\frac{sf^2}{1+f}\right). \end{aligned}$$

This finishes the proof. ■

Lemma 24 *We have the following expansion for the mean curvature of* Γ_s :

$$\frac{H_{\Gamma_s}}{1+f} = \frac{H_{\Gamma_0}}{1+f} + t\left|A\right|^2 + I_2,$$

where

$$I_2 = \frac{1}{1+f}\sum_{i=1}^{2}\frac{s^2 k_i^3}{1-sk_i}.$$

Proof. The mean curvature of the surface Γ_s has the form (see [16]):

$$H_{\Gamma_s} = \sum_{i=1}^{2}\frac{k_i}{1-sk_i} = H_{\Gamma_0} + \sum_{i=1}^{2} sk_i^2 + \sum_{i=1}^{2}\frac{s^2k_i^3}{1-sk_i}.$$

Hence

$$\begin{aligned}\frac{H_{\Gamma_s}}{1+f} &= \frac{H_{\Gamma_0}}{1+f} + \frac{\left|A\right|^2}{1+f}s + \frac{1}{1+f}\sum_{i=1}^{2}\frac{s^2k_i^3}{1-sk_i} \\ &= \frac{H_{\Gamma_0}}{1+f} + t\left|A\right|^2 + \frac{1}{1+f}\sum_{i=1}^{2}\frac{s^2k_i^3}{1-sk_i}.\end{aligned}$$

This finishes the proof. ■

The function ϕ satisfies $\phi = 0$ on $\digamma^{\pm}$. By Lemma 23 and Lemma 24, we have

$$\begin{aligned}\Delta\phi = -\Delta\bar{W} = \frac{H_{\Gamma_0}}{1+f} + \left(\Delta_{\Gamma_0} f + \left|A\right|^2\right)t \\ - I_1 + I_2 + \Delta_{\Gamma_0}\bar{W} - \Delta_{\Gamma_s}\bar{W} \quad \text{in } \left\{\left|W_k\right| < 1\right\}.\end{aligned} \tag{42}$$

Our next purpose is to analyze the boundary condition $\left|\nabla W\right| = 1$. We use $\mathfrak{g}_s^{i,j} = \mathfrak{g}^{i,j}$ to denote the entries of inverse matrix of the metric tensor on Γ_s.

Lemma 25 *On* Γ_0, *we have*

$$\partial_t\phi - f = I_{3,-},$$

where

$$I_{3,-} = \frac{f^2}{2} - \frac{\left(\partial_t\phi\right)^2}{2} - \mathfrak{g}_0^{1,1}\frac{\left(t+1\right)^2 f'^2}{2}. \tag{43}$$

Similarly,

$$\partial_t\phi - f = I_{3,+}, \ \textit{ on } \Gamma_h,$$

where

$$I_{3,+} = -\frac{1}{2}\left(1+\mathfrak{g}^{1,1}h'^2\right)\left(\partial_t\phi\right)^2 + \frac{\mathfrak{g}_h^{1,1}h'}{1+f}\partial_t\phi + \frac{1}{2}f^2 - \frac{1}{2}\mathfrak{g}_h^{1,1}\left(\left(t+1\right)f'\right)^2. \tag{44}$$

Proof. We have

$$\left|\nabla\left(\bar{W}+\phi\right)\right|^2=\left(\partial_s\bar{W}+\partial_s\phi\right)^2+\mathfrak{g}^{1,1}\left(\partial_l\bar{W}+\partial_l\phi\right)^2. \tag{45}$$

Since $\partial_s\bar{W}=\frac{1}{1+f}$, $\partial_l\bar{W}=-\frac{sf'}{(1+f)^2}$, and $\partial_l\phi=0$ on Γ_0, we obtain from $|\nabla W|=1$ that

$$(\partial_s\phi)^2+\frac{2}{1+f}\partial_s\phi+\frac{1}{(1+f)^2}+\mathfrak{g}_0^{1,1}\frac{s^2f'^2}{(1+f)^4}=1,\ \text{ on }\Gamma_0. \tag{46}$$

Observe that $\partial_s\phi=\frac{\partial_t\phi}{1+f}$. Hence

$$(\partial_t\phi)^2+2\partial_t\phi+\mathfrak{g}_0^{1,1}\frac{s^2f'^2}{(1+f)^2}=2f+f^2,\ \text{ on }\Gamma_0.$$

This is (43).

On Γ_h, since $\phi(l,h(l))=0$, we have $\partial_l\phi=-\partial_s\phi h'$. Hence from (45) and $|\nabla W|=1$, we deduce

$$\left(1+\mathfrak{g}_h^{1,1}h'^2\right)(\partial_s\phi)^2+\left(\frac{2}{1+f}-2\mathfrak{g}_h^{1,1}\partial_l\bar{W}h'\right)\partial_s\phi+\frac{1}{(1+f)^2}+\mathfrak{g}_h^{1,1}\frac{s^2f'^2}{(1+f)^4}=1.$$

This is (44). ■

It is expected that the functions f and p'' decays like $O\left(l^{-2}\right)$ as $l\to+\infty$. To prove this, we need to work in a suitable (exponentially weighted, rather than algebraically weighted) functional spaces. Fix an $\alpha\in(0,1)$.

Definition 26 *For $\mu=0,1,2,\ \beta\geq 0,$ the space $\mathcal{B}_{\beta,\mu}$ consists of those functions $\eta=\eta(l)\,,l\in[0,+\infty),$ such that*

$$\|\eta\|_{\beta,\mu}:=\sup_l\left[e^{\beta l}\|\eta\|_{C^{\mu,\alpha}([l,l+1])}\right]<+\infty.$$

Definition 27 *For $\mu=0,1,2,\ \beta\geq 0,$ the space $\mathcal{B}_{\beta,\mu;*}$ consists of those functions $\varphi=\phi(t,l)\,,\,(t,l)\in[-1,1]\times[0,+\infty),$ such that*

$$\|\varphi\|_{\beta,\mu;*}:=\sup_{(t,l)}\left[e^{\beta l}\|\varphi\|_{C^{\mu,\alpha}([-1,1]\times[l,l+1])}\right]<+\infty.$$

Lemma 28 *Let $\delta>0$ be a fixed small constant. Assume $\beta\in[0,\delta]$. Suppose $\eta\in\mathcal{B}_{\beta,0;*}$ and $\Phi\in\mathcal{B}_{\beta,2;*}$ satisfying*

$$\begin{cases}\partial_t^2\Phi+\partial_l^2\Phi+\frac{1}{l}\partial_l\Phi=\eta, & [-1,1]\times[0,+\infty),\\ \Phi(t,l)=0,\ \textit{for } t=\pm1.\end{cases}$$

Then $\|\Phi\|_{\beta,2;*}\leq C\|\eta\|_{\beta,0;*}$.

The proof of Lemma 28 follows from standard arguments, see for instance Lemma 5.1 of [14], where a more complicated situation for the Allen-Cahn equation is studied. We omit the details.

The solution W resembles the one dimensional profile only when r is large, say $r > r_0$. Therefore the function ϕ is only well defined in the region $r > r_0$. Note that r_0 can be chosen to be independent of k. We introduce a cutoff function ζ such that

$$\zeta(s) = \begin{cases} 1, s > 1, \\ 0, s < 0. \end{cases}$$

Slightly abusing the notation, we still write the function ϕ in the (t, l) coordinate as $\phi(t, l)$. For $a > r_0$, let $\Psi_a(t, l) := \zeta_a(l)\,\phi(t, l)$ and $\bar{f}_a = \zeta_a f$, where $\zeta_a(l) = \zeta(l - a)$. Using (42) and Lemma 25, we find that Ψ_a satisfies

$$\begin{cases} \Delta\Psi_a = \zeta_a H_{\Gamma_0} + \left(\Delta_{\Gamma_0} f + |A|^2\right) t\zeta_a + P, \ \ (t, l) \in [-1, 1] \times [0, +\infty), \\ \Psi_a(\pm 1, l) = 0, \\ \partial_t \Psi_a - \bar{f}_a = \gamma_-, \text{for } t = -1, \\ \partial_t \Psi_a - \bar{f}_a = \gamma_+, \ \text{for } t = 1. \end{cases} \tag{47}$$

Here P is a perturbation term and explicitly,

$$P = \left(I_2 - I_1 + \Delta_{\Gamma_0}\bar{W} - \Delta_{\Gamma_s}\bar{W}\right)\zeta_a + 2\nabla\zeta_a\nabla\phi + \Delta\zeta_a\phi,$$

and $\gamma_+ = \zeta_a I_{3,+}$, $\gamma_- = \zeta_a I_{3,-}$.

We need the following linear theory.

Lemma 29 *Suppose $\beta \in [0, \delta]$, with $\delta > 0$ being small. Assume $\eta^{\pm} \in \mathcal{B}_{\beta,1}$, $g_1, g_2 \in \mathcal{B}_{\beta,0}$, and $\vartheta \in \mathcal{B}_{\beta,0,*}$. If Φ satisfies*

$$\begin{cases} \partial_t^2\Phi + \partial_l^2\Phi + \frac{1}{l}\partial_l\Phi = g_1 + t\Delta_{\Gamma_0} g_2 + \vartheta, \ \ (t, l) \in [-1, 1] \times [0, +\infty). \\ \Phi(\pm 1, l) = 0, \\ \partial_t\Phi - g_2 = \eta_-, \textit{for } t = -1, \\ \partial_t\Phi - g_2 = \eta_+, \ \textit{for } t = 1. \end{cases} \tag{48}$$

Then

$$\begin{aligned} \|g_1\|_{\beta,0} &\le C\,\|\eta_+ - \eta_-\|_{\beta,1} + C\,\|\vartheta\|_{\beta,0,*}, \\ \|g_2\|_{\beta,2} &\le C\,\|\eta_+ + \eta_-\|_{\beta,1} + C\,\|\vartheta\|_{\beta,0,*}. \end{aligned}$$

Proof. The proof of this lemma is similar as [30, Proposition 17], using Fourier transform. We sketch the proof for completeness. For each $\xi \in \mathbb{R}^2$, let $q_{1,\xi}, q_{2,\xi}$ solve

$$\begin{cases} q_{1,\xi}''(t) - |\xi|^2 q_{1,\xi}(t) = 1, \\ q_{1,\xi}(-1) = q_{1,\xi}(1) = 0, \end{cases}$$

and

$$\begin{cases} q_{2,\xi}''(t) - |\xi|^2 q_{2,\xi}(t) = t, \\ q_{2,\xi}(-1) = q_{2,\xi}(1) = 0. \end{cases}$$

Explicitly, $q_{1,\xi}$ and $q_{2,\xi}$ are given by

$$q_{1,\xi}(t) = \frac{\cosh(|\xi|\,t)}{|\xi|^2\cosh|\xi|} - \frac{1}{|\xi|^2},$$

$$q_{2,\xi}(t) = \frac{\sinh(|\xi| t)}{|\xi|^2 \sinh|\xi|} - \frac{t}{|\xi|^2}.$$

We first deal with the case of $\vartheta = 0$. Taking Fourier transform in (48)in $\mathbb{R}^2$ with respect to the z_1, z_2, where $l = \sqrt{z_1^2 + z_2^2}$, variable, we are lead to

$$\begin{cases} \partial_t^2 \hat{\Phi} - |\xi|^2 \hat{\Phi} = \hat{g}_1 + \hat{g}_2 t, \ t \in [-1,1], \\ \hat{\Phi}(-1,\xi) = \hat{\Phi}(1,\xi) = 0, \\ \partial_t \hat{\Phi}(-1,\xi) - \hat{g}_2(\xi) = \hat{\gamma}_{-1}(\xi), \\ \partial_t \hat{\Phi}(1,\xi) - \hat{g}_2(\xi) = \hat{\gamma}_1(\xi). \end{cases}$$

It follows that the function $\hat{\Phi}$ has the form

$$\hat{\Phi}(t,\xi) = \hat{g}_1(\xi) q_{1,\xi}(t) + (\Delta_{\Gamma_0} g_2)^{\hat{}}(\xi) q_{2,\xi}(t).$$

In view of the boundary condition at $t = \pm 1$, we get

$$\begin{cases} \hat{g}_1(\xi) q_{1,\xi}'(-1) + (\Delta_{\Gamma_0} g_2)^{\hat{}}(\xi) q_{2,\xi}(-1) - \hat{g}_2(\xi) = \hat{\gamma}_{-1}(\xi), \\ \hat{g}_1(\xi) q_{1,\xi}'(1) + (\Delta_{\Gamma_0} g_2)^{\hat{}} q_{2,\xi}'(1) - \hat{g}_2(\xi) = \hat{\gamma}_1(\xi). \end{cases}$$

Using the symmetry of $q_{1,\xi}$ and $q_{2,\xi}$, we obtain

$$\begin{cases} \hat{g}_1(\xi) = \frac{\hat{\gamma}_1(\xi) - \hat{\gamma}_{-1}(\xi)}{2q_{1,\xi}'(1)}, \\ (\Delta_{\Gamma_0} g_2)^{\hat{}}(\xi) = \frac{2\hat{g}_2(\xi) + \hat{\gamma}_{-1}(\xi) + \hat{\gamma}_1(\xi)}{2q_{2,\xi}'(1)}. \end{cases}$$

Taking inverse Fourier transform, using the analyticity and asymptotic behavior of $q_{1,\xi}'(1)$ and $q_{2,\xi}'(1)$ (it behaves like $|\xi|^{-1}$ as $|\xi| \to +\infty$, see Lemma 16 of [31]), we get the desired weighted norm estimate. Note that in [31], we have considered the algebraically weighted norms, here we are dealing with exponentially weighted norms.

We now turn to the general case $\vartheta \neq 0$. Let us use Φ^* to denote the solution of the problem

$$\begin{cases} \partial_t^2 \Phi^* + \partial_l^2 \Phi^* + \frac{1}{l} \partial_l \Phi^* = \vartheta, \ \ (t,l) \in [-1,1] \times [0,+\infty). \\ \Phi^*(\pm 1, l) = 0. \end{cases}$$

Note that Φ^* decays like $O\left(e^{-\beta l}\right)$. Then we can write $\Phi = \Phi^* + \tilde{\Phi}$ and proceed similarly as before. ■

For function η define on $\mathbb{R}^+$, we use the notation

$$\|\eta\|_s := \|\eta\|_{C^{2,\alpha}([s,+\infty])}, \ \ \|\eta\|_{[c,d]} := \|\eta\|_{C^{2,\alpha}([c,d])}.$$

Moreover, for function $\tilde{\eta}$ define on $[-1,1] \times \mathbb{R}^+$, we set

$$\|\tilde{\eta}\|_{s,\hat{}} := \|g\|_{C^{2,\alpha}([-1,1]\times[s,+\infty))},$$
$$\|\tilde{\eta}\|_{[c,d],\hat{}} := \|g\|_{C^{2,\alpha}([-1,1]\times[c,d))}.$$

Proof of Proposition 22. Let $\beta > 0$ be a fixed small constant. We first observe that in the perturbation term P, the term $2\nabla\zeta_a\nabla\phi + \Delta\zeta_a\phi$ is compactly supported. On the other hand, by the estimate (non-optimal) $|p'| \leq C$, and the formula

$$H_{\Gamma_0} = \frac{1}{l}\left(\frac{lp'}{\sqrt{1+p'^2}}\right)',$$

we find that the rest terms in P can be estimated as

$$\left(I_2 - I_1 + \Delta_{\Gamma_0}\bar{W} - \Delta_{\Gamma_s}\bar{W}\right)\zeta_a = o\left(|H_{\Gamma_0}| + |f| + |\phi| + |f'|\right) + O\left(l^{-2}\right).$$

We use ϕ^* to denote the solution of the problem

$$\begin{cases} \Delta\phi^* = 2\nabla\zeta_a\nabla\phi + \Delta\zeta_a\phi, \\ \phi^*(t,l) = 0, t = \pm 1. \end{cases}$$

Writing Ψ_a as $\phi^* + \tilde{\Psi}_a$ and applying Lemma 28, we obtain

$$\|\phi\|_{a+s,\hat{}} = O\left(e^{-\beta s}\|\phi\|_{[a,a+1],\hat{}}\right) + O\left(\|f\|_a + \|H_{\Gamma_0}\|_{C^{0,\alpha}([a,+\infty])} + a^{-2}\right), s \geq 0. \tag{49}$$

On the other hand, from equation (43) and (44), we get

$$\gamma^- = O\left(f^2 + f'^2 + (\partial_t\phi)^2\right) \text{ and } \gamma^+ = O\left((\partial_t\phi)^2\right) + o(f).$$

Combining this with Lemma 29 and (49), we infer

$$\begin{aligned} &\|f\|_{a+s} + \|H_{\Gamma_0}\|_{C^{0,\alpha}([a+s,+\infty])} \\ &= O\left[e^{-\beta s}\left(\|f\|_{[a,a+1]} + \|\phi\|_{[a,a+1],\hat{}}\right)\right] + O\left(a^{-3}\right), s \geq 0. \end{aligned} \tag{50}$$

Now let us define

$$\theta(a) := \|f\|_a + \|H_{\Gamma_0}\|_{C^{0,\alpha}([a,+\infty])} + \|\phi\|_{a,\hat{}}.$$

Fix a constant d. From (49) and (50), we are led to

$$\theta(a+d) \leq Ce^{-\beta d}\left(\theta(a) + \theta(a-d)\right) + Ca^{-2}. \tag{51}$$

Applying this estimate at $a - dj, j = 0, 1, 2, ..., \left[\frac{a-r_0}{d}\right]$, we get

$$\begin{aligned} \theta(a) &\leq Ce^{-\beta d}\left(\theta(a-d) + \theta(a-2d)\right) + Ca^{-2} \\ &\leq C^2 e^{-\beta d}\left[e^{-\beta d}\left(\theta(a-2d) + \theta(a-3d)\right) + a^{-2}\right] \\ &+ C^2 e^{-\beta d}\left[e^{-\beta d}\left(\theta(a-3d) + \theta(a-4d)\right) + a^{-2}\right] + Ca^{-2} \\ &\leq ... \leq \bar{C}a^{-2}, \end{aligned}$$

for another constant $\bar{C}$, provided that d is sufficiently large. With this decay information at hand, repeating the above arguments, we find that $|f'(a)| +$

$|p'(a)| = O\left(a^{-1}\right)$. Hence the term a^{-2} in (49) can be improved to a^{-3}, and (51) can be refined to

$$\theta(a+d) \leq Ce^{-\beta d}\left(\theta(a) + \theta(a-d)\right) + Ca^{-3}.$$

Similar arguments as before yield $\theta(a) \leq Ca^{-3}$, which in particular implies

$$\|H_{\Gamma_0}\|_{C^{0,\alpha}([a,+\infty])} \leq Ca^{-3}, \text{ for any } a > r_0.$$

Hence the function p_k satisfies

$$\left(\frac{lp_k'}{\sqrt{1+p_k'^2}}\right)' = O\left(l^{-3}\right).$$

Integrating this equation once, we get $p_k'(l) = \frac{\bar{k}}{l} + O\left(l^{-3}\right)$, for some constant $\bar{k}$. Necessarily $\bar{k} = k$. Integrating once more, we get

$$p_k(l) = k \ln l + b_k + O\left(l^{-1}\right).$$

To show that $|b_k| \leq C$, it remains to prove that $|p_k(r_0)| \leq C$. If this were not true, then after suitable translation along the z axis, a subsequence of W_k converges to a solution w of the free boundary problem (6), with $\{(r,z) : |w(r,z)| < 1\} = \{(r,z) : c_1 < r < c_2 < +\infty\}$. This is not possible. The proof is thus completed. ■

Lemma 30 *As $k \to 0$, W_k converges in $C^{0,\alpha}_{loc}$ to the function $\mathcal{H}(z-1)$ in the upper half space.*

Proof. Similar as Lemma 19, we can prove that the distance of the free boundary of W_k to the origin is uniformly bounded for k.

Using Lemma 22, we deduce that W_k converges to a solution W_∞. Since for each W_k, its free boundary point is nondegenerated, the free boundary of W_∞ is smooth away from the origin, We use $z = f(r)$ to represent the curve

$$\partial\{|W_\infty| < 1\} \cap \{W_\infty = 1\}.$$

The estimate in Lemma 22 tells us that f is bounded. Since the surface $z = f(r)$ has nonnegative mean curvature, with respect to the unit normal pointing towards the positive z direction, we get

$$\left[\frac{rf'(r)}{\sqrt{1+f'(r)^2}}\right]' \geq 0. \tag{52}$$

On the other hand, we know that $f(r) \to C_0$ for some constant C_0. This together with (52) implies that

$$\lim_{r\to+\infty} f(r) \leq \max_{r\in[0,+\infty)} f(r).$$

Then a sliding plane type argument using solutions of the form $\mathcal{H}(z-\beta)$ implies that f is a constant and

$$W_{\infty}(z)=\mathcal{H}(z-\alpha),$$

for some $\alpha\geq 1$. This argument also tells us that for k small, $\partial\{W_k<1\}\cap\{W_k=-1\}$ intersects with the r axis and $\partial\{W_k<1\}\cap\{W_k=1\}$ does not intersect with the r axis.

If $\alpha>1$, then we could use a blow up argument for W_k near the point $(0,\alpha-1)$ to get a contradiction. This completes the proof. ■

With all these preparation, we are in a position to prove Theorem 1 in $3D$.

Proof of Theorem 1 in $\mathbb{R}^3$. Using Lemma 30, we would like to perform a blow up analysis on W_k to get a solution to the one phase free boundary problem. Indeed, let ρ_k be the distance of the free boundary (the part where $W_k=-1$) to the origin. Then by Lemma 30, $\rho_k\to 0$ as $k\to 0$. Let us define

$$\psi_k(X)=\frac{W_k(\rho_k X)+1}{\rho_k}.$$

Then ψ_k converges to a solution u of the one-phase free boundary problem. This is the desired solution. The asymptotic behavior (5) follows from the positivity of the mean curvature of the free boundary. ■

Remark 31 *The Hauswirth-Helein-Pacard solution in 2D can also be constructed using our variational and blow up technique. Note that in 2D, we need to construct solutions to* (6) *with nodal set asymptotic to straight lines at infinity.*

6 The proof of Theorem 1 for dimension $n>3$

In this section, we assume $n>3$. The proof is essentially same as before, except that at some points we need to modify certain estimates.

As we already show in the $3D$ case, the construction of solutions to our free boundary problem is closely related to the geometry of the catenoids. Let us first of all recall the definition of catenoids in $\mathbb{R}^n$, which are codimension one minimal submanifolds. Let ϕ be the solution of

$$\begin{cases}\frac{\phi''}{1+\phi'^2}-\frac{n-2}{\phi}=0,\\ \phi(0)=1,\phi'(0)=0.\end{cases}\tag{53}$$

Then the manifold given by $r:=\phi(z)$ is a minimal submanifold, called catenoid. The principle curvatures are given by

$$k_1=...=k_{n-2}=\frac{1}{\phi(1+\phi'^2)^{\frac{1}{2}}},\quad k_{n-1}=-\frac{\phi''}{(1+\phi'^2)^{\frac{3}{2}}}.$$

Introducing a parametrization:

$$r=(\eta(s))^{\frac{1}{n-2}},z=\int_0^s(\eta(t))^{\frac{3-n}{n-2}}\,dt,\tag{54}$$

we find that η satisfies

$$\frac{1}{n-2}\eta''\frac{1}{1+\left(\frac{1}{n-2}\eta'\right)^2}-\frac{n-2}{\eta}=0.$$

From this we get $\eta(s)=\cosh((n-2)s)$.

In the upper z space, we can also write this catenoid as

$$z=\bar{\phi}(r), r\in[1,+\infty). \tag{55}$$

Then there are constants c_n, c_n' such that as $r\to+\infty$,

$$\bar{\phi}(r)\sim c_n-c_n' r^{3-n}.$$

In terms of $\bar{\phi}$, the equation in (53) can be written as

$$\frac{\bar{\phi}''(r)}{1+\bar{\phi}'^2(r)}+\frac{(n-2)\bar{\phi}'(r)}{r}=0.$$

Note that for each $\rho>0$, the rescaled function $z=\rho\bar{\phi}\left(\rho^{-1}r\right)$ also gives us a catenoid. We refer to [34] for more detailed properties on catenoids, including their Morse index.

For each $\alpha>0$, we shall use $r=\phi_\alpha(z)$ to represent the catenoid which satisfies $\phi_\alpha(0)=\alpha$. This catenoid will also be written as $z=\bar{\phi}_\alpha(r)$. On the other hand, we use $z=\bar{\phi}^*_\alpha(r)$ to represent the catenoid with

$$\lim_{r\to+\infty}\bar{\phi}^*_\alpha(r)=\alpha.$$

This catenoid will also be written as $r=\phi^*_\alpha(z)$.

Let $k>1$ be a parameter. For each a large, let

$$\Omega_a:=\{(r,z):r\in[0,a], z\in[0,b_\varepsilon]\},$$

where $b_\varepsilon=\bar{\phi}^*_k(a)+2+\delta_\varepsilon$ and δ_ε is defined by (12). Set $L_a:=L_{1,a}\cup L_{2,a}$, where

$$L_{1,a}:=\{(a,z):z\in[0,b_\varepsilon]\},\quad \text{and}\quad L_{2,a}:=\{(r,b_\varepsilon):r\in[0,a]\}.$$

We then define a function $\omega=\omega(r,z)$, depending on the parameter ε and a, to be

$$\omega(r,z)=w_{\varepsilon,\bar{\phi}^*_k(a)-\varepsilon}\left(z-\bar{\phi}^*_k(a)\right).$$

Here, same as before, $w_{\varepsilon,l}$ is the function appeared in Lemma 5.

For ε sufficiently small, we need to construct mountain pass solutions for the problem

$$\begin{cases} -\partial_r^2 u-\frac{1}{r}\partial_r u-\partial_z^2 u+\frac{1}{2}F_\varepsilon'(u)=0 \text{ in } \Omega_a,\\ \partial_r u(0,z)=0, \partial_z u(r,0)=0,\\ u=\omega, \text{ on } L_a.\end{cases}$$

Similarly as before, using parabolic flow, we can first of all construct two solutions u_1, u_2, where u_1 has almost horizontal nodal set and u_2 has almost vertical nodal set. Moreover, $\partial_r u_i < 0$ and $\partial_z u_i > 0$ in Ω_a. Furthermore, we can assume

$$\int_{\Omega_a} \left(|\nabla u_i|^2 + F_\varepsilon (u_i) \right) \leq \frac{4}{n-1} a^{n-1} + o(1),$$

where $o(1)$ is a term tending to 0 as $\varepsilon \to 0$.

Let $\mathcal{E}$ be the set of C^1 functions g satisfying the following properties:
(I) $u_1 < g < u_2$ in Ω_a,
(II) $\partial_z g > 0; \partial_r g < 0$, in Ω_a,
(III) $g|_{L_a} = \omega$,
(IV)$\partial_r g(0, z) = 0, \partial_z g(r, 0) = 0$.

The following geometric property of the catenoids will be used later on.

Lemma 32 *Let $k > 1$ be a fixed constant. Suppose a is large. Then for each $c < \phi_k^*(0)$,*

$$\int_c^a \sqrt{1 + \left(\bar{\phi}_c'(r)\right)^2} r^{n-2} dr - \frac{a^{n-1}}{n-1} \geq \frac{\delta}{2} c^{n-1},$$

where

$$\delta = \frac{1}{2(n-2)} \left(1 - \left(\frac{1}{2} \right)^{\frac{1}{n-2}} \right).$$

Proof. We first consider the case of $c = 1$. Let s_0 be the constant defined by

$$\cosh((n-2) s_0) = a^{n-2}.$$

Using the parametrization (54), we compute

$$\begin{aligned}
\int_1^a \sqrt{1 + \bar{\phi}_1'^2(r)} r^{n-2} dr &= \int_0^{\bar{\phi}_1(a)} \sqrt{1 + \phi_1'^2(z)} \phi_1^{n-2}(z) dz \\
&= \int_0^{s_0} \sqrt{1 + \sinh^2((n-2)s)} \cosh^{\frac{1}{n-2}}((n-2)s) ds \\
&= \frac{1}{n-2} \int_0^{\sinh((n-2)s_0)} \left(1 + x^2\right)^{\frac{1}{2(n-2)}} dx \\
&\geq \frac{1}{n-2} \int_{\frac{1}{2}}^{\sinh((n-2)s_0)} x^{\frac{1}{n-2}} dx + \frac{1}{2(n-2)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_1^a \sqrt{1 + \bar{\phi}_1'^2(r)} r^{n-2} dr \\
&\geq \frac{1}{n-1} a^{n-1} + \delta + O\left(a^{3-n}\right),
\end{aligned}$$

where $\delta = \frac{1}{2(n-2)} \left(1 - \left(\frac{1}{2}\right)^{\frac{1}{n-2}}\right)$.

Now since $\bar{\phi}_c(r) = c\bar{\phi}_1(c^{-1}r)$, we have

$$\begin{aligned}\int_c^a \sqrt{1+\bar{\phi}_c'^2(r)}r^{n-2}dr &= \int_c^a \sqrt{1+\bar{\phi}_1'^2(c^{-1}r)}r^{n-2}dr \\ &= c^{n-1}\int_1^{c^{-1}a} \sqrt{1+\bar{\phi}_1'^2(r)}r^{n-2}dr \\ &\geq c^{n-1}\left(\frac{1}{n-1}c^{1-n}a^{n-1}+\delta+O\left(a^{3-n}c^{n-3}\right)\right) \\ &= \frac{1}{n-1}a^{n-1}+\frac{\delta}{2}c^{n-1}.\end{aligned}$$

This is the desired estimate. ∎

Lemma 33 *Let a be a large constant. There exists $\varepsilon_0 > 0$ depending on a, such that for $\varepsilon < \varepsilon_0$, the following is true: Suppose $\xi \in \mathcal{E}$ and $\xi(\phi_{k-1}(0), 0) = -1+\varepsilon$. Then*

$$\int_{\Omega_a}\left(|\nabla\xi|^2 + F_\varepsilon(\xi)\right) \geq \frac{4}{n-1}a^{n-1}+\delta_0,$$

where $\delta_0 > 0$ is a constant independent of ξ and ε.

Proof. We still use $A(s)$ to be denote the area of the surface $\{(r,z):\xi(r,z)=s\}$. Consider the points P_1, P_2 whose (r,z) coordinates are given by $(q,0)$ and $\left(q,\frac{k+1}{2}\right)$ respectively, where $q = \frac{1}{4}\phi_{k-1}(0)$. Let $\sigma > 0$ be a small positive constant(indepedent of ε) to be determined later on. There are two possibilities.

Case 1. $\xi(P_1) > -1+\sigma$ or $\xi(P_2) > 0$.

Subcase 1. $\xi(P_1) > -1+\sigma$.

Using Lemma 33 and the fact that the catenoid is a minimal surface, we find that for $s \in (-1+\varepsilon, \xi(P_1))$,

$$A(s) \geq \frac{1}{n-1}a^{n-1}+\frac{\delta}{2}q^{n-1}.$$

Hence by the coerea formula we get

$$\begin{aligned}\int_{\Omega_a}\left(|\nabla\xi|^2+F_\varepsilon(\xi)\right) &\geq 2\int_{-1}^{1} A(s)\sqrt{F_\varepsilon(s)}ds. \\ &\geq \left(\int_{\xi(P_1)}^{1}+\int_{-1}^{\xi(P_1)}\right)\left(A(s)\sqrt{F_\varepsilon(s)}\right)ds \\ &\geq \frac{a^{n-1}}{n-1}(1-\xi(P_1))(1+O(\varepsilon)) \\ &+\left(\frac{a^{n-1}}{n-1}+\frac{\delta}{2}q^{n-1}\right)(\xi(P_1)-1)(1+O(\varepsilon)) \\ &\geq \frac{1}{n-1}a^{n-1}+\frac{\delta\sigma}{2}q^{n-1}+O(\varepsilon). \qquad (56)\end{aligned}$$

provided that ε is sufficiently small.

Subcase 2. $\xi(P_2) > 0$.

Similarly as Subcase 1, from Lemma 33, we deduce that for $s \in \left(\frac{1-k}{5}, 0\right)$,

$$A(s) \geq \frac{1}{n-1}a^{n-1} + \frac{\delta}{2}q^{n-1}.$$

Using this lower bound, under the assumption that ε is small, we can estimate

$$\begin{aligned}\int_{\Omega_a}\left(|\nabla\xi|^2 + F_\varepsilon(\xi)\right) &\geq \left(\int_{-1}^{\frac{1-k}{5}} + \int_{\frac{1-k}{5}}^{0} + \int_0^1\right)\left(A(s)\sqrt{F_\varepsilon(s)}\right)ds \\ &\geq \frac{2}{n-1}a^{n-1} + \frac{\delta(k-1)}{10}q^{n-1} + O(\varepsilon). \qquad (57)\end{aligned}$$

Case 2. $\xi(P_1) + 1 < \sigma$ and $\xi(P_2) < 0$.

Let us define

$$\begin{aligned}\Omega_1^* &= \{(r,z) \in \Omega_a : r > \phi_{k-1}(0)\}, \\ \Omega_2^* &= \{(r,z) \in \Omega_a : r < \phi_{k-1}(0)\}.\end{aligned}$$

Then the energy in the region Ω_1^* has the estimate

$$\begin{aligned}\int_{\Omega_1^*}\left(|\nabla\xi|^2 + F_\varepsilon(\xi)\right) &\geq \int_{\phi_{k-1}(0)}^{a}\left(\int_0^{b_\varepsilon}\left((\partial_z\xi)^2 + F_\varepsilon(\xi)\right)dz\right)r^{n-2}dr \\ &\geq \frac{4+O(\varepsilon)}{n-1}\left(a^{n-1} - (\phi_{k-1}(0))^{n-1}\right). \qquad (58)\end{aligned}$$

On the other hand, for $r \in (q, \phi_{k-1}(0))$, using the fact that $\xi\left(r, \frac{k+1}{2}\right) < 0$ and

$$-1+\varepsilon < \xi(r,0) < -1+\sigma,$$

we obtain

$$\int_0^{b_\varepsilon}\left((\partial_z\xi)^2 + F_\varepsilon(\xi)\right)dz \geq 2 + \left(\left(\frac{2(-1+\sigma)}{k+1}\right)^2 + 1\right)\frac{k+1}{2} + O(\varepsilon). \qquad (59)$$

Then we can choose a small constant $\sigma > 0$ such that the right hand side of (59) is bounded below by a constant $4+\delta_1$, where $\delta_1 > 0$ is independent of ε. Then we can estimate

$$\begin{aligned}\int_{\Omega_2^*}\left(|\nabla\xi|^2 + F_\varepsilon(\xi)\right) &\geq \int_q^{\phi_{k-1}(0)}\left(\int_0^{b_\varepsilon}\left((\partial_z\xi)^2 + F_\varepsilon(\xi)\right)dz\right)r^{n-2}dr \\ &\geq \frac{4+\delta_2}{n-1}\left((\phi_{k-1}(0))^{n-1} - q^{n-1}\right). \qquad (60)\end{aligned}$$

Combining (58) and (60), we deduce that when ε is sufficiently small,

$$\int_{\Omega_a}\left(|\nabla\xi|^2 + F_\varepsilon(\xi)\right) \geq \frac{4}{n-1}\left(a^{n-1} - (\phi_{k-1}(0))^{n-1}\right)$$

$$+\frac{4+\delta_1}{n-1}\left((\phi_{k-1}(0))^{n-1}-q^{n-1}\right)+O(\varepsilon)$$
$$\geq \frac{4a^{n-1}}{n-1}+\frac{\delta_1}{2(n-1)}\left(1-\frac{1}{4^{n-1}}\right)(\phi_{k-1}(0))^{n-1}. \quad (61)$$

From equations $(56), (57)$ and (61), we conclude the proof. ■

For each fixed $k>0$ and large a, when ε is small, with the help of Lemma 33 and the parabolic flow, we then get a family of mountain pass type solutions $U_{\varepsilon,a}$(depending on k), with the energy estimate

$$\frac{4}{n-1}a^{n-1}+\delta_0 \leq \int_{\Omega_a}\left(|\nabla U_{\varepsilon,a}|^2+F_\varepsilon(U_{\varepsilon,a})\right) \leq \frac{4}{n-1}a^{n-1}+C, \quad (62)$$

for some constant C independent of ε, a.

Letting $\varepsilon \to 0$, up to a subsequence, $U_{\varepsilon,a}$ converges to a function V_a solving

$$\begin{cases} \Delta V_a = 0, \text{ in } \Omega_a \cap \{|V_a|<1\}, \\ |\nabla V_a| = 1, \text{ on } \Omega_a \cap \partial\{|V_a|<1\}. \end{cases}$$

Moreover, on $\partial\Omega_a$, V_a satisfies the boundary condition inherited from $U_{\varepsilon,a}$.

As a tends to infinity, up to a subsequence, V_a converges to a solution W of the free boundary problem

$$\begin{cases} \Delta W = 0, \text{ in } \{|W|<1\}, \\ |\nabla W| = 1, \text{ on } \partial\{|W|<1\}. \end{cases}$$

The next lemma states that W behaves like a catenoid at infinity.

Lemma 34 *Let $\Omega = \{(r,z) : z>0 \text{ and } |W(r,z)|<1\}$. Let r_0 be a large constant. Suppose that in the region where $r>r_0$,*

$$\partial\Omega \cap \{(r,z) : W(r,z)=1\} = \{(r,z) : z = f_1(r)\},$$
$$\partial\Omega \cap \{(r,z) : W(r,z)=-1\} = \{(r,z) : z = f_2(r)\}.$$

Then there exists $k' \geq 1$ such that

$$f_1(r)-k'-1 \to 0,$$
$$f_2(r)-k'+1 \to 0,$$

as $r \to +\infty$.

Proof. The mean curvatures of the surfaces $z=f_1(r)$ and $z=f_2(r)$ have a sign. That is,

$$\left(\frac{r^{n-2}f_1'}{\sqrt{1+f_1'^2}}\right)' \geq 0,$$

and

$$\left(\frac{r^{n-2}f_2'}{\sqrt{1+f_2'^2}}\right)' \leq 0.$$

Then the proof of this lemma is similar as that of Lemma 18. ■

Our next purpose is to show that W has the desired asymptotic behavior, that is, $k' = k$. To prove this, we need the following lemma, which is a result parallel to Lemma 21.

Lemma 35 *Suppose $k \neq k'$. Assume A is large and $A < a$. Let $\xi = \xi(r)$ be a C^1 monotone increasing function satisfying $\xi(A) = k'$ and $\xi(a) = k$. Then*

$$\int_A^a \sqrt{1 + (\xi'(r))^2} r^{n-2} dr \geq \frac{a^{n-1} - A^{n-1}}{n-1} + \frac{1}{2}\sqrt{A}(k - k').$$

Proof. We compute

$$\int_A^a \left(\sqrt{1 + (\xi'(r))^2} - 1\right) r^{n-2} dr = \int_A^a \frac{\xi'(r)^2}{\sqrt{1 + (\xi'(r))^2} + 1} r^{n-2} dr.$$

Let $S \subset [A, a]$ be the set where

$$\left| \frac{\xi'(r)}{\sqrt{1 + (\xi'(r))^2} + 1} r^{n - \frac{5}{2}} \right| \leq 1.$$

Then using the fact that A is large, which implies that in S, ξ' is small, we obtain

$$\left| \int_S \xi'(r) \, dr \right| \leq \frac{8}{2n-7} A^{\frac{7}{2} - n}.$$

Therefore, when A is sufficiently large, from $\int_{[A,a]} \xi'(r) \, dr = k - k'$, we get

$$\begin{aligned}
&\int_{[A,a] \setminus S} \frac{\xi'(r)^2}{\sqrt{1 + (\xi'(r))^2} + 1} r^{n-2} dr \\
&\geq \int_{[A,a] \setminus S} \xi'(r) \, r^{\frac{1}{2}} dr \\
&\geq \sqrt{A} \int_{[A,a] \setminus S} \xi'(r) \, dr \\
&\geq \frac{1}{2}\sqrt{A}(k - k'),
\end{aligned}$$

This implies that

$$\int_A^a \left(\sqrt{1 + (\xi'(r))^2} - 1\right) r^{n-2} dr \geq \frac{1}{2}\sqrt{A}(k - k').$$

The proof is then completed. ■

With Lemma 35 at hand, we can use the energy upper bound (62) and proceed similarly as the proof of Proposition 20, to conclude that for the solution

W, there holds $k' = k$. That is, the nodal set of W is asymptotic to $z = k$ at infinity. Denote this solution by W_k.

The next step is to analyze the precise asymptotic behavior of W_k, uniformly in k as $k \to 0$. This can be achieved from similar arguments as that of Section 5, with straightforward modifications(The decay rate of the principle curvatures are different). Finally, same as the 3D case, suitable blow up sequence of W_k near the origin then converges to a desired solution of the one phase free boundary problem.

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