The Existence and Stability of Spikes in the One-Dimensional Keller–Segel Model with Logistic Growth

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Abstract

It is well known that Keller–Segel models serve as a paradigm to describe the self aggregation phenomenon, which exists in a variety of biological processes such as wound healing, tumor growth, etc. In this paper, we study the existence of monotone decreasing spiky steady states and its linear stability property in Keller–Segel models with logistic growth over one-dimensional bounded domain subject to homogeneous Neumann boundary conditions. Under the assumption that chemo-attractive coefficient is asymptotically large, we construct the single boundary spike and next show this non-constant steady state is locally linear stable via Lyapunov-Schmidt reduction method. As a consequence, the multi-symmetric spikes are obtained by reflection and periodic extension. In particular, we present the formal analysis to illustrate our rigorous theoretical results.

Keywords: Keller–Segel Models; Logistic Growth; Single Boundary Spike; Reduction Method

1 Introduction and Main results

Chemotaxis is a process in which the uni-cellular and multi-cellular organisms are stimulated by chemical signal, then direct their movements along the gradient of stimulus. One of the most important findings in biological processes involving chemotactic movement is the self-organized aggregation, which refers to a situation, for instance initially evenly distributed bacteria together move towards high concentrations of the chemical stimulus and finally form several groups of spatial aggregates. Since this interesting phenomenon widely exists in physiological and pathological activities of organisms, numerous researchers would like to investigate it from the viewpoint of mathematics. Before achieving this goal, some appropriate mathematical models need to be initiated so as to describe the phenomenon.

1.1 Keller–Segel Models

To study the traveling bands of *E. coli*, Keller and Segel [20,21] in 1971 proposed one type of reaction-diffusion systems which consist of two coupled partial differential equations (PDEs), and the forms read as follows:

$$\begin{cases} u_t = \nabla\cdot(\overbrace{d_1\nabla u}^{\text{random (flux)}} - \overbrace{\chi\rho(u,v)\nabla v}^{\text{chemotactic (flux)}}) + \overbrace{f(u)}^{\text{source}}, & x\in\Omega, t>0,\\ v_t = \overbrace{d_2\Delta v}^{\text{chemical diffusion}} + \overbrace{g(u,v)}^{\text{chemical creation/consumption}}, & x\in\Omega, t>0, \end{cases} \tag{K-S}$$

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where spatial region Ω is usually taken to be the whole space $\mathbb{R}^N$, $N \geq 1$, or a bounded domain in $\mathbb{R}^N$. Here u and v represent the cellular density and chemical concentration at location x and time t, respectively. In addition, ρ is the so-called sensitivity function, which measures the chemotactic fluctuation, while constants $d_1 > 0$ and $d_2 > 0$ denote the self-diffusion effect of u and v, respectively; χ describes the strength of chemotactic movement. In particular, the environment is assumed to be enclosed, and hence one imposes the no-flux boundary conditions on cellular density u and chemical concentration v.

System (K-S) have been well studied over the past few decades and we refer the readers to survey papers [12, 14–16]. It is worthy mentioning that the seminal work of Nanjudiah [30] and Childress and Percus [5] stimulate scholars to qualitatively analyze the rich properties of global solutions, blow-up solutions, stationary solutions and traveling waves within (K-S) [11,17,27,28,33,36]. In the absence of the source term $f(u)$, there are a variety of results focusing on the global well-posedness and dynamics. The classical form of (K-S) in this situation is the so-called minimal chemotaxis model, which reads

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}}(x,t) = \frac{\partial v}{\partial \mathbf{n}}(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $d_1 = d_2 = 1$, $(u^0(x), v^0(x))$ are non-negative initial data and $\mathbf{n}$ is the unit outer normal vector. The famous phenomenon in the 2D case of (1.1) is commonly referred to as "chemotactic collapse", which has been fairly understood. In fact, there exists some critical mass $M_0 := \frac{4\pi}{\chi}$ for the bounded domain or $\frac{8\pi}{\chi}$ for the whole space $\mathbb{R}^N$ such that if $\int_\Omega u(x,0)dx < M_0$, the solution to (1.1) will globally exist [29]; otherwise (1.1) will admit finite time blow-up solutions [5,11,30,37,40]. Moreover, focusing on the stationary counterpart of (1.1) in 2D, Wei and Delpino employed the "localized energy method" to construct the multi-spiky solutions [6]. It is necessary to point out that the groundbreaking work for the construction of large amplitude steady states in Keller–Segel models was completed by Lin, Ni and Takagi [24, 31, 32]. Based on these results, researchers comprehensively studied the non-constant stationary solutions with a single boundary or interior spike or multiple spikes, which refer to [1–3, 8, 9, 25]. In contrast with 2D case, it was shown that the solution to (1.1) in 1D globally exists for all time [27, 34]. Furthermore, the results involving the formal and rigorous construction of the spiky steady states can be found in [4, 13, 19]. In particular, Wang and Xu [41] adopted an innovative method to directly tackle the stationary problem of (1.1) without converting it into a single equation.

In the presence of the source term $f(u)$, the results for the global existence and large time behavior of solutions in (1.1) are distinct. The common form of $f(u)$ is the logistic growth satisfying $f(u) = u(\bar{u} - \kappa u)$, which are usually used to model population dynamics. Hhere $\bar{u} > 0$ measures the carrying capacity of the habitat for cells and $\kappa > 0$ represents the strength of cellular degradation. It is well-known that the logistic growth can inhibit the occurrence of blow-up phenomenon in any dimension when κ is large [26, 45–49]. For the stationary counterpart of (1.1) with the logistic source, there are also a few results related to the construction of non-constant stationary solutions [18, 35, 38]. Recently, we have rigorously constructed multi-spikes in 2D [23]. Our successful analysis applied in 2D gives us motivation to perform the similar argument for one-dimensional minimal models with the logistic growth. Actually, Kolokolnikov et al. in 2014 [38] did the formal computation to show the existence of spiky steady states in 1D. Our goal in this paper is also to give the rigorous proof for these existence results.

1.2 Motivations and Main Results

In this paper, we focus on the 1D version of (1.1) with the logistic source in $(0, L)$, which is

$$\begin{cases} u_t = u_{xx} - \chi(uv_x)_x + \mu u(\bar{u} - u), & x \in (0, L), t > 0, \\ v_t = v_{xx} - v + u, & x \in (0, L), t > 0, \\ u_x(0, t) = u_x(L, t) = v_x(0, t) = v_x(L, t) = 0, & t > 0, \\ u(x, 0) = u^0(x), \quad v(x, 0) = v^0(x), & x \in (0, L). \end{cases} \tag{1.2}$$

According to the results of Osaki and Yagi [34], the solution globally exists and is uniformly bounded for all the time. However, it might admit the bounded non-constant steady states which possess the striking structures such as spikes, transition layers, etc. To confirm our conjecture, it is necessary to study the stationary system of (1.2), which is

$$\begin{cases} u_{xx} - \chi(uv_x)_x + \mu u(\bar{u} - u) = 0, & x \in (0, L), \\ v_{xx} - v + u = 0, & x \in (0, L), \\ u_x(0) = u_x(L) = v_x(0) = v_x(L) = 0. \end{cases} \tag{1.3}$$

It is well-known that when $\mu = 0$, there exists the boundary layer to (1.3) under the asymptotically limit of $L \gg 1$ [19]. The natural generalization is to construct the boundary spikes with the similar profile for arbitrary $\mu > 0$. Under the assumption that the chemotactic coefficient χ is sufficiently large, we show the existence of single boundary spikes to (1.3) via the standard Lyapunov-schmidt reduction and the results are stated in the following Theorem:

Theorem 1.1. *Let $\chi := \frac{1}{\epsilon^2}$. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, (1.3) admits the non-constant solution (u^-, v^-) which has the following asymptotical form:*

$$\begin{cases} u^- = U_0(\frac{x}{\epsilon}) + o(1), & x \in (0, L), \\ v^- = v_0(x; L) + o(1), & x \in (0, L), \end{cases} \tag{1.4}$$

where $U_0(y) := \frac{a}{2}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)$ and $v_0(x; L)$ is given by

$$v_0(x; L) = \epsilon^2 \log\Big(\frac{1}{4}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}\frac{x}{\epsilon}\Big)\Big) + \epsilon\frac{\sqrt{a}}{\sinh L}\cosh(x - L) + \sqrt{a}\epsilon x. \tag{1.5}$$

In particular, $a = 3\bar{u} + O(\epsilon)$.

Theorem 1.1 illustrates that (1.2) admits non-constant steady states for χ large enough in which cellular density $u = O(1)$ and chemical concentration $v = O(\frac{1}{\sqrt{\chi}})$. The natural question is whether there exist other nontrivial stationary solutions to (1.3). Indeed, it follows from the reflection and periodic extension that there are symmetric multi-spikes for $\chi \gg 1$, which is summarized as

Corollary 1.1. *Suppose $m \geq 1$ be any fixed integer, then we can find constant ϵ_0 such that for $\epsilon \in (0, \epsilon_0)$, there exist the following two types of m-symmetric spikes to (1.3):*

$$(\mathbb{U}_m^-, \mathbb{V}_m^-) = \sum_{j=0}^{[\frac{m}{2}]}(U^m, V^m)\Big(\frac{2jL}{m} - x\Big) + (U^m, V^m)\Big(x - \frac{2jL}{m}\Big), \ x \in (0, L), \tag{1.6}$$

and

$$(\mathbb{U}_m^+, \mathbb{V}_m^+) = \sum_{j=1}^{[\frac{m}{2}]+1}(U^m, V^m)\Big(\frac{(2j-1)L}{m} - x\Big) + (U^m, V^m)\Big(x - \frac{(2j-1)L}{m}\Big), x \in (0, L), \tag{1.7}$$

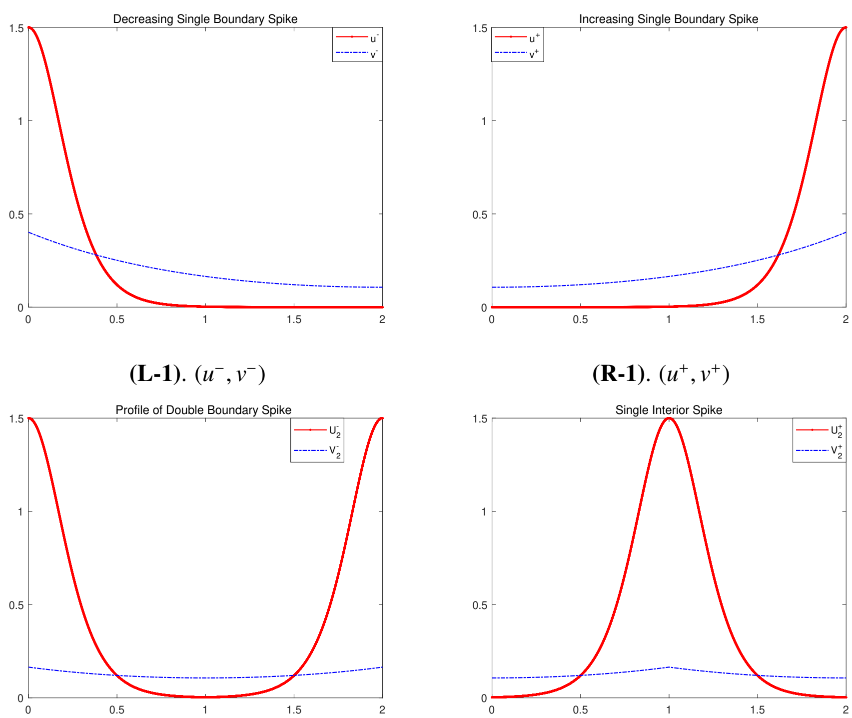
where U^m and V^m are defined as

$$U^m = \begin{cases} U_0(\frac{x}{\epsilon}) + O(\epsilon), & x \in (0, \frac{L}{m}), \\ 0, & x \notin (0, \frac{L}{m}), \end{cases} \quad \text{and} \quad V^m = \begin{cases} v_0\Big(x; \frac{L}{m}\Big) + O(\epsilon), & x \in (0, \frac{L}{m}), \\ 0, & x \notin (0, \frac{L}{m}). \end{cases}$$

The other important finding is the stability property of (1.4). We state the relevant results in the following Theorem:

Theorem 1.2. *When $\epsilon \ll 1$, (u^-, v^-) defined by (1.4) is locally linear stable.*

From the perspective of energy, Theorem 1.2 implies the energy of (u^-, v^-) is the smallest among a class of solutions given in Corollary 1.1. Next, we exhibit the profile of these single and multiple spikes in Figure 1. Moreover, as is shown in Figure 2, the solution $(u, v)(\cdot, t)$ of (1.2) converges to (u^-, v^-) after we impose some small perturbation on the initial data.

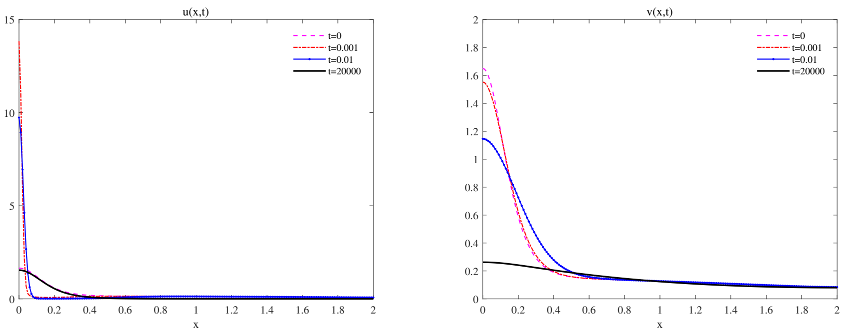


(L-1). (u^-, v^-) **(R-1)**. (u^+, v^+)

(L-2). $(\mathbb{U}_2^-, \mathbb{V}_2^-)$ **(R-2)**. $(\mathbb{U}_2^+, \mathbb{V}_2^+)$

Figure 1: *Given $L = 2$, $\bar{u} = 1$ and $\chi = 20$. Top: Profiles of increasing and decreasing single spikes. Bottom: Profiles of double boundary and single interior spikes.* We find cellular density u is localized and chemical concentration v globally varies over $(0, L)$.

The remaining parts of this paper are organized as follows: In Section 2, we shall firstly convert (1.3) into the single nonlocal equation, then show our idea for the choice of the ansatz to this single equation. Section 3 is devoted to the existence of spiky stationary solutions via Lyapunov-schmidt reduction. In this section, we mainly focus on the investigation of linear and nonlinear projected problems. In Section 4, we will linearize (1.2) around the single boundary spike then study the linearized eigenvalue problem



(L-1). cellular density u **(R-1)**. chemical concentration v

Figure 2: *Given $L = 2$, $\bar{u} = 1$ and $\chi = 50$. The dynamics of single boundary spikes defined in Theorem 1.1.* We can regard time-dependent solution $(u, v)(\cdot, t)$ at $t \approx 20000s$ as the stationary solution and obtain $(u, v)(\cdot, t)$ finally converges to (u^-, v^-).

to prove this steady state is locally asymptotical stable. Finally, we exhibit the formal computation which complements our rigorous analysis in Appendix A and Appendix B.

2 The Approximate Solution of u and v

In section 2, we shall introduce the selection of the ansatz. First of all, we set $\mu = 0$ in the first equation of (1.3) to obtain

$$u = Ce^{\chi v}, \tag{2.1}$$

where $C > 0$ is some constant need to be determined. Next, we define $\bar{v} := \chi v$ and substitute it into the second equation to arrive at

$$\bar{v}_{xx} - \bar{v} + C\chi e^{\bar{v}} = 0, \quad x \in (0, L). \tag{2.2}$$

Define $\tilde{\epsilon} := \frac{1}{\sqrt{C\chi}}$ and $y = \frac{x}{\tilde{\epsilon}}$, then we have (2.2) can be simplified as

$$\bar{v}_{yy} - \tilde{\epsilon}^2 \bar{v} + e^{\bar{v}} = 0, \quad y \in (0, L/\tilde{\epsilon}). \tag{2.3}$$

When $\epsilon \ll 1$, we note that (2.3) can be approximated by the following Liouville equation with the Neumann boundary condition in the half space:

$$w_{yy} + e^{w} = 0, \ y \in (0, \infty), \quad w_y(0) = 0. \tag{2.4}$$

It is well-known that (2.4) has a family of solutions satisfying

$$w(y; \bar{a}) = \log\Big[\frac{\bar{a}}{2}\text{sech}^2\Big(\frac{\sqrt{\bar{a}}}{2}y\Big)\Big], \quad \int_0^\infty e^{w} dy < +\infty, \tag{2.5}$$

where $\bar{a} > 0$ is a free parameter. Hence, we regard $w(y, \bar{a})$ as the basic ansatz of $\bar{v}$, then obtain from $u = Ce^{\bar{v}}$ and $\tilde{\epsilon} := \frac{1}{\sqrt{C\chi}}$ that u_0, the ansatz of u, can be defined as

$$u_0 := \frac{\bar{a}C}{2}\text{sech}^2\Big(\frac{\sqrt{\bar{a}C}}{2}y\Big) = \frac{\bar{a}C}{2}\text{sech}^2\Big(\frac{\sqrt{\bar{a}C}}{2}\frac{x}{\epsilon}\Big) = \frac{a}{2}\text{sech}^2\Big(\frac{\sqrt{a}}{2}\frac{x}{\epsilon}\Big), \tag{2.6}$$

where $a > 0$ is a free parameter and $\epsilon := \sqrt{\frac{1}{\chi}}$. It is necessary to determine the constant a so as to establish the explicit form of u_0. To this end, we integrate the u-equation in (1.3), then find from the Neumann boundary condition that

$$\int_0^L u(\bar{u} - u)dx = 0. \tag{2.7}$$

Since u_0 is defined as the approximate solution of u, one further substitutes (2.6) into (2.7) to get $a \sim 3\bar{u}$. Now, we have established the following rough ansatz of u and $\bar{v}$:

$$u_0 = \frac{a}{2}\text{sech}^2\left(\frac{\sqrt{a}}{2}\frac{x}{\epsilon}\right) \text{ and } \bar{v}_0 = \log\left[\frac{a}{2}\text{sech}^2\left(\frac{\sqrt{a}}{2}\frac{x}{\epsilon}\right)\right] - \log C. \tag{2.8}$$

Step 1: Construction of (u_0, v_0)

It is necessary to point out that $\bar{v}_0$ is not the uniform expansion of $\bar{v}$ since it does not satisfy the Neumann boundary condition at $x = L$. To solve this issue and obtain the uniform expansion of v, we define v_0 as the following form in terms of the Neumann Green's function:

$$v_0(x) = \int_0^L G(z; x)u_0(z)dz, \tag{2.9}$$

where u_0 is given in (2.8) and $G(z; x)$ satisfies

$$G(z; x) = \begin{cases} \frac{\cosh(x-L)}{\sinh L}\cosh z, & z \in (0, x), \\ \frac{\cosh x}{\sinh L}\cosh(z - L), & z \in (x, L). \end{cases} \tag{2.10}$$

According to the potential theory, we have v_0 is the solution to

$$\begin{cases} v_{0xx} - v_0 + u_0 = 0, & x \in (0, L), \\ v_{0x}(0) = v_{0x}(L) = 0, \end{cases}$$

which can be regarded as the uniform approximation of v.

To analyze the error generated by (u_0, v_0) comprehensively, we shall decompose v_0 given by (2.9) into $v_{00} + v_{01} + v_{02}$. To begin with, we define v_{01} as the solution to

$$\begin{cases} v_{xx} - v = 0, & x \in (0, L), \\ v_x(L) = 0, \end{cases} \tag{2.11}$$

which satisfies

$$v_{01}(x) = \epsilon\frac{\sqrt{a}}{\sinh L}\cosh(x - L). \tag{2.12}$$

Noting that v_{01} does not satisfy the homogeneous Neumann boundary condition at $x = 0$, we define the correction term ω as $\epsilon^2\omega(y) = v - v_{01}$ and substitute it into (2.2) to find the equation of ω is

$$\omega_{yy} - \epsilon^2\omega + Ce^{\bar{V}_{01}+\omega} = 0, \quad y \in \left(0, \frac{L}{\epsilon}\right), \tag{2.13}$$

where $\bar{V}_{01}(y) = \chi v_{01}(x)$. We further define ω_0 as the approximate solution of ω, then expand $\bar{V}_{01}$ as the polynomial form and use $C \sim e^{-\sqrt{a}\epsilon^{-1}\coth L}$ to get

$$\omega_{0yy} + e^{\omega_0 - \sqrt{a}y} = 0, \quad y \in (0, \infty). \tag{2.14}$$

In light of the Neumann boundary condition at $x = 0, L$, we have $\omega_0(\infty) = 0$ and $\omega_{0y} = \sqrt{a}$. By solving the equation of ω_0, one finds

$$\omega_0 = \sqrt{a}y + \log\Big(\frac{1}{4}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\Big). \tag{2.15}$$

Combining (2.12) and (2.15), we simplify (2.9) as

$$v_0 = \epsilon^2 \log\Big(\frac{1}{4}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\Big) + \epsilon\frac{\sqrt{a}}{\sinh L}\cosh(x-L) + \sqrt{a}\epsilon^2 y := v_{00} + v_{01} + v_{02}, \tag{2.16}$$

where $a \sim 3\bar{u}$. Now, we have established the basic ansatz (u_0, v_0) of (u, v), which are given by (2.8) and (2.16).

It is natural to study the error generated by u_0 and v_0. Before this, we rewrite v as $v = -(\Delta_x - 1)^{-1}u$ and convert (1.3) into the following single nonlocal equation:

$$S(u) := \Big[u_x + \frac{1}{\epsilon^2}u\Big((\Delta_x - 1)^{-1}u\Big)_x\Big]_x + \mu u(\bar{u} - u) = 0, \tag{2.17}$$

where $\epsilon \ll 1$, $\mu > 0$ and $\bar{u} > 0$ are constants. We define ϕ and ψ as $\phi = u - u_0$ and $\psi = v - v_0$, then substitute them into (2.17) to obtain

$$S(u) = (u_0 + \phi)_{xx} - \frac{1}{\epsilon^2}[(u_0 + \phi)(v_0 + \psi)_x]_x + \mu(u_0 + \phi)(\bar{u} - u_0 - \phi). \tag{2.18}$$

Then we rearrange (2.18) and find

$$\begin{aligned} S(u) =& u_{0xx} - \frac{1}{\epsilon^2}(u_0 v_{00x})_x \overbrace{- \frac{1}{\epsilon^2}(u_0(v_{01} + v_{02})_x)_x}^{I_{11}} \\ &+ \phi_{xx} - \frac{1}{\epsilon^2}(u_0\psi_x)_x - \frac{1}{\epsilon^2}(\phi v_{00x})_x \\ &\overbrace{- \frac{1}{\epsilon^2}(\phi v_{01x})_x - \frac{1}{\epsilon^2}(\phi v_{02x})_x - \frac{1}{\epsilon^2}(\phi\psi_x)_x}^{I_{12}} \\ &+ \overbrace{\mu u_0(\bar{u} - u_0) - \mu u_0\phi + \mu\phi(\bar{u} - u_0 - \phi)}^{I_{13}}. \end{aligned} \tag{2.19}$$

It can be seen that I_{11} makes the main contribution on the error generated by u_0 and v_0. We further use (2.16) to calculate I_{11} and find

$$\begin{aligned} I_{11} =& -\frac{1}{\epsilon^2}u_{0x}v_{01x} - \frac{1}{\epsilon^2}u_0 v_{01xx} - \frac{1}{\epsilon^2}u_{0x}v_{02x} - \frac{1}{\epsilon^2}u_0 v_{02xx} \\ =& -\frac{1}{\epsilon}\frac{a^{\frac{3}{2}}}{2\sinh L}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}\frac{x}{\epsilon}\Big)\cosh(x-L) - \frac{1}{\epsilon^2}u_{0x}(v_{01} + v_{02})_x \\ =& -\frac{1}{\epsilon}\frac{a^{\frac{3}{2}}}{2\sinh L}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}\frac{x}{\epsilon}\Big)\cosh(x-L) \\ &+ \frac{a^{\frac{3}{2}}}{2\epsilon^2}\Big[\frac{\sqrt{a}}{\sinh L}\sinh(x-L) + \sqrt{a}\Big]\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}\frac{x}{\epsilon}\Big)\tanh\Big(\frac{\sqrt{a}}{2}\frac{x}{\epsilon}\Big). \end{aligned} \tag{2.20}$$

Define $\Phi(y) := \phi(x)$, $\Psi(y) := \psi(x)$ and $L[\Phi] = \Phi_{yy} - \frac{1}{\epsilon^2}(U_0\Psi_y)_y - \frac{1}{\epsilon^2}(\Phi V_{00y})_y$ with $U_0(y) := u_0(x)$ and

$V_{00}(y) := v_{00}(x)$, then we utilize (2.20) to simplify (2.19) as

$$\begin{aligned} S(u) =& \frac{1}{\epsilon^2} L[\Phi] - \frac{1}{\epsilon} \frac{a^{\frac{3}{2}}}{2\sinh L} \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2} y\Big) \cosh(\epsilon y - L) \\ &+ \frac{a^{\frac{3}{2}}}{2} \frac{1}{\epsilon^2} \Big[\frac{\sqrt{a}}{\sinh L} \sinh(\epsilon y - L) + \sqrt{a} \Big] \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2} y\Big) \tanh\Big(\frac{\sqrt{a}}{2} y\Big) \\ &+ I_{12} + I_{13} \\ =& \frac{1}{\epsilon^2} L[\Phi] - \frac{1}{\epsilon} \frac{a^{\frac{3}{2}}}{2\sinh L} \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2} y\Big) [\cosh L - \epsilon \sinh L y + O(\epsilon^2) y^2] \\ &+ \frac{a^{\frac{3}{2}}}{2} \frac{1}{\epsilon} \Big[\sqrt{a} \coth L y + O(\epsilon) y^2 \Big] \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2} y\Big) \tanh\Big(\frac{\sqrt{a}}{2} y\Big) \\ &+ I_{12} + I_{13}. \end{aligned} \tag{2.21}$$

It is shown in (2.21) that the leading term is $O\Big(\frac{y}{\epsilon} e^{-\sqrt{a} y}\Big)$.

Step 2: Construction of (u_1, v_1)

To balance this leading term and reduce the error, we shall add the $O(\epsilon)$ term in the ansatz of u and v. To be more precisely, we set the refined ansatz as $u = u_0 + \epsilon u_1 + \phi$ and $v = v_0 + \epsilon v_1 + \psi$, where u_1 and v_1 will be determined later on. It is convenient to calculate them in y-variable. Define $U_1(y) := u_1(x)$, $V_1(y) := v_1(x)$, $\bar{V}_{00} := \frac{1}{\epsilon^2} V_{00}$ and $\bar{V}_{10} := \frac{1}{\epsilon^2} V_{10}$, then we have U_1 and $\bar{V}_{10}$ satisfies

$$\begin{aligned} & U_{1yy} - (U_1 \bar{V}_{00y})_y - (U_0 \bar{V}_{10y})_y \\ =& \cosh L \frac{a^{\frac{3}{2}}}{2\sinh L} \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2} y\Big) - \frac{a^2}{2} \coth L \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2} y\Big) \tanh\Big(\frac{\sqrt{a}}{2} y\Big) y, \end{aligned} \tag{2.22}$$

where $-\bar{V}_{10yy} = U_1$. Since $U_{0y} = U_0 \bar{V}_{00y}$, one has (2.22) can be transformed into the following quadratic form:

$$\begin{aligned} (U_0 g_{1y})_y =& \frac{a^{\frac{3}{2}} \coth L}{2} \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2} y\Big) \\ & - \frac{a^2 \coth L}{2} y \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2} y\Big) \tanh\Big(\frac{\sqrt{a}}{2} y\Big), \end{aligned} \tag{2.23}$$

where

$$g_1 = \frac{U_1}{U_0} - \bar{V}_{10}, \quad -\bar{V}_{10yy} = U_1. \tag{2.24}$$

By integrating (2.23), we find

$$\begin{aligned} U_0 g_{1y} =& a \coth L \tanh\Big(\frac{\sqrt{a}}{2} y\Big) \\ & - \frac{a}{2} \coth L [\sinh(\sqrt{a} y) - \sqrt{a} y] \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2} y\Big) + c_1, \end{aligned} \tag{2.25}$$

where c_1 is any arbitrary constant. We further solve (2.25) to get

$$g_1 = \frac{\coth L}{\sqrt{a}} \cosh(\sqrt{a} y) + \frac{\sqrt{a} y^2}{2} \coth L + c_2 \Big[\frac{\sinh(\sqrt{a} y)}{\sqrt{a}} + y \Big] + c_3, \tag{2.26}$$

where c_2 and c_3 are arbitrary constants. Thus, one obtains the following equation of $\bar{V}_{10}$:

$$\bar{V}_{10yy} + U_0 \bar{V}_{10} = -U_0 g_1,$$

where $-U_0 g_1$ is chosen to satisfy

$$-U_0 g_1 = -\frac{a^{\frac{3}{2}}}{4}\operatorname{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\coth Ly^2.$$

By applying the variation of parameters formula, we choose $\bar{V}_{10}$ as

$$\bar{V}_1(y) = z(y)\int_0^y U_0(r)g_1(r)\tilde{z}(r)\mathrm{d}r - \tilde{z}(y)\int_0^y U_0(r)g_1(r)z(r)\mathrm{d}r + c_4 z(y) + c_5\tilde{z}(y), \tag{2.27}$$

where c_4 and c_5 need to be determined later on. In particular, z and $\tilde{z}$ are defined as the linearly independent kernels of the equation

$$\Psi_{yy} + \frac{a}{2}\operatorname{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\Psi = 0.$$

To guarantee z and $\tilde{z}$ satisfies $z\tilde{z}_y - \tilde{z}z_y = 1$, we set the form of kernel z and $\tilde{z}$ as

$$z(y) = \frac{e^{\sqrt{a}y} - 1}{e^{\sqrt{a}y} + 1}, \quad \tilde{z}(y) = \frac{\sqrt{a}y(e^{\sqrt{a}y} - 1) - 4}{\sqrt{a}(e^{\sqrt{a}y} + 1)}. \tag{2.28}$$

Noting that there are linear growth terms in (2.27), we would like to eliminate them by choosing appropriate constants c_4 and c_5. Define c_5 as

$$c_5 = \int_0^\infty U_0(r)g_1(r)z(r)\mathrm{d}r$$

and $c_4 = -c_5\frac{\tilde{z}_y(0)}{z_y(0)}$, then we obtain $\bar{V}_{10}$ is uniformly bounded and satisfies the Neumann boundary condition at $y = 0$. Moreover, U_1 is defined by

$$U_1 = U_0 g_1 + U_0\bar{V}_{10}. \tag{2.29}$$

We next rewrite $U_1(y)$ in the x-variable as $u_1(x)$ then employ the representation formula in terms of the Neumann Green's function $G(z;x)$ to conclude that

$$v_1(x) = \int_0^L G(z;x)u_1(z)dz, \tag{2.30}$$

where $G(z;x)$ is given by (2.10). In summary, we have established the form of u_1 and v_1, which are defined as (2.29) and (2.30), respectively. It can be seen that there exists constant $C > 0$ such that $\bar{V}_{10} \le C$ and $U_1 \le Cy^2 e^{-\sqrt{a}y}$.

Step 3: Construction of (u_2, v_2)

Similarly, our next goal is to balance the error generated by the logistic term. By defining $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \phi$ and $v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \psi$, we have $U_2(y) := u(x)$ and $\bar{V}_{20}(y) := \frac{1}{\epsilon^2}V_{20}(y)$ satisfy

$$U_{2yy} - (U_0\bar{V}_{20y})_y - (U_2\bar{V}_{00y})_y + \mu U_0(\bar{u} - U_0) = 0,$$

where $\bar{V}_{20yy} = -U_2$. Denote g_2 as $g_2 := \frac{U_2}{U_0} - \bar{V}_{20}$, then one similarly gets

$$(U_0 g_{2y})_y = -\mu\frac{a}{2}\operatorname{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\Big[\bar{u} - \frac{a}{2}\operatorname{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\Big]. \tag{2.31}$$

We solve (2.31) to arrive at

$$\begin{aligned} g_2 = &-\frac{\mu}{a}\bar{u}\cosh(\sqrt{a}y) + \frac{\mu}{3}\Big(\cosh(\sqrt{a}y) + 2\log\Big[\cosh\Big(\frac{\sqrt{a}y}{2}\Big)\Big] + 1\Big) \\ &+ c_6\Big(\frac{\sinh(\sqrt{a}y)}{\sqrt{a}} + y\Big) + c_7, \end{aligned} \tag{2.32}$$

where c_6 and c_7 are arbitrary constants. Letting $c_6 = c_7 = 0$, we utilize $a \sim 3\bar{u}$ to simplify (2.32) as

$$g_2 = \frac{2\mu}{3} \log\Big[\cosh\Big(\frac{\sqrt{3\bar{u}}}{2}y\Big)\Big] + \frac{\mu}{3} + O(\epsilon). \tag{2.33}$$

Noting that $\bar{V}_{20}$ satisfies

$$-\bar{V}_{20yy} - \frac{a}{2}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\bar{V}_{20} = \frac{a}{2}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)g_2 := U_0 g_2, \tag{2.34}$$

we employ the variation of parameters formula to choose $\bar{V}_{20}$ as

$$\begin{aligned}\bar{V}_{20} =& z\int_0^y U_0(r)g(r)\tilde{z}(r)\mathrm{d}r - \tilde{z}\int_0^y U_0(r)g(r)z(r)\mathrm{d}r + c_3 z + c_4\tilde{z}\\ =& z(y)\int_0^y U_0(r)g(r)\tilde{z}(r)\mathrm{d}r - \tilde{z}(y)\Big[\int_0^\infty U_0(r)g(r)z(r)\mathrm{d}r - \int_y^\infty U_0(r)g(r)z(r)\mathrm{d}r\Big]\\ &+ c_8 z + c_9\tilde{z},\end{aligned} \tag{2.35}$$

where $U_0 g$ is given in (2.34). To establish the uniformly boundedness of $\bar{V}_{20}$, we similarly choose c_9 such that

$$c_9 = \int_0^\infty U_0(r)g(r)z(r)\mathrm{d}r. \tag{2.36}$$

Meanwhile, c_8 is defined as $c_8 = -c_9\frac{\tilde{z}_y(0)}{z_y(0)}$ to satisfy the Neumann boundary condition at $y = 0$. Thus, U_2 can be expressed as

$$U_2 = U_0 g_2 + U_0\bar{V}_{20}. \tag{2.37}$$

We transform $U_2(y)$ into $u_2(x)$ in the x-variable and define v_2 as

$$v_2(x) = \int_0^L G(z;x)u_2(z)dz \tag{2.38}$$

in terms of the Neumann Green's function $G(z;x)$. Now, we obtain there exist some constant $C > 0$ such that $U_2 \le Cye^{-\sqrt{a}y}$ and $\bar{V}_{20} \le C$. By collecting (2.29), (2.30), (2.37) and (2.38), one finds u and v can be decomposed as

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \phi, \quad v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \psi. \tag{2.39}$$

It is similar to (2.16) that v_1 and v_2 can be decomposed as $v_1 = v_{10} + v_{11} + v_{12}$ and $v_2 = v_{20} + v_{21} + v_{22}$, respectively. By substituting (2.39) into $S(u)$ and defining $\Phi(y) := \phi(x)$, $\Psi(y) = \psi(x)$, we obtain

$$\begin{aligned}S(u) =& \frac{1}{\epsilon^2}\bar{L}[\Phi] + \sinh Ly\frac{a^{\frac{3}{2}}}{2\sinh L}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\\ &+ O(1)a^{\frac{3}{2}}y\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\tanh\Big(\frac{\sqrt{a}}{2}y\Big)\\ &- \epsilon^{-1}(u_0(v_{11}+v_{12})_x)_x - (u_0(v_{21}+v_{22})_x)_x - \epsilon^{-2}(u_0\psi_x)_x\\ &- \epsilon^{-1}(u_1(v_{01}+v_{02})_x)_x - (u_1 v_{1x})_x - \epsilon(u_1 v_{2x})_x - \epsilon^{-1}(u_1\psi_x)_x\\ &- (u_2(v_{01}+v_{02})_x)_x - \epsilon(u_2 v_{1x})_x - (u_2\psi_x)_x\\ &- \epsilon^{-1}(\phi v_{1x})_x - (\phi v_{2x})_x - \epsilon^{-2}(\phi\psi_x)_x\\ &- \mu u_0(\epsilon u_1 + \epsilon^2 u_2 + \phi) + \mu(\epsilon u_1 + \epsilon^2 u_2 + \phi)(\bar{u} - u_0 - \epsilon u_1 - \epsilon^2 u_2 - \phi),\end{aligned} \tag{2.40}$$

where $\bar{L}[\Phi] := \Phi_{yy} - \frac{1}{\epsilon^2}(U_0\Psi_y)_y - \frac{1}{\epsilon^2}(\Phi V_{0y})_y$. Next, we establish the equation of Φ by analyzing (2.40). By defining

$$\bar{L}_1[\Phi] = \bar{L}[\Phi] + \epsilon^2\mu\bar{u}\Phi, \tag{2.41}$$

one has from (2.40) and $S(u) = 0$ that

$$\bar{L}_1[\Phi] + F(\Phi, \Psi; \epsilon) = 0, \tag{2.42}$$

where Ψ is uniquely determined by $\Psi_{yy} - \epsilon^2\Psi + \epsilon^2\Phi = 0$ and F is defined as

$$\begin{aligned}
F(\Phi, \Psi; \epsilon) :=& \epsilon^2 \sinh Ly \frac{a^{\frac{3}{2}}}{2\sinh L}\text{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big) \\
&+ O(1)\epsilon^2 a^{\frac{3}{2}} y\text{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\tanh\Big(\frac{\sqrt{a}}{2}y\Big) \\
&- \epsilon^{-1}(U_0(V_{11} + V_{12})_y)_y - (U_0(v_{21} + v_{22})_y)_y - \epsilon^{-2}(U_0\Psi_y)_y \\
&- \epsilon^{-1}(U_1(V_{01} + V_{02})_y)_y - (U_1V_{1y})_y - \epsilon(U_1V_{2y})_y - \epsilon^{-1}(U_1\Psi_y)_y \\
&- (U_2(V_{01} + V_{02})_y)_y - \epsilon(U_2V_{1y})_y - (U_2\Psi_y)_y \\
&- \epsilon^{-1}(\Phi V_{1y})_y - (\Phi V_{2y})_y - \epsilon^{-2}(\Phi\Psi_y)_y \\
&- \epsilon^2\mu U_0(\epsilon U_1 + \epsilon^2 U_2 + \Phi) \\
&+ \epsilon^2\mu(\epsilon U_1 + \epsilon^2 U_2 + \Phi)(\bar{u} - U_0 - \epsilon U_1 - \epsilon^2 U_2 - \Phi) - \epsilon^2\mu\bar{u}\Phi.
\end{aligned} \tag{2.43}$$

Now, we have found it is equivalent to prove the existence of solution Φ to (2.42) so as to obtain the construction results stated in Theorem 1.1. Before proving this via the Lyapunov-schmidt reduction method, we need to study the property of operator $\bar{L}_1$.

3 Lyapunov-schmidt Reduction Method

In Section 3, we proceed to prove the existence of spikes to (1.3) with the ansatz given by (2.39). The approach what we shall employ is the well-known Lyapunov-schmidt reduction method [7,42,43]. This method is well-used to study the boundary layer and interior spikes arising from reaction-diffusion systems [9,10,44].

3.1 Linearized Projected Problem

Subsection 3.1 is devoted to the linear theory of $\bar{L}_1$. To begin with, we consider the following space:

$$\mathcal{H} := \Big\{u \in H_N^2\Big(\Big(0, \frac{L}{\epsilon}\Big)\Big) : \int_0^{\frac{L}{\epsilon}} u dy = 0\Big\}, \tag{3.1}$$

where $H_N^2\Big(\Big(0, \frac{L}{\epsilon}\Big)\Big)$ is defined as

$$H_N^2\Big(\Big(0, \frac{L}{\epsilon}\Big)\Big) := \Big\{u \in H^2\Big(\Big(0, \frac{L}{\epsilon}\Big)\Big) : u_y(0) = u_y\Big(\frac{L}{\epsilon}\Big) = 0\Big\}.$$

Now, we turn our attention to the following linearized problem:

$$\begin{cases} \Phi_{yy} - \frac{1}{\epsilon^2}(V_{0y}\Phi)_y - \frac{1}{\epsilon^2}(U_0\Psi_y)_y + \epsilon^2\mu\bar{u}\Phi = h + m_1W_0 & \text{in } (0, \frac{L}{\epsilon}), \\ \Psi_{yy} - \epsilon^2\Psi = -\epsilon^2\Phi & \text{in } (0, \frac{L}{\epsilon}), \\ \Phi_y(0) = \Phi_y(\frac{L}{\epsilon}) = 0, \quad \Phi \in \mathcal{H}, \end{cases} \tag{3.2}$$

where U_0, V_0 are given by (2.8) and (2.16). In addition, compactly supported function W_0 is chosen to satisfy $\int_0^{\frac{L}{\epsilon}} W_0 dy = 1$ and constant m_1 is defined as $m_1 = -\int_0^{L/\epsilon} h dy$. Moreover, the norm $\|\cdot\|_\sigma$ is defined as

$$\|h\|_\sigma = \sup_{y\in(0,\frac{L}{\epsilon})} |h(y)|e^{\sigma y} \tag{3.3}$$

for some constant $\sigma > 0$. We consider the space $\mathcal{H}$ equipped with norm (3.3), then analyze (3.2) to obtain the following proposition:

Proposition 3.1. *Assume $\|h\|_\sigma \leq +\infty$ for some constant $\sigma > 0$, then we have there exist constants $\epsilon_0 > 0$ and $C > 0$ such that (3.2) admits the unique solution (Φ, m_1) with $\Phi \in \mathcal{H}$. In particular, m_1 is defined by $m_1 = -\int_0^{\frac{L}{\epsilon}} h dy$ and Φ satisfies the following estimate:*

$$\|\Phi\|_{\sigma_1} \leq C\|\tilde{h}\|_\sigma, \tag{3.4}$$

where σ_1 is a constant and $0 < \sigma_1 < \sigma$.

Proof. We shall give an indirect proof and divide it into two steps.

Step 1: A Priori Estimates

We argue by contradiction and assume there exist sequences ϵ_n, R_n, h_n, m_1^n and Φ_n such that $\epsilon_n \to 0$, $R_n \to \infty$, $\|h_n\|_\sigma \to 0$ but $\|\Phi_n\|_{\sigma_1} \equiv 1$ as $n \to \infty$. First of all, we focus on the case of any bounded domain $B_R(0) \cap \left(0, \frac{L}{\epsilon}\right)$, where $R > 0$ is any constant. We have Φ_n and Ψ_n satisfy

$$\bar{L}_1[\Phi_n] = h_n - m_1^n W_0, \tag{3.5}$$

where $\Psi_{nyy} - \epsilon^2\Psi_n = -\epsilon^2\Phi_n$. It is similar to rewrite (3.5) as the following divergence form:

$$\begin{cases} (U_0 g_{ny})_y = h_n + \frac{1}{\epsilon^2}(\Phi_n V_{01y})_y + \frac{1}{\epsilon^2}(\Phi_n V_{02y})_y - \epsilon^2\mu\bar{u}\Phi_n + m_1^n W_0 \ \text{ in } \ B_{R_n}(0) \cap \left(0, \frac{L}{\epsilon}\right), \\ -\Psi_{nyy} + \epsilon^2\Psi_n = \epsilon^2\Phi_n, \ \ g_n := \frac{\Phi_n}{U_0} - \frac{1}{\epsilon^2}\Psi_n. \end{cases} \tag{3.6}$$

We claim that $m_1^n \to 0$ as $n \to \infty$. To prove this, we integrate the first equation in (3.6) over $(0, R_n)$ to get

$$U_0 g_{ny}\Big|_0^{R_n} = \int_0^{R_n} h_n dy + \frac{1}{\epsilon^2}[\Phi_n(V_{01} + V_{02})_y]\Big|_0^{R_n} - \epsilon^2\mu\bar{u}\int_0^{R_n}\Phi_n dy + m_1^n\int_0^{R_n} W_0 dy. \tag{3.7}$$

We further simplify (3.7) as

$$\begin{aligned} U_0(R_n)g_{ny}(R_n) - U_0(0)g_{ny}(0) = &\int_0^{R_n} h_n dy + \overbrace{\frac{1}{\epsilon^2}\Phi_n(R_n)(V_{01} + V_{02})_y(R_n)}^{I_{21}} \\ &\overbrace{-\frac{1}{\epsilon^2}(\Phi_n(0)V_{01} + V_{02})_y(0)}^{I_{22}} - \epsilon^2\mu\bar{u}\int_0^{R_n}\Phi_n dy \\ &+ m_1^n\int_0^{R_n} W_0 dy. \end{aligned} \tag{3.8}$$

It is necessary to estimate I_{21} and I_{22}. In light of $(V_{01} + V_{02})_y(0) = 0$, one has $I_{22} = 0$. On the other hand, since $V_{01} + V_{02} \lesssim \epsilon^3 y^2$, we find there exists constant $C > 0$ such that

$$|I_{21}| \leq C\epsilon(|\Phi_n(R_n)|e^{\sigma_1 R_n})e^{-\sigma_1 R_n}R_n \leq C\epsilon\|\Phi\|_{\sigma_1}e^{-\sigma_2 R_n} \to 0, \tag{3.9}$$

where $0 < \sigma_2 < \sigma_1$ is a constant. Moreover, it follows from $\|h_n\|_\sigma < +\infty$ and $h_n \to 0$ as $n \to \infty$ that $\int_0^{R_n} h_n dy \to 0$. Furthermore, $g_{ny}(0) = 0$ due to the Neumann boundary condition of g_n at $y = 0$ and $\epsilon^2\mu\bar{u}\int_0^{R_n} \Phi_n dy \to 0$ thanks to $\|\Phi_n\|_{\sigma_1} < +\infty$. Thus, we conclude from (3.8) that

$$\lim_{n\to\infty} U_0(R_n)g_{ny}(R_n) = \lim_{n\to\infty} m_1^n.$$

By using the orthogonality condition of h_n, which is $\int_0^{R_n}(h_n + m_1^n W_0)dy \to 0$, one can further obtain

$$U_0 g_{ny}|_{y=R_n} \to 0 \text{ as } n \to \infty.$$

This finishes the proof of our claim.

Noting that the right hand side of the first equation in (3.6) has perturbation terms, we need to establish the good estimate for them. By straightforward computation, we have

$$\begin{aligned}
I_{23} :=& \frac{1}{\epsilon^2}(\Phi_n V_{01y})_y + \frac{1}{\epsilon^2}(\Phi_n V_{02y})_y \\
=& -\frac{1}{\epsilon^2}\Phi_{ny}V_{01y} - \frac{1}{\epsilon^2}\Phi_n V_{01yy} - \frac{1}{\epsilon^2}\Phi_{ny}V_{02y} - \frac{1}{\epsilon^2}\Phi_n V_{02yy} \\
=& -\epsilon\frac{\sqrt{a}}{\sinh L}\cosh(\epsilon y - L)\Phi_n - \frac{1}{\epsilon^2}\Phi_{ny}(V_{01} + V_{02})_y \\
=& -\epsilon\frac{\sqrt{a}}{\sinh L}\cosh(\epsilon y - L)\Phi_n - \left[\frac{\sqrt{a}}{\sinh L}\sinh(\epsilon y - L) + \sqrt{a}\right]\Phi_{ny} \\
=& -\epsilon\frac{\sqrt{a}}{\sinh L}\cosh(\epsilon y - L)\Phi_n - [\epsilon\sqrt{a}(\coth L)y + O(\epsilon^2)y^2]\Phi_{ny}.
\end{aligned} \tag{3.10}$$

It follows from (3.10) that there exists some constant $C > 0$ such that

$$\|I_{23}\|_\sigma \le C\epsilon\|\Phi_n\|_\sigma. \tag{3.11}$$

In addition, we estimate $\epsilon^2\mu\bar{u}\Phi_n$ to obtain

$$\|\epsilon^2\mu\bar{u}\Phi_n\|_\sigma \le C\epsilon^2\|\Phi_n\|_\sigma.$$

Our next aim is to establish the limiting problem of Φ_n. To this end, we apply the standard elliptic estimate into (3.6) over any compact set in $\left(0, \frac{L}{\epsilon}\right)$. In fact, $\Phi_n \in \mathcal{H}$ implies $\Phi_n \in L^\infty$ and $\Psi_n \in L^\infty$. According to the definition of g_n, one has $g_n \in L^\infty$. It immediately follows from the interior L^p estimate that $g_n \in W^{2,p}$ for $p \in [1, +\infty)$. Moreover, the embedding Theorem tells us that $g_n \in C^{0,\alpha}$, which yields $\Psi_n \in C^{2,\alpha}$ thanks to the second equation in (3.6). Then, we again obtain from the g_n-equation that $g_{ny} \in L^\infty$. By solving the g_n-equation in (3.6), we get

$$\begin{aligned}
g_n(y) =& \frac{2}{\sqrt{a}}\int_0^y \cosh^2\left(\frac{\sqrt{a}}{2}\rho\right)\int_0^\rho [h_n(r) + m_1^n W_0]\mathrm{d}r\mathrm{d}\rho \\
&+ \frac{2}{\sqrt{a}}\int_0^y \cosh^2\left(\frac{\sqrt{a}}{2}\rho\right)\int_0^\rho [h_n(r) + m_1^n W_0]\mathrm{d}r\mathrm{d}\rho.
\end{aligned} \tag{3.12}$$

With the help of the orthogonality condition, one can estimate g_n to find

$$|g_n| \le C\|h_n\|_\sigma(e^{(\sqrt{3\bar{u}}-\sigma)y} + 1) + C\epsilon\|\Phi_n\|_\sigma e^{(\sqrt{3\bar{u}}-\sigma)y} \tag{3.13}$$

for some constant $C > 0$. Noting $|I_{21}| \to 0$ shown in (3.9), we have for n large,

$$|g_n| = o\left(e^{(\sqrt{3\bar{u}}-\sigma)y}\right) + O(\epsilon)\|\Phi_n\|_\sigma e^{(\sqrt{3\bar{u}}-\sigma)y}. \tag{3.14}$$

According to the regularity of g_n and Ψ_n, we utilize the Arzela-Ascoli's theorem to prove that for n large, Ψ_n satisfies

$$\Psi_{nyy} + U_0\Psi_n = o(\epsilon^2)e^{-\sigma y}, \ \ \Psi_{ny}(0) = 0. \tag{3.15}$$

Recall the following representation formula of Ψ_n:

$$\begin{aligned}\Psi_n =&\epsilon^2 z\int_0^y [U_0(r)g_n(r) - \Psi_n]\tilde{z}\mathrm{d}r - \epsilon^2\tilde{z}\Big[\int_0^\infty U_0(r)g_n(r)z(r)\mathrm{d}r - \int_y^\infty U_0(r)g_n(r)z(r)\mathrm{d}r\Big]\\ &- \epsilon^2\tilde{z}\int_0^y \Psi_n(r)z(r)dr + \epsilon^2 c_1^n z + \epsilon^2 c_2^n\tilde{z},\end{aligned} \tag{3.16}$$

where c_1^n and c_2^n are chosen as

$$c_1^n = -c_2^n\frac{\tilde{z}_y(0)}{z_y(0)}, \ \ c_2^n = \int_0^\infty U_0(r)g_n(r)z(r)\mathrm{d}r, \tag{3.17}$$

then we have from (3.14) that $g_n = o(e^{-\sigma y})$, which yields $c_1^n = o(1)$ and $c_2^n = o(1)$. By considering $\bar{\Psi}_n := \frac{1}{\epsilon^2}\Psi_n$ and using (3.16), it is easy to find

$$|\bar{\Psi}_n| = o(1) + O(\epsilon^2)y^2|\bar{\Psi}_n| + O(\epsilon)\|\Phi_n\|_\sigma, \tag{3.18}$$

which implies $|\bar{\Psi}_n| = o(1) + O(1)\epsilon\|\Phi_n\|_\sigma$. Noting $\Phi_n = U_0 g_n + U_0\bar{\Psi}_n$, we combine (3.14) and (3.18) to obtain $\|\Phi_n\|_{\sigma(0,R_n)} = o(1)$ for R_n large.

We next develop the outer estimate of Φ_n and focus on the region $\left(0, \frac{L}{\epsilon}\right)\backslash B_R(0)$ for R sufficiently large but fixed. It is convenient to rewrite the Neumann Green's function $G(z;x)$ in the y-variable. Recall that $G(z;y)$ satisfies

$$\begin{cases} G_{yy} - \epsilon^2 G = -\delta(y - y_0), & y \in (0, \frac{L}{\epsilon}),\\ G_y(0) = G_y(\frac{L}{\epsilon}) = 0,\end{cases}$$

then we solve it to get the following explicit form of G:

$$G = \begin{cases} \frac{\cosh(\epsilon y - L)}{\epsilon\sinh L}\cosh(\epsilon y), & y \in (0, y_0),\\ \frac{\cosh(\epsilon y_0)}{\epsilon\sinh L}\cosh(\epsilon y - L), & y \in (y_0, \frac{L}{\epsilon}).\end{cases} \tag{3.19}$$

Thus, we employ the representation formula to estimate Ψ_n and obtain

$$\begin{aligned}\Psi_n(y) =&\epsilon^2\int_0^{\frac{L}{\epsilon}} G(z;y)\Phi_n(z)dz\\ \lesssim&\frac{\epsilon\cosh(\epsilon y - L)}{\sinh L}\int_0^y \cosh(\epsilon z)e^{-\sigma_1 z}dz\|\Phi_n\|_{\sigma_1} + \frac{\epsilon\cosh(\epsilon y)}{\sinh L}\int_y^{\frac{L}{\epsilon}}\cosh(\epsilon z - L)e^{-\sigma_1 z}dz\|\Phi_n\|_{\sigma_1}\\ \lesssim&\epsilon\frac{\cosh(\epsilon y - L)}{\sinh L}\|\Phi_n\|_{\sigma_1} + O(\epsilon)\|\Phi_n\|_{\sigma_1}\\ =&O(1)(\epsilon + \epsilon^2 y)\|\Phi_n\|_{\sigma_1}.\end{aligned}$$

It is similar to estimate Ψ_{ny} and Ψ_{nyy}, which are

$$\begin{aligned}\Psi_{ny} \lesssim&\frac{\epsilon^2\sinh(\epsilon y - L)}{\sinh L}\int_0^y \cosh(\epsilon z)e^{-\sigma_1 z}dz\|\Phi_n\|_{\sigma_1} + \frac{\epsilon\cosh(\epsilon y - L)}{\sinh L}\cosh(\epsilon y)e^{-\sigma_1 y}\|\Phi_n\|_{\sigma_1}\\ &+ \frac{\epsilon^2\sinh(\epsilon y)}{\sinh L}\int_y^{\frac{L}{\epsilon}}\cosh(\epsilon z - L)\epsilon^{-\sigma_1 z}dz\|\Phi\|_{\sigma_1} - \frac{\epsilon\cosh(\epsilon y)}{\sinh L}\cosh(\epsilon y - L)\epsilon^{-\sigma_1 y}\|\Phi\|_{\sigma_1}\\ =&O(1)\epsilon^2\|\Phi\|_{\sigma_1},\end{aligned}$$

and $\Psi_{yy} = O(1)\epsilon^2\|\Phi\|_{\sigma_1}$. Thanks to these inequalities, we establish the estimate for the nonlocal term $U_0\Psi_{ny}$, which is shown as follows:

$$\begin{aligned}\frac{1}{\epsilon^2}|(U_0\Psi_{ny})_y| =&\frac{1}{\epsilon^2}|(U_{0y}\Psi_{ny} + U_0\Psi_{nyy})| \\ \lesssim& e^{-\sqrt{3\bar{u}}y}\|\Phi_n\|_{\sigma_1} \lesssim e^{-\sqrt{\bar{u}}R}e^{-\sigma y}\|\Phi_n\|_{\sigma_1}, \quad y \in \left(R, \frac{L}{\epsilon}\right).\end{aligned}$$

Then, we simplify the first equation in (3.6) as

$$\begin{cases} \left(\Phi_{ny} - \frac{1}{\epsilon^2}V_{0y}\Phi_n\right)_y + \epsilon^2\mu\bar{u}\Phi_n = O(1)h_n, \quad y \in (R, \frac{L}{\epsilon}), \\ \Phi_n(\frac{L}{\epsilon}) = 0. \end{cases} \tag{3.20}$$

The inner estimate tells us $\|\Phi_n\|_{\sigma_1(0,R)} \to 0$ for any $R > 0$. Hence, we have for R sufficiently large but fixed,

$$|\Phi_n(R)| = o(1)e^{-\sigma_1 R},$$

where $\sigma_1 < \sigma$ is some constant. We next integrate the Φ-equation in (3.20) over $(R, \frac{L}{\epsilon})$ and use the Neumman boundary condition to obtain

$$\Phi_{ny} - \frac{1}{\epsilon^2}V_{0y}\Phi_n + \epsilon^2 O(1)e^{-\sigma y}\|\Phi_n\|_\sigma = O(1)h_n, \tag{3.21}$$

hence we solve (3.21) to get

$$\begin{aligned}\Phi_n(y) =&\Phi_n(R)e^{\frac{V_0(\epsilon y)-V_0(\epsilon R)}{\epsilon^2}} + O(1)\int_R^y e^{\frac{V_0(\epsilon y)-V_0(\epsilon s)}{\epsilon^2}}e^{-\sigma s}ds\|h_n\|_\sigma \\ &+ O(\epsilon^2)\int_R^y e^{\frac{V_0(\epsilon y)-V_0(\epsilon s)}{\epsilon^2}}e^{-\sigma s}ds\|\Phi_n\|_\sigma.\end{aligned} \tag{3.22}$$

To estimate Φ_n in (3.22), we first recall V_0 satisfies

$$V_{0x}(\epsilon y) = \frac{\sqrt{a}}{\sinh L}\epsilon\sinh(\epsilon y - L) + O(\epsilon^2)$$

and

$$\frac{V_0(\epsilon y) - V_0(\epsilon R)}{\epsilon^2} = \frac{V_{0x}(\epsilon\xi)}{\epsilon}(y - y_0)$$

where $\xi \in (R, y)$, then we find when $L - \epsilon y > C\sigma$ for some constant $C > 0$,

$$\frac{V_{0x}(\epsilon\xi)}{\epsilon} = \frac{\sinh(\epsilon\xi - L)}{\sinh L} + O(\epsilon) \leq -2\sigma \text{ for } \epsilon \ll 1,$$

and thereby one has the first term in (3.22) satisfies

$$\left|\Phi_n(R)e^{\frac{V_0(\epsilon y)-V_0(\epsilon R)}{\epsilon^2}}\right| = o(1)e^{-\sigma_1 R}e^{-2\sigma(y-R)}.$$

Concerning the second term in (3.22), we similarly obtain when $L - \epsilon y > C\sigma$,

$$\int_R^y e^{\frac{V_0(\epsilon y)-V_0(\epsilon s)}{\epsilon^2}}e^{-\sigma s}ds\|h_n\|_\sigma \lesssim \int_{y_0}^y e^{-2\sigma(y-s)}e^{-\sigma s}ds\|h_n\|_\sigma \lesssim e^{-\sigma y}\|h_n\|_\sigma.$$

For the third term, one can show

$$\epsilon^2\int_R^y e^{\frac{V_0(\epsilon y)-V_0(\epsilon s)}{\epsilon^2}}e^{-\sigma s}ds\|\Phi_n\|_\sigma \lesssim \epsilon^2\int_{y_0}^y e^{-2\sigma(y-s)}e^{-\sigma s}ds\|\Phi_n\|_\sigma \lesssim \epsilon^2 e^{-\sigma y}\|\Phi_n\|_\sigma.$$

By summarizing the above estimates, we have for $L - \epsilon y > C\sigma$,

$$\|\Phi_n\|_\sigma = o(1)e^{-\sigma_1 R}e^{\sigma y}e^{-2\sigma(y-R)} + o(1) + \epsilon^2\|\Phi_n\|_\sigma, \tag{3.23}$$

where we use the fact that $\|h_n\|_\sigma \to 0$ and $|\Phi_n(R)| = o(1)$. It follows that

$$\|\Phi_n\|_\sigma = o(1)e^{-\sigma_1 R}e^{\sigma y}e^{-2\sigma(y-R)} + o(1) \to 0. \tag{3.24}$$

We next discuss the region for $y \in (\frac{L-C\sigma}{\epsilon}, \frac{L}{\epsilon})$. In this case, we have from (3.24) and continuity that

$$\left|\Phi_n\Big(\frac{L-C\sigma}{\epsilon}\Big)\right| = o(1)e^{-\sigma_1 R}e^{-2\sigma(\frac{L-C\sigma}{\epsilon}-R)} + o(1)e^{-\sigma\frac{L-C\sigma}{\epsilon}}.$$

Then we rewrite the Φ-equation (3.20) in $(\frac{L-\bar{C}\sigma}{\epsilon}, \frac{L}{\epsilon})$ as

$$\begin{aligned}&\Big(\Phi_{ny} - \frac{1}{\epsilon^2}V_{0y}\Phi_n\Big)_y + \epsilon^2\mu\bar{u}\Phi_n + O(e^{-\sigma y})\|h_n\|_\sigma\\ =&\Phi_{nyy} - \frac{1}{\epsilon^2}V_{0y}\Phi_{ny} - \frac{1}{\epsilon^2}V_{0yy}\Phi_n + \epsilon^2\mu\bar{u}\Phi_n + o(e^{-\sigma\frac{L-C\sigma}{\epsilon}}) = 0.\end{aligned} \tag{3.25}$$

Noting that V_0 satisfies

$$\frac{1}{\epsilon^2}V_{0yy} = V_0 = O(\epsilon)\cosh(x-L),$$

we apply the Maximum Principle into (3.25) to obtain for $\frac{L-C\sigma}{\epsilon} < y < \frac{L}{\epsilon}$,

$$|\Phi_n(y)| = o(1)e^{-\sigma_1 R}e^{-2\sigma(\frac{L-C\sigma}{\epsilon}-R)} + o(1)e^{-\sigma\frac{L-C\sigma}{\epsilon}} + o\Big(\frac{1}{\epsilon}\Big)e^{-\sigma\frac{L-C\sigma}{\epsilon}}.$$

which yields

$$\|\Phi_n\|_{\sigma_1} = o(1) \;\text{ for }\; \frac{L-C\sigma}{\epsilon} < y < \frac{L}{\epsilon}.$$

In summary, we have $\|\Phi_n\|_{\sigma_1((0,L/\epsilon)\setminus B_R(0))} \to 0$ as $n \to \infty$.

Therefore, by combining the arguments for the inner and outer estimates, we can find $\|\Phi_n\|_{\sigma(0,L/\epsilon)} \to 0$, which is contradicted to $\|\Phi_n\|_{\sigma(0,L/\epsilon)} \equiv 1$. This gives the desired estimate (3.4).

Step 2: Existence of Φ

Assume $\Phi \in \mathcal{H}$, then the equation of Φ stated in (3.6) can be rewritten as

$$\Phi + K(\Phi) = \bar{h} \;\text{ in }\; \mathcal{H},$$

where $\bar{h}$ is the duality and $K : \mathcal{H} \to \mathcal{H}$ is a linear compact operator. Thanks to the Fredholm alternative theorem, we have there exists a unique solution for each $\bar{h}$ is equivalent to show the homogeneous equation has a unique solution with $\bar{h} = 0$. A priori estimates shown as above yield the homogeneous problem admits the unique solution $\Phi \equiv 0$. This indicates the existence of Φ to 3.6, then finishes the proof of this Proposition. □

Remark 3.1. *Proposition 3.1 demonstrates that for any $\Phi \in \mathcal{H}$ being the unique solution to (3.6), we can write Φ as $\Phi = \mathcal{A}(h)$, which satisfies*

$$\|\mathcal{A}(h)\|_{\sigma_1} \le C\|h\|_\sigma \tag{3.26}$$

for some constant $C > 0$.

In the following subsection, we shall develop the nonlinear theory for operator $\bar{L}_1$ shown in (3.6).

3.2 Nonlinear Projected Problem

We are concerned with the following nonlinear problem:

$$\begin{cases} \bar{L}_1[\Phi] := \Phi_{yy} - \frac{1}{\epsilon^2}(U_0\Psi_y)_y - \frac{1}{\epsilon^2}(\Phi V_{0y})_y = m_1 W_0 - F, \\ \Phi \in \mathcal{H}, \end{cases} \tag{3.27}$$

where W_0 is a compact supported function satisfying $\int_0^{L/\epsilon} W_0 dy = 1$, constant m_1 is chosen such that $m_1 = \int_0^{\frac{L}{\epsilon}} F dy$ and

$$\begin{aligned} F :=& \epsilon^2 \frac{a^{\frac{3}{2}}}{2\sinh L} \sinh L y \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big) \\ &+ O(\epsilon^2) a^{\frac{3}{2}} y \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\tanh\Big(\frac{\sqrt{a}}{2}y\Big) \\ &- \epsilon^{-1}(U_0(V_{11}+V_{12})_y)_y - (U_0(V_{21}+V_{22})_y)_y - \epsilon^{-2}(U_0\Psi_y)_y \\ &- \epsilon^{-1}(U_1(V_{01}+V_{02})_y)_y - (U_1 V_{1y})_y - \epsilon(U_1 V_{2y})_y - \epsilon^{-1}(U_1\Psi_y)_y \\ &- (U_2(V_{01}+V_{02})_y)_y - \epsilon(U_2 V_{1y})_y - (U_2\Psi_y)_y \\ &- \epsilon^{-1}(\Phi V_{1y})_y - (\Phi V_{2y})_y - \epsilon^{-2}(\Phi\Psi_y)_y \\ &- \epsilon^2\mu U_0(\epsilon U_1 + \epsilon^2 U_2 + \Phi) + \epsilon^2\mu(\epsilon U_1 + \epsilon^2 U_2 + \Phi)(\bar{u} - U_0 - \epsilon U_1 - \epsilon^2 U_2 - \phi) - \epsilon^2\mu\bar{u}\Phi. \end{aligned}$$

Before establishing the good estimate for F, we decompose F as $F = E + N(\Phi)$, where E and $N(\Phi)$ are defined as

$$\begin{aligned} E :=& \epsilon^2 \frac{a^{\frac{3}{2}}}{2\sinh L} \sinh L y \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big) + O(\epsilon^2) a^{\frac{3}{2}} y \mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\tanh\Big(\frac{\sqrt{a}}{2}y\Big) \\ &- \epsilon^{-1}(U_0(V_{11}+V_{12})_y)_y - (U_0(V_{21}+V_{22})_y)_y \\ &- \epsilon^{-1}(U_1(V_{01}+V_{02})_y)_y - (U_1 V_{1y})_y - \epsilon(U_1 V_{2y})_y \\ &- \epsilon^2\mu U_0(\epsilon U_1 + \epsilon^2 U_2) + \epsilon^2\mu(\epsilon U_1 + \epsilon^2 U_2)(\bar{u} - U_0 - \epsilon U_1 - \epsilon^2 U_2), \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} N(\Phi) =& -\epsilon^{-2}(U_0\Psi_y)_y - \epsilon^{-1}(U_1\Psi_y)_y - (U_2\Psi_y)_y \\ &- \epsilon^{-1}(\Phi V_{1y})_y - (\Phi V_{2y})_y - \epsilon^{-2}(\Phi\Psi_y)_y \\ &- \epsilon^2\mu U_0\Phi - \epsilon^2\mu(\epsilon U_1 + \epsilon^2 U_2 + \Phi)\Phi + \epsilon^2\mu\Phi(-U_0 - \epsilon U_1 - \epsilon^2 U_2 - \Phi). \end{aligned} \tag{3.29}$$

With the help of Proposition 3.1, we rewrite the solution Φ to (3.27) as

$$\Phi = -\mathcal{A}(F(\Phi)) = -\mathcal{A}(E + N(\Phi)), \tag{3.30}$$

where $\mathcal{A}$ is defined as (3.26). Now, we transform our problem to find a fixed point for the operator T, which is given by

$$T(\Phi) := -\mathcal{A}(E + N(\Phi)). \tag{3.31}$$

Our next goal is to show that the operator T is a contraction mapping on

$$\mathcal{B} = \Big\{\Phi \in H_N^2((0, L/\epsilon)) : \|\Phi\|_{\sigma_1} \le C\epsilon^2, \int_0^{\frac{L}{\epsilon}} \Phi dy = 0\Big\} \tag{3.32}$$

for some large $C > 0$. Before proving it, we need to show the following Lemma:

Lemma 3.1. *There exist constants $\epsilon_0 > 0$, $C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the following estimates hold:*

$$\|E\|_\sigma \leq C\epsilon^2, \tag{3.33}$$

and

$$\|N(\Phi)\|_\sigma = o(1)\|\Phi\|_{\sigma_1}. \tag{3.34}$$

Proof. We analyze (3.28) and find the worse term is $U_0(V_{11} + V_{12})_y)_y$. Since $|V_{11} + V_{12}| \lesssim \epsilon^3 y^2$, we estimate this term to obtain

$$\epsilon^{-1}|(U_0(V_{11} + V_{12})_y)_y| \lesssim \epsilon^2 e^{-\sigma y}$$

for some constant $C > 0$. Thus, we have there exists constant $C > 0$ such that

$$\|E\|_\sigma \leq C\epsilon^2,$$

which proves estimate (3.33).

We next investigate the nonlinear error $N(\Phi)$. It can be observed that the terms involving U_0, U_1 and U_2 have good estimates since U_0, U_1 and U_2 have fast decay properties. Focusing on the error shown in the divergence form, we find the one involving Φ need to be estimated in a delicate way. Taking $\epsilon^{-1}(\Phi V_{1y})_y$ as an example, we find from $V_1 + V_2 \lesssim \epsilon^3 y^2$ that when $y \in (0, R)$ with R being large but fixed,

$$\epsilon^{-1}(\Phi V_{1y})_y = O(1)\epsilon^2(\Phi y)_y = O(1)\epsilon^2(\Phi_y y + \Phi). \tag{3.35}$$

By choosing σ and σ_1 such that $C\sigma^2 + \epsilon|\ln\epsilon| < (\sigma - \sigma_1)L < 2\epsilon|\ln\epsilon|$ and $\epsilon|\ln\epsilon| \lesssim \sigma \lesssim \sqrt{\epsilon|\ln\epsilon|}$, we then have from (3.35) that

$$\epsilon^{-1}\|(\Phi V_{1y})_y\|_\sigma = o(1)\|\Phi\|_{\sigma_1} \text{ for } y \in (0, R). \tag{3.36}$$

It is similar to tackle the other terms only involving Φ and Ψ in the error. Finally, we are concentrated at the nonlinear growth and find the only worse term is $\epsilon^2\mu\Phi^2$. By choosing σ and σ_1 such that $0 < 2\sigma_1 \leq \sigma$, we obtain

$$\epsilon^2\mu\|\Phi^2\|_\sigma = o(1)\|\Phi\|_{\sigma_1}.$$

In summary, we have proved (3.34) and finished the proof of this Lemma. □

Now, we are well prepared to study operator T and the results are summarized as follows:

Proposition 3.2. *There exist positive numbers ϵ_0 and C such that for all $\epsilon \in (0, \epsilon_0)$, there is a unique solution (Φ, m_1) to (3.27). Moreover, Φ satisfies*

$$\|\Phi\|_{\sigma_1} \leq C\epsilon^2. \tag{3.37}$$

Proof. We firstly show that T is a mapping from $\mathcal{B}$ to $\mathcal{B}$. Indeed, for any $\mathcal{B}$, we have from Lemma 3.1 that

$$\|T(\Phi)\|_{\sigma_1} \leq C\|F\|_\sigma \leq C\|E + N(\Phi)\|_\sigma \leq C(\epsilon^2 + o(1)\|\Phi\|_{\sigma_1}) \leq C\epsilon^2, \tag{3.38}$$

which indicates T maps on $\mathcal{B}$ into itself. Next, we will prove operator T is a contraction. By taking Φ_1 and Φ_2 in $\mathcal{B}$, thanks to Lemma 3.1, one finds there exists constant $C > 0$ such that

$$\begin{aligned}\|T(\Phi_1) - T(\Phi_2)\|_{\sigma_1} \leq& C\|N(\Phi_1) - N(\Phi_2)\|_\sigma \\ =& o(1)\|\Phi_1 - \Phi_2\|_{\sigma_1},\end{aligned} \tag{3.39}$$

which implies T is a contraction mapping from $\mathcal{B}$ to itself. Therefore, we obtain the existence result according to the contraction mapping theorem. □

Proposition 3.2 demonstrates that if the right hand side h satisfy the orthogonality condition, (3.6) admits the unique solution Φ with the good estimate. It is necessary to verify this orthogonality condition, which are discussed in the following subsection.

3.3 Reduced Problem

Now, we look for the next order of a given in (2.8) to guarantee the orthogonality condition. Noting that $\bar{L}_1[\Phi] = F$ given by (2.42), we integrate F over $(0, \frac{L}{\epsilon})$ and find the leading term in the divergence form is from $\frac{1}{\epsilon^2}(U_0(V_{01}+V_{02})_y)_y$. Since $V_{01}+V_{02} = \epsilon\sqrt{a}\coth L + O(\epsilon^3)y^2 + O(\epsilon^4)y^3$, we rewrite it as

$$\frac{1}{\epsilon^2}(U_0(V_{01}+V_{02})_y)_y = O(\epsilon)(U_0(y^2+\epsilon y^3)_y)_y. \tag{3.40}$$

Since the term $(U_0(y^2)_y)_y$ in (3.40) was balanced by U_1, we find the next order term is $(U_0(y^3)_y)_y$ and the corresponding integral satisfies

$$O(\epsilon^2)\int_0^{\frac{L}{\epsilon}}(U_0(y^3)_y)_y dy = O(\epsilon^2)\Big(\frac{L}{\epsilon}\Big)^2\operatorname{sech}^2\Big(\frac{L}{\epsilon}\Big) = o(\epsilon^3). \tag{3.41}$$

Similarly, we use the divergence theorem, then obtain from the exponential decay property of U_0 and Neumann boundary condition that all the terms in the divergence form are both $o(\epsilon^3)$. On the other hand, focusing on the logistic growth, we find

$$\begin{aligned}\epsilon^2\int_0^{\frac{L}{\epsilon}} f(U)dy =&\epsilon^2\int_0^{\frac{L}{\epsilon}} U_0(\bar{u}-U_0)dy + O(\epsilon^3)\\ =&\frac{a\epsilon^2}{2}\int_0^{\frac{L}{\epsilon}}\operatorname{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\Big(\bar{u}-\frac{a}{2}\operatorname{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big)\Big)dy + O(\epsilon^3)\\ =&\sqrt{a}\epsilon^2\int_0^{\frac{\sqrt{a}L}{2\epsilon}}\operatorname{sech}^2 z\Big(\bar{u}-\frac{\sqrt{a}}{2}\operatorname{sech}^2 z\Big)dz + O(\epsilon^3).\end{aligned} \tag{3.42}$$

Define a as $a = a_0 + a_1$ with $a_0 = 3\bar{u}$, then we obtain

$$\epsilon^2\int_0^{\frac{L}{\epsilon}} f(U)dy = \epsilon^2\sqrt{3\bar{u}+a_1}\int_0^{\frac{\sqrt{3\bar{u}+a_1}L}{2\epsilon}}\operatorname{sech}^2 z\Big(\bar{u}-\frac{\sqrt{3\bar{u}+c_1}}{2}\operatorname{sech}^2 z\Big)dz. \tag{3.43}$$

Since $\int_0^\infty \operatorname{sech}^2 z\big(\bar{u}-\frac{\sqrt{3\bar{u}}}{2}\operatorname{sech}^2 z\big)dz = 0$, we combine (3.41) and (3.43) to get

$$a_1 = O(\epsilon), \tag{3.44}$$

which implies $a = 3\bar{u} + O(\epsilon)$.

In summary, we have finished the rigorous proof of Theorem 1.1 via Lyapunov-schmidt reduction method. A natural question is whether the non-constant steady state constructed in Theorem 1.1 is linear stable.

4 Stability Analysis

In this section, we study the linearized stability of the single boundary spike (1.4). Define

$$u(x,t) = u^-(x) + e^{\lambda t}\phi(x),\quad v(x,t) = v^-(x) + e^{\lambda t}\psi(x),\quad |\phi|,\ |\psi| \ll 1,$$

then we substitute them into (1.2) to collect the following linearized eigenvalue problem:

$$\begin{cases} \lambda\phi = \phi_{xx} - \chi(u\psi_x + \phi v_x)_x + \mu(\bar{u} - 2u)\phi, & x \in (0, L), \\ \lambda\psi = \psi_{xx} - \psi + \phi, & x \in (0, L), \\ \phi_x(0) = \psi_x(0) = \phi_x(L) = \psi_x(L) = 0, \end{cases} \tag{4.1}$$

where $(\phi, \psi) \in H^2_N((0, L)) \times H^2_N((0, L))$ and $H^2_N((0, L))$ is defined as

$$H^2_N((0, L)) = \{w \in H^2((0, L)) | w_x(0) = w_x(L) = 0\}.$$

We define

$$y := \frac{x}{\epsilon},\ \ \tau := \epsilon^2\lambda,\ \ , U(y) := u(x),\ \ \bar{V}(y) := \frac{1}{\epsilon^2}v(x),\ \ \Phi(y) := \phi(x),\ \ \Psi(y) := \psi(x),\ \ \bar{\Psi}(y) := \frac{1}{\epsilon^2}\Psi(y)$$

and rescale (4.1) to obtain the following system:

$$\begin{cases} \tau\Phi = \Phi_{yy} - (U\bar{\Psi}_y + \Phi\bar{V}_y)_y + \epsilon^2\mu(\bar{u} - 2U)\Phi, & y \in (0, L/\epsilon), \\ \tau\bar{\Psi} = \bar{\Psi}_{yy} - \epsilon^2\bar{\Psi} + \Phi, & y \in (0, L/\epsilon), \\ \Phi(0) = \Phi(L/\epsilon) = \bar{\Psi}(0) = \bar{\Psi}(L/\epsilon). \end{cases} \tag{4.2}$$

First of all, we give a priori estimates for $|\tau|$, which is

Proposition 4.1. *Assume $Re(\tau) \geq 0$, then we have $|\tau| \leq C$, where C is a large constant independent of ϵ.*

Proof. First of all, we multiply the first equation of (4.2) by Φ^* and integrate it over $(0, L/\epsilon)$ to obtain

$$\tau\int_0^{\frac{L}{\epsilon}} |\Phi|^2 dy = -\int_0^{\frac{L}{\epsilon}} |\Phi_y|^2 dy + \int_0^{\frac{L}{\epsilon}} \Phi_y(U\bar{\Psi}_y + \Phi\bar{V}_y)dy + \epsilon^2\mu\int_0^{\frac{L}{\epsilon}} (\bar{u} - 2U)|\Phi|^2 dy. \tag{4.3}$$

We rearrange (4.3) to get from Young's inequality that

$$\left|\tau\int_0^{\frac{L}{\epsilon}} |\Phi|^2 dy + \int_0^{\frac{L}{\epsilon}} |\Phi_y|^2 dy\right| \leq \frac{1}{2}\int_0^{\frac{L}{\epsilon}} |\Phi_y|^2 dy + C\int_0^{\frac{L}{\epsilon}} |\Phi|^2 dy + C\int |\bar{\Psi}_y|^2 dy + C\int_0^{\frac{L}{\epsilon}} |\Phi|^2 dy, \tag{4.4}$$

where $C > 0$ is a constant. Similarly, we test the second equation of (4.2) against $\bar{\Psi}^*$, then integrate by parts over $(0, L/\epsilon)$ to arrive at

$$\left|(\epsilon^2 + \tau)\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 dy + \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 dy\right| \leq \frac{1}{2}\int_0^{\frac{L}{\epsilon}} |\Phi|^2 dy + \frac{1}{2}\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 dy. \tag{4.5}$$

Since $Re(\tau) \geq 0$, we claim that

$$\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 dy \leq C\int_0^{\frac{L}{\epsilon}} |\Phi|^2 dy,$$

where $C > 0$ is a constant independent of ϵ. To prove this, we analyze the left hand side of (4.5) then obtain from Young's inequality that

$$\begin{aligned} &\left|(\tau_R + \epsilon^2)\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 dy + \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 dy + i\tau_I\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 dy\right| \\ \geq&\left[(\tau_R + \epsilon^2)^2\Big(\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 dy\Big)^2 + \Big(\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 dy\Big)^2 + \tau_I^2\Big(\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 dy\Big)^2\right]^{\frac{1}{2}} \\ \geq&\frac{|\tau|}{2}\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 dy + \frac{1}{2}\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 dy. \end{aligned} \tag{4.6}$$

Suppose $|\tau| \geq 1$, we have from (4.6) that

$$\frac{1}{2}\int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 \mathrm{d}y \leq \frac{1}{2}\int_0^{\frac{L}{\epsilon}} |\Phi|^2 \mathrm{d}y, \tag{4.7}$$

which finishes the proof of our claim. Otherwise if $|\tau| \leq 1$, we have proved the uniformly boundedness of $|\tau|$.

Now, we focus on (4.4) and similarly find the left hand side satisfies

$$\begin{aligned} \Big|\tau\int_0^{\frac{L}{\epsilon}} |\Phi|^2 \mathrm{d}y + \int_0^{\frac{L}{\epsilon}} |\Phi_y|^2 \mathrm{d}y\Big| =& \Big[\Big(\tau_R\int_0^{\frac{L}{\epsilon}} |\Phi|^2 \mathrm{d}y + \int_0^{\frac{L}{\epsilon}} |\Phi_y|^2 \mathrm{d}y\Big)^2 + \tau_I^2\Big(\int_0^{\frac{L}{\epsilon}} |\Phi|^2 \mathrm{d}y\Big)^2\Big]^{\frac{1}{2}} \\ \geq& \Big[|\tau|^2\Big(\int_0^{\frac{L}{\epsilon}} |\Phi|^2 \mathrm{d}y\Big)^2 + \Big(\int_0^{\frac{L}{\epsilon}} |\Phi_y|^2 \mathrm{d}y\Big)^2\Big]^{\frac{1}{2}} \\ \geq& \frac{|\tau|}{2}\int_0^{\frac{L}{\epsilon}} |\Phi|^2 \mathrm{d}y + \frac{1}{2}\int_0^{\frac{L}{\epsilon}} |\Phi_y|^2 \mathrm{d}y. \end{aligned} \tag{4.8}$$

Combining (4.4), (4.7) and (4.8), we show that for $C > 0$ independent of ϵ,

$$|\tau|\int_0^{\frac{L}{\epsilon}} |\Phi|^2 \mathrm{d}y \leq C\int_0^{\frac{L}{\epsilon}} |\Phi|^2 \mathrm{d}y, \tag{4.9}$$

which proves $|\tau| \leq C$. □

We next rule out the large eigenvalue case for (4.2). In other words, we shall prove the instability of (1.4) only probably occurs when $\tau \to 0$ as $\epsilon \to 0$. In fact, Proposition 4.1 indicates $|\tau|$ is uniformly bounded in ϵ, then one can choose the subsequence U_n, $\bar{V}_n := \chi V_n$, Φ_n, $\bar{\Psi}_n := \chi\Psi_n$ and τ_n such that as $\epsilon_n \to 0$, $U_n \to U_0$, $\bar{V}_n \to \bar{V}_0$, $\Phi_n \to \Phi_0$, $\bar{\Psi}_n \to \bar{\Psi}_0$ and $\tau_n \to \tau_0$ over any compact set. Moreover, they satisfy the following limiting problem:

$$\begin{cases} \tau_0 U_0(g_0 + \bar{\Psi}_0) = (U_0 g_{0y})_y, & y \in (0,\infty), \\ \tau_0\bar{\Psi}_0 = \bar{\Psi}_{0yy} + U_0(g_0 + \bar{\Psi}_0), & y \in (0,\infty), \\ \Phi_{0y}(0) = \bar{\Psi}_{0y}(0) = \Phi_{0y}(\infty) = \bar{\Psi}_{0y}(\infty) = 0, \quad g_0 = \frac{\Phi_0}{U_0} - \bar{\Psi}_0. \end{cases} \tag{4.10}$$

Since the structure of (4.10) is the same as the minimal Keller–Segel models, we follow the same approach in [22, 50] to analyze it. As a consequence, we have the following Proposition:

Proposition 4.2. *Assume that $(\tau_0, < g_0, \bar{\Psi}_0 >)$ is the solution to (4.10) and $\tau_0 \neq 0$, then we have τ_0 is real.*

Proof. First of all, we multiply the first equation in (4.10) by g_0^* and integrate it over $(0,\infty)$ to obtain

$$-\int_0^\infty U_0|g_{0y}|^2 \mathrm{d}y = \tau_0\int_0^\infty U_0|g_0|^2 \mathrm{d}y + \tau_0\int_0^\infty U_0 g_0^*\bar{\Psi}_0 \mathrm{d}y. \tag{4.11}$$

Then, we test the conjugate of the second equation in (4.10) against $\tau_0\bar{\Psi}_0$ and integrate by parts to get

$$|\tau_0|^2\int_0^\infty |\bar{\Psi}_0|^2 \mathrm{d}y = -\tau_0\int_0^\infty |\bar{\Psi}_{0y}|^2 \mathrm{d}y + \tau_0\int_0^\infty U_0 g_0^*\bar{\Psi}_0 \mathrm{d}y + \tau_0\int_0^\infty U_0|\bar{\Psi}_0|^2 \mathrm{d}y. \tag{4.12}$$

We further subtract (4.12) from (4.11) to find

$$\begin{aligned} &|\tau_0|^2\int_0^\infty |\bar{\Psi}_0|^2 \mathrm{d}y + \int_0^\infty U_0|g_{0y}|^2 \mathrm{d}y \\ =&\tau_0\Big[-\int_0^\infty |\bar{\Psi}_{0y}|^2 \mathrm{d}y + \int_0^\infty U_0|\bar{\Psi}_0|^2 \mathrm{d}y\Big] - \tau_0\int_0^\infty U_0|g_0|^2 \mathrm{d}y. \end{aligned} \tag{4.13}$$

To simplify (4.13), we define $\mathcal{L}\bar{\Psi}_0$ such that

$$< \mathcal{L}\bar{\Psi}_0, \bar{\Psi}_0 >:= \int_0^\infty U_0|\bar{\Psi}_0|^2 \mathrm{d}y - \int_0^\infty |\bar{\Psi}_{0y}|^2 \mathrm{d}y - \frac{|\int_0^\infty \bar{\Psi}_0 U_0 \mathrm{d}y|^2}{\int_0^\infty U_0 \mathrm{d}y}, \tag{4.14}$$

then we rewrite (4.13) as

$$\begin{aligned}
&\tau_0\Big[-\int_0^\infty |\bar{\Psi}_{0y}|^2 \mathrm{d}y + \int_0^\infty U_0|\bar{\Psi}_0|^2 \mathrm{d}y\Big] - \tau_0 \int_0^\infty U_0|g_0|^2 \mathrm{d}y \\
=&\tau_0 < \mathcal{L}\bar{\Psi}_0, \bar{\Psi}_0 > +\tau_0 \frac{\Big|\int_0^\infty \bar{\Psi}_0 U_0 \mathrm{d}y\Big|^2}{\int_0^\infty U_0 \mathrm{d}y} - \tau_0 \int_0^\infty U_0|g_0|^2 \mathrm{d}y.
\end{aligned} \tag{4.15}$$

On the other hand, according to the first equation shown in (4.10), we have $\int_0^\infty \Phi_0 dy = 0$ since $\tau_0 \neq 0$. Hence, we find

$$\int_0^\infty \Phi_0 \mathrm{d}y = \int_0^\infty U_0\bar{\Psi}_0 + U_0 g_0 \mathrm{d}y = 0, \tag{4.16}$$

which implies

$$\int_0^\infty U_0\bar{\Psi}_0 \mathrm{d}y = -\int_0^\infty U_0 g_0 \mathrm{d}y. \tag{4.17}$$

It follows that

$$\int_0^\infty U_0|g_0|^2 \mathrm{d}y = \int_0^\infty U_0\Big|g_0 - \frac{\int_0^\infty U_0 g_0 \mathrm{d}y}{\int_0^\infty U_0 \mathrm{d}y}\Big|^2 \mathrm{d}y + \frac{\Big|\int_0^\infty U_0\bar{\Psi} \mathrm{d}y\Big|^2}{\int_0^\infty U_0 \mathrm{d}y}. \tag{4.18}$$

We next substitute (4.18) into (4.15) to get

$$\begin{aligned}
&\tau_0\Big[-\int_0^\infty |\bar{\Psi}_{0y}|^2 \mathrm{d}y + \int_0^\infty U_0|\bar{\Psi}_0|^2 \mathrm{d}y\Big] - \tau_0 \int_0^\infty U_0|g_0|^2 \mathrm{d}y \\
=&\tau_0\Bigg[< \mathcal{L}\bar{\Psi}_0, \bar{\Psi}_0 > - \int_0^\infty U_0\Big|g_0 - \frac{\int_0^\infty U_0 g_0 \mathrm{d}y}{\int_0^\infty U_0\, dy}\Big|^2 \mathrm{d}y\Bigg].
\end{aligned} \tag{4.19}$$

Combining (4.13) and (4.19), we obtain the following key identity:

$$\begin{aligned}
&|\tau_0|^2 \int_0^\infty |\bar{\Psi}_0|^2 \mathrm{d}y + \int_0^\infty U_0|g_{0y}|^2 \mathrm{d}y \\
=&\tau_0\Bigg[< \mathcal{L}\bar{\Psi}_0, \bar{\Psi}_0 > - \int_0^\infty U_0\Big|g_0 - \frac{\int_0^\infty U_0 g_0 \mathrm{d}y}{\int_0^\infty U_0 \mathrm{d}y}\Big|^2 \mathrm{d}y\Bigg].
\end{aligned} \tag{4.20}$$

In light of (4.14), one has from (4.20) that τ_0 is real. □

Proposition (4.2) implies τ_0 is real and satisfies the important identity (4.20). We can see from (4.20) that if $\langle \mathcal{L}\bar{\Psi}, \bar{\Psi} \rangle \leq 0$, $\tau_0 \leq 0$. Thus, the critical step is to determine the sign of $\langle \mathcal{L}\bar{\Psi}, \bar{\Psi} \rangle$. In other words, we need to analyze the following nonlocal eigenvalue problem (NLEP):

$$\begin{cases} \bar{\Psi}_{0yy} + U_0\bar{\Psi}_0 - U_0\frac{(\bar{\Psi}_0, U_0)}{(U_0, 1)} = \mu\bar{\Psi}_0, \quad y \in (0, \infty), \\ \bar{\Psi}_0 \in H^2_{N,loc}((0, \infty)). \end{cases} \tag{4.21}$$

In fact, the eigenvalue problem (4.21) was well-studied in [39]. For completeness, we present the proof of the following Lemma:

Lemma 4.1 (Cf. Lemma 5)**.** *Assume* $(\mu, \bar{\Psi}_0)$ *is the solution to (4.21), then the largest eigenvalue of (4.21) is 0, i. e. for* $\bar{\Psi}_0 \in H^2_{N,loc}((0,\infty))$*, we have* $\langle \mathcal{L}\bar{\Psi}_0, \bar{\Psi}_0\rangle \le 0$*. Moreover,*

$$\int_0^\infty U_0|\bar{\Psi}_0|^2 dy - \int_0^\infty |\bar{\Psi}_{0y}|^2 dy - \frac{\left|\int_0^\infty \bar{\Psi}_0 U_0 dy\right|^2}{\int_0^\infty U_0 dy} \le 0. \tag{4.22}$$

Proof. First of all, we consider the following eigenvalue problem of local operator $\mathcal{L}_0$:

$$\begin{cases} \mathcal{L}_0\bar{\Psi}_0 := \bar{\Psi}_{0yy} + U_0\bar{\Psi}_0, \ \ y \in (0,\infty), \\ \bar{\Psi}_{0y}(0) = 0, \end{cases}$$

where $U_0 = \frac{a}{2}\operatorname{sech}^2(\frac{\sqrt{a}}{2}y)$ with $a = 3\bar{u} + O(\epsilon)$. It is convenient to let $\sqrt{a}y = z$, then we find $\mathcal{L}_0[\bar{\Psi}_0(z)]$ can be rewritten as

$$\mathcal{L}_0[\bar{\Psi}_0(z)] = a(\bar{\Psi}_{0zz} + \hat{U}_0\bar{\Psi}_0), \ \ z \in (0,\infty), \tag{4.23}$$

where $\hat{U}_0 = \frac{1}{2}\operatorname{sech}^2(\frac{z}{2})$. The results for the spectrum of $\hat{\mathcal{L}}_0\bar{\Psi}_0 := \bar{\Psi}_{0zz} + \hat{U}_0\bar{\Psi}_0$ are well-known. Since $\hat{\mathcal{L}}_0$ is self-adjoint, we have the discrete spectrum of $\hat{\mathcal{L}}_0$ is countable, which satisfies $\hat{\lambda}_1 = \frac{1}{4}, \hat{\lambda}_2 < 0, \cdots$ with respect to even eigenfunctions. Since $(U_0, 1) = \sqrt{a}$, we rewrite (4.21) as

$$\mathcal{L}_0\bar{\Psi}_0 - \frac{1}{\sqrt{a}}U_0\int_0^\infty \bar{\Psi}_0 U_0 dy = \mu\bar{\Psi}_0, \tag{4.24}$$

where $\bar{\Psi}_{0y}(0) = 0$. Similarly, we have in the z-variable, (4.24) becomes

$$a\hat{\mathcal{L}}_0\bar{\Psi}_0 - a\hat{U}_0\int_0^\infty \bar{\Psi}_0\hat{U}_0 dz = \mu\bar{\Psi}_0. \tag{4.25}$$

Now, we investigate the following nonlocal operator:

$$\bar{\mathcal{L}}\bar{\Psi}_0 := \hat{\mathcal{L}}_0\bar{\Psi}_0 - \hat{U}_0\int_0^\infty \bar{\Psi}_0\hat{U}_0 dz,$$

which satisfies $\bar{\mathcal{L}}\bar{\Psi}_0 = \bar{\mu}\bar{\Psi}_0$ with $\mu = a\bar{\mu}$. Then we transform this eigenvalue problem into the following form:

$$(\hat{\mathcal{L}}_0 - \bar{\mu})\bar{\Psi}_0 = \hat{U}_0, \quad \int_0^\infty \bar{\Psi}_0\hat{U}_0 dz = 1.$$

To study the sign of $\bar{\mu}$, we only need to investigate the algebraic equation $\varphi(\bar{\mu}) = 1$, where

$$\varphi(\bar{\mu}) := \int_0^\infty \hat{U}_0(\hat{\mathcal{L}}_0 - \bar{\mu})^{-1}\hat{U}_0 dz. \tag{4.26}$$

Since $\bar{\mathcal{L}}$ is self-adjoint, we have $\bar{\mu}$ is real. Thus, it suffices to show $h(\bar{\mu}) \neq 1$ for all $\bar{\mu} > 0$ so as to obtain the desired conclusion.

Firstly, we have the fact that $\bar{\mu} = 0$ is an eigenvalue of (4.21), and the corresponding eigenfunction is $\bar{\Psi}_0 = 1$. Hence, $\varphi(0) = 1$. In addition, we differentiate φ to find

$$\varphi'(\bar{\mu}) = \int_0^\infty \hat{U}_0(\hat{\mathcal{L}}_0 - \bar{\mu})^{-2}\hat{U}_0 dz = \int_0^\infty [(\hat{\mathcal{L}}_0 - \bar{\mu})^{-1}\hat{U}_0]^2 dz > 0, \tag{4.27}$$

which implies φ is an increasing function. According to our above analysis, we find $\hat{\mathcal{L}}_0$ only admits a single positive eigenvalue $\hat{\lambda}_0 = \frac{1}{4}$, and others are negative. Thus, we obtain $\varphi(\bar{\mu})$ only has a single pole at $\bar{\mu} = \frac{1}{4}$ for $\bar{\mu} > 0$. In addition, when $\bar{\mu}$ is large, one has

$$\varphi(\bar{\mu}) \sim -\frac{1}{\bar{\mu}} \int_0^\infty \hat{U}_0^2 dy \to 0^- \text{ as } \bar{\mu} \to \infty. \tag{4.28}$$

Now, we summarize our results to obtain $\varphi(\bar{\mu})$ has a vertical asymptote at $\bar{\mu} = \frac{1}{4}$, $\varphi(0) = 1$, $\varphi \to 0^-$ as $\bar{\mu} \to \infty$ and φ is increasing for $\bar{\mu} \neq \frac{1}{4}$. Therefore, we have $\bar{\mu} \neq 1$ for all $\bar{\mu} > 0$, which proves this Lemma. □

Now, we apply Lemma 4.1 into identity (4.20), then find $\tau_0 \leq 0$. Noting that τ_0 is assumed to satisfy $\tau_0 \neq 0$, one can further obtain $\tau_0 < 0$. By summarizing our argument, we establish the following Proposition:

Proposition 4.3. *Assume there exists a subsequence $\epsilon_n \to 0$ as $n \to \infty$, $\{(\tau_n, < g_n, \bar{\Psi}_n >)\}_{n=1}^\infty$ are the solutions to (4.2) and $\tau_n \to \tau_0$ as $n \to \infty$, then we have $\tau_0 < 0$.*

Proposition (4.3) demonstrates that when there exist large eigenvalues τ in (4.2), the real part of τ must be negative for ϵ small enough, which indicates (1.4) is stable with respect to these modes. In the following, we shall focus on the small eigenvalue case. Since we plan to employ the Lyapunov-schmidt reduction method to compute the asymptotics of $(\tau_\epsilon, \Phi_\epsilon)$ for ϵ small enough then determine the sign of eigenvalues by using these asymptotics, we need to develop the exact order estimate of τ.

4.1 A Priori Estimates of Small Eigenvalues

In this subsection, we suppose $\tau_0 = 0$, which implies $\tau = o(1)$ for ϵ sufficiently small. However, it is not enough for us to finish the subsequent analysis. Thus, we shall show $|\tau| = O(\epsilon^2)$ with the assumption that $Re(\tau) \geq 0$ or $Re(\tau) \geq -c_0\epsilon^2$ for some $c_0 > 0$. The approach what I shall employ is similar as that in [22, 50] with few necessary modifications. We define $g = \frac{\Phi}{U} - \bar{\Psi}$, then transform the first equation of (4.2) as

$$\tau(Ug + U\bar{\Psi}) = (Ug_y)_y + ((\ln U - \bar{V})_y(Ug + U\bar{\Psi}))_y + \epsilon^2\mu(\bar{u} - 2U)(Ug + U\bar{\Psi}), \tag{4.29}$$

where $\Phi = Ug + U\bar{\Psi}$. Recall the steady state U satisfies

$$U_{yy} - (U\bar{V}_y)_y + \epsilon^2\mu U(\bar{u} - U) = 0, \quad y \in (0, L/\epsilon), \tag{4.30}$$

then we have

$$(\ln U - \bar{V})_y = -\frac{\epsilon^2\mu}{U} \int_0^y (\bar{u}U - U^2) dx. \tag{4.31}$$

Thanks to (1.4), we conclude

$$U^{-1} \int_0^y (\bar{u}U - U^2) dx = O(1) \tanh\left(\frac{\sqrt{3\bar{u}}}{2} y\right) + O(\epsilon) e^{-\sigma_1 y}. \tag{4.32}$$

Upon substituting (4.31) into (4.29), we rewrite (4.2) as the following system of $(\tau, g, \bar{\Psi})$:

$$\begin{cases} \tau(Ug + U\bar{\Psi}) = (Ug_y)_y - \epsilon^2\Big[\Big(\mu \int_0^y (\bar{u}U - U^2)dx\Big)(g + \bar{\Psi})\Big]_y + \epsilon^2\mu(\bar{u} - 2U)(Ug + U\bar{\Psi}), \\ \tau\bar{\Psi} = \bar{\Psi}_{yy} - \epsilon^2\bar{\Psi} + (Ug + U\bar{\Psi}). \end{cases} \tag{4.33}$$

We multiply the first equation of (4.33) by g^* and integrate it over $(0, L/\epsilon)$ to get

$$\tau \int_0^{\frac{L}{\epsilon}} U|g|^2 \mathrm{dy} + \tau \int_0^{\frac{L}{\epsilon}} U\bar{\Psi}g^* \mathrm{dy} = -\int_0^{\frac{L}{\epsilon}} U|g_y|^2 \mathrm{dy} + \epsilon^2 \int_0^{\frac{L}{\epsilon}} \mu(\bar{u} - 2U)U|g|^2 \mathrm{dy} + \epsilon^2 \int_0^{\frac{L}{\epsilon}} \mu(\bar{u} - 2U)U\bar{\Psi}g^* \mathrm{dy}$$
$$- \int_0^{\frac{L}{\epsilon}} (\ln U - \bar{V})_y (Ug + U\bar{\Psi}) g_y^* \mathrm{dy}. \tag{4.34}$$

Similarly, we multiply the conjugate of the second equation of (4.33) by $\tau\bar{\Psi}$ and integrate it to obtain

$$|\tau|^2 \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{dy} = -\tau \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 \mathrm{dy} - \epsilon^2 \tau \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{dy} + \tau \int_0^{\frac{L}{\epsilon}} U\bar{\Psi}g^* \mathrm{dy} + \tau \int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2 \mathrm{dy}. \tag{4.35}$$

We sum (4.34) and (4.35) to find

$$\tau \int_0^{\frac{L}{\epsilon}} U|g|^2 \mathrm{dy} + |\tau|^2 \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{dy} = -\int_0^{\frac{L}{\epsilon}} U|g_y|^2 \mathrm{dy} + \epsilon^2 \int_0^{\frac{L}{\epsilon}} \mu(\bar{u} - 2U)U|g|^2 \mathrm{dy} + \epsilon^2 \int_0^{\frac{L}{\epsilon}} \mu(\bar{u} - 2U)U\bar{\Psi}g^* \mathrm{dy}$$
$$- \int_0^{\frac{L}{\epsilon}} (\ln U - \bar{V})_y (Ug + U\bar{\Psi}) g_y^* \mathrm{dy}$$
$$- \tau \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 \mathrm{dy} - \epsilon^2 \tau \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{dy} + \tau \int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2 \mathrm{dy}. \tag{4.36}$$

Upon substituting (4.31) into (4.36), we obtain

$$\tau \int_0^{\frac{L}{\epsilon}} U|g|^2 \mathrm{dy} + |\tau|^2 \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{dy} = -\int_0^{\frac{L}{\epsilon}} U|g_y|^2 \mathrm{dy} + \epsilon^2 \int_0^{\frac{L}{\epsilon}} \mu(\bar{u} - 2U)U|g|^2 \mathrm{dy} + \epsilon^2 \int_0^{\frac{L}{\epsilon}} \mu(\bar{u} - 2U)U\bar{\Psi}g^* \mathrm{dy}$$
$$+ \epsilon^2 \mu \int_0^{\frac{L}{\epsilon}} \left(\int_0^y (\bar{u}U - U^2) dx \right) (g + \bar{\Psi}) g_y^* \mathrm{dy}$$
$$- \tau \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 \mathrm{dy} - \epsilon^2 \tau \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{dy} + \tau \int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2 \mathrm{dy}. \tag{4.37}$$

We rearrange (4.37) and rewrite it as

$$\tau \int_0^{\frac{L}{\epsilon}} U|g|^2 \mathrm{dy} + \tau \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 \mathrm{dy} + \epsilon^2 \tau \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{dy} - \tau \int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2 \mathrm{dy} + |\tau|^2 \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{dy}$$
$$= -\int_0^{\frac{L}{\epsilon}} U|g_y|^2 \mathrm{dy} + \epsilon^2 \int_0^{\frac{L}{\epsilon}} \mu(\bar{u} - 2U)U|g|^2 \mathrm{dy} + \epsilon^2 \int_0^{\frac{L}{\epsilon}} \mu(\bar{u} - 2U)U\bar{\Psi}g^* \mathrm{dy}$$
$$+ \epsilon^2 \mu \int_0^{\frac{L}{\epsilon}} \left(\int_0^y (\bar{u}U - U^2) dx \right) (g + \bar{\Psi}) g_y^* \mathrm{dy}. \tag{4.38}$$

To further simplify (4.38), we integrate the first equation of (4.2) and find from the fact $\Phi = Ug + U\bar{\Psi}$ that

$$\tau \int_0^{\frac{L}{\epsilon}} \Phi \mathrm{dy} = \epsilon^2 \mu \int_0^{\frac{L}{\epsilon}} (\bar{u} - 2U)\Phi \mathrm{dy} = \tau \int_0^{\frac{L}{\epsilon}} (Ug + U\bar{\Psi}) \mathrm{dy}. \tag{4.39}$$

It follows that

$$
\begin{aligned}
\tau\int_0^{\frac{L}{\epsilon}} U|g|^2\mathrm{dy} =&\tau\int_0^{\frac{L}{\epsilon}} U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{dy}}{\int_0^{\frac{L}{\epsilon}} U\mathrm{dy}}\Big|^2\mathrm{dy}+\tau\frac{\Big|\int_0^{\frac{L}{\epsilon}} Ug\mathrm{dy}\Big|^2}{\int_0^{\frac{L}{\epsilon}} U\mathrm{dy}}\\
=&\tau\int_0^{\frac{L}{\epsilon}} U\Big|g-\frac{\int Ugdy}{\int Udy}\Big|^2dy+\tau\frac{\Big|\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}\mathrm{dy}-\frac{\epsilon^2}{\tau}\mu\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{dy}\Big|^2}{\int_0^{\frac{L}{\epsilon}} Udy}\\
=&\tau\int_0^{\frac{L}{\epsilon}} U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{dy}}{\int_0^{\frac{L}{\epsilon}} U\mathrm{dy}}\Big|^2\mathrm{dy}\\
&+\tau\frac{\Big|\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}\mathrm{dy}\Big|^2-Re\Big[\frac{2\epsilon^2\mu}{\tau}\Big(\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{dy}\Big)\Big(\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}^*\mathrm{dy}\Big)\Big]+\frac{\epsilon^4\mu^2}{|\tau|^2}\Big|\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{dy}\Big|^2}{\int_0^{\frac{L}{\epsilon}} Udy}.
\end{aligned}
\tag{4.40}
$$

We substitute (4.40) into (4.38) then find the left hand side of (4.38) becomes

$$
\begin{aligned}
&\tau\int_0^{\frac{L}{\epsilon}} U|g|^2\mathrm{dy}+\tau\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}_y|^2\mathrm{dy}+\epsilon^2\tau\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{dy}-\tau\int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2\mathrm{dy}+|\tau|^2\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{dy}\\
=&\tau\int_0^{\frac{L}{\epsilon}} U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{dy}}{\int_0^{\frac{L}{\epsilon}} U\mathrm{dy}}\Big|^2\mathrm{dy}+\tau\frac{-Re\Big[\frac{2\epsilon^2\mu}{\tau}\Big(\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{dy}\Big)\Big(\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}^*\mathrm{dy}\Big)\Big]+\frac{\epsilon^4\mu^2}{|\tau|^2}\Big|\Big(\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{dy}\Big|^2}{\int_0^{\frac{L}{\epsilon}} U\mathrm{dy}}\\
&+\tau\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}_y|^2\mathrm{dy}+\epsilon^2\tau\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{dy}-\tau\int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2\mathrm{dy}+\tau\frac{\Big|\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}\mathrm{dy}\Big|^2}{\int_0^{\frac{L}{\epsilon}} U\mathrm{dy}}\\
&+|\tau|^2\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{dy}.
\end{aligned}
\tag{4.41}
$$

Define $\bar{\mathcal{L}}$ such that

$$
<\bar{\mathcal{L}}\bar{\Psi}_0,\bar{\Psi}_0>:=\int_0^\infty U|\bar{\Psi}_0|^2\mathrm{dy}-\int_0^\infty|\bar{\Psi}_{0y}|^2\mathrm{dy}-\frac{|\int_0^\infty\bar{\Psi}_0U\mathrm{dy}|^2}{\int_0^\infty U\mathrm{dy}},
\tag{4.42}
$$

then we use (4.41) to rewrite (4.38) as

$$
\begin{aligned}
&\tau\int_0^{\frac{L}{\epsilon}} U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{dy}}{\int_0^{\frac{L}{\epsilon}} U\mathrm{dy}}\Big|^2\mathrm{dy}+\tau\frac{-Re\Big[\frac{2\epsilon^2\mu}{\tau}\Big(\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{dy}\Big)\Big(\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}^*\mathrm{dy}\Big)\Big]+\frac{\epsilon^4\mu^2}{|\tau|^2}\Big|\Big(\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{dy}\Big|^2}{\int_0^{\frac{L}{\epsilon}} U\mathrm{dy}}\\
&-\tau\langle\bar{\mathcal{L}}\bar{\Psi},\bar{\Psi}\rangle+|\tau|^2\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{dy}+\epsilon^2\tau\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{dy}\\
=&-\int_0^{\frac{L}{\epsilon}} U|g_y|^2\mathrm{dy}+\epsilon^2\mu\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)U|g|^2\mathrm{dy}+\epsilon^2\mu\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)U\bar{\Psi}g^*\mathrm{dy}\\
&+\epsilon^2\mu\int_0^{\frac{L}{\epsilon}}\Big(\int_0^y(\bar{u}U-U^2)dx\Big)(g+\bar{\Psi})g_y^*\mathrm{dy}.
\end{aligned}
\tag{4.43}
$$

We rearrange (4.43) to get

$$
\begin{aligned}
&\tau\int_0^{\frac{L}{\epsilon}}U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}}Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}}U\mathrm{d}y}\Big|^2\mathrm{d}y-\tau\langle\bar{\mathcal{L}}\bar{\Psi},\bar{\Psi}\rangle+|\tau|^2\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2+\epsilon^2\tau\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{d}y\\
=&-\int_0^{\frac{L}{\epsilon}}U|g_y|^2\mathrm{d}y+\epsilon^2\mu\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)U|g|^2\mathrm{d}y+\epsilon^2\mu\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)U\bar{\Psi}g^*\mathrm{d}y\\
&+\epsilon^2\mu\int_0^{\frac{L}{\epsilon}}\Big(\int_0^y(\bar{u}U-U^2)dx\Big)(g+\bar{\Psi})g_y^*\mathrm{d}y\\
&-\tau\frac{-Re\Big[\frac{2\epsilon^2\mu}{\tau}\Big(\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{d}y\Big)\Big(\int_0^{\frac{L}{\epsilon}}U\bar{\Psi}^*\mathrm{d}y\Big)\Big]+\frac{\epsilon^4\mu^2}{|\tau|^2}\Big|\Big(\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{d}y\Big|^2}{\int_0^{\frac{L}{\epsilon}}U\mathrm{d}y}.
\end{aligned}
\tag{4.44}
$$

Assume $Re(\tau)\geq 0$, then it follows from (4.44) and $\Phi = Ug + U\bar{\Psi}$ that if $1 \gg |\tau| \gg \epsilon^2$,

$$
\begin{aligned}
&\Big|\tau\int_0^{\frac{L}{\epsilon}}U\Big|g-\frac{\int Ugdy}{\int Udy}\Big|^2\mathrm{d}y-\tau\langle\bar{\mathcal{L}}\bar{\Psi},\bar{\Psi}\rangle+|\tau|^2\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{d}y+\epsilon^2\tau\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{d}y+\int_0^{\frac{L}{\epsilon}}U|g_y|^2\mathrm{d}y\Big|\\
\leq& C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|g|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|\bar{\Psi}|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|g_y|^2\mathrm{d}y\\
&+C\epsilon^2\Big[\Big(\int_0^{\frac{L}{\epsilon}}U|g+\bar{\Psi}|\mathrm{d}y\Big)\Big(\int_0^{\frac{L}{\epsilon}}U|\bar{\Psi}^*|\mathrm{d}y\Big)\Big]+C\epsilon^2\Big(\int_0^{\frac{L}{\epsilon}}U|g+\bar{\Psi}|\mathrm{d}y\Big)^2\\
\leq& C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|g|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|\bar{\Psi}|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|g_y|^2\mathrm{d}y.
\end{aligned}
\tag{4.45}
$$

Otherwise if $|\tau| \lesssim \epsilon^2$, we obtain the desired conclusion. It is necessary to mention that the only difference between $\mathcal{L}$ and $\bar{\mathcal{L}}$ is U_0 is replaced by U. However, we have the fact that $U = U_0 + o(1)e^{-\sigma_1 y}$ for some $\sigma_1 > 0$. Noting that Lemma 4.1 implies $\langle\mathcal{L}\bar{\Psi},\bar{\Psi}\rangle \leq 0$ and $\langle\mathcal{L}\bar{\Psi},\bar{\Psi}\rangle \leq -\nu\int|\bar{\Psi}|^2$ for some $\nu > 0$ independent of ϵ when $\bar{\Psi} \not\equiv C$, where C is any nonzero constant, we have $\bar{\mathcal{L}}$ possesses the same property as $\mathcal{L}$. Thus, if $\bar{\Psi} \equiv C$, we obtain from the second equation of (4.2) that $\Phi = (\tau + \epsilon^2)C$, which is a constant. Then we integrate the first equation of (4.2) over $(0, L/\epsilon)$ and use the Neumann boundary condition to arrive at

$$
\tau\frac{\Phi L}{\epsilon}=\epsilon^2\frac{\Phi L}{\epsilon}-2\epsilon^2\Phi\int_0^{\frac{L}{\epsilon}}U\mathrm{d}y.
$$

This implies $|\tau| = O(\epsilon^2)$, which gives the desired conclusion.

Next, we consider the case that $\bar{\Psi} \not\equiv C$. Assume $Re(\tau) \geq 0$, then we find from (4.45) that

$$
\begin{aligned}
&|\tau|\int_0^{\frac{L}{\epsilon}}U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}}Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}}U\mathrm{d}y}\Big|^2\mathrm{d}y+\nu|\tau|\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2+|\tau|^2\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{d}y\\
\leq& C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|g|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|\bar{\Psi}|^2\mathrm{d}y\\
\leq& C\epsilon^2\Big|\int_0^{\frac{L}{\epsilon}}Ug\mathrm{d}y\Big|^2+C\epsilon^2\int_0^{\frac{L}{\epsilon}}U\Big|g-\frac{\int Ugdy}{\int Udy}\Big|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|\bar{\Psi}|^2\mathrm{d}y\\
\leq& C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|g|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}}U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}}Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}}U\mathrm{d}y}\Big|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}}U|\bar{\Psi}|^2\mathrm{d}y,
\end{aligned}
\tag{4.46}
$$

where $\nu > 0$ is a constant independent of ϵ. We claim when $Re(\tau) \geq 0$, there exists some constant $C > 0$ such that

$$\int_0^{\frac{L}{\epsilon}} U|g|^2 \mathrm{d}y \leq C \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{d}y. \tag{4.47}$$

To prove this, one finds from (4.38) that

$$\begin{aligned}
&|\tau| \int_0^{\frac{L}{\epsilon}} U|g|^2 \mathrm{d}y + |\tau| \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}_y|^2 \mathrm{d}y + \int_0^{\frac{L}{\epsilon}} U|g_y|^2 \mathrm{d}y \\
\leq& \epsilon^2 C \int_0^{\frac{L}{\epsilon}} U|g|^2 \mathrm{d}y + \epsilon^2 C \int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2 \mathrm{d}y + \epsilon^2 C \int_0^{\frac{L}{\epsilon}} U|g_y|^2 \mathrm{d}y + |\tau| C \int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2 \mathrm{d}y \\
\leq& \epsilon^2 C \int_0^{\frac{L}{\epsilon}} U|g|^2 \mathrm{d}y + \epsilon^2 C \int_0^{\frac{L}{\epsilon}} U|g_y|^2 \mathrm{d}y + |\tau| C \int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2 \mathrm{d}y,
\end{aligned} \tag{4.48}$$

which proves (4.47). We next apply (4.47) into (4.46) and obtain

$$\begin{aligned}
&|\tau| \int_0^{\frac{L}{\epsilon}} U\Big|g - \frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y}\Big|^2 \mathrm{d}y + |\tau| \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{d}y \\
\leq& C\epsilon^2 \int_0^{\frac{L}{\epsilon}} U\Big|g - \frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y}\Big|^2 \mathrm{d}y + C\epsilon^2 \int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2 \mathrm{d}y.
\end{aligned} \tag{4.49}$$

This implies $|\tau| = O(\epsilon^2)$ with $Re(\tau) \geq 0$.

Next, we would like to prove $Im(\tau) = O(\epsilon^2)$ when $Re(\tau)$ is assumed to satisfy $Re(\tau) = O(\epsilon^2)$ but $Re(\tau)$ is negative. Define $Re(\tau) = \tau_R$ and $Im(\tau) = \tau_I$, then we take the real part of (4.44) to obtain

$$\begin{aligned}
&\tau_R \int_0^{\frac{L}{\epsilon}} U\Big|g - \frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y}\Big|^2 \mathrm{d}y - \tau_R \langle \bar{\mathcal{L}}\bar{\Psi}, \bar{\Psi}\rangle + |\tau|^2 \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{d}y + \epsilon^2 \tau_R \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{d}y \\
=& -\int_0^{\frac{L}{\epsilon}} U|g_y|^2 \mathrm{d}y + \epsilon^2 \mu \int_0^{\frac{L}{\epsilon}} (\bar{u} - 2U)U|g|^2 \mathrm{d}y \\
&+ \mu\epsilon^2 \int_0^{\frac{L}{\epsilon}} (\bar{u} - 2U)U Re(\bar{\Psi} g^*) \mathrm{d}y + \epsilon^2 \mu \int_0^{\frac{L}{\epsilon}} \Big(\int_0^y (\bar{u}U - U^2) \mathrm{d}x \Big) Re[(g + \bar{\Psi}) g_y^*] \mathrm{d}y \\
&- \tau_R \Big(\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y \Big)^{-1} \Big[- Re\Big[\frac{2\epsilon^2 \mu}{\tau} \Big(\int_0^{\frac{L}{\epsilon}} (\bar{u} - 2U)\Phi \mathrm{d}y \Big) \Big(\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}^* \mathrm{d}y \Big) \Big] + \frac{\epsilon^4 \mu^2}{|\tau|^2} \Big| \int (\bar{u} - 2U)\Phi \mathrm{d}y \Big|^2 \Big],
\end{aligned} \tag{4.50}$$

which implies that

$$\int_0^{\frac{L}{\epsilon}} U|g_y|^2 \mathrm{d}y \leq \epsilon^2 C \Big(\int_0^{\frac{L}{\epsilon}} U\Big|g - \frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y}\Big|^2 \mathrm{d}y - \langle \bar{\mathcal{L}}\bar{\Psi}, \bar{\Psi}\rangle + \int_0^{\frac{L}{\epsilon}} U|g|^2 \mathrm{d}y + \int_0^{\frac{L}{\epsilon}} U|\Psi|^2 \mathrm{d}y \Big). \tag{4.51}$$

On the other hand, by taking the imaginary part of (4.44), we have

$$\begin{aligned}
&\tau_I \Big(\int_0^{\frac{L}{\epsilon}} U\Big|g - \frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y}\Big|^2 \mathrm{d}y - \langle \bar{\mathcal{L}}\bar{\Psi}, \bar{\Psi}\rangle + \epsilon^2 \int_0^{\frac{L}{\epsilon}} |\bar{\Psi}|^2 \mathrm{d}y \Big) \\
=& \epsilon^2 \mu \int_0^{\frac{L}{\epsilon}} (\bar{u} - 2U)U Im(\bar{\Psi} g^*) \mathrm{d}y + \epsilon^2 \mu \int_0^{\frac{L}{\epsilon}} \Big(\int_0^y (\bar{u}U - U^2) dx \Big) Im[(g + \bar{\Psi}) g_y^*] \mathrm{d}y,
\end{aligned} \tag{4.52}$$

which concludes that

$$
\begin{aligned}
&|\tau_I|\Big(\int_0^{\frac{L}{\epsilon}} U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y}\Big|^2\mathrm{d}y-\langle\bar{\mathcal{L}}\bar{\Psi},\bar{\Psi}\rangle+\epsilon^2\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{d}y\Big)\\
\leq& C\epsilon^2\int_0^{\frac{L}{\epsilon}} U|g|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2\mathrm{d}y+C\epsilon^2\int_0^{\frac{L}{\epsilon}} U|g_y|^2\mathrm{d}y.
\end{aligned}
\tag{4.53}
$$

Assume $|\tau_I|\gg\epsilon^2$, then we combine (4.51) and (4.53) to obtain

$$
\begin{aligned}
&|\tau_I|\Big(\int_0^{\frac{L}{\epsilon}} U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y}\Big|^2\mathrm{d}y-\langle\bar{\mathcal{L}}\bar{\Psi},\bar{\Psi}\rangle+\epsilon^2\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{d}y\Big)\\
\leq& C\epsilon^2\Big(\int_0^{\frac{L}{\epsilon}} U|g|^2\mathrm{d}y+\int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2\mathrm{d}y\Big).
\end{aligned}
$$

We claim that

$$
\int_0^{\frac{L}{\epsilon}} U|g|^2\mathrm{d}y\leq C\int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2\mathrm{d}y
$$

for some constant $C>0$ independent of ϵ. Then we have for $\bar{\Psi}\not\equiv C$,

$$
\nu|\tau_I|\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{d}y\leq C\epsilon^2\Big(\int_0^{\frac{L}{\epsilon}} U|g|^2\mathrm{d}y+\int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2\mathrm{d}y\Big)\leq C\epsilon^2\int_0^{\frac{L}{\epsilon}}|\bar{\Psi}|^2\mathrm{d}y \tag{4.54}
$$

for constant $\nu>0$ independent of ϵ. Thus, $|\tau_I|=O(\epsilon^2)$.

To show our claim, we recall

$$
\begin{aligned}
\int_0^{\frac{L}{\epsilon}} U|g|^2\mathrm{d}y=&\int_0^{\frac{L}{\epsilon}} U\Big|g-\frac{\int_0^{\frac{L}{\epsilon}} Ug\mathrm{d}y}{\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y}\Big|^2\mathrm{d}y\\
&+\frac{\Big|\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}\mathrm{d}y\Big|^2-Re\Big[\frac{2\epsilon^2\mu}{\tau}\Big(\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{d}y\Big)\Big(\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}^*\mathrm{d}y\Big)\Big]+\frac{\epsilon^4\mu^2}{|\tau|^2}\Big|\int_0^{\frac{L}{\epsilon}}(\bar{u}-2U)\Phi\mathrm{d}y\Big|^2}{\int_0^{\frac{L}{\epsilon}} U\mathrm{d}y}.
\end{aligned}
$$

Then if $|\tau_I|\gg\epsilon^2$, one has

$$
\int_0^{\frac{L}{\epsilon}} U|g|^2\mathrm{d}y\leq C\Big|\int_0^{\frac{L}{\epsilon}} U\bar{\Psi}\mathrm{d}y\Big|^2+o(1)\Big(\int_0^{\frac{L}{\epsilon}} U|g|^2\mathrm{d}y+\int_0^{\frac{L}{\epsilon}} U|\bar{\Psi}|^2\mathrm{d}y\Big), \tag{4.55}
$$

which implies our claim. Otherwise if there exists $C>0$ such that $|\tau_I|\leq C\epsilon^2$, we obtain $|\tau_I|\lesssim\epsilon^2$, which is our desired conclusion. we summarize our discussion shown as above to obtain the following Proposition:

Proposition 4.4. *Assume $(\tau,<\Phi,\bar{\Psi}>)$ is the solution to (4.2), then we have $|\tau|=O(\epsilon^2)$ when $Re(\tau)\geq -c_0\epsilon^2$ for some $c_0>0$ independent of ϵ.*

Proposition 4.4 indicates that when Re(τ) is positive or sufficiently small, $|\tau|$ can be regarded as the perturbation. Now, we have the sufficient context to construct the asymptotics of small eigen-pairs.

4.2 Eigenvalue Asymptotics

This subsection is devoted to the case of $|\tau| = o(1)$ for $\epsilon \ll 1$. According to Proposition 4.4, we further have $|\tau| = O(\epsilon^2)$. Since we linearize (1.2) around (u^-, v^-), we need to recall the property of this steady state. It is convenient to consider u and v in the y-variable. Moreover, thanks to Theorem 1.1, one can find

$$u(x) = U(y), \quad U(y) := U_0(y) + \tilde{U}_1(y;\epsilon),$$

and

$$v(x) = V(y), \quad V(y) := V_0(y) + \tilde{V}_1(y;\epsilon),$$

where (U_0, V_0) and $(\tilde{U}_1, \tilde{V}_1)$ have the following properties:

$$U_0(y) = \frac{a}{2}\mathrm{sech}^2\Big(\frac{\sqrt{a}}{2}y\Big), \quad |\tilde{U}_1(y;\epsilon)| \le M\epsilon e^{-\sigma_1 y}, \quad a = 3\bar{u} + O(\epsilon), \tag{4.56}$$

and

$$V_0 = \epsilon^2 \int_0^{\frac{L}{\epsilon}} G(y;z)U_0(z)dz := T(U_0), \quad |\tilde{V}_1(y;\epsilon)| \le M\epsilon^3.$$

Here $G(y;z)$ is given by (3.19) and constants $M > 0$, $\sigma_1 > 0$ are independent of ϵ and y. Recall we define the stretched variable $y = \frac{x}{\epsilon}$, eigenfunctions $\Phi(y) = \phi(x)$, $\Psi(y) = \psi(y)$, eigenvalues $\tau = \epsilon^2\lambda$, then transform (4.1) into the following rescaled problem:

$$\begin{cases} \tau\Phi = \Phi_{yy} - \frac{1}{\epsilon^2}(U\Psi_y + \Phi V_y)_y + \epsilon^2\mu(\bar{u} - 2U)\Phi, & y \in (0, \frac{L}{\epsilon}), \\ \tau\Psi = \Psi_{yy} - \epsilon^2\Psi + \epsilon^2\Phi, & y \in (0, \frac{L}{\epsilon}), \\ \Phi_y(0) = \Psi_y(0) = \Phi_y(\frac{L}{\epsilon}) = \Psi_y(\frac{L}{\epsilon}) = 0. \end{cases} \tag{4.57}$$

We relabel $\bar{L}_1$ given by (2.41) as $\bar{L}_\epsilon$. Noting that $\Psi := T(\Phi)$ in terms of the Neumann Green's function, we rewrite $\bar{L}_\epsilon$ as

$$\bar{L}_\epsilon(\Phi; U_0) = \Phi_{yy} - \frac{1}{\epsilon^2}(U_0[T(\Phi)]_y)_y - \frac{1}{\epsilon^2}(\Phi[T(U_0)]_y)_y \tag{4.58}$$

and find $\bar{L}_\epsilon$ is defined by:

$$\bar{L}_\epsilon : \mathcal{K}_\epsilon^\perp \to C_\epsilon^\perp,$$

where $\mathcal{K}_\epsilon^\perp$ and $C_\epsilon^\perp$ are introduced as

$$\mathcal{K}_\epsilon^\perp := \left\{u \in H_N^2((0, L/\epsilon)) : \int_0^{\frac{L}{\epsilon}} u dy = 0\right\} \subset H_N^2((0, L/\epsilon)), \tag{4.59}$$

and

$$C_\epsilon^\perp := \left\{u \in H_N^2((0, L/\epsilon)) : \int_0^{\frac{L}{\epsilon}} u dy = 0\right\} \subset L^2((0, L/\epsilon)). \tag{4.60}$$

Proposition 3.1 implies $\bar{L}_\epsilon$ is uniformly invertible for ϵ small enough, which can be summarized as

Proposition 4.5. *There exist positive numbers ϵ_0, C such that for all $\epsilon \in (0, \epsilon_0)$, we have*

$$\|\bar{L}_\epsilon\Phi\|_\sigma \ge C\|\Phi\|_{\sigma_1} \text{ for all } \Phi \in \mathcal{K}_\epsilon^\perp, \tag{4.61}$$

where $\|\cdot\|_\sigma$ is given by (3.3). Furthermore, operator $\bar{L}_\epsilon$ is surjective.

Similarly, Proposition 3.2 can be restated as

Proposition 4.6. *There exist constants $\epsilon_0 > 0$, $C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, we have the unique solution $\Phi_\epsilon \in \mathcal{K}_\epsilon^\perp \otimes H_N^2((0, L/\epsilon))$ such that $S_\epsilon(U_0 + \epsilon U_1 + \epsilon^2 U_2 + \Phi_\epsilon) = 0$. Moreover, Φ_ϵ satisfies*

$$\|\Phi_\epsilon\|_{\sigma_1} \le C\epsilon^2, \tag{4.62}$$

for some constant $\sigma_1 > 0$.

Now, we have rewritten the results in Section 3 obtained via the Lyapunov-schimdt reduction method as Proposition 4.5 and Proposition 4.6. Next, we will establish the asymptotics of eigen-pairs $(\tau, <\Phi, \Psi>)$ and our results are summarized as follows:

Proposition 4.7. *Let $(\tau, <\Phi, \Psi>)$ be the solutions to (4.57) and suppose $\tau = o(1)$ for $\epsilon \ll 1$, then we have there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$,*

$$\tau_\epsilon = -2\mu\bar{u}\epsilon^2 + O(\epsilon^3), \tag{4.63}$$

and

$$\Phi_\epsilon = [e_0 + o(1)]\Phi_0 + o(1),$$

where e_0 is any nonzero constant and Φ_0 is defined as

$$\Phi_0 = \left(1 - \frac{\sqrt{a}}{2} y \tanh\left(\frac{\sqrt{a}}{2} y\right)\right)\operatorname{sech}^2\left(\frac{\sqrt{a}}{2} y\right), \quad a = 3\bar{u} + O(\epsilon). \tag{4.64}$$

As a consequence, λ_ϵ satisfies

$$\lambda_\epsilon = -2\mu\bar{u} + O(\epsilon). \tag{4.65}$$

Proof. Before using Proposition 4.5 and Proposition 4.6, it is necessary to set the ansatz of $(\Phi_\epsilon, \Psi_\epsilon)$ to system (4.57). To this end, we similarly take subsequences ϵ_n, τ_n, U_n, $\bar{V}_n$, Φ_n and $\bar{\Psi}_n$ such that $\epsilon_n \to 0$, $U_n \to U_0$, $\bar{V}_n \to \bar{V}_{00}$, $\Phi_n \to \Phi_0$ and $\bar{\Psi}_n \to \bar{\Psi}_{00}$ over any compact subset. Then we consider $\tau_0 = 0$ and obtain the following limiting problem satisfied by $(\Phi_0, \bar{\Psi}_0)$:

$$\begin{cases} 0 = \Phi_{0yy} - (U_0\bar{\Psi}_{00y} + \Phi_0\bar{V}_{00y})_y, & y \in (0, \infty), \\ 0 = \bar{\Psi}_{00yy} + \Phi_0, & y \in (0, \infty), \\ \Phi_{0y}(0) = \bar{\Psi}_{00y}(0) = 0, \quad \Phi_0(\infty) = 0. \end{cases} \tag{4.66}$$

We have (4.66) can be rewritten as the following divergence form:

$$\begin{cases} (U_0 g_{0y})_y = 0, \\ g_0 = \frac{\Phi_0}{U_0} - \bar{\Psi}_{00}, \quad \bar{\Psi}_{00yy} + \Phi_0 = 0. \end{cases} \tag{4.67}$$

In light of the first equation in (4.67), we choose $g_0 \equiv C$ for some constant C, then we substitute $g_0 \equiv C$ into the second equation to obtain

$$\begin{cases} \bar{\Psi}_{00yy} + \frac{a}{2}\operatorname{sech}^2(\frac{\sqrt{a}}{2}y)\bar{\Psi}_{00} + C_1\operatorname{sech}^2(\frac{\sqrt{a}}{2}y) = 0, & y \in (0, \infty), \\ \bar{\Psi}_{00y}(0) = 0, \end{cases} \tag{4.68}$$

where C_1 is a different constant. By noting the scaling invariance and nondegeneracy properties, we have from $\bar{\Psi}_{00yy} + U_0\bar{\Psi}_{00} = 0$ that the solution to (4.68) is

$$\bar{\Psi}_{00} = \frac{1}{\sqrt{a}} y \tanh\left(\frac{\sqrt{a}}{2} y\right). \tag{4.69}$$

It follows that Φ_0 satisfies

$$\begin{aligned}\Phi_0 =& CU_0 + \bar{\Psi}_{00}U_0\\ =&\left(1-\frac{\sqrt{a}}{2}y\tanh\left(\frac{\sqrt{a}}{2}y\right)\right)\mathrm{sech}^2\left(\frac{\sqrt{a}}{2}y\right).\end{aligned} \tag{4.70}$$

Now, we regard Φ_0 as the basic ansatz of Φ. Similarly, define Ψ_0 as the basic approximation of Ψ, then we have

$$\Psi_0 = \epsilon^2\int_0^{\frac{L}{\epsilon}} G(z;y)\Phi_0(z)dz := T(\Phi_0), \tag{4.71}$$

where $G(z;y)$ is given by (3.19). Define $\Phi = e_0\Phi_0 + \Phi^\perp$ and $\Psi = e_0\Psi_0 + \Psi^\perp$. It is natural to analyze the error generated by Φ_0 and Ψ_0, which is

$$\begin{aligned}S(\Phi^\perp) =& -\frac{e_0}{\epsilon^2}(U_0(\Psi_{01}+\Psi_{02})_y)_y - \frac{e_0}{\epsilon^2}(\tilde{U}_1\Psi_{0y})_y - \frac{1}{\epsilon^2}(\tilde{U}_1\Psi_y^\perp)_y - \frac{e_0}{\epsilon^2}(\Phi_0\tilde{V}_{1y})_y - \frac{1}{\epsilon^2}(\Phi^\perp\tilde{V}_{1y})_y\\ &+\Phi^\perp_{yy} - \frac{1}{\epsilon^2}(U_0\Psi^\perp_y)_y - \frac{1}{\epsilon^2}(\Phi^\perp V_{0y})_y\\ &-\tau(e_0\Phi_0+\Phi^\perp) + \epsilon^2\mu[\bar{u}-2(U_0+\tilde{U}_1)](e_0\Phi_0+\Phi^\perp).\end{aligned} \tag{4.72}$$

Noting that $\bar{L}_\epsilon$ is given by (4.58), we have

$$\begin{aligned}S(\Phi^\perp) =& -\frac{e_0}{\epsilon^2}(U_0(\Psi_{01}+\Psi_{02})_y)_y - \frac{e_0}{\epsilon^2}(\tilde{U}_1\Psi_{0y})_y - \frac{1}{\epsilon^2}(\tilde{U}_1\Psi_y^\perp)_y - \frac{e_0}{\epsilon^2}(\Phi_0\tilde{V}_{1y})_y - \frac{1}{\epsilon^2}(\Phi^\perp\tilde{V}_{1y})_y\\ &+\bar{L}_\epsilon[\Phi^\perp] - \tau(e_0\Phi_0+\Phi^\perp) + \epsilon^2\mu[\bar{u}-2(U_0+\tilde{U}_1)](e_0\Phi_0+\Phi^\perp) - \epsilon^2\mu\bar{u}\Phi^\perp.\end{aligned} \tag{4.73}$$

It is similar to find that $V_{01}+V_{02} \lesssim \epsilon^3y^2$, $\Psi_{01}+\Psi_{02} \lesssim \epsilon^3y^2$ and $\tilde{V}_{10} \lesssim \epsilon^3 y$, hence the leading order error in (4.73) is from the combination of $(U_0(\Psi_{01}+\Psi_{02})_y)_y$, $(U_0(\Psi_{01}+\Psi_{02})_y)_y$ and $(\Phi_0(V_{01}+V_{02})_y)_y$, where the order is $O(\epsilon)$. To eliminate this error, we add the correction term Φ_1, then similarly solve the quadratic form to obtain $\Phi_1 \lesssim y^2e^{-\sqrt{a}y}$. Then, we also express Ψ_1 as $\Psi_1 = T(\Phi_1)$ in terms of the Neumann Green's function. In summary, we decompose Φ and Ψ as

$$\Phi = e_0\Phi_0 + \epsilon\Phi_1 + \Phi_\epsilon^\perp, \quad \Psi = e_0\Psi_0 + \epsilon\Psi_1 + \Psi_\epsilon^\perp, \tag{4.74}$$

where e_0 is any nonzero constant, $\Phi_\epsilon^\perp \in \mathcal{K}_\epsilon^\perp$, where $\mathcal{K}_\epsilon^\perp$ is given by (4.59) and $\Psi_\epsilon^\perp = T(\Phi_\epsilon^\perp)$. We substitute (4.74) into $S(u) = 0$, then obtain

$$\begin{aligned}S(\Phi_\epsilon^\perp) =& \overbrace{-\epsilon^{-2}e_0(U_0(\Psi_{01}+\Psi_{02})_y)_y + 2e_0\epsilon(U_0y)_y}^{II_1}\\ &\overbrace{-\epsilon^{-2}e_0(\Phi_0(V_{01}+V_{02})_y)_y + 2\epsilon e_0(\Phi_0 y)_y}^{II_2}\\ &-\epsilon^{-1}(U_0(\Psi_{11}+\Psi_{12})_y)_y - \frac{e_0}{\epsilon^2}(\Phi_0(\tilde{V}_{11}+\tilde{V}_{12})_y)_y\\ &-\frac{1}{\epsilon^2}(\tilde{U}_1(e_0\Psi_0+\epsilon\Psi_1+\Psi_\epsilon^\perp)_y)_y - \frac{1}{\epsilon^2}(\Phi_\epsilon^\perp\tilde{V}_{1y})_y\\ &-\epsilon^{-1}(\Phi_1(V_{01}+V_{02})_y)_y - \epsilon^{-1}(\Phi_1\tilde{V}_{1y})_y\\ &+\bar{L}_\epsilon[\Phi_\epsilon^\perp] - \tau(e_0\Phi_0+\epsilon\Phi_1+\Phi_\epsilon^\perp)\\ &+\epsilon^2\mu[\bar{u}-2(U_0+\tilde{U}_1)](e_0\Phi_0+\epsilon\Phi_1+\Phi_\epsilon^\perp) - \epsilon^2\mu\bar{u}\Phi_\epsilon^\perp,\end{aligned} \tag{4.75}$$

where $|II_1|$ and $|II_2|$ are the leading terms with the order being $O(\epsilon^2)$ since we add the correction term Φ_1 to eliminate the $O(\epsilon)$ error.

Since $S(\Phi_\epsilon^\perp) = 0$, we collect from (4.75) that $\Phi_\epsilon^\perp$ satisfies

$$\begin{cases} \bar{L}_\epsilon[\Phi_\epsilon^\perp] = \Phi_{\epsilon yy}^\perp - \frac{1}{\epsilon^2}(U_0\Psi_{\epsilon y}^\perp)_y - \frac{1}{\epsilon^2}(\Phi_\epsilon^\perp V_{0y})_y + F_1(\Phi_\epsilon^\perp, \Psi_\epsilon^\perp; \epsilon) = 0, \quad y \in (0, \frac{L}{\epsilon}), \\ \Phi_{\epsilon y}^\perp(0) = \Phi_{\epsilon y}^\perp(\frac{L}{\epsilon}) = 0, \end{cases} \tag{4.76}$$

where $F_1(\Phi_\epsilon^\perp, \Psi_\epsilon^\perp; \epsilon)$ is defined as

$$\begin{aligned} F_1(\Phi_\epsilon^\perp, \Psi_\epsilon^\perp; \epsilon) = &- \epsilon^{-2}e_0(U_0(\Psi_{01} + \Psi_{02})_y)_y + 2e_0\epsilon(U_0 y)_y \\ &- \epsilon^{-2}e_0(\Phi_0(V_{01} + V_{02})_y)_y + 2\epsilon e_0(\Phi_0 y)_y \\ &- \epsilon^{-1}(U_0(\Psi_{11} + \Psi_{12})_y)_y - \frac{e_0}{\epsilon^2}(\Phi_0(\tilde{V}_{11} + \tilde{V}_{12})_y)_y \\ &- \frac{1}{\epsilon^2}(\tilde{U}_1(e_0\Psi_0 + \epsilon\Psi_1 + \Psi_\epsilon^\perp)_y)_y - \frac{1}{\epsilon^2}(\Phi_\epsilon^\perp \tilde{V}_{1y})_y \\ &- \epsilon^{-1}(\Phi_1(V_{01} + V_{02})_y)_y - \epsilon^{-1}(\Phi_1\tilde{V}_{1y})_y \\ &- \tau(e_0\Phi_0 + \epsilon\Phi_1 + \Phi_\epsilon^\perp) \\ &+ \epsilon^2\mu[\bar{u} - 2(U_0 + \tilde{U}_1)](e_0\Phi_0 + \epsilon\Phi_1 + \Phi_\epsilon^\perp) - \epsilon^2\mu\bar{u}\Phi_\epsilon^\perp. \end{aligned} \tag{4.77}$$

If $F_1(\Phi_\epsilon^\perp, \Psi_\epsilon^\perp; \epsilon)$ satisfies the orthogonality condition $\int_0^{\frac{L}{\epsilon}} F_1 dy = 0$, Proposition 4.5 tells us $\bar{L}_\epsilon$ is uniformly invertible for ϵ small enough. Moreover, $\Phi_\epsilon^\perp$ satisfies

$$\|\Phi_\epsilon^\perp\|_{\sigma_1} \le C\|F_1\|_\sigma \tag{4.78}$$

for constant $C > 0$ independent of ϵ. Thus, we rewrite $\Phi_\epsilon^\perp$ as $\Phi^\perp = -\mathcal{A}(F_1)$. It is similar to decompose F_1 as the linear and nonlinear error, which is $F_1 = E_1 + N_1(\Phi_\epsilon^\perp)$, where

$$\begin{aligned} E_1 := &- \epsilon^{-2}e_0(U_0(\Psi_{01} + \Psi_{02})_y)_y + 2e_0\epsilon(U_0 y)_y \\ &- \epsilon^{-2}e_0(\Phi_0(V_{01} + V_{02})_y)_y + 2\epsilon e_0(\Phi_0 y)_y \\ &- \epsilon^{-1}(U_0(\Psi_{11} + \Psi_{12})_y)_y - \frac{e_0}{\epsilon^2}(\Phi_0(\tilde{V}_{11} + \tilde{V}_{12})_y)_y \\ &- \frac{1}{\epsilon^2}(\tilde{U}_1(e_0\Psi_0 + \epsilon\Psi_1)_y)_y - \epsilon^{-1}(\Phi_1(V_{01} + V_{02})_y)_y - \epsilon^{-1}(\Phi_1\tilde{V}_{1y})_y \\ &- \tau(e_0\Phi_0 + \epsilon\Phi_1) + \epsilon^2\mu[\bar{u} - 2(U_0 + \tilde{U}_1)](e_0\Phi_0 + \epsilon\Phi_1), \end{aligned} \tag{4.79}$$

and

$$\begin{aligned} N_1(\Phi^\perp) = &- \frac{1}{\epsilon^2}(\tilde{U}_1\Psi_{\epsilon y}^\perp)_y - \frac{1}{\epsilon^2}(\Phi_\epsilon^\perp\tilde{V}_{1y})_y - \tau\Phi_\epsilon^\perp \\ &- 2\epsilon^2\mu(U_0 + \tilde{U}_1)\Phi_\epsilon^\perp. \end{aligned} \tag{4.80}$$

Next, we estimate E_1 and $N_1(\Phi^\perp)$. Focusing on the linear error E_1, we find the worse term in (4.79) is $O(\epsilon^2)$ since we set Φ_1 to eliminate the $O(\epsilon)$ error. Hence,

$$\|E_1\|_\sigma = O(\epsilon^2).$$

For the nonlinear error $N_1(\Phi_\epsilon^\perp)$, by noting that U_0 and $\tilde{U}_1$ have the fast decay property, we have the worse term is $\tau\Phi^\perp$. It is necessary to mention that $(\Phi_\epsilon^\perp\tilde{V}_{10y})_y$ is also the leading term in (4.80). However, we can add it into the operator $\bar{L}_\epsilon$ and the new operator have the same property with the old one. The reason is $\tilde{V}_1$ is $O(\epsilon^4)$ in the outer region, then we can still use the Maximum Principle to give the good estimate for $\Phi^{\perp}$. Now, we invoke Proposition 4.4 to get $|\tau| = O(\epsilon^2)$. Hence, we can regard it as the perturbation. Recall σ and σ_1 are chosen such that $C\sigma^2 + \epsilon|\ln\epsilon| < (\sigma - \sigma_1)L < 2\epsilon|\ln\epsilon|$ and $\epsilon|\ln\epsilon| \lesssim \sigma \lesssim \sqrt{\epsilon|\ln\epsilon|}$, then we have $\|\tau\Phi^{\perp}\|_{\sigma} = o(1)\|\Phi^{\perp}\|_{\sigma_1}$. In summary, the nonlinear error $N_1(\Phi_\epsilon^{\perp})$ satisfies

$$\|N_1(\Phi_\epsilon^{\perp})\|_{\sigma} = o(1)\|\Phi_\epsilon^{\perp}\|_{\sigma_1}. \tag{4.81}$$

Proposition 4.6 implies there exists $\Phi_\epsilon^{\perp}$ satisfying $\|\Phi_\epsilon^{\perp}\|_{\sigma_1} = O(\epsilon^2|e_0|)$. Moreover,

$$\Phi_\epsilon = e_0\Phi_0 + \Phi_1 + \Phi_\epsilon^{\perp}. \tag{4.82}$$

Furthermore, we can normalize Φ_ϵ such that $\|\Phi_\epsilon\|_{L^2} = 1$, which implies $|e_0| = O(1)$.

Now, we have obtained the asymptotics of the eigenfunction Φ_ϵ given by (4.82). We shall use it to derive the asymptotical form of τ_ϵ. In fact, τ_ϵ can be established by considering the orthogonality condition.

We integrate the Φ-equation in (4.57) over $\left(0, \frac{L}{\epsilon}\right)$ and use the Neumann boundary condition to get

$$\tau_\epsilon \int_0^{\frac{L}{\epsilon}} \Phi \mathrm{d}y = \epsilon^2\mu \int_0^{\frac{L}{\epsilon}} (\bar{u} - 2U)\Phi \mathrm{d}y. \tag{4.83}$$

We have from (4.82) that

$$\tau \int_0^{\frac{L}{\epsilon}} \Phi \mathrm{d}y = \tau_\epsilon e_0 \int_0^{\frac{L}{\epsilon}} \Phi_0 \mathrm{d}y + \tau\epsilon \int_0^{\frac{L}{\epsilon}} \Phi_1 \mathrm{d}y + \tau \int_0^{\frac{L}{\epsilon}} \Phi^{\perp} \mathrm{d}y := I_{51} + I_{52} + I_{53}. \tag{4.84}$$

Define $z := \frac{\sqrt{a}}{2} y$, then by straightforward computation, one can show that

$$\begin{aligned} I_{51} = \tau e_0 \int_0^{\frac{L}{\epsilon}} \Phi_0 \mathrm{d}y =& \tau e_0 \int_0^{\infty} \Phi_0 \mathrm{d}y + O(|\tau||e_0|\epsilon) \\ =& \frac{2\tau e_0}{\sqrt{a}} \int_0^{\infty} (z\tanh z - 1)\mathrm{sech}^2 z \mathrm{d}z + O(|\tau||e_0|\epsilon) \\ =& -\frac{\tau e_0}{\sqrt{a}} + O(|\tau||e_0|\epsilon). \end{aligned} \tag{4.85}$$

It is similar to obtain that $|I_{52}| = O(|\tau_\epsilon|\epsilon)$ and $|I_{53}| = O(|\tau_\epsilon|\epsilon^2)$ since $|\Phi_1| \lesssim e^{-(\sqrt{3\bar{u}}-\delta)y}$ for some δ sufficiently small. On the other hand, we similarly find from (4.56) that

$$\begin{aligned} &\int_0^{\frac{L}{\epsilon}} (U_0 + \tilde{U}_1)(e_0\Phi_0 + \epsilon\Phi_1 + \Phi^{\perp})\mathrm{d}y \\ =& \int_0^{\frac{L}{\epsilon}} U_0(e_0\Phi_0 + \epsilon\Phi_1 + \Phi^{\perp})\mathrm{d}y + \int_0^{\frac{L}{\epsilon}} \tilde{U}_1(e_0\Phi_0 + \epsilon\Phi_1 + \Phi^{\perp})\mathrm{d}y \\ =& \int_0^{\frac{L}{\epsilon}} U_0(e_0\Phi_0 + \epsilon\Phi_1 + \Phi^{\perp})\mathrm{d}y + O(|e_0|\epsilon), \end{aligned} \tag{4.86}$$

where

$$\begin{aligned} e_0 \int_0^{\frac{L}{\epsilon}} U_0\Phi_0 \mathrm{d}y =& e_0 \int_0^{\infty} U_0\Phi_0 \mathrm{d}y + O(\epsilon|e_0|) \\ =& \sqrt{a} e_0 \int_0^{\infty} (z\tanh z - 1)\mathrm{sech}^4 z dz + O(\epsilon|e_0|) \\ =& -\frac{1}{2}\sqrt{a} e_0 + O(\epsilon|e_0|). \end{aligned} \tag{4.87}$$

The other terms such as $U_0\Phi_1$ in (4.86) are both $O(\epsilon)$. Hence, we combine (4.86) and (4.87) to get

$$\int_0^{\frac{L}{\epsilon}} (U_0 + \tilde{U}_1)(e_0\Phi_0 + \epsilon\Phi_1 + \Phi^\perp)\mathrm{d}y = -\frac{1}{2}\sqrt{a}e_0 + O(|e_0|\epsilon). \tag{4.88}$$

Since we normalize Φ_ϵ, we have $|e_0| = O(1)$. Moreover, with the help of Proposition 4.4, we can see $|\tau_\epsilon| = O(\epsilon^2)$ if we assume $Re(\tau_\epsilon) \geq -c_0\epsilon^2$ for some constant c_0 independent of ϵ. Therefore, by collecting (4.83), (4.85) and (4.88), one has from $a = 3\bar{u} + O(\epsilon)$ that

$$\begin{aligned}\tau_\epsilon =& - \epsilon^2\mu a + \epsilon^2\mu\bar{u} + O(\epsilon^3)\\ =& - 2\mu\bar{u}\epsilon^2 + O(\epsilon^3),\end{aligned} \tag{4.89}$$

which proves (4.65), then this Proposition.

□

Combining Proposition 4.3, Proposition 4.4 and Proposition 4.7, we have the steady state (1.4) is locally linear stable for $\epsilon \ll 1$, which proves Theorem 1.2.

5 Conclusion and Open Problems

In this paper, we have investigated the Keller–Segel models with logistic growth and large advection. Our main contributions are the existence and stability analysis of spiky steady states to (1.2). Motivated by the formal computation in [38], we have established the rigorous proof of the construction and stability results via Lyapunov-schmidt reduction. In particular, we perform a priori estimates of linearized eigenvalues to rigorously rule out the large eigenvalue case.

The behavior of system (1.2) differs significantly from the minimal Keller–Segel models (1.1). It is well-known that one-dimensional minimal models admit monotone decreasing spiky steady states with the height of cellular density u being $O(1/\chi)$ [4, 19]. Whereas, the logistic source prevents the height of u from being $O(1/\chi)$, and thereby it becomes $O(1)$. The stability property of monotone spiky solution is similar as its counterpart in minimal models. To be more precisely, it is locally linear stable with respect to even perturbations.

There are also some intriguing questions arising from the pattern formation within system (1.2) that deserve future explorations. Our numerical experiment indicates that the double boundary spike is linearly stable with respect to translation modes, which is a distinct phenomenon we have not met in the minimal models. Thus, it is worthy utilizing either formal or rigorous method to verify this phenomenon. The other interesting direction for future explorations is to study the existence and stability properties of non-constant steady states in the regime $d_2 \ll 1$. Noting the numerical results shown in [23], we conjecture that system in 1D admits the stable interior spike when $d_2 \ll 1$. But it is open to study or prove it via matched asymptotic analysis or Lyapunov-schmidt reduction.

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Appendix A Formal Expansion of Single Boundary Spikes

In Appendix A, we shall employ the matched asymptotic analysis to reconstruct the non-constant steady state (1.4), which can support our rigorous argument.

We firstly multiply the both hand side of the u-equation by $\frac{1}{\chi}$ in (1.3) to obtain

$$\begin{cases} 0 = \frac{1}{\chi}u_{xx} - (uv_x)_x + \frac{\mu}{\chi}u(\bar{u} - u), & x \in (0, L), \\ 0 = v_{xx} - v + u, & x \in (0, L). \end{cases} \tag{1.1}$$

Our aim is to look for a localized pattern with the centre being 0. Recall $\epsilon := \sqrt{1/\chi}$, then in the inner region, we introduce

$$U(y) := u(x), \quad V(y) := v(x) \text{ with } y := \frac{x}{\epsilon}. \tag{1.2}$$

Upon substituting (1.2) into (1.1), one has

$$\begin{cases} 0 = U_{yy} - \epsilon^{-2}(UV_y)_y + \epsilon^2\mu U(\bar{u} - U), & y \in (0, \frac{L}{\epsilon}), \\ 0 = \epsilon^{-2}V_{yy} - V + U, & y \in (0, \frac{L}{\epsilon}). \end{cases} \tag{1.3}$$

We expand

$$U(y) = U_0 + \epsilon^2 U_1 + \cdots, \quad V(y) = V_0 + \epsilon^2 V_1 \cdots \tag{1.4}$$

and substitute it into (1.3). To the leading order, we see that $V_{0yy} = 0$. Since we would like to find the uniformly bounded solution in $(0, \infty)$, one takes $V_0 = V_{00}$, where V_{00} is an undetermined constant. Moreover, we collect the following hierarchy from (1.3) and (1.4):

$$\begin{cases} U_{0yy} - (U_0 V_{1y})_y = 0, & y \in (0, \infty), \\ V_{1yy} - V_{00} + U_0 = 0, & y \in (0, \infty), \end{cases} \tag{1.5}$$

and

$$\begin{cases} U_{1yy} - (U_1 V_{1y})_y - (U_0 V_{2y})_y = -\mu U_0(\bar{u} - U_0), & y \in (0, \infty), \\ V_{2yy} = V_1 - U_1, & y \in (0, \infty). \end{cases} \tag{1.6}$$

Noting $U \ll 1$ in the outer region, one has $U_0 \to 0$ as $|y| \to 0$. Thus, we infer from the first equation of (1.5) that $U_0(y) = U_{00} e^{V_1(y)}$, where U_{00} is an unknown constant. Now in the outer region, we can replace U in sense of distribution by

$$U \to \epsilon U_{00} \int_0^{\infty} e^{V_1} d\rho \delta_0(x). \tag{1.7}$$

As such we find the outer problem for v is

$$\begin{cases} v_{yy} - v = -\epsilon U_{00} C \delta_0(x), & x \in (0, L), \\ v_y(0) = v_y(L) = 0, \end{cases} \tag{1.8}$$

where $C := \int_0^{\infty} e^{V_1} d\rho$ and we impose that C is a finite integral. To express v in the outer region, we introduce the following one-dimensional Neumann Green's function $G(x; \xi)$:

$$\begin{cases} G_{xx} - G = -\delta(x - \xi), & x \in (0, L), \\ G_x(0; \xi) = G_x(L; \xi) = 0, \end{cases}$$

where G has the following explicit form:

$$G(x; \xi) = \begin{cases} \frac{\cosh(L-\xi)}{\sinh L} \cosh x, & x \in (0, \xi), \\ \frac{\cosh \xi}{\sinh L} \cosh(L - x), & x \in (\xi, L). \end{cases}$$

Hence, we find v satisfies $v \sim \epsilon U_{00} C G(x; 0)$. It follows that as $x \to 0$ and $\epsilon \to 0^+$, $v \to 0$. Recall in the inner expansion, $V = V_{00} + \epsilon^2 V_1 + \cdots$, then we conclude from the matching that $V_{00} = 0$.

We next solve (1.5) to find the inner solution. Upon substituting $U_0 = U_{00} e^{V_1}$ and $V_{00} = 0$ into the second equation, we establish the following core problem:

$$\begin{cases} V_{1yy} + U_{00} e^{V_1} = 0, \ y \in (0, \infty), \\ V_{1y}(0) = 0. \end{cases} \tag{1.9}$$

Solving equation (1.9) gives rise to

$$V_1(y) = \log\left(\frac{a}{2U_{00} \cosh^2(\frac{\sqrt{a}}{2} y)}\right),$$

where a is a free parameter. Since $V_{1y} = -\sqrt{a} \tanh\left(\frac{\sqrt{a}}{2} y\right)$, one has from the relationship between V_1 and U_0 that

$$U_0 = \frac{a}{2} \operatorname{sech}^2\left(\frac{\sqrt{a}}{2} y\right). \tag{1.10}$$

To determine the constants, we next apply the integral constraint $\int_0^L u(\bar{u} - u)dx = 0$ thanks to the Neumann boundary condition. Noting $U \sim U_0$, we can arrive at

$$\int_0^\infty U_0(\bar{u} - U_0)dy = 0.$$

By using $U_0(y) = U_{00}e^{V_1}$, one has $\int_0^\infty (e^{V_1} - U_{00}e^{2V_1})dy = 0$, which then yields that

$$\int_0^\infty \left(\bar{u}\mathrm{sech}^2\left(\frac{\sqrt{a}}{2}y\right) - \frac{a}{2}\mathrm{sech}^4\left(\frac{\sqrt{a}}{2}y\right)\right)dy = 0. \tag{1.11}$$

Let $z = \frac{\sqrt{a}}{2}y$, then we solve (1.11) to get $a \sim 3\bar{u}$. Therefore, we have from (1.10) that for $y \in (0, \infty)$,

$$U_0 = \frac{a}{2}\mathrm{sech}^2\left(\frac{\sqrt{a}}{2}y\right)$$

with $a \sim 3\bar{u}$. Now, we have obtained the inner solution.

Focusing on the outer region, one has $u \sim 0$ and $v_{xx} - v \sim 0$ with $v_{0y}(L) = 0$. By solving it, we get

$$v \sim C_v \cosh(x - L), \quad x \in (0, L),$$

where C_v is an unknown constant to be determined. We next use the matching condition to determine constant C_v. From the inner solution, one finds

$$\frac{dV}{dy} = -\epsilon^2 \sqrt{a} \tanh\left(\frac{\sqrt{a}}{2}y\right), \tag{1.12}$$

which yields

$$\frac{dV}{dy} \to -\epsilon^2 \sqrt{a} \text{ as } y \to +\infty. \tag{1.13}$$

On the other hand, from the outer solution, we conclude for $0 < x < L$,

$$\frac{dv}{dy} = \frac{dv}{dx}\epsilon \sim \epsilon C_v \sinh(x - L).$$

It follows that

$$\frac{dv}{dy} \to -C_v\epsilon \sinh L \text{ as } x \to 0. \tag{1.14}$$

After matching (1.13) and (1.14), one can get $C_v = \epsilon\sqrt{a}/\sinh L$.

In summary, the single boundary spike (u^-, v^-) can be asymptotically written as

$$u^- \sim \frac{a}{2}\mathrm{sech}^2\left(\frac{\sqrt{a}}{2} \cdot \frac{x}{\epsilon}\right), \text{ for } x \in (0, L),$$

and

$$v^- \sim \epsilon\frac{\sqrt{a}}{\sinh L}\cosh(x - L), \text{ for } x \in (0, L),$$

where $a \sim 3\bar{u}$. Next, we use the Van Dyke's matching principle $v_{\text{unif}} = v_{\text{inner}} + v_{\text{outer}} - v_{\text{overlap}}$ to find the composite expansion of v, which is

$$v^- \sim \epsilon^2 \log\left(\frac{1}{4}\mathrm{sech}^2\left(\frac{\sqrt{a}}{2}\frac{x}{\epsilon}\right)\right) + \epsilon\frac{\sqrt{a}}{\sinh L}\cosh(x - L) + \sqrt{a}\epsilon x.$$

These results agree with those stated in Theorem 1.1.

Appendix B Formal Analysis of the Eigenvalue Problem

This section is devoted to the study of linearized eigenvalue problem (4.1) via the matched asymptotic analysis. Similarly, in the inner Region, we introduce the following rescaled functions:

$$\Phi(y) := \phi(x), \ \ \Psi(y) := \psi(x) \text{ with } y = \frac{x}{\epsilon}.$$

By using it together with (1.2), one can rewrite (4.1) as

$$\begin{cases} \epsilon^2\lambda\Phi = \Phi_{yy} - \epsilon^{-2}(U\Psi_y + \Phi V_y)_y + \epsilon^2\mu(\bar{u} - 2U)\Phi, & y \in (0, \frac{L}{\epsilon}), \\ \lambda\Psi = \epsilon^{-2}\Psi_{yy} - \Psi + \Phi, & y \in (0, \frac{L}{\epsilon}), \\ \Phi_y(0) = \Psi_y(0) = \Phi_y(\frac{L}{\epsilon}) = \Psi_y(\frac{L}{\epsilon}) = 0. \end{cases} \tag{2.1}$$

Similarly as above, we expand

$$\lambda = \lambda_0 + \cdots, \ \ \Phi(y) = \Phi_0(y) + \epsilon^2\Phi_1(y) + \cdots \ \text{ and } \ \Psi = \Psi_0(y) + \epsilon^2\Psi_1(y) + \cdots \tag{2.2}$$

and substitute them together with (1.4) into (2.1). Then one can find $\Psi_{0y} = 0$, and thereby $\Psi_0(y) := \Psi_{00}$ with Ψ_{00} being a constant. Moreover, with the help of matching condition between the inner and the outer solution, we obtain $\Psi_{00} = 0$.

We further collect the following leading order system:

$$\begin{cases} 0 = \Phi_{0yy} - (U_0\Psi_{1y} + \Phi_0 V_{1y})_y, & y \in (0, \infty), \\ 0 = \Psi_{1yy} + \Phi_0, & y \in (0, \infty), \\ \Phi_{0y}(0) = \Psi_{1y}(0) = 0, \ \ \Phi_0(\infty) = 0. \end{cases} \tag{2.3}$$

The first equation in (2.3) implies that $\left(\frac{\Phi_0}{U_0}\right)_y = \Psi_{1y}$, hence $\Phi_0 = U_0\Psi_1 + CU_0$ thanks to the boundary conditions, where C is some constant to be determined later on. Therefore, (2.3) yields that

$$\begin{cases} \Psi_{1yy} + U_0\Psi_1 + CU_0 = 0, & y \in (0, \infty), \\ \Psi_{1y}(0) = 0. \end{cases} \tag{2.4}$$

Since $U_0 = \frac{a}{2}\text{sech}^2\left(\frac{\sqrt{a}}{2}y\right)$ and $a \sim 3\bar{u}$, we further solve (2.4) to get the eigenfunctions are unique up to a constant multiplier of the following

$$\Psi_1(y) = \frac{1}{\sqrt{a}}y\tanh\left(\frac{\sqrt{a}}{2}y\right), \Phi_0(y) = \left(1 - \frac{\sqrt{a}}{2}y\tanh\left(\frac{\sqrt{a}}{2}y\right)\right)\text{sech}^2\left(\frac{\sqrt{a}}{2}y\right), \ \ a \sim 3\bar{u}. \tag{2.5}$$

Next, we proceed to show the corresponding leading eigenvalue $\lambda_0 < 0$, which tells us that steady state (1.4) is linearly stable.

Proof. We integrate the ϕ-equation in (4.1) over $(0, L)$ to get

$$\lambda\int_0^L \phi(x)dx = \mu\int_0^L (\bar{u} - 2u)\phi(x)dx. \tag{2.6}$$

Upon substituting (2.5) into (2.6), one has the left hand side and right hand side satisfy

$$\lambda\int_0^L \phi(x)dx \sim \epsilon\lambda_0\int_0^\infty \Phi_0(y)dy \tag{2.7}$$

and

$$\int_0^L \mu(\bar{u} - 2u)\phi(x)dx \sim \epsilon\int_0^\infty \mu(\bar{u} - 2U_0)\Phi_0(y)dy, \tag{2.8}$$

respectively. By straightforward calculation, we obtain

$$\int_0^\infty \Phi_0 dy = -\frac{1}{\sqrt{a}}, \tag{2.9}$$

and

$$\int_0^\infty U_0 \Phi_0 dy = -\frac{1}{2}\sqrt{a}. \tag{2.10}$$

Combining (2.9) and (2.10), we have from (2.6), (2.7) and (2.8) that $\lambda_0 \sim -2\mu\bar{u}$. This gives us (1.4) is linearly stable with respect to the even eigenfunction (2.5), then formally verifies Theorem 1.2. □