NONDEGENERACY, MORSE INDEX AND ORBITAL STABILITY OF THE KP-I LUMP SOLUTION

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ABSTRACT. Using Bäcklund transformation, we analyze the spectral property of the KP-I lump solution. It is proved that the lump is nondegenerate and has Morse index one. As a consequence, we show that it is orbitally stable.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The KP equation, introduced by Kadomtsev and Petviashvili [32], is a classical nonlinear dispersive equation appearing in many physical contexts, including the motion of shallow water waves. It is actually a universal model and arises as a modulation equation for a large class of PDEs(See for instance [57]). The KP equation has the form

$$\partial_x \left(\partial_t u+\partial_x^3 u+3\partial_x \left(u^2\right)\right)-\sigma\partial_y^2 u=0, \tag{1.1}$$

where σ is a parameter, usually chosen to be ± 1. In this paper, we will consider the case of $\sigma=1$, which is called KP-I equation. It is worth pointing out that the KP-II equation, corresponding to $\sigma=-1$, has different dispersion relation and its property is quite different from that of KP-I.

If the function u is y independent, then equation (1.1) will be reduced to the KdV equation, a well known integrable system which has been extensively studied for about half a century. Dryuma [17] in 1974 found a Lax pair for the KP equation. The inverse scattering transform for KP-I equation then has been carried out in [18, 39, 67]. (See [9] for a more detailed review on KP equation. See also [1, 2] and the references therein for more discussions on the results concerning this topic obtained before 1991.)

The soliton solutions of the KP equation can be obtained through various different methods, both for the KP-I and KP-II equations. An important feature of the KP-I equation is that it has the so called lump solutions which are also of travelling wave type. They were found in [40, 61]. Explicitly, the KP-I equation has a family of lump solutions of the form

$$u_c=4\frac{-\left(x-ct\right)^2+cy^2+\frac{3}{c}}{\left(\left(x-ct\right)^2+cy^2+\frac{3}{c}\right)^2}. \tag{1.2}$$

It particular, for $c=1$, it can be written as $Q\left(x-t,y\right)$, where

$$Q\left(x,y\right)=4\frac{y^2-x^2+3}{\left(x^2+y^2+3\right)^2}. \tag{1.3}$$

Note that Q is non-radial and decays like $O\left(\frac{1}{x^2+y^2}\right)$ at infinity. Because of this slow decaying property, the inverse scattering transform of KP-I turns out to be quite delicate. Indeed, it is shown in [64] that a winding number can be associated to the lump solutions. We also mention that the interaction of multi-lump solutions has been investigated in [20, 37].

The KP-I equation and its lump solutions actually appear in many other physical models. For instance, formal asymptotic analysis shows that in the context of the motion in a Bose condensate, the transonic limit of certain traveling wave solutions to the GP equation is related to the lump solutions ([6, 31]). This is the so called Roberts's Program ([30,31]). A rigorous verification is given recently in [7] and [12]. The KP-I equation has other travelling wave solutions with energy higher than the lump Q. Indeed, using binary Darboux transformation and other methods, families of rational type solutions have been studied in [3, 4, 20, 53–55] (for instance, see P. 139 of [3]). Explicitly, one of the families of these solutions has the form of $2\partial_x^2 \ln F$, with

$$F = \left|f^3 - 4f + g\right|^2 + 27\left|f^2 - \frac{8}{3}\sqrt{3} - 3\right|^2 + 162\left|f\right|^2 + 1593,$$

where $f = x + yi + \gamma$ and $g = 12yi + \delta$, with γ, δ being complex parameters. In particular, if $\gamma = \delta = 0$, then we get a solution with F being even in both x and y variables.

Regarding to traveling wave type solutions, it is worth mentioning that the generalized KP-I equations

$$\partial_x^2\left(\partial_x^2 u - u + u^p\right) - \partial_y^2 u = 0 \tag{1.4}$$

also have lump type solutions for suitable $p \in (1,5)$([13, 36]). Note that for general $p \neq 2$, this is believed to be not an integrable system. Hence no explicit formula is available. These lump type solutions are obtained via variational methods (concentration compactness). The stability(or instability) and other properties of these ground state solutions have been studied in [13–15, 34, 35, 62, 63, 65]. It is known that for $p \in \left(1, \frac{7}{3}\right)$, they are orbitally stable, while for $p \in \left(\frac{7}{3}, 5\right)$, they are unstable. However, it is not known whether or not Q has this variational characterization. It is conjectured that for $p = 2$, there should be a unique (up to translation) ground state, which should be the lump solution Q. We refer to the interesting paper [34] for a review and numerical study of the lumps.

Our first result in this paper is

Theorem 1. *Suppose ϕ is a smooth solution to the equation*

$$\partial_x^2\left(\partial_x^2 \phi - \phi + 6Q\phi\right) - \partial_y^2 \phi = 0. \tag{1.5}$$

Assume that

$$\phi(x,y) \to 0, \text{ as } x^2 + y^2 \to +\infty.$$

Then $\phi = c_1\partial_x Q + c_2 \partial_y Q$, for some constants c_1, c_2.

Theorem 1 has long been conjectured to be true. See the remark after Lemma 7 of [65], concerning the spectral property and its relation to stability.

Our second result concerns the Morse index of Q. By definition, the Morse index of Q is the number of negative eigenvalues (counted with multiplicity) of the self-adjoint operator $\mathcal{L}$ acting on the space E. Here

$$\mathcal{L}\phi := -\partial_x^2\phi + \phi - 6Q\phi + \partial_x^{-2}\partial_y^2\phi.$$

E is the closure of the space of smooth compactly supported functions under the norm

$$\|f\|^2 = \int_{\mathbb{R}^2} \left(|\partial_x f|^2 + f^2 + \left(\partial_x^{-1}\partial_y f\right)^2 \right).$$

This is the energy space for the KP-I equation.

Theorem 2. *The operator $\mathcal{L}$ has exactly one negative eigenvalue and hence has Morse index one.*

We remark that if η is an eigenfunction corresponding to the negative eigenvalue, then indeed $\int_{-\infty}^{+\infty} \eta(x,y)\, dx = 0$ for each y and

$$\partial_x^{-2}\partial_y^2 \eta = \int_{-\infty}^{x} \int_{-\infty}^{s} \partial_y^2 \eta(v,y)\, dv ds.$$

With the spectral information of the lump at hand, it is natural to investigate the nonlinear stability of the lump solution. By a result of Ionescu-Kenig-Tataru [29], the KP-I equation is well-posed in the natural energy space. The Cauchy problem of KP equation has attracted many attention. For instance, the initial value problem of KP-II is studied in [11]. The well-posedness (or ill-posedness) of the KP-I equation is considered [33, 44–46]. It is worth pointing out that KP-I and KP-II have quite different nature, and the methods to solve the Cauchy problem in low regularity spaces are quite different. We also refer to the references in the above mentioned papers for more detailed discussion on this topic. We prove in this paper:

Theorem 3. *The lump solution u_c is orbitally stable in the energy space E.*

We refer to Theorem 45 in Section 5 for a more precisely statement of orbital stability. We note that the Morse index of the lump solution has been numerically shown to be one [16]. Here we give a rigorous proof. It is also worth mentioning that the stability or instability of the line solitons of the KP-I or KP-II equation has already been studied. The instability of the line soliton of the KP-I equation has been proved in [68]. The spectral instability of the line solitons for the generalized KP-I equation is proved in [5]. In [59, 60], the orbital stability or instability of line solitons of KP-I equation in y-periodic space is studied. It is shown there that the stability is determined by the travelling speed of the line solitons. The case of critical speed is then investigated in a recent paper [66]. The orbital and asymptotic stability of the line solitons for the KP-II equation is studied by Mizumachi and Tzvetkov in [49, 51]. We also refer to [42] for a review on stability of solitons for the gKdV equations.

As we mentioned before, the KP-I equation has close relation with travelling waves of the GP equation. For solutions of the form $\Phi(z - ct, x)$, the GP equation is

$$ic\partial_z \Phi = \Delta_{(z,x)}\Phi - \Phi\left(|\Phi|^2 - 1\right), \quad (z,x) \in \mathbb{R}^2. \tag{1.6}$$

Formal computation performed in the appendix of [6] suggests that this equation has a family of solutions with c close to $\sqrt{2}$, and their asymptotic profiles are determined by the lump solution. Our nondegeneracy results in this paper will have potential applications in the rigorous construction of these solutions. For more discussions on the traveling waves of Gross-Pitaevskii equation, we refer to [7, 8, 16, 41] and the references therein.

Now let us briefly describe the main ideas of the proof of our main results. One of our tools will be the Bäcklund transformation. We first show that the lump solution can be obtained from the trivial solution by performing Bäcklund transformation twice. Note that while this fact should be known to experts in the field, we have not been able to find a reference for this result. Next we consider these transformations in the linearized level. We show that a kernel of the linearized operator around Q can be transformed to a kernel of the operator $\partial_x^2+\partial_y^2-\partial_x^4$, which is the linearized operator around the trivial solution. With some information on the growth rate of the kernel function, we are able to conclude that the only decaying solutions to (1.5) are corresponding to translations in the x and y axes. This proves Theorem 1. Note that the general idea of using linearized Bäcklund transformation to investigate the spectral property has already been used in [48] in the case of Toda lattice. We also refer to [47, 50] for related discussion about n-soliton solutions for the KdV equation and the NLS equation.

To prove Theorem 2, we consider the nondegeneracy of a family of y-periodic traveling wave solutions to KP-I connecting the one dimensional soliton and the two-dimensional lump solution. They are given as follows: let $k, b>0$ with $k^2+b^2=1$. Define

$$\Gamma_k := \cosh(kx) - \sqrt{\frac{1-4k^2}{1-k^2}}\cosh(kbiy),$$
$$T_{2,k}(x,y) := 2\partial_x^2 \ln \Gamma_k. \tag{1.7}$$

Then $T_{2,k}(x-t,y)$ are traveling wave solutions to KP-I (with speed $c=1$). They are periodic in y with period $t_k=\frac{2\pi}{k\sqrt{1-k^2}}$. As $k\to 0$, $T_{2,k}$ converges to the lump solution. As $k\to\frac{1}{2}$, $T_{2,k}$ converges to the one-dimensional soliton solution $\frac{1}{2}\operatorname{sech}^2(\frac{x}{2})$.

Theorem 4. *The solution $T_{2,k}$ is nondegenerate in the following sense: Suppose φ is a smooth function decaying in the x variable and t_k-periodic in the y variable, satisfying*

$$\partial_x^2\left(\partial_x^2\varphi-\varphi+6T_{2,k}\varphi\right)-\partial_y^2\varphi=0.$$

Then $\varphi=c_1\partial_x T_{2,k}+c_2\partial_y T_{2,k}$ for some constants c_1 and c_2.

With the help of Theorem 4 we prove Theorem 2 by a continuation argument. By nondegeneracy along the path $k\in\left(0,\frac{1}{2}\right)$, the Morse index is invariant along the periodic solution. Since the Morse index of the one dimensional soliton is 1, we conclude that the Morse index of the lump is also 1. Note that this type of continuation argument is in spirit similar to that of [19], where the fractional Laplacian has been studied.

Finally to prove Theorem 3 we show the convexity of the energy

$$d(c) := \int_{\mathbb{R}^2}\left(\frac{1}{2}\left(\partial_x u_c\right)^2-u_c^3+\frac{1}{2}\left(\partial_y\partial_x^{-1}u_c\right)^2+\frac{1}{2}cu_c^2\right).$$

and the classical result of Grillakis-Shatah-Strauss [23] yields Theorem 3.

This paper is organized in the following way. In Section 2, we recall some basic facts about the bilinear derivatives. In Section 3, we prove the nondegenracy of lump. In Section 4, we prove the nondegeneracy of a family of periodic solutions naturally associated to the lump solution. In Section 5, we prove the orbital stability of lump.

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2. Preliminaries

In this section we describe the bilinear form of KP-I and its associated Bäcklund transform.

We will use D to denote the bilinear derivative. Explicitly,

$$D_x^m D_t^n f \cdot g = \partial_y^m \partial_s^n f(x+y, t+s)\, g(x-y, t-s)\,|_{s=0, y=0}.$$

In particular,

$$\begin{aligned} D_x f \cdot g &= \partial_x f g - f \partial_x g, \\ D_x^2 f \cdot g &= \partial_x^2 f g - 2\partial_x f \partial_x g + f \partial_x^2 g, \\ D_x D_t f \cdot g &= \partial_x \partial_t f g - \partial_x f \partial_t g - \partial_t f \partial_x g + f \partial_x \partial_t g. \end{aligned}$$

The KP-I equation (1.1) can be written in the following bilinear form [25]:

$$\left(D_x D_t + D_x^4 - D_y^2\right) \tau \cdot \tau = 0. \tag{2.1}$$

To be more precise, we introduce the τ-function:

$$u = 2\partial_x^2 \ln \tau. \tag{2.2}$$

Then equation (1.1) becomes

$$\partial_x^2 \left[\left(\partial_x \partial_t \ln \tau + \partial_x^4 \ln \tau + 6\left(\partial_x^2 \ln \tau\right)^2 \right) - \partial_y^2 \ln \tau \right] = 0.$$

Using the identities (see Section 1.7.2 of [25])

$$2\partial_x^2 \ln \tau = \frac{D_x^2 \tau \cdot \tau}{\tau^2},\ 2\partial_x \partial_t \ln \tau = \frac{D_x D_t \tau \cdot \tau}{\tau^2},$$

and

$$2\partial_x^4 \ln \tau = \frac{D_x^4 \tau \cdot \tau}{\tau^2} - 3\left(\frac{D_x^2 \tau \cdot \tau}{\tau^2}\right)^2,$$

we get

$$\partial_x^2 \left(\frac{D_x D_t \tau \cdot \tau + D_x^4 \tau \cdot \tau - D_y^2 \tau \cdot \tau}{\tau^2} \right) = 0. \tag{2.3}$$

Therefore, if τ is positive and satisfies the bilinear equation (2.1), then u satisfies the KP-I equation (1.1).

Remark 5. *Suppose u is a solution of the KP-I equation (1.1) and $u = 2\partial_x^2 \ln \tau$. Then, a priori, from (2.3), we can only say that*

$$D_x D_t \tau \cdot \tau + D_x^4 \tau \cdot \tau - D_y^2 \tau \cdot \tau = (a(y)\, x + b(y))\, \tau^2,$$

for some functions a, b. However, in the special case that u is the lump solution, actually we have $a = b = 0$. We also remark that in this paper, a solution τ of (2.1) is allowed to be complex valued(see τ_1 and ι_2). But if τ is not positive everywhere, then the corresponding function $u = 2\partial_x^2 \ln \tau$ is not a well defined solution of the KP-I equation.

Let i be the imaginary unit. We will investigate properties of the following three functions:

$$\begin{aligned}\tau_0(x,y) &= 1,\\ \tau_1(x,y) &= x+iy+\sqrt{3},\\ \tau_2(x,y) &= x^2+y^2+3.\end{aligned}$$

Let

$$\begin{aligned}\tilde{\tau}_0(x,y,t) &= \tau_0(x-t,y),\\ \tilde{\tau}_1(x,y,t) &= \tau_1(x-t,y),\\ \tilde{\tau}_2(x,y,t) &= \tau_2(x-t,y).\end{aligned}$$

Then $\tilde{\tau}_0, \tilde{\tau}_1, \tilde{\tau}_2$ are solutions of (2.1).

2.1. **The Bäcklund transformation.** Under the transformation (2.2), the function $\tilde{\tau}_2$ is corresponding to the lump solution $Q(x-t,y)$. Now let μ be a constant. We first recall the following bilinear operator identity which can be found in [52]. We also refer to [56] for a simplified version of this identity(See also [28,38] and the references therein for the construction of more general rational solutions, for KP and other related equations). Let μ, ν, λ be arbitrary parameters. Then we have

$$\begin{aligned}&\frac{1}{2}\left[\left(D_xD_t+D_x^4-D_y^2\right)f\cdot f\right]gg-\frac{1}{2}\left[\left(D_xD_t+D_x^4-D_y^2\right)g\cdot g\right]ff\\ &=D_x\left[\left(D_t+3\lambda D_x-\sqrt{3}i\mu D_y+D_x^3-\sqrt{3}iD_xD_y+\nu\right)f\cdot g\right]\cdot(fg)\\ &+3D_x\left[\left(D_x^2+\mu D_x+\frac{1}{\sqrt{3}}iD_y-\lambda\right)f\cdot g\right]\cdot(D_xg\cdot f)\\ &+\sqrt{3}iD_y\left[\left(D_x^2+\mu D_x+\frac{1}{\sqrt{3}}iD_y-\lambda\right)f\cdot g\right]\cdot(fg).\end{aligned} \tag{2.4}$$

By this identity, we can consider the Bäcklund transformation from $\tilde{\tau}_0$ to $\tilde{\tau}_1$($\mu=\frac{1}{\sqrt{3}}$):

$$\begin{cases}\left(D_x^2+\frac{1}{\sqrt{3}}D_x+\frac{1}{\sqrt{3}}iD_y\right)\tilde{\tau}_0\cdot\tilde{\tau}_1=0,\\ \left(D_t-iD_y+D_x^3-\sqrt{3}iD_xD_y\right)\tilde{\tau}_0\cdot\tilde{\tau}_1=0.\end{cases} \tag{2.5}$$

The Bäcklund transformation from $\tilde{\tau}_1$ to $\tilde{\tau}_2$ is given by$\left(\mu=-\frac{1}{\sqrt{3}}\right)$

$$\begin{cases}\left(D_x^2-\frac{1}{\sqrt{3}}D_x+\frac{1}{\sqrt{3}}iD_y\right)\tilde{\tau}_1\cdot\tilde{\tau}_2=0,\\ \left(D_t+iD_y+D_x^3-\sqrt{3}iD_xD_y\right)\tilde{\tau}_1\cdot\tilde{\tau}_2=0.\end{cases} \tag{2.6}$$

Throughout the paper, we set $r=\sqrt{x^2+y^2}$. We would like to relate the kernel of the linearized KP-I equation to that of the linearized KP-I equation in the bilinear form.

Lemma 6. *Suppose ϕ is a smooth function satisfying the linearized KP-I equation*

$$\partial_x^2\left(\partial_x^2\phi-\phi+6Q\phi\right)-\partial_y^2\phi=0. \tag{2.7}$$

Assume

$$\phi(x,y)\to 0, \ \ \textit{as } r\to+\infty.$$

Let

$$\eta(x,y)=\tau_2\int_0^x\int_{-\infty}^t\phi(s,y)\,dsdt.$$

Then η satisfies

$$\left(-D_x^2+D_x^4-D_y^2\right)\eta\cdot\tau_2=0.$$

Moreover,

$$|\eta|+\left(|\partial_x\eta|+|\partial_y\eta|+\left|\partial_x^2\eta\right|+|\partial_x\partial_y\eta|+\left|\partial_x^3\eta\right|\right)(1+r)\le C(1+r)^{\frac{5}{2}}. \tag{2.8}$$

Proof. We write equation (2.7) in the form

$$\partial_x^4\phi-\partial_x^2\phi-\partial_y^2\phi=-6\partial_x^2(Q\phi).$$

Since $\phi\to0$ as $r\to+\infty$, it follows from the a priori estimate of the operator $\partial_x^2-\partial_x^4+\partial_y^2$(see Lemma 3.6 of [14]) that

$$|\phi|+\left(|\partial_x\phi|+|\partial_y\phi|+\left|\partial_x^2\phi\right|+\left|\partial_x^3\phi\right|\right)(1+r)\le C(1+r)^{-2}. \tag{2.9}$$

Now with this estimate at hand, we may integrate equation (2.7) in x from $-\infty$ to $+\infty$ and obtain

$$\partial_y^2\left(\int_{-\infty}^{+\infty}\phi(x,y)\,dx\right)=0.$$

Hence $\int_{-\infty}^{+\infty}\phi(x,y)\,dx=a_1y+a_2$ for some constants a_1,a_2. On the other hand, in view of the estimate (2.9),

$$\int_{-\infty}^{+\infty}\phi(x,y)\,dx\to0,\ \text{as } y\to\infty.$$

It then follows that $a_1=a_2=0$ and $\int_{-\infty}^{+\infty}\phi(x,y)\,dx=0$, for all y. Therefore,

$$|\eta|+\left(|\partial_x\eta|+|\partial_y\eta|+\left|\partial_x^2\eta\right|+|\partial_x\partial_y\eta|+\left|\partial_x^3\eta\right|\right)(1+r)\le C(1+r)^{\frac{5}{2}}.$$

As a consequence,

$$\frac{-D_x^2\eta\cdot\tau_2+D_x^4\eta\cdot\tau_2-D_y^2\eta\cdot\tau_2}{\tau_2^2}\to0,\ \text{as } r\to+\infty. \tag{2.10}$$

Since ϕ satisfies the linearized KP-I equation, η satisfies

$$\partial_x^2\left(\frac{-D_x^2\eta\cdot\tau_2+D_x^4\eta\cdot\tau_2-D_y^2\eta\cdot\tau_2}{\tau_2^2}\right)=0.$$

This together with (2.10) implies

$$-D_x^2\eta\cdot\tau_2+D_x^4\eta\cdot\tau_2-D_y^2\eta\cdot\tau_2=0.$$

The proof is completed. □

3. Nongeneracy of the lump solution

In this section, we will analyze the linearized Bäcklund transformation and use it to prove the nondegeneracy of the lump solution.

3.1. **Linearized Bäcklund transformation between τ_0 and τ_1.** In terms of τ_0 and τ_1, the Bäcklund transformation (2.5) can be written as

$$\begin{cases} \left(D_x^2 + \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} i D_y\right) \tau_0 \cdot \tau_1 = 0, \\ \left(-D_x - i D_y + D_x^3 - \sqrt{3} i D_x D_y\right) \tau_0 \cdot \tau_1 = 0. \end{cases} \tag{3.1}$$

Linearizing this system at (τ_0, τ_1), we obtain

$$\begin{cases} L_1 \phi = G_1 \eta, \\ M_1 \phi = N_1 \eta. \end{cases} \tag{3.2}$$

Here for notational simplicity, we have defined

$$L_1 \phi = \left(D_x^2 + \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} i D_y\right) \phi \cdot \tau_1,$$
$$M_1 \phi = \left(-D_x - i D_y + D_x^3 - \sqrt{3} i D_x D_y\right) \phi \cdot \tau_1,$$

and

$$G_1 \eta = -\left(D_x^2 + \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} i D_y\right) \tau_0 \cdot \eta,$$
$$N_1 \eta = -\left(-D_x - i D_y + D_x^3 - \sqrt{3} i D_x D_y\right) \tau_0 \cdot \eta.$$

Proposition 7. *Let η be a solution of the linearized bilinear KP-I equation at τ_1 :*

$$-D_x^2 \eta \cdot \tau_1 + D_x^4 \eta \cdot \tau_1 - D_y^2 \eta \cdot \tau_1 = 0. \tag{3.3}$$

Suppose η satisfies (2.8). Then in the region $\Omega_* := \mathbb{R}^2 \backslash \left\{(x,0) : x \geq -\sqrt{3}\right\}$, the system (3.2) has a solution ϕ satisfying

$$|\phi| + |\partial_x \phi| + |\partial_y \phi| \leq C (1+r)^{\frac{5}{2}},$$

and

$$-D_x^2 \phi \cdot \tau_0 + D_x^4 \phi \cdot \tau_0 = D_y^2 \phi \cdot \tau_0. \tag{3.4}$$

Moreover, there exists a constant c such that

$$\lim_{y \to 0^+} \phi(x,y) - \lim_{y \to 0^-} \phi(x,y) = c \xi_2(x,0), \text{ for } x > -\sqrt{3},$$

where ξ_2 is the function defined in (3.12).

The rest of this subsection will be devoted to the proof of Proposition 7.

First of all, from the first equation in (3.2), we get

$$\partial_y \phi \tau_1 = i \left[\partial_x \phi \tau_1 + \sqrt{3}\left(\partial_x^2 \phi \tau_1 - 2 \partial_x \phi\right)\right] - \sqrt{3} i G_1 \eta. \tag{3.5}$$

Inserting (3.5) into the right hand side of the second equation of (3.2), we get

$$4 \partial_x^3 \phi \tau_1 + \left(2\sqrt{3} \tau_1 - 12\right) \partial_x^2 \phi + \left(-4\sqrt{3} + \frac{12}{\tau_1}\right) \partial_x \phi = F_1. \tag{3.6}$$

Here the right hand side is defined by

$$F_1 = 3 \partial_x (G_1 \eta) + \sqrt{3} G_1 \eta + N_1 \eta - \frac{6}{\tau_1} G_1 \eta$$
$$= -2 \partial_x^3 \eta + 2\sqrt{3} i \partial_x \partial_y \eta - \frac{6}{\tau_1} G_1 \eta.$$

To solve the equation (3.6), we shall analyze the solutions of the equation

$$2\tau_1^2\partial_x^2 g + \left(\sqrt{3}\tau_1 - 6\right)\tau_1\partial_x g + \left(6 - 2\sqrt{3}\tau_1\right) g = 0. \tag{3.7}$$

For each fixed y, this is a second order ODE with respect to x.

Lemma 8. *The equation* (3.7) *has two linearly independent solutions*

$$g_1 = \tau_1 - \frac{\sqrt{3}}{2}\tau_1^2, \text{ and } g_2 = \tau_1 e^{-\frac{\sqrt{3}}{2}\tau_1}.$$

Proof. This can be verified directly. Indeed, equation (3.7) can be exactly solved using software such as *Maple.* □

Let W be the Wronskian of the two solutions g_1 and g_2. That is,

$$W := g_1\partial_x g_2 - g_2\partial_x g_1 = \frac{3}{4}\tau_1^3 e^{-\frac{\sqrt{3}}{2}\tau_1}.$$

By the variation of parameter formula, the equation

$$4\tau_1\partial_x^2 g + \left(2\sqrt{3}\tau_1 - 12\right)\partial_x g + \left(\frac{12}{\tau_1} - 4\sqrt{3}\right) g = F_1$$

has a solution

$$g^*(x, y) := g_2(x, y)\int_{-\infty}^x \frac{g_1 F_1}{4\tau_1 W} - g_1(x, y)\int_{-\infty}^x \frac{g_2 F_1}{4\tau_1 W}.$$

It follows that for each fixed y, the equation

$$4\tau_1\partial_x^3\phi + \left(2\sqrt{3}\tau_1 - 12\right)\partial_x^2\phi + \left(\frac{12}{\tau_1} - 4\sqrt{3}\right)\partial_x\phi = F$$

has a solution

$$w_0(x, y) := \int_{-\sqrt{3}}^x g^*(s, y)\, ds. \tag{3.8}$$

We emphasize that $\frac{1}{\tau_1}$ has a singularity at the point $(x, y) = \left(-\sqrt{3}, 0\right)$. Therefore we need to be very careful about the behavior of w_0 around this singular point.

Lemma 9. *Suppose η satisfies* (2.8) *and* (3.3). *Let w_0 be defined by* (3.8). *Then in Ω_*, w_0 is continuous and*

$$|w_0| \le C(1 + r)^{\frac{5}{2}}, \text{ for } x \le 10.$$

Proof. We denote the ball of radius one centered at the singularity $\left(-\sqrt{3}, 0\right)$ by $B_1\left(-\sqrt{3}, 0\right)$. Outside this ball, since η satisfies (2.8), we get

$$|F_1| \le C(1 + r)^{\frac{3}{2}}.$$

Hence using the asymptotic behavior of W and g_1, we find that for r large,

$$\left|\frac{g_1 F_1}{\tau_1 W}\right| \le C e^{\frac{\sqrt{3}}{2}x}(1 + r)^{-\frac{1}{2}}.$$

Similarly, using the asymptotic behavior of g_2, we get

$$\left|\frac{g_2 F_1}{\tau_1 W}\right| \le C(1 + r)^{-\frac{3}{2}}.$$

Therefore, for $x \leq -2\sqrt{3}$, we have

$$\left| \int_{-\infty}^{x} \frac{g_1 F_1}{\tau_1 W} \right| \leq C e^{\frac{\sqrt{3}}{2} x} (1+r)^{-\frac{1}{2}},$$
$$\left| \int_{-\infty}^{x} \frac{g_2 F_1}{\tau_1 W} \right| \leq C (1+r)^{-\frac{1}{2}}.$$

These estimates then imply that for $x \leq -2\sqrt{3}$,

$$|g^*| \leq C (1+r)^{\frac{3}{2}}. \tag{3.9}$$

On the other hand, in $B_1\left(\sqrt{3}, 0\right)$, we have

$$g_3 := g_2 - g_1 = \tau_1 e^{-\frac{\sqrt{3}}{2}\tau_1} - \left(\tau_1 - \frac{\sqrt{3}}{2}\tau_1^2\right) = O\left(\tau_1^3\right).$$

Using the fact that $g_1 = O(\tau_1), W = O\left(\tau_1^3\right)$ and $F_1 = O\left(\tau_1^{-1}\right)$, we then infer that in $B_1\left(-\sqrt{3}, 0\right) \cap \Omega_*$,

$$\left| g_3(x,y) \int_{-2}^{x} \frac{g_1 F_1}{4\tau_1 W} - g_1(x,y) \int_{-2}^{x} \frac{g_3 F_1}{4\tau_1 W} \right| \leq C. \tag{3.10}$$

As a consequence of the estimates (3.9) and (3.10), for $x \leq 10$, we obtain

$$|g^*| \leq C (1+r)^{\frac{3}{2}}.$$

It then follows that,

$$|w_0| \leq C (1+r)^{\frac{5}{2}}. \tag{3.11}$$

We remark that as $x \to +\infty$, since $g_1 = O\left(\tau_1^2\right)$, w_0 may not satisfy (3.11). □

Lemma 10. *The functions* $\xi_0(x,y) := 1$,

$$\xi_1 := \frac{1}{2}\tau_1^2 - \frac{\sqrt{3}}{6}\tau_1^3,$$

and

$$\xi_2 := \left(\frac{\sqrt{3}}{2}\tau_1 + 1\right) e^{-\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{4}yi} \tag{3.12}$$

solve the homogeneous system

$$\begin{cases} L_1 \phi = 0, \\ M_1 \phi = 0. \end{cases}$$

Proof. This can be checked directly(The fact that ξ_0 is a solution follows automatically from the invariance of the Bäcklund transformation under a multiplication of τ_0 by any constant). Alternatively, denoting

$$\partial_x^{-1} g_1 = \frac{1}{2}\tau_1^2 - \frac{\sqrt{3}}{6}\tau_1^3,$$
$$\partial_x^{-1} g_2 = -\frac{2}{3} e^{-\frac{\sqrt{3}}{2}\tau_1} \left(\sqrt{3}\tau_1 + 2\right),$$

we can look for solutions of $L_1\phi = 0$ in the form $f_1(y)\,\partial_x^{-1} g_1$ and $f_2(y)\,\partial_x^{-1} g_2$, where f_1, f_2 are two unknown functions to be determined. Then fortunately, it happens that in each case, $L_1\phi = 0$ reduces to an ODE in y and it can be solved. □

For given function η, we have seen from (3.8) that the ODE (3.6) for the unknown function ϕ can be solved for each fixed y. To solve the whole system (3.2) , we define

$$\Phi_0(x,y) := L_1\phi - G_1\eta,$$

and

$$\Phi_1 = \partial_x \Phi_0, \Phi_2 = \partial_x^2 \Phi_0.$$

Note that Φ_i actually depends on the function ϕ.

Consider the system of equations

$$\begin{cases} \Phi_0(x,y) = 0, \\ \Phi_1(x,y) = 0, \quad \text{for } x = -2. \\ \Phi_2(x,y) = 0, \end{cases} \tag{3.13}$$

We seek a solution ϕ of (3.13) in the form $w_0 + w_1$, where w_0 is defined in (3.8) and

$$w_1(x,y) = \rho_0(y)\xi_0(x,y) + \rho_1(y)\xi_1(x,y) + \rho_2(y)\xi_2(x,y),$$

for some unknown functions ρ_0, ρ_1, ρ_2.

Lemma 11. *System (3.13) has a solution (ρ_0, ρ_1, ρ_2) with the initial condition*

$$\rho_i(0) = 0, \ i = 0, 1, 2.$$

Proof. The equation $\Phi_0 = 0$ can be written as

$$L_1 w_1 = -L_1 w_0 + G_1\eta := H_0.$$

Similarly, we write $\Phi_1 = 0$ as

$$\partial_x[L_1 w_1] = -\partial_x[L_1 w_0] + \partial_x[G_1\eta] := H_1.$$

The equation $\Phi_2 = 0$ can be written as

$$\partial_x^2[L_1 w_1] = -\partial_x^2[L_1 w_0] + \partial_x^2[G_1\eta] := H_2.$$

Consider the system

$$\begin{cases} L_1 w_1 = 0, \\ \partial_x[L_1 w_1] = 0, \ x = -2. \\ \partial_x^2[L_1 w_1] = 0, \end{cases} \tag{3.14}$$

In view of the definition of w_1, we know that (3.14) is a homogeneous system of first order differential equations for the functions ρ_0, ρ_1, ρ_2. Explicitly, (3.14) has the form

$$\mathcal{A}\begin{pmatrix} \rho_0' \\ \rho_1' \\ \rho_2' \end{pmatrix} = 0,$$

where

$$\mathcal{A} = \begin{pmatrix} \frac{i\tau_1\xi_0}{\sqrt{3}} & \frac{i\tau_1\xi_1}{\sqrt{3}} & \frac{i\tau_1\xi_2}{\sqrt{3}} \\ \frac{i\tau_1\partial_x\xi_0}{\sqrt{3}} & \frac{i\tau_1\partial_x\xi_1}{\sqrt{3}} & \frac{i\tau_1\partial_x\xi_2}{\sqrt{3}} \\ \frac{i\tau_1\partial_x^2\xi_0}{\sqrt{3}} & \frac{i\tau_1\partial_x^2\xi_1}{\sqrt{3}} & \frac{i\tau_1\partial_x^2\xi_2}{\sqrt{3}} \end{pmatrix},$$

and the functions are evaluated at $x = -2$. Hence (3.13) has a solution

$$\int_0^y \mathcal{A}^{-1}(1,s)\begin{pmatrix} H_0(1,s) \\ H_1(1,s) \\ H_2(1,s) \end{pmatrix} ds.$$

This completes the proof. □

Proposition 12. *Suppose η satisfies* (3.3) *and ϕ satisfies* (3.6)*. Then*

$$\partial_x^3\Phi_0 = \left(-\frac{\sqrt{3}}{2}+\frac{6}{\tau_1}\right)\partial_x^2\Phi_0 + \frac{1}{\tau_1}\left(2\sqrt{3}-\frac{15}{\tau_1}\right)\partial_x\Phi_0 + \frac{1}{\tau_1^2}\left(\frac{15}{\tau_1}-2\sqrt{3}\right)\Phi_0.$$

Proof. One can verify this identity by direct computation. Although this tedious calculation can be done by hand, we suggest to do it using computer softwares such as *Maple* or *Mathematica*.

Intuitively, we expect that this identity follows from the compatibility properties of suitable Lax pair of the KP-I equation. But up to now we have not been able to rigorously show this. □

Lemma 13. *The function $\phi = w_0 + w_1$ satisfies* (3.2) *for all $(x, y) \in \Omega_*$. As a consequence, it satisfies the linearized bilinear KP-I equation at τ_0 :*

$$-D_x^2\phi\cdot\tau_0 + D_x^4\phi\cdot\tau_0 - D_y^2\phi\cdot\tau_0 = 0. \tag{3.15}$$

Proof. By the definition of Φ_0, Φ_1, Φ_2 and Proposition 12, we have

$$\partial_x\begin{pmatrix}\Phi_0\\ \Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ \frac{\frac{15}{\tau_1}-2\sqrt{3}}{\tau_1^2} & \frac{2\sqrt{3}-\frac{15}{\tau_1}}{\tau_1} & \frac{6}{\tau_1}-\frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}\Phi_0\\ \Phi_1\\ \Phi_2\end{pmatrix}.$$

For each fixed $y \neq 0$, by Lemma 11, $\Phi_0(1, y) = \Phi_1(1, y) = \Phi_2(1, y) = 0$. By the uniqueness of solution to ODE, we obtain $\Phi_0(x, y) = \Phi_1(x, y) = \Phi_2(x, y) = 0$, for all $x \in \mathbb{R}$.

For each fixed y, since w_1 is a linear combination of ξ_0, ξ_1, ξ_2, $\phi = w_0 + w_1$ satisfies (3.6). From $\Phi_0 = 0$, we know that ϕ satisfies the first equation of (3.2). Observe that (3.6) is obtained by inserting the first equation of (3.2) into the second one. Hence the second equation of (3.2) is also satisfied by ϕ for $(x, y) \in \Omega_*$.

Linearizing the bilinear identity (2.4)(with t replaced by $-x$) at (τ_0, τ_1), we get

$$\begin{aligned}&\left[\left(-D_x^2 + D_x^4 - D_y^2\right)\tau_0\cdot\phi\right]\tau_1 - \left[\left(-D_x^2 + D_x^4 - D_y^2\right)\tau_1\cdot\eta\right]\tau_0^2\\ &= D_x\left[M_1\phi - N_1\eta\right]\cdot(\tau_0\tau_1)\\ &+ 3D_x\left[L_1\phi - G_1\eta\right]\cdot(D_x\tau_1\cdot\tau_0)\\ &+ \sqrt{3}iD_y\left[L_1\phi - G_1\eta\right]\cdot(\tau_0\tau_1).\end{aligned}$$

Since (ϕ, η) satisfies the linearized Backlund transformation, we obtain (3.15). □

Lemma 14. *Suppose η satisfies* (2.8)*. Let $\rho_i, i = 0, 1, 2$, be functions given by Lemma* 11*. Then*

$$\rho_1(y) = \rho_2(y) = 0, \text{ for all } y \in \mathbb{R}.$$

Proof. Dividing the equation $\Phi_0 = 0$ by ξ_2, we get

$$\frac{i}{\sqrt{3}\xi_2}\left(\rho_0'\xi_0 + \rho_1'\xi_1 + \rho_2'\xi_2\right)\tau_1 = \frac{1}{\xi_2}H_0. \tag{3.16}$$

Note that $\xi_2 = 0$ when $(x, y) = \left(-\frac{5\sqrt{3}}{3}, 0\right)$. But actually we are only interested the equation (3.16) when $-x$ is large. In this region, using (2.8) and the estimate of

w_0, we have

$$\begin{aligned} |H_0| &= |G_1\eta - L_1 w_0| \leq |G_1\eta| + |L_1 w_0| \\ &\leq C(1+r)^{\frac{7}{2}}. \end{aligned}$$

For each fixed $y \in \mathbb{R}$, sending $x \to -\infty$ in (3.16) and using the fact that ξ_2 is exponentially growing in this direction, we infer

$$\rho_2'(y) = 0.$$

This together with the initial condition $\rho_2(0) = 0$ tell us that $\rho_2 = 0$.

Now $\Phi_0 = 0$ becomes

$$\frac{i}{\sqrt{3}}(\rho_0'\xi_0 + \rho_1'\xi_1)\tau_1 = H_0.$$

Dividing both sides of this equation by $\xi_1\tau_1$ and letting $x \to -\infty$, we get $\rho_1' = 0$. Hence $\rho_1 = 0$. The proof is completed. □

Proof of Proposition 7. We have proved that ρ_1 and ρ_2 are both zero. Now let us define

$$k(y) = \int_{-\infty}^{+\infty} \frac{g_2 F_1}{4\tau_1 W} ds, \text{ for } y \neq 0.$$

For each fixed y, note that due to the definition of w_0, as $x \to +\infty$, the main order term of w_0 is $-k(y)\xi_1(x,y)$, which behaves like x^3. Let us write

$$w_0 + k(y)\xi_1(x,y) := w^*.$$

Then

$$|w^*| \leq C(1+r)^{\frac{5}{2}}, \text{ for } x \text{ large.} \tag{3.17}$$

We also have $\phi = w^* - k(y)\xi_1(x,y) + \rho_0\xi_0$. Inserting this into the equation $L_1\phi = G_1\eta$, we get

$$-\frac{i}{\sqrt{3}}k'(y)\xi_1\tau_1 = G_1\eta - L_1(w^* + \rho_0\xi_0).$$

Dividing this equation by $\xi_1\tau_1$ and sending x to $-\infty$, we see that $k'(y) = 0$. But $k(y) \to 0$ as $|y| \to +\infty$. Hence $k(y) = 0$. In particular, the function $g_1(x,y)\int_{-\infty}^{x} \frac{g_2 F_1}{4\tau_1 W}$ can be extended continuously to the whole plane. For the term $J := g_2(x,y)\int_{-\infty}^{x} \frac{g_1 F_1}{4\tau_1 W}$, in general we don't have $\int_{-\infty}^{+\infty} \frac{g_1 F_1}{4\tau_1 W} = 0$, because the function $\frac{g_1 F_1}{4\tau_1 W}$ may not even be integrable. Recall that $\int_{-2\sqrt{3}}^{x} \frac{ds}{s+\sqrt{3}+iy} = \ln\left(x+\sqrt{3}+iy\right) - \ln\left(-\sqrt{3}+iy\right)$ and it has a jump across the half line $\left\{(x,0) : x > -\sqrt{3}\right\}$(but its derivatives have no jump). Hence there exists a constant c such that

$$\lim_{y\to 0^+} w_0(x,y) - \lim_{y\to 0^-} w_0(x,y) = c\xi_2(x,0). \tag{3.18}$$

Moreover, in view of (3.17), we have

$$|w_0| = |w^*| \leq C(1+r)^{\frac{5}{2}}.$$

Since ρ_1, ρ_2 are both identically zero, we find that ρ_0 satisfies

$$\rho_0'(y) = -\frac{\sqrt{3}}{\tau_1} i H_0.$$

Integrating this equation in terms of y, we find that

$$|\rho_0(y)| = \sqrt{3}\left|\int_0^y \frac{H_0}{\tau_1}\right| \leq C(1+r)^{\frac{5}{2}}. \tag{3.19}$$

With the estimate for ρ_0 and w_0, we then infer from Lemma 13 that the function $\phi = w_0 + \rho_0\xi_0$ is the desired solution. Note that the equation (3.4) follows from the linearization of the bilinear identity (2.4). □

3.2. **Linearized Bäcklund transformation between τ_1 and τ_2.** In terms of τ_1 and τ_2, the Bäcklund transformation (2.6) can be written as

$$\begin{cases} \left(D_x^2 - \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\tau_1\cdot\tau_2 = 0, \\ \left(-D_x + iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\tau_1\cdot\tau_2 = 0. \end{cases}$$

The linearization of this system is

$$\begin{cases} L_2\phi = G_2\eta, \\ M_2\phi = N_2\eta. \end{cases} \tag{3.20}$$

Here

$$L_2\phi = \left(D_x^2 - \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\phi\cdot\tau_2,$$
$$M_2\phi = \left(-D_x + iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\phi\cdot\tau_2,$$

and

$$G_2\eta = -\left(D_x^2 - \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\tau_1\cdot\eta,$$
$$N_2\eta = -\left(-D_x + iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\tau_1\cdot\eta.$$

Proposition 15. *Let $\eta = \eta(x,y)$ be a function solving the linearized bilinear KP-I equation at τ_2:*

$$-D_x^2\eta\cdot\tau_2 + D_x^4\eta\cdot\tau_2 = D_y^2\eta\cdot\tau_2. \tag{3.21}$$

Suppose η satisfies (2.8). Then the system (3.2) has a solution ϕ with

$$|\phi| + \left(|\partial_x\phi| + |\partial_y\phi| + \left|\partial_x^2\phi\right| + |\partial_x\partial_y\phi| + \partial_x^3\phi\right)(1+r) \leq C(1+r)^{\frac{5}{2}}. \tag{3.22}$$

Moreover, ϕ satisfies

$$-D_x^2\phi\cdot\tau_1 + D_x^4\phi\cdot\tau_1 = D_y^2\phi\cdot\tau_1.$$

From the first equation in (3.20), we get

$$\partial_y\phi\tau_2 = i\left[\sqrt{3}\left(\partial_x^2\phi\tau_2 - 2\partial_x\phi\partial_x\tau_2\right) - \partial_x\phi\tau_2\right] + 2i\phi\bar{\tau}_1 - \sqrt{3}iG\eta. \tag{3.23}$$

Here we have used $\bar{\tau}_1$ to denote the complex conjugate of τ_1. Inserting (3.23) into the second equation of (3.20), we obtain

$$\partial_x^3\phi\tau_2 + \left(-\frac{\sqrt{3}}{2}\tau_2 - 6x\right)\partial_x^2\phi + \left(2\sqrt{3}x + \frac{12x^2}{\tau_2}\right)\partial_x\phi - \frac{2\sqrt{3}x}{\tau_2}\bar{\tau}_1\phi = \frac{F_2}{4}. \tag{3.24}$$

Here F_2 is defined to be

$$F_2 = N_2\eta - \left(\frac{6\partial_x\tau_2}{\tau_2}G_2\eta - 3\partial_x(G_2\eta) + \sqrt{3}G_2\eta\right).$$

To solve equation (3.24), we set

$$\phi = \tau_1 \kappa \text{ and } h = \partial_x \kappa.$$

Equation (3.24) is transformed into the equation

$$\begin{aligned} T(h) &:= \tau_1 \tau_2 \partial_x^2 h + \left[3\tau_2 + \left(-\frac{\sqrt{3}}{2}\tau_2 - 6x \right) \tau_1 \right] \partial_x h \\ &+ \left[2\left(-\frac{\sqrt{3}}{2}\tau_2 - 6x \right) + \tau_1 \left(2\sqrt{3}x + \frac{12x^2}{\tau_2} \right) \right] h \\ &= \frac{F_2}{4}. \end{aligned}$$

Lemma 16. *The homogenous equation $T(h) = 0$ has two solutions h_1, h_2, given by*

$$\begin{aligned} h_1(x,y) &= (x - yi)^2 + \frac{2}{\sqrt{3}}(x - yi) + 3 + \frac{12\left(y - \sqrt{3}i\right)\left(y + \frac{i}{\sqrt{3}}\right)}{\tau_1^2} \\ &= \frac{\tau_2}{3\tau_1^2}\left(3\tau_2 + 4\sqrt{3}\left(x + \bar{\tau}_1\right)\right), \end{aligned}$$

and

$$h_2(x,y) = \frac{\tau_2}{\tau_1^2}\left(x + yi - \sqrt{3}\right) e^{\frac{\sqrt{3}}{2}x}.$$

Proof. This equation can be solved using *Maple.* □

Lemma 17. *The system*

$$\begin{cases} \left(D_x^2 - \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\phi \cdot \tau_2 = 0, \\ \left(-D_x + iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\phi \cdot \tau_2 = 0 \end{cases}$$

has three solutions $\zeta_0, \zeta_1, \zeta_2$, given by $\zeta_0 = \tau_1$,

$$\begin{aligned} \zeta_1 &= \tau_1 \partial_x^{-1} h_1 \\ &= \tau_1 \left(\frac{\bar{z}^3}{3} + \frac{\bar{z}^2}{\sqrt{3}} + 3x - 11yi \right) - 12\left(y - \sqrt{3}i\right)\left(y + \frac{i}{\sqrt{3}}\right), \end{aligned}$$

and

$$\zeta_2 = e^{\sqrt{3}yi}\tau_1 \partial_x^{-1} h_2 = \left(x^2 - \frac{8}{\sqrt{3}}x + y^2 + \frac{4yi}{\sqrt{3}} + 7 \right) e^{\frac{\sqrt{3}}{2}x + \sqrt{3}yi}.$$

Proof. This is similar to Lemma 10 and can be checked by direct computation. □

Let $\tilde{W}$ be the Wronskian of h_1, h_2. That is

$$\tilde{W} = h_1 \partial_x h_2 - h_2 \partial_x h_1 = e^{\frac{\sqrt{3}}{2}x} \frac{\tau_2^3}{\tau_1^3}.$$

Variation of parameter formula gives us a solution $\tau_1 \int_0^x h^*$ to the equation (3.24), where

$$h^* = h_2 \int_{+\infty}^{x} \frac{h_1 F_2}{4\tau_1 \tau_2 \tilde{W}} ds - h_1 \int_{+\infty}^{x} \frac{h_2 F_2}{4\tau_1 \tau_2 \tilde{W}} ds.$$

Lemma 18. *Suppose η satisfies (2.8). Let $\tilde{w}_0 = \tau_1 \int_0^x h^*$. Then*

$$|\tilde{w}_0| \le C(1 + r)^{\frac{5}{2}}, \text{ for } x \ge -10.$$

Proof. Since η satisfies (2.8), we have

$$|F_2| \leq C(1+r)^{\frac{5}{2}}.$$

This together with the explicit formula of τ_1, τ_2 imply that

$$\left|\frac{h_1 F_2}{\tilde{W}\tau_1\tau_2}\right| \leq Ce^{-\frac{\sqrt{3}x}{2}}(1+r)^{-\frac{3}{2}},$$
$$\left|\frac{h_2 F_2}{\tilde{W}\tau_1\tau_2}\right| \leq C(1+r)^{-\frac{5}{2}}.$$

We then deduce

$$|h^*(x,y)| \leq (1+r)^{\frac{1}{2}}, \text{ for } x \geq -10. \tag{3.25}$$

From this upper bound, it follows that

$$|\tilde{w}_0| \leq C(1+r)^{\frac{5}{2}}, \text{ for } x \geq -10.$$

□

Let η be a function satisfying (3.21). Slightly abusing the notation, we define

$$\Phi_0 = L_2\phi - G_2\eta.$$

Also define $\Phi_1 = \partial_x \Phi_0$ and $\Phi_2 = \partial_x^2 \Phi_0$. The precise form of the function ϕ is to be defined below.

We would like to find a solution ϕ for the system

$$\begin{cases} \Phi_0 = 0 \\ \Phi_1 = 0 \\ \Phi_2 = 0 \end{cases}, \text{ for } x = 0. \tag{3.26}$$

Similarly as before, we seek a solution ϕ of this problem with the form

$$\phi := \tilde{w}_0 + \tilde{w}_1, \tag{3.27}$$

with

$$\tilde{w}_1 = \beta_0(y)\zeta_0 + \beta_1(y)\zeta_1 + \beta_2(y)\zeta_2,$$

where $\beta_0, \beta_1, \beta_2$ are functions of y, to be determined.

The problem (3.26) can be written as

$$\begin{cases} L_2\tilde{w}_1 = H_0, \\ \partial_x(L_2\tilde{w}_1) = H_1, \text{ for } x = 0. \\ \partial_x^2(L_2\tilde{w}_1) = H_2, \end{cases} \tag{3.28}$$

Here

$$H_0 = G_2\eta - L_2\tilde{w}_0,$$
$$H_1 = \partial_x(G_2\eta - L_2\tilde{w}_0),$$
$$H_2 = \partial_x^2(G_2\eta - L_2\tilde{w}_0).$$

Lemma 19. *The equation* (3.28) *has a solution* $(\beta_0, \beta_1, \beta_2)$ *satisfying the initial condition*

$$\beta_i(0) = 0, i = 0, 1, 2.$$

Proof. This is a system of ODE for $\beta_1, \beta_2, \beta_3$. The proof is similar to that of Lemma 11. We omit the details. □

Proposition 20. *Suppose η satisfies (3.21) and ϕ is defined by (3.27). Then*

$$\partial_x^3\Phi_0 = \left(\frac{12x}{\tau_2} + \frac{\sqrt{3}}{2}\right)\partial_x^2\Phi_0 + \left(\frac{6}{\tau_2} - \frac{4\sqrt{3}x}{\tau_2} - \frac{60x^2}{\tau_2^2}\right)\partial_x\Phi_0$$
$$+ \left(2\sqrt{3}\frac{x\bar{\tau}_1}{\tau_2^2} - \frac{\sqrt{3}}{\tau_2} - \frac{36x}{\tau_2^2} + \frac{8\sqrt{3}x^2}{\tau_2^2} + \frac{120x^3}{\tau_2^3}\right)\Phi_0.$$

Proof. This is similar to Propositon 12 and can be checked directly using Maple. We remark that the property of ϕ used here is essentially equation (3.24). □

Lemma 21. *The function $\phi = \tilde{w}_0 + \tilde{w}_1$ solves the system (3.20) for all $(x, y) \in \mathbb{R}^2$.*

Proof. By Proposition 20,

$$\partial_x \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix},$$

where

$$a_{31} = 2\sqrt{3}\frac{x\bar{\tau}_1}{\tau_2^2} - \frac{\sqrt{3}}{\tau_2} - \frac{36x}{\tau_2^2} + \frac{8\sqrt{3}x^2}{\tau_2^2} + \frac{120x^3}{\tau_2^3},$$
$$a_{32} = \frac{6}{\tau_2} - \frac{4\sqrt{3}x}{\tau_2} - \frac{60x^2}{\tau_2^2},$$

and

$$a_{33} = \frac{12x}{\tau_2} + \frac{\sqrt{3}}{2}.$$

For each fixed y, since $\Phi_i(0, y) = 0, i = 0, 1, 2$, we deduce from the uniqueness of solutions to ODE that $\Phi_i(x, y) = 0$, for all $x \in \mathbb{R}$. This finishes the proof. □

Lemma 22. *Let $\beta_i, i = 0, 1, 2$ be the functions given by Lemma 19. Then $\beta_1 = \beta_2 = 0$.*

Proof. The proof is similar to that of Lemma 14, using the asymptotic behavior of ζ_1, ζ_2 as $x \to +\infty$. □

Proof of Proposition 15. The proof is similar as that of Proposition 7. Here we briefly sketch the proof of the estimates for the derivatives of ϕ.

By Lemma 22,

$$\phi = \tilde{w}_0 + \tilde{w}_1 = \tilde{w}_0 + \beta_0\tau_1.$$

We compute

$$\partial_x\phi = \partial_x\tilde{w}_0 + \beta_0$$
$$= \int_0^x h^* + \tau_1 h^* + \beta_0.$$

By (3.25), for $x \geq -10$,

$$\left|\int_0^x h^* + \tau_1 h^*\right| \leq C(1 + r)^{\frac{3}{2}}.$$

On the other hand, similarly as (3.19), we have $|\beta_0| \leq C(1 + r)^{\frac{3}{2}}$. Hence

$$|\partial_x\phi| \leq C(1 + r)^{\frac{3}{2}}, \text{ for } x \geq -10.$$

To get this same estimate for $x \leq -10$, we consider the function

$$k_2(y) = \int_{-\infty}^{+\infty} \frac{h_2 F_2}{\tilde{W}\tau_1\tau_2}, y \neq 0.$$

Then similarly as the function k appeared in the proof of Proposition 7, we have $k_2' = 0$ and $k_2 = 0$. From this, we then get

$$|\partial_x \phi| \leq C(1+r)^{\frac{3}{2}}, \text{ for all } x.$$

The estimates of $\partial_x^2\phi, \partial_x^3\phi$ and $\partial_x\partial_y\phi$ are similar. To estimate $\partial_y\phi$, we simply use the equation $L_2\phi = G_2\eta$ and the already obtained estimates of $\partial_x^2\phi$ and $\partial_x\phi, \phi$. □

3.3. **Proof of the nondegeneracy of the lump.** With the linearized Bäcklund transformation being understood, we now proceed to the proof of the nondegeneracy of the lump solution. In this section, we denote $x + yi$ by z.

Lemma 23. *Suppose η satisfies*

$$\begin{cases} L_1\phi = G_1\eta, \\ M_1\phi = N_1\eta, \end{cases}$$

Then

$$-4\partial_x^3\eta + 2\sqrt{3}\partial_x^2\eta = \Theta_1\phi,$$

where

$$\Theta_1\phi := -M_1\phi - \sqrt{3}L_1\phi + 3\partial_x(L_1\phi)$$

In particular, if

$$G_1\eta = N_1\eta = 0,$$

and

$$|\eta| \leq C(1+r)^{\frac{5}{2}}. \tag{3.29}$$

Then $\eta = c_1 + c_2\tau_1$, for some constants c_1, c_2.

Proof. The equation $G_1\eta = L_1\phi$ is

$$\left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{i}{\sqrt{3}}D_y\right)\tau_0 \cdot \eta = -L_1\phi.$$

That is,

$$\partial_x^2\eta + \frac{1}{\sqrt{3}}(-\partial_x\eta) + \frac{i}{\sqrt{3}}(-\partial_y\eta) = -L_1\phi.$$

Inserting this identity into the equation

$$\left(-D_x - iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\tau_0 \cdot \eta = -M_1\phi,$$

we get

$$\sqrt{3}\partial_x^2\eta - \partial_x^3\eta - \sqrt{3}i\partial_x\left(-\sqrt{3}i\left(\partial_x^2\eta - \frac{1}{\sqrt{3}}\partial_x\eta + L_1\phi\right)\right) = -M_1\phi - \sqrt{3}L_1\phi.$$

Hence

$$-4\partial_x^3\eta + 2\sqrt{3}\partial_x^2\eta = -M_1\phi - \sqrt{3}L_1\phi + 3\partial_x(L_1\phi).$$

If $L_1\phi = M_1\phi = 0$, then

$$-4\partial_x^3\eta + 2\sqrt{3}\partial_x^2\eta = 0.$$

The solutions of this equation are given by

$$c_1 + c_2x + c_3e^{\frac{\sqrt{3}}{2}x},$$

where c_1, c_2, c_3 are constants which may depend on y. Due to the growth estimate of η, we find that

$$\eta = c_1 + c_2 x.$$

Inserting this into the equation $G_1 \eta = 0$, we find that η is a linear combination of 1 and τ_1. □

Lemma 24. *We have*

$$\Theta_1 (x) = 4\sqrt{3},\ \Theta_1 (y) = 0.$$

and

$$\Theta_1 \left(x^2 - y^2\right) = 8\sqrt{3}x,\ \Theta_1 (xy) = 2\sqrt{3}i\bar{\tau}.$$

Proof. This follows from direct computation. For instances, we compute

$$\begin{aligned} L_1 \left(x^2 - y^2\right) &= \left(D_x^2 + \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} i D_y\right) \left(x^2 - y^2\right) \cdot \tau_1 \\ &= 2\tau_1 - 4x + \frac{1}{\sqrt{3}} \left(2x\tau_1 - \left(x^2 - y^2\right)\right) + \frac{i}{\sqrt{3}} \left(-2y\tau_1 - i\left(x^2 - y^2\right)\right) \\ &= \frac{2}{\sqrt{3}} \tau_2. \end{aligned}$$

$$\begin{aligned} M_1 \left(x^2 - y^2\right) &= \left(-D_x - iD_y + D_x^3 - \sqrt{3} i D_x D_y\right) \left(x^2 - y^2\right) \cdot \tau_1 \\ &= -\left(2x\tau_1 - \left(x^2 - y^2\right)\right) - i\left(-2y\tau_1 - \left(x^2 - y^2\right) i\right) \\ &\quad + (-3)\, 2 - \sqrt{3} i \left(-2xi - (-2y)\right) \\ &= -2\tau_2 - 4\sqrt{3}x. \end{aligned}$$

It follows immediately that

$$\begin{aligned} \Theta_1 \left(x^2 - y^2\right) &= 2\tau_2 + 4\sqrt{3}x - \sqrt{3} \frac{2}{\sqrt{3}} \tau_2 + 3 \frac{4x}{\sqrt{3}} \\ &= 8\sqrt{3}x. \end{aligned}$$

□

Lemma 25. *Define* $\digamma (\phi) := (L_1 (\phi), M_1 (\phi))$, $\mathcal{J} (\phi) := (G_1 (\phi), N_1 (\phi))$. *Then*

$$\digamma (x) = \mathcal{J} \left(x\tau_1 - \sqrt{3} z\right),$$

$$\digamma (y) = \mathcal{J} (y\tau_1),$$

and

$$\begin{aligned} \digamma \left(x^2 - y^2\right) &= \mathcal{J} (\Pi_1), \qquad (3.30) \\ \digamma (xy) &= \mathcal{J} (\Pi_2), \end{aligned}$$

where

$$\begin{aligned} \Pi_1 &= \frac{2}{3} x^3 + \frac{4}{3} \sqrt{3} x^2 - \frac{4\sqrt{3}}{3} yix - \frac{2i}{3} y^3 + \frac{2}{3} \sqrt{3} y^2 - 14yi, \\ \Pi_2 &= \frac{1}{2} x^2 y + \frac{5}{6} i \sqrt{3} x^2 + \frac{1}{6} i x^3 + \left(\frac{y^2}{2} i + \frac{\sqrt{3}}{3} y\right) x + \frac{y^3}{6} + \frac{\sqrt{3}}{6} y^2 i + 5y. \end{aligned}$$

Proof. We only prove (3.30). The proof of other cases are similar. Consider the equation

$$-4\partial_x^3\eta+2\sqrt{3}\partial_x^2\eta=\Theta_1\left(x^2-y^2\right)=8\sqrt{3}x.$$

This equation has a solution

$$\Pi_1=\frac{2}{3}x^3+\frac{4}{3}\sqrt{3}x^2+a\left(y\right)x+b\left(y\right),$$

where $a\left(y\right)$ and $b\left(y\right)$ are functions to be determined. Since

$$\partial_x^2\Pi_1+\frac{1}{\sqrt{3}}\left(-\partial_x\Pi_1\right)+\frac{i}{\sqrt{3}}\left(-\partial_y\Pi_1\right)=-L_1\left(x^2-y^2\right)=-2\frac{x^2+y^2+3}{\sqrt{3}},$$

we get

$$4x+\frac{8\sqrt{3}}{3}-\frac{1}{\sqrt{3}}\left(2x^2+\frac{8}{3}\sqrt{3}x+a\right)-\frac{i}{\sqrt{3}}\left(a'x+b'\right)=-2\frac{x^2+y^2+3}{\sqrt{3}}.$$

Hence

$$a\left(y\right)=-\frac{4\sqrt{3}}{3}yi,b\left(y\right)=-\frac{2i}{3}y^3+\frac{2}{3}\sqrt{3}y^2-14yi.$$

From here we get Π_1 immediately. □

Lemma 26. *Suppose η satisfies*

$$\begin{cases} L_2\phi=G_2\eta,\\ M_2\phi=N_2\eta.\end{cases}$$

Then

$$\partial_x^3\eta\tau_1+\left(\frac{\sqrt{3}}{2}\tau_1-3\right)\partial_x^2\eta+\left(\frac{3}{\tau_1}-\sqrt{3}\right)\partial_x\eta+\frac{\sqrt{3}}{\tau_1}\eta=\Theta_2\left(\phi\right).$$

Here

$$\Theta_2\left(\phi\right):=\frac{1}{4}\left(\frac{6}{\tau_1}L_2\phi-3\partial_x\left(L_2\phi\right)+M_2\phi+\sqrt{3}L_2\phi\right).$$

Proof. Explicitly, η satisfies

$$\begin{cases}\left(D_x^2-\frac{1}{\sqrt{3}}D_x+\frac{1}{\sqrt{3}}iD_y\right)\tau_1\cdot\eta=-L_2\phi,\\ \left(-D_x+iD_y+D_x^3-\sqrt{3}iD_xD_y\right)\tau_1\cdot\eta=-M_2\phi.\end{cases}\tag{3.31}$$

Let us write the first equation in this system as

$$\partial_y\eta\tau_1=-i\left[\partial_x\eta\tau_1+\sqrt{3}\left(\partial_x^2\eta\tau_1-2\partial_x\eta\right)-2\eta\right]-\sqrt{3}iL_2\phi.$$

Inserting this identity into the right hand side of the second equation the system (3.31), we get

$$\partial_x^3\eta\tau_1+\left(\frac{\sqrt{3}}{2}\tau_1-3\right)\partial_x^2\eta+\left(\frac{3}{\tau_1}-\sqrt{3}\right)\partial_x\eta+\frac{\sqrt{3}}{\tau_1}\eta=\Theta_2.$$

When $\phi=0$, we get a third order ODE for η. We remark that this ODE has a singularity at the points where $\tau_1=0$. That is the point $\left(-\sqrt{3},0\right)$. However, as we will see below, it has three linearly independent smooth solutions($\tau_2,z,\tau_2\partial_x^{-1}p_2$). □

Note that $\eta = \tau_2$ satisfies the homogeneous equation

$$\partial_x^3 \eta \tau_1 + \left(\frac{\sqrt{3}}{2} \tau_1 - 3 \right) \partial_x^2 \eta + \left(\frac{3}{\tau_1} - \sqrt{3} \right) \partial_x \eta + \frac{\sqrt{3}}{\tau_1} \eta = 0. \tag{3.32}$$

Letting $\eta = \tau_2 \kappa$ and $p = \partial_x \kappa$, equation (3.32) becomes

$$\tau_1 \tau_2 \partial_x^2 p + \left(6x\tau_1 + \left(\frac{\sqrt{3}}{2} \tau_1 - 3 \right) \tau_2 \right) \partial_x p + \left(6\tau_1 + 2x \left(\sqrt{3}\tau_1 - 6 \right) + \left(\frac{3}{\tau_1} - \sqrt{3} \right) \tau_2 \right) p = 0. \tag{3.33}$$

Lemma 27. *The equation* (3.33) *has two solutions given by*

$$p_1 := \frac{(x + yi)^2 - 3}{\tau_2^2}$$

and

$$p_2 := \left(8\sqrt{3}x - 4i\sqrt{3}y + 3x^2 + 3y^2 + 21 \right) e^{-\frac{1}{2}\sqrt{3}x} \frac{\tau_1}{\tau_2^2}.$$

In particular, if η satisfies (3.29) *and*

$$G_2 \eta = N_2 \eta = 0,$$

Then $\eta = c_1 z + c_2 \tau_2$.

Note that $\partial_x^{-1} p_1 = -\frac{z}{\tau_2}$. Hence we get a solution $\eta = -z$ for the equation (3.32). This solution is corresponding to the translation of τ_2 along the x and y axes.

Lemma 28. *We have*

$$M_2 (y\tau_1) = -2\sqrt{3}xy + 3x^2 y + 3xi + 2\sqrt{3}y^2 i - x^3 i + 3xy^2 i + 9y - y^3 - 6\sqrt{3}i,$$

$$M_2 \left(x\tau_1 - \sqrt{3}z \right) = x^3 + 3ix^2 y - 2\sqrt{3}x^2 - 3xy^2 + 2i\sqrt{3}xy + 3x - iy^3 + 3iy,$$

$$\begin{aligned} L_2 \Pi_1 = {} & 24x + 12\sqrt{3}xyi - \frac{4\sqrt{3}}{9} x^3 yi + 4xy^2 + \frac{4\sqrt{3}}{3} x^2 y^2 - \frac{8}{3} y^3 i + 22\sqrt{3} + \frac{4\sqrt{3}}{3} x^2 \\ & + \frac{4\sqrt{3}}{9} xy^3 i + \frac{4\sqrt{3}}{3} y^2 - 28yi + \frac{2\sqrt{3}}{9} x^4 - \frac{4}{3} x^3 + \frac{2\sqrt{3}}{9} y^4, \end{aligned}$$

$$\begin{aligned} M_2 \Pi_1 = {} & -42 + \frac{4\sqrt{3}}{3} y^3 i - \frac{4\sqrt{3}}{3} x^3 + 4\sqrt{3}x^2 yi - \frac{4}{3} xy^3 i + \frac{4}{3} yx^3 i \\ & - 4xyi + 4\sqrt{3}xy^2 - 8\sqrt{3}yi - \frac{2}{3} y^4 - \frac{2}{3} x^4 - 4x^2 y^2 + 16\sqrt{3}x + 16y^2 - 24x^2, \end{aligned}$$

$$\begin{aligned} L_2 \Pi_2 = {} & 13y + 10\sqrt{3}i - \frac{2\sqrt{3}}{9} x^3 y - 4\sqrt{3}xy - x^2 y + xy^2 i + \frac{5}{3} y^3 + \frac{2\sqrt{3}}{9} x^4 i \\ & + \frac{2\sqrt{3}}{9} y^4 i + \frac{2\sqrt{3}}{9} xy^3 + \frac{4\sqrt{3}}{3} y^2 i - \frac{1}{3} x^3 i + 9xi - \frac{2\sqrt{3}}{3} x^2 i, \end{aligned}$$

$$\begin{aligned} M_2 \Pi_2 = {} & 7y^2 i - 3\sqrt{3}x^2 y - 2xy + \frac{2}{3} x^3 y + 5\sqrt{3}y - 9x^2 i + \sqrt{3}xy^2 i \\ & - 15i - \frac{2}{3} y^4 i - \frac{2}{3} x^4 i - \frac{2}{3} xy^3 + \sqrt{3}xi - \frac{\sqrt{3}}{3} ix^3 - \frac{\sqrt{3}}{3} y^3. \end{aligned}$$

Proof. Direct computation. We omit the details. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let ϕ be a solution(real valued) of (1.5) satisfying the assumption of Theorem 1. Then using Lemma 6, we can find η_2(also real valued), a solution of (3.21), satisfying (2.8). In view of Lemma 27, if $G_2\eta_2 = N_2\eta_2 = 0$, then $\eta_2 = c_1 z + c_2\tau_2$ for some constants c_1, c_2. In this case, since η_2 is real valued, we must have $\eta_2 = c_2\tau_2$. But then the corresponding $\phi = 0$. Therefore, to prove the theorem, from now on, we can assume $G_2\eta_2 \neq 0$ or $N_2\eta_2 \neq 0$.

By Proposition 15, there exists a solution η_1 of the equation

$$\left(D_x^2 - D_x^4 + D_y^2\right)\eta_1 \cdot \tau_1 = 0,$$

satisfying the estimate (2.8).

Case 1. $G_1\eta_1 = N_1\eta_1 = 0$.

In this case, by Lemma 23, $\eta_1 = a_1 + a_2\tau_1$ for some complex valued constants a_1, a_2. Observe that

$$\begin{cases} L_2(1) = G_2(\partial_x\tau_2), \\ M_2(1) = N_2(\partial_x\tau_2), \end{cases}$$

and

$$\begin{cases} L_2(i) = G_2(\partial_y\tau_2), \\ M_2(i) = N_2(\partial_y\tau_2). \end{cases}$$

Accordingly, using the fact that η_2 is real valued, we obtain

$$\eta_2 = d_1\partial_x\tau_2 + d_2\partial_y\tau_2 + d_3\tau_2,$$

for some real numbers d_1, d_2, d_3. This then implies that $\phi = d_1\partial_x Q + d_2\partial_y Q$.

Case 2. $G_1\eta_1 \neq 0$ or $N_1\eta_1 \neq 0$.

In this case, by Proposition 7, in the region Ω_*, there exists a solution η_0 of

$$\left(D_x^2 - D_x^4 + D_y^2\right)\eta_0 \cdot \tau_0 = 0, \tag{3.34}$$

satisfying

$$|\eta_0| + |\partial_x\eta_0| + |\partial_y\eta_0| \leq C(1+r)^{\frac{5}{2}}. \tag{3.35}$$

Moreover, η_0 satisfies

$$\lim_{y\to 0^+}\eta_0(x,y) - \lim_{y\to 0^-}\eta_0(x,y) = c\xi_2(x,0), \text{ for } x > 0. \tag{3.36}$$

Since ϕ is a kernel of the linearized KP-I equation at the lump, it is real analytic. Hence η_1 is also real analytic. From this, we infer that for $j = 1, 2$,

$$\lim_{\varepsilon\to 0^+}\left(\partial_y^j\eta_0(x,\varepsilon) - \partial_y^j\eta_0(x,-\varepsilon)\right) = c\partial_y^j\xi_2(x,y)|_{y=0}, \text{ for all } x \in \mathbb{R}. \tag{3.37}$$

The equation (3.34) reads as

$$\partial_x^2\eta_0 + \partial_y^2\eta_0 - \partial_x^4\eta_0 = 0. \tag{3.38}$$

In this equation, we can take Fourier transform in the x variable and solve the derived ODE of y for each frequency. Using (3.36) and (3.37), we find that $c = 0$. Now let $\hat{\eta}_0(\cdot,\cdot)$ be the Fourier transform of η_0 in both x and y variables. We get

$$\left(s^2 + s^4 + t^2\right)\hat{\eta}_0(s,t) = 0.$$

Hence $\hat{\eta}_0$ is a distribution supported at the origin. Applying Theorem 2.3.4 of [27], we deduce that η_0 is a polynomial in the x and y variables. Now using the estimate (3.35), we infer that $\partial_x^4\eta_0 = 0$ and thus

$$\partial_x^2\eta_0 + \partial_y^2\eta_0 = 0.$$

Therefore, for some constants $c_1, ..., c_5$,

$$\eta_0 = c_1 + c_2 x + c_3 y + c_4 \left(x^2 - y^2\right) + c_5 xy.$$

We claim that $c_2 = c_3 = c_4 = c_5 = 0$. Indeed, if c_4 or c_5 is nonzero, then using Lemma 25 and Lemma 28, we find that $L_2(\eta_1)$ and $M_2(\eta_1)$ will grow like x^4. This contradicts with the asymptotic behavior (2.8) of η_2. On the other hand, if c_2 or c_3 is nonzero, then still by Lemma 28, $M_2(\eta_1)$ will grow like x^3 or $x^2 y$. This also contradicts with the asymptotic behavior of η_2. Hence η_0 is a constant. Then from the previous discussion, we deduce that

$$\eta_2 = d_1 \partial_x \tau_2 + d_2 \partial_y \tau_2 + d_3 \tau_2,$$

for some constants d_1, d_2, d_3, which implies $\phi = d_1 \partial_x Q + d_2 \partial_y Q$. The proof is thus finished. □

4. Nondegeneracy of a family of y-periodic solutions

Following similar arguments as in the previous sections, we would like to show the nondegeneracy of a family of y-periodic solutions naturally associated to the lump solution.

Let $k \in (0,1)$ and $p = \sqrt{1-k^2}i$. In some cases, we also denote $\sqrt{1-k^2}$ by b. Define

$$\tilde{\iota}_1 = re^{\frac{k}{2}(x-py-t)} - e^{-\frac{k}{2}(x-py-t)}.$$

Here $r > 0$ is a constant to be determined later on. Throughout the paper, we use the notation $x^* = k^{-1} \ln r$. Then we can also write

$$\tilde{\iota}_1 = \sqrt{r} \left(e^{\frac{k}{2}(x+x^*-py-t)} - e^{-\frac{k}{2}(x+x^*-py-t)} \right).$$

Note that actually $\tilde{\iota}_1$ is depending on k and p and of travelling wave type. We now define ι_1 through the relation $\tilde{\iota}_1(t,x,y) = \iota_1(x-t,y)$.

Lemma 29. *$\tilde{\iota}_1$ satisfies the bilinear KP-I equation.*

Proof. This follows from direct computation. See, for instance, Page 197 in [1]. □

4.1. The linearized Bäcklund transformation between ι_0 and ι_1.

Next we consider the Bäcklund transformation between $\iota_0 = 1$ and ι_1. Note that indeed ι_0 is same as τ_0. Since many ideas are similar as in the previous sections, in the sequel, we will omit some of the details of the proof.

Define the parameters

$$\lambda = \frac{k^2}{4}, \mu = \frac{pi}{\sqrt{3}}. \tag{4.1}$$

Lemma 30. *The Bäcklund transformation between ι_0 and $\tilde{\iota}_1$ is given by*

$$\begin{cases} \left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right) \iota_0 \cdot \tilde{\iota}_1 - \lambda \iota_0 \tilde{\iota}_1 = 0, \\ \left(D_t + 3\lambda D_x - \sqrt{3}\mu i D_y + D_x^3 - \sqrt{3} i D_x D_y - \frac{3k^2\mu}{4}\right) \iota_0 \cdot \tilde{\iota}_1 = 0. \end{cases} \tag{4.2}$$

Proof. Let $\eta_1 = k\left(x - py - t\right)$. We have

$$\begin{aligned}
&\left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right)\iota_0 \cdot \tilde{\iota}_1 - \lambda \iota_0 \tilde{\iota}_1\\
&= \partial_x^2 \tilde{\iota}_1 + \mu\left(-\partial_x \tilde{\iota}_1\right) + \frac{1}{\sqrt{3}} i\left(-\partial_y \tilde{\iota}_1\right) - \lambda \tilde{\iota}_1\\
&= e^{-\frac{\eta_1}{2}}\left(\frac{k^2}{4} + \frac{\mu k}{2} - \frac{pki}{2\sqrt{3}} - \lambda\right)\\
&- r e^{\frac{\eta_1}{2}}\left(\frac{k^2}{4} - \frac{\mu k}{2} + \frac{pki}{2\sqrt{3}} - \lambda\right).
\end{aligned}$$

Since λ, μ are given by (4.1), we get

$$\begin{aligned}
\frac{k^2}{4} + \frac{\mu k}{2} - \frac{pki}{2\sqrt{3}} - \lambda = 0,\\
\frac{k^2}{4} - \frac{\mu k}{2} + \frac{pki}{2\sqrt{3}} - \lambda = 0.
\end{aligned}$$

Hence

$$\left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right)\iota_0 \cdot \tilde{\iota}_1 - \lambda \iota_0 \tilde{\iota}_1 = 0. \tag{4.3}$$

The second equation

$$\left(D_t + 3\lambda D_x - \sqrt{3}\mu i D_y + D_x^3 - \sqrt{3} i D_x D_y - \frac{3k^2\mu}{4}\right)\iota_0 \cdot \tilde{\iota}_1 = 0 \tag{4.4}$$

can also be verified directly. Alternatively, once (4.3) is proved, we can use the bilinear identity (2.4) to prove (4.4). □

Similarly as before, we need to investigate the linearized Bäcklund transformation between ι_0 and ι_1. That is

$$\begin{cases}
\left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right)\phi \cdot \iota_1 - \lambda \phi \iota_1 = \mathcal{G}_1 \eta,\\
\left(-D_x + 3\lambda D_x - \sqrt{3}\mu i D_y + D_x^3 - \sqrt{3} i D_x D_y - \frac{3k^2\mu}{4}\right)\phi \cdot \iota_1 = \mathcal{N}_1 \eta.
\end{cases} \tag{4.5}$$

Here

$$\mathcal{G}_1 \eta = -\left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right)\iota_0 \cdot \eta + \lambda \iota_0 \eta$$

and

$$\mathcal{N}_1 \eta = -\left(-D_x + 3\lambda D_x - \sqrt{3}\mu i D_y + D_x^3 - \sqrt{3} i D_x D_y - \frac{3k^2\mu}{4}\right)\iota_0 \cdot \eta.$$

Let us write the first equation as

$$i D_y \phi \cdot \iota_1 = -\left(\sqrt{3} D_x^2 + \sqrt{3}\mu D_x\right)\phi \cdot \iota_1 + \sqrt{3}\lambda \phi \iota_1 + \sqrt{3}\mathcal{G}_1 \eta.$$

From this, we get

$$\begin{aligned}
\partial_y \phi = &\frac{i}{\iota_1}\left[\sqrt{3}\mu\left(\partial_x \phi \iota_1 - \phi \partial_x \iota_1\right) + \sqrt{3}\left(\partial_x^2 \phi \iota_1 - 2\partial_x \phi \partial_x \iota_1 + \phi \partial_x^2 \iota_1\right)\right]\\
&+ \frac{\partial_y \iota_1}{\iota_1}\phi - \sqrt{3}\lambda i \phi - \frac{\sqrt{3} i}{\iota_1}\mathcal{G}_1 \eta.
\end{aligned}$$

Inserting (3.5) into the right hand side of the second equation, we obtain the following third order ODE:

$$4\partial_x^3\phi+\left(6\mu-12\frac{\partial_x\iota_1}{\iota_1}\right)\partial_x^2\phi+\left(3\mu^2-1-12\mu\frac{\partial_x\iota_1}{\iota_1}+12\frac{(\partial_x\iota_1)^2}{\iota_1^2}\right)\partial_x\phi$$
$$=\frac{1}{\iota_1}\left(3\partial_x\left(\mathcal{G}_1\eta\right)-\frac{6\partial_x\iota_1}{\iota_1}\mathcal{G}_1\eta+\mathcal{N}_1\eta+3\mu\mathcal{G}_1\eta\right). \tag{4.6}$$

Letting $g=\partial_x\phi$, we are lead to the corresponding homogeneous equation

$$4\partial_x^2 g+\left(6\mu-6k\coth\left(\frac{k}{2}\left(x-py\right)\right)\right)\partial_x g$$
$$+\left(-k^2-6\mu k\coth\left(\frac{k}{2}\left(x-py\right)\right)+3k^2\coth^2\left(\frac{k}{2}\left(x-py\right)\right)\right)g=0. \tag{4.7}$$

Next, we would like to find solutions for the equation (4.7). Introduce the new variable $z=\coth\frac{k(x+x^*-py)}{2}$. The equation (4.7) becomes

$$k^2\left(1-z^2\right)^2 g_{zz}+\left(\left(3\mu-3kz\right)k\left(1-z^2\right)-2k^2\left(1-z^2\right)z\right)g_z$$
$$+\left(-k^2-6\mu kz+3k^2z^2\right)g=0. \tag{4.8}$$

This equation can be regarded as a second order ODE for the variable z. Since our aim is to solve the ODE (4.6) for the x variable, the equation (4.8) is introduced here to find the fundamental solutions of (4.6).

Recall that $\mu=-\sqrt{\frac{1-k^2}{3}}$. We have the following

Lemma 31. *The equation (4.8) has solutions of the form*

$$q_1=\frac{z-\frac{3\mu}{k}}{z^2-1},\ q_2=\frac{1}{\sqrt{z^2-1}}\left(\frac{z-1}{z+1}\right)^{\frac{3\mu}{2k}}.$$

One crucial fact is that as $|z|\to+\infty$, $q_1=O\left(\frac{1}{z}\right)$ and $q_2=O\left(\frac{1}{z}\right)$. This implies that at the singularities of (4.6), where $\iota_1=0$, the solutions q_1,q_2 are actually smooth.

For any given function η, with the help of these explicit fundamental solutions, the solutions of the inhomogeneous third order ODE (4.6) can be written down using the variation of parameter formula, as have been done for the τ_0,τ_1 case. Note that as $x\to+\infty$,

$$q_1\sim c_1(y)\,e^{kx},q_2\sim c_2(y)\,e^{\frac{k-3\mu}{2}x}.$$

On the other hand, as $x\to-\infty$,

$$q_1\sim d_1(y)\,e^{-kx},q_2\sim d_2(y)\,e^{-\frac{k+3\mu}{2}x}.$$

Hence in this case the Wronskian of q_1 and q_2 behaves like $e^{\left(-\frac{3k}{2}-\frac{3}{2}\mu\right)x}$. Since $\coth\frac{k(x+x^*-py)}{2}$ is $\frac{2\pi}{kb}$ periodic in y, we know that q_1 and q_2 are both $\frac{2\pi}{kb}$-periodic in y. However, ι_1 is only $\frac{4\pi}{kb}$-periodic.

For later purpose, let us define the function

$$\Phi_0:=\left(D_x^2+\mu D_x+\frac{1}{\sqrt{3}}iD_y\right)\phi\cdot\iota_1-\lambda\phi\iota_1+\left(D_x^2+\mu D_x+\frac{1}{\sqrt{3}}iD_y\right)\iota_0\cdot\eta-\lambda\iota_0\eta. \tag{4.9}$$

The choice of ϕ is to be determined later on. We have the following

Proposition 32. *Suppose ϕ satisfies (4.6) and η satisfies the linearized bilinear KP-I equation at ι_1. Let Φ_0 be defined by (4.9). Then*

$$\partial_x^3\Phi_0 = \left(-\frac{3\mu}{2} + \frac{6}{\iota_1}\partial_x\iota_1\right)\partial_x^2\Phi_0 + b\partial_x\Phi_0 + c\Phi_0,$$

where

$$b := \frac{1}{4}k^2 + 6\mu\frac{\partial_x\iota_1}{\iota_1} + 3\frac{\partial_x^2\iota_1}{\iota_1} - 15\left(\frac{\partial_x\iota_1}{\iota_1}\right)^2,$$

$$\begin{aligned} c := &-\frac{1}{4}k^2\frac{\partial_x\iota_1}{\iota_1} - 6\mu\left(\frac{\partial_x\iota_1}{\iota_1}\right)^2 + \frac{3\mu}{2}\frac{\partial_x^2\iota_1}{\iota_1} + 15\left(\frac{\partial_x\iota_1}{\iota_1}\right)^3 \\ &- 9\frac{\partial_x\iota_1\partial_x^2\iota_1}{\iota_1^2} + \frac{\partial_x^3\iota_1}{\iota_1}. \end{aligned}$$

Proof. This is a result similar to Porposition 12 and can be directly verified with the help of Maple. □

4.2. **The linearized Bäcklund transformation between ι_1 and ι_2.** Let $k \in \left(0, \frac{1}{2}\right)$. Throughout the paper, we denote by A the constant $\sqrt{\frac{1-4k^2}{1-k^2}}$. Then we define

$$\begin{aligned} \tilde{\iota}_2 &= e^{\frac{\eta_1+\eta_2}{2}} + e^{-\frac{\eta_1+\eta_2}{2}} - A\left(e^{\frac{\eta_1-\eta_2}{2}} + e^{\frac{\eta_2-\eta_1}{2}}\right) \\ &= 2\left(\cosh\left(k\left(x-t\right)\right) - A\cos\left(kby\right)\right), \end{aligned}$$

where $\eta_1 = k\left(x-py-t\right), \eta_2 = k\left(x+py-t\right)$. As is well known, $\tilde{\iota}_2$ is a solution to the bilinear KP-I equation, as can be checked by hand. Define ι_2 through the relation $\tilde{\iota}_2\left(t,x,y\right) = \iota_2\left(x-t,y\right)$. Let

$$\mu^* = -\frac{pi}{\sqrt{3}}, r = \frac{\mu^* A}{k+\mu^*}.$$

We emphasize that ι_2 is $\frac{2\pi}{kb}$-periodic in y, while the minimal period of ι_1 in the y direction is equal to $\frac{4\pi}{kb}$.

Lemma 33. *We have the following Bäcklund transformation between $\tilde{\iota}_1$ and $\tilde{\iota}_2$:*

$$\begin{cases} \left(D_x^2 + \mu^* D_x + \frac{1}{\sqrt{3}}iD_y\right)\tilde{\iota}_1\cdot\tilde{\iota}_2 = \lambda\tilde{\iota}_1\tilde{\iota}_2, \\ \left(D_t + 3\lambda D_x - \sqrt{3}\mu^* iD_y + D_x^3 - \sqrt{3}iD_xD_y - \frac{3k^2\mu^*}{4}\right)\tilde{\iota}_1\cdot\tilde{\iota}_2 = 0. \end{cases} \tag{4.10}$$

Proof. We have not been able to locate a reference in the literature for this result. Therefore let us sketch the proof below.

We compute

$$\begin{aligned} I := &\left(D_x^2 + \mu^* D_x + \frac{iD_y}{\sqrt{3}}\right)\tilde{\iota}_1\cdot\tilde{\iota}_2 - \lambda\tilde{\iota}_1\tilde{\iota}_2 \\ &= \partial_x^2\tilde{\iota}_1\tilde{\iota}_2 - 2\partial_x\tilde{\iota}_1\partial_x\tilde{\iota}_2 + \tilde{\iota}_1\partial_x^2\tilde{\iota}_2 \\ &+ \mu^*\left(\partial_x\tilde{\iota}_1\iota_2 - \tilde{\iota}_1\partial_x\tilde{\iota}_2\right) + \frac{i}{\sqrt{3}}\left(\partial_y\tilde{\iota}_1\tilde{\iota}_2 - \tilde{\iota}_1\partial_y\tilde{\iota}_2\right) - \frac{k^2}{4}\tilde{\iota}_1\tilde{\iota}_2. \end{aligned}$$

Note that $\partial_x^2 \tilde{\iota}_1 \tilde{\iota}_2 = \frac{k^2}{4} \tilde{\iota}_1 \tilde{\iota}_2$. Hence I is equal to

$$
\begin{aligned}
& -k^2 \left(e^{-\frac{\eta_1}{2}} + re^{\frac{\eta_1}{2}}\right)\left(-e^{-\frac{\eta_1+\eta_2}{2}} + e^{\frac{\eta_1+\eta_2}{2}}\right) + k^2 \left(-e^{-\frac{\eta_1}{2}} + re^{\frac{\eta_1}{2}}\right)\left(e^{-\frac{\eta_1+\eta_2}{2}} + e^{\frac{\eta_1+\eta_2}{2}}\right) \\
& - \frac{pi}{\sqrt{3}} \frac{k}{2} \left(e^{-\frac{\eta_1}{2}} + re^{\frac{\eta_1}{2}}\right)\left(e^{-\frac{\eta_1+\eta_2}{2}} - Ae^{\frac{\eta_1-\eta_2}{2}} - Ae^{\frac{-\eta_1+\eta_2}{2}} + e^{\frac{\eta_1+\eta_2}{2}}\right) \\
& + \frac{pi}{\sqrt{3}} k \left(-e^{-\frac{\eta_1}{2}} + re^{\frac{\eta_1}{2}}\right)\left(-e^{-\frac{\eta_1+\eta_2}{2}} + e^{\frac{\eta_1+\eta_2}{2}}\right) \\
& + \frac{i}{\sqrt{3}} \frac{pk}{2} \left(-e^{-\frac{\eta_1}{2}} - re^{\frac{\eta_1}{2}}\right)\left(e^{-\frac{\eta_1+\eta_2}{2}} - Ae^{\frac{\eta_1-\eta_2}{2}} - Ae^{\frac{-\eta_1+\eta_2}{2}} + e^{\frac{\eta_1+\eta_2}{2}}\right) \\
& - \frac{ipk}{\sqrt{3}} \left(-e^{-\frac{\eta_1}{2}} + re^{\frac{\eta_1}{2}}\right)\left(Ae^{\frac{\eta_1-\eta_2}{2}} - Ae^{\frac{-\eta_1+\eta_2}{2}}\right).
\end{aligned}
$$

It can be simplified to

$$
2k\left(rk - \frac{\sqrt{3}ipr}{3} + \frac{\sqrt{3}ipA}{3}\right)e^{-\frac{\eta_2}{2}} - 2k\left(k - \frac{\sqrt{3}i}{3}Apr + \frac{\sqrt{3}ip}{3}\right)e^{\frac{\eta_2}{2}}. \tag{4.11}
$$

Due to the choice of the constant r, (4.11) is equal to zero. This proves the first identity in (4.10).

The second equation of (4.10) then follows from the first one and the bilinear identity (2.4). $\square$

Consider the linearized Bäcklund transformation

$$
\begin{cases} \left(D_x^2 + \mu^* D_x + \frac{1}{\sqrt{3}} iD_y\right)\phi \cdot \iota_2 - \lambda\phi\iota_2 = \mathcal{G}_2\eta, \\ \left(-D_x + 3\lambda D_x - \sqrt{3}\mu^* iD_y + D_x^3 - \sqrt{3}iD_xD_y - \frac{3}{4}k^2\mu^*\right)\phi \cdot \iota_2 = \mathcal{N}_2\eta. \end{cases} \tag{4.12}
$$

Here

$$
\mathcal{G}_2\eta := -\left(D_x^2 + \mu^* D_x + \frac{1}{\sqrt{3}} iD_y\right)\iota_1 \cdot \eta + \lambda\iota_1\eta,
$$

$$
\mathcal{N}_2\eta := -\left(-D_x + 3\lambda D_x - \sqrt{3}\mu^* iD_y + D_x^3 - \sqrt{3}iD_xD_y - \frac{3}{4}k^2\mu^*\right)\iota_1 \cdot \eta.
$$

Similarly as before, after we plug the $\partial_y\phi$ term into the first equation of (4.12) into the second one, we get the following third order ODE:

$$
\begin{aligned}
& 4\iota_2\partial_x^3\phi + (6\mu^*\iota_2 - 12\partial_x\iota_2)\partial_x^2\phi \\
& + \left(-k^2\iota_2 - 12\mu^*\partial_x\iota_2 + 12\frac{(\partial_x\iota_2)^2}{\iota_2}\right)\partial_x\phi + B\phi \\
& = 3\partial_x(\mathcal{G}_2\eta) - \frac{6\partial_x\iota_2}{\iota_2}\mathcal{G}_1\eta + \mathcal{N}_2\eta + 3\mu^*\mathcal{G}_2\eta,
\end{aligned} \tag{4.13}
$$

where

$$
\begin{aligned}
B := {} & 2\partial_x^3\iota_2 - 6\frac{\partial_x\iota_2}{\iota_2}\partial_x^2\iota_2 + k^2\partial_x\iota_2 \\
& + 2\sqrt{3}i\frac{-\sqrt{3}i\mu^*\partial_x\iota_2 + \partial_y\iota_2}{\iota_2}\partial_x\iota_2 \\
& - 2\sqrt{3}i\partial_x\partial_y\iota_2 - 6\mu^*\lambda\iota_2.
\end{aligned}
$$

Note that if we replace ι_2, μ^* by ι_1, μ respectively, B will be 0, as we already computed.

The homogeneous version of (4.13) is

$$\begin{aligned} &4\iota_2\partial_x^3\phi+(6\mu^*\iota_2-12\partial_x\iota_2)\,\partial_x^2\phi \\ &+\left(-k^2\iota_2-12\mu^*\partial_x\iota_2+12\frac{(\partial_x\iota_2)^2}{\iota_2}\right)\partial_x\phi+B\phi \\ &=0. \end{aligned} \tag{4.14}$$

Observe that ι_1 is automatically a solution of this equation. We write $\phi=\iota_1 h$, and $g=\partial_x h$. Then g should satisfy

$$\begin{aligned} &\partial_x^2 g+\left(3\frac{\partial_x\iota_1}{\iota_1}+\frac{3\mu^*}{2}-3\frac{\partial_x\iota_2}{\iota_2}\right)\partial_x g \\ &+\left(\frac{k^2}{2}+\left(3\mu^*-6\frac{\partial_x\iota_2}{\iota_2}\right)\frac{\partial_x\iota_1}{\iota_1}-3\mu^*\frac{\partial_x\iota_2}{\iota_2}+3\frac{(\partial_x\iota_2)^2}{\iota_2^2}\right)g=0. \end{aligned} \tag{4.15}$$

If we introduce the new variable $z=e^{k(x-py)}$, then we get the following second order ODE for g in the z variable:

$$\begin{aligned} &k^2z^2\partial_z^2 g+\left(k^2z+\left(3\frac{\partial_x\iota_1}{\iota_1}+\frac{3\mu^*}{2}-3\frac{\partial_x\iota_2}{\iota_2}\right)kz\right)\partial_z g \\ &+\left(\frac{k^2}{2}+\left(3\mu^*-6\frac{\partial_x\iota_2}{\iota_2}\right)\frac{\partial_x\iota_1}{\iota_1}-3\mu^*\frac{\partial_x\iota_2}{\iota_2}+3\frac{(\partial_x\iota_2)^2}{\iota_2^2}\right)g \\ &=0. \end{aligned}$$

To have an idea for what happened in this equation, we consider the special case of $y=0$. In this case, the equation has the form

$$\begin{aligned} &z^2\partial_z^2 g+z\left(1+\frac{3}{2}\frac{rz-1}{rz+1}+\frac{3\mu^*}{2k}-3\frac{z^2-1}{z^2+2Az+1}\right)\partial_z g \\ &+\left(\frac{1}{2}+\left(3\frac{\mu^*}{k}-6\frac{z^2-1}{z^2+2Az+1}\right)\frac{1}{2}\frac{rz-1}{rz+1}-3\frac{\mu^*}{k^2}\frac{z^2-1}{z^2+2Az+1}+\frac{3}{k^2}\left(\frac{z^2-1}{z^2+2Az+1}\right)^2\right)g \\ &=0. \end{aligned}$$

If we choose $k=\frac{1}{2}$, then $\mu^*=\frac{1}{2}$, $A=\sqrt{\frac{1-4k^2}{1-k^2}}=0, r=0$. Hence the equation reduces to

$$\partial_z^2 g+\frac{4-2z^2}{(z^2+1)\,z}\partial_z g+\frac{2\left(4z^4-13z^2+7\right)}{z^2\left(z^2+1\right)^2}g=0. \tag{4.16}$$

The solutions of (4.16) are given by hypergeometric functions. For general $k\in\left(0,\frac{1}{2}\right)$, it seems to us that the solutions of (4.16) do not have explicit expressions. This contrasts with the previous cases and is an interesting issue.

The next lemma gives the asymptotic behavior at infinity of solutions to (4.15). Recall that $\mu^*=\sqrt{\frac{1-k^2}{3}}$.

Lemma 34. *The indicial roots of* (4.15) *are given by*

$$\lambda_1^+=k,\lambda_2^+=\frac{k}{2}-\frac{3}{2}\mu^*,$$

and

$$\lambda_1^-=-k,\lambda_2^-=-\frac{k}{2}-\frac{3}{2}\mu^*.$$

Consequently, the indicial roots of (4.14) *are*

$$\alpha_0^+ = \frac{k}{2}, \alpha_1^+ = \frac{3k}{2}, \alpha_2^+ = k - \frac{3}{2}\mu^*,$$
$$\alpha_0^- = -\frac{k}{2}, \alpha_1^- = -\frac{3k}{2}, \alpha_2^- = -k - \frac{3}{2}\mu^*.$$

Proof. Recall that as $x \to \pm\infty$, $\frac{\partial_x \iota_1}{\iota_1} \to \pm\frac{k}{2}$. On the other hand,

$$\iota_2 = 2\left(\cosh\left(kx\right) - A\cosh\left(kbiy\right)\right).$$

Therefore,

$$\frac{\partial_x \iota_2}{\iota_2} = \frac{k\sinh\left(kx\right)}{\cosh\left(kx\right) - A\cosh\left(kbiy\right)}.$$

From this we see that

$$\frac{\partial_x \iota_2}{\iota_2} \to \pm k, \text{ as } x \to \pm\infty.$$

It follows that the limiting equation of (4.15) at $+\infty$ is

$$\eta'' + \left(\frac{3\mu^*}{2} - \frac{3}{2}k\right)\eta' + \left(\frac{k^2}{2} - \frac{3}{2}\mu^* k\right)\eta = 0.$$

Hence the indicial roots are

$$\lambda_{1,2}^+ = \frac{-\left(\frac{3\mu^*}{2} - \frac{3}{2}k\right) \pm \sqrt{\left(\frac{3\mu^*}{2} - \frac{3}{2}k\right)^2 - 4\left(\frac{k^2}{2} - \frac{3}{2}\mu^* k\right)}}{2}.$$

Using the fact that $\mu^* = \frac{b}{\sqrt{3}}$, we get $\lambda_1^+ = k, \lambda_2^+ = \frac{k}{2} - \frac{3}{2}\mu^*$.

Similarly, as $x \to -\infty$, we get

$$\eta'' + \left(\frac{3\mu^*}{2} + \frac{3}{2}k\right)\eta' + \left(\frac{k^2}{2} + \frac{3}{2}\mu^* k\right)\eta = 0.$$

Hence the indicial roots are

$$\frac{-\left(\frac{3\mu^*}{2} + \frac{3}{2}k\right) \pm \sqrt{\left(\frac{3\mu^*}{2} + \frac{3}{2}k\right)^2 - 4\left(\frac{k^2}{2} + \frac{3}{2}\mu^* k\right)}}{2}.$$

Hence $\lambda_1^- = -k, \lambda_2^- = -\frac{k}{2} - \frac{3}{2}\mu^*$. The proof is thus completed. ∎

For each fixed y, we use $\beta_j, j = 1, 2$, to denote the fundamental solutions of (4.14) with the asymptotic behavior $e^{\alpha_j^- x}$ as $x \to -\infty$. Moreover, we use $f_j, j = 1, 2$, to denote the fundamental solutions of (4.15) with the asymptotic behavior $e^{\lambda_j^- x}$ as $x \to -\infty$.

Next we define the function

$$\Phi_0^* := \left(D_x^2 + \mu^* D_x + \frac{1}{\sqrt{3}}iD_y\right)\phi \cdot \iota_2 - \lambda\phi\iota_2 + \left(D_x^2 + \mu^* D_x + \frac{1}{\sqrt{3}}iD_y\right)\iota_1 \cdot \eta - \lambda\iota_1\eta.$$

A result parallel to Proposition 32 is:

Proposition 35. *Suppose* ϕ *satisfies* (4.13) *and* η *satisfies the linearized bilinear KP-I equation at* ι_2. *Then*

$$\partial_x^3 \Phi_0^* = \left(-\frac{3\mu^*}{2} + \frac{6}{\iota_2}\partial_x\left(\iota_2\right)\right)\partial_x^2 \Phi_0^* + b\partial_x \Phi_0^* + c\Phi_0^*,$$

where

$$b=\frac{1}{4}k^{2}+6\mu^{*}\frac{\partial_{x}\iota_{2}}{\iota_{2}}+3\frac{\partial_{x}^{2}\iota_{2}}{\iota_{2}}-15\left(\frac{\partial_{x}\iota_{2}}{\iota_{2}}\right)^{2},$$

$$\begin{aligned}c=&-\frac{15\mu^{*}}{2}\left(\frac{\partial_{x}\iota_{2}}{\iota_{2}}\right)^{2}-\frac{1}{2}k^{2}\frac{\partial_{x}\iota_{2}}{\iota_{2}}+\frac{3}{2}\mu^{*}\lambda+\frac{\sqrt{3}i}{2}\frac{\partial_{x}\partial_{y}\iota_{2}}{\iota_{2}}\\&-\frac{\sqrt{3}i}{2\iota_{2}^{2}}\partial_{x}\iota_{2}\partial_{y}\iota_{2}+\frac{3\mu^{*}}{2}\frac{\partial_{x}^{2}\iota_{2}}{\iota_{2}}+\frac{1}{2}\frac{\partial_{x}^{3}\iota_{2}}{\iota_{2}}\\&-\frac{15}{2}\frac{\partial_{x}\iota_{2}\partial_{x}^{2}\iota_{2}}{\iota_{2}^{2}}+15\left(\frac{\partial_{x}\iota_{2}}{\iota_{2}}\right)^{3}.\end{aligned}$$

Remark 36. *The coefficients before $\partial_x^2\Phi_0^*$ and $\partial_x\Phi_0^*$ are essentially the same as that of (32). The coefficient before Φ_0^* is different, due to the fact that the coefficient B in the equation (4.14) is nonzero in this case. Indeed, Proposition 12, Proposition 20, and Proposition 32 can regarded as special cases of this identity, with different choices of μ,λ,ι.*

4.3. **Proof of the nondegeneracy of the periodic solutions.** Recall that ι_2 is periodic in y with period $\frac{2\pi}{kb}$. When $k=\frac{1}{2}$, $\frac{2\pi}{kb}=\frac{8\pi}{\sqrt{3}}$. As $k\to 0$, the period tends to $+\infty$. The KP-I equation has a family of solutions $2\partial_x^2\ln\tilde{\iota}_2$. In this part, we prove that $T_2=2\partial_x^2\ln\iota_2$ is nondegenerated in the class of $\frac{2\pi}{kb}$-periodic(in y) solutions.

Lemma 37. *Let $k\in\left(0,\frac{1}{2}\right)$ be fixed. Let φ be a function $\frac{2\pi}{kb}$-periodic in y. Suppose for each fixed y,*

$$\varphi(x,y)\to 0,\ \ as\ \ |x|\to+\infty.$$

Assume φ solves the linearized KP-I equation at T_2 :

$$\partial_x^2\left(\partial_x^2\varphi-\varphi+6T_2\varphi\right)-\partial_y^2\varphi=0.$$

Then φ has the estimate:

$$|\varphi(x,y)|\le Ce^{-k|x|}. \tag{4.17}$$

Proof. Since φ is $\frac{2\pi}{kb}$-periodic in y, we can write it in the form of a Fourier series. Suppose

$$\varphi=Z_0(x)+\sum_{n=1}^{+\infty}\left(Z_n(x)\cos(kbny)+Z_n^*(x)\sin(kbny)\right).$$

The function φ satisfies

$$\partial_x^2\left(\partial_x^2\varphi-\varphi+6T_2\varphi\right)-\partial_y^2\varphi=-\partial_x^2\left(6T_2\varphi\right).$$

Note that $|T_2|\le C_ke^{-k|x|}$. Regarding the right hand side of the above equation as a perturbation term(for $|x|$ large), we find that $Z_n(x)$ satisfies

$$\partial_x^2\left(\partial_x^2Z_n-Z_n+6T_2Z_n\right)+k^2b^2n^2Z_n=O\left(e^{-k|x|}\right). \tag{4.18}$$

Note that the indicial roots λ of the equation

$$Z^{(4)}-Z^{(2)}+k^2b^2n^2Z=0$$

are given by

$$\tilde{\lambda}^2=\frac{1\pm\sqrt{1-4k^2b^2n^2}}{2}.$$

We obtain

$$\tilde{\lambda} = \pm\sqrt{\frac{1 \pm \sqrt{1-4k^2b^2n^2}}{2}}.$$

We claim that for $n \in \mathbb{N}$, $\left|\operatorname{Re}\tilde{\lambda}\right| > k$. There are two cases.

Case 1: $4k^2b^2n^2 < 1$.

We need to show

$$\frac{1-\sqrt{1-4k^2b^2n^2}}{2} \geq k^2.$$

The validity of this inequality follows from the fact that

$$1-4k^2b^2 = \left(1-2k^2\right)^2.$$

Case 2: $4k^2b^2n^2 > 1$.

Writing $\tilde{\lambda} = c + di$, we have, for some $\alpha \in \mathbb{R}$,

$$c^2 - d^2 + 2cdi = \tilde{\lambda}^2 = \frac{1 \pm \alpha i}{2}.$$

This implies $c^2 - d^2 = \frac{1}{2}$ and $2cd = \frac{\alpha}{2}$. Since $k \leq \frac{1}{2}$, we find that $|c| \geq \frac{1}{\sqrt{2}} > k$.

Once we know the asymptotic behavior of the homogeneous equation of (4.18), we can use the variation of parameter formula to get the desired estimate of φ. □

Let φ be a function satisfying the assumption of Lemma 37. We differentiate the equation $T_2 = 2\partial_x^2 \ln \iota_2$ and get a function η:

$$\varphi = 2\partial_x^2 \frac{\eta}{\iota_2}.$$

Then we get a corresponding solution η of the linearized bilinear KP-I equation, with the form

$$\eta = \iota_2 \partial_x^{-2} \frac{\varphi}{2}.$$

Here $\partial_x^{-1} := \int_{-\infty}^{x}$. Observe that as $|x| \to \infty$, $\iota_2 = O\left(e^{k|x|}\right)$. Due to the estimate of φ, we conclude that $\eta \leq C$, for $x \leq 0$. The next step is to solve the linearized Bäcklund transformation between ι_1 and ι_2.

Lemma 38. *Suppose η is a function solving the linearized bilinear KP-I equation at ι_2 :*

$$-D_x^2\eta \cdot \iota_2 + D_x^4 \eta \cdot \iota_2 = D_y^2 \eta \cdot \iota_2.$$

Assume η is $\frac{2\pi}{kb}$-periodic in y and satisfies $|\eta(x,y)| \leq (1+|x|)e^{k|x|}$, and

$$|\eta(x,y)| \leq C, \text{ for } x \leq 0.$$

Then the system (4.12) has a solution ϕ, $\frac{4\pi}{kb}$-periodic in y, with

$$|\phi(x,y)| \leq Ce^{\frac{kx}{2}}, \text{ for } x \leq 0.$$

Moreover, there exists a constant c^* such that

$$\left|\phi(x,y) - c^* e^{\frac{3}{2}kx + \frac{kb}{2}yi}\right| \leq C(1+|x|)e^{k|x|}.$$

Additionally, ϕ satisfies the linearized bilinear KP-I equation at ι_1

$$-D_x^2\phi \cdot \iota_1 + D_x^4 \phi \cdot \iota_1 = D_y^2 \phi \cdot \iota_1.$$

Proof. The proof is similar to the case of the linearized Bäcklund transformation between τ_1 and τ_2. We use W to denote the Wronskian of f_1 and f_2, introduced after Lemma 34. As $x \to -\infty$, W behaves like $e^{-\frac{3}{2}(k+\mu^*)x}$.

For each fixed y, the inhomogeneous equation we need to solve is

$$4\iota_2 \partial_x^3 \phi + (6\mu^* \iota_2 - 12\partial_x \iota_2)\, \partial_x^2 \phi + \left(\left(3\mu^{*2} - 1\right) \iota_2 - 12\mu^* \partial_x \iota_2 + 12 \frac{(\partial_x \iota_2)^2}{\iota_2} \right) \partial_x \phi + B\phi$$

$$= \left(\mathcal{N}_2 \eta + \frac{6\partial_x \tau_2}{\tau_2} \mathcal{G}_2 \eta + 3\partial_x (\mathcal{G}_2 \eta) + 3\mu \mathcal{G}_2 \eta \right).$$

Let us define

$$M := \frac{1}{4\iota_2} \left(\mathcal{N}_2 \eta + \frac{6\partial_x \iota_2}{\iota_2} \mathcal{G}_2 \eta + 3\partial_x (\mathcal{G}_2 \eta) + 3\mu \mathcal{G}_2 \eta \right).$$

This equation has a solution of the form

$$\phi_0 (x, y) := \iota_1 \int_{-\infty}^{x} \left[f_2 \int_{-\infty}^{s} \left(\frac{f_1 M}{\iota_1 W} \right) - f_1 \int_{-\infty}^{s} \left(\frac{f_2 M}{\iota_1 W} \right) \right] ds. \tag{4.19}$$

Using the estimate of η, we obtain

$$|\mathcal{N}_2 \eta| \le C (1 + |x|)\, e^{\frac{3}{2} k |x|},$$

and for $x \le 0$, $|\mathcal{N}_2 \eta| \le C e^{\frac{k}{2}|x|}$. The term $\mathcal{G}_2 \eta$ satisfies similar estimates. From these bounds and (4.19), we infer that for $x \le 0$,

$$|\phi_0| \le C e^{\frac{kx}{2}}.$$

Note that although ι_1 is only $\frac{4\pi}{kb}$ periodic in y, ϕ_0 is indeed $\frac{2\pi}{kb}$ periodic, due to fact that ι_1 appears both in the nominator and denominator.

The next step is to find a solution ϕ satisfying the first equation of (4.12) on one line parallel to the y axis, say $x = 0$. This can be achieved by finding a solution of the form

$$\phi = \phi_0 + \rho_0 (y)\, \iota_1 + \rho_1 (y)\, \beta_1 + \rho_2 (y)\, \beta_2,$$

with the initial condition $\rho_0 (0) = \rho_1 (0) = \rho_2 (0) = 0$. Then applying Proposition 35, we conclude that ϕ solves the whole system (4.12). Using the asymptotic behavior of $\iota_1, \beta_1, \beta_2$ at $-\infty$(observe that $\alpha_1^- = -\frac{3}{2}k < 0$ and $\alpha_2^- = -k - \frac{3}{2}\mu^* < 0$), we infer that $\rho_j (y) = 0, j = 0, 1, 2$. Moreover, for x positive and large, in view of the growth rate of $\beta_1, \beta_2, \iota_1$, as $x \to +\infty$, for each y, ϕ_0 can be written as the sum of a function $a (y)\, e^{\frac{3}{2} kx}$ and a function ϕ_0^* satisfying

$$|\phi_0^*| \le C (1 + |x|)\, e^{k|x|}. \tag{4.20}$$

The function $a (y)$ should satisfy

$$\left(D_x^2 + \mu^* D_x + \frac{1}{\sqrt{3}} i D_y \right) \left(a (y)\, e^{\frac{3}{2} kx} \right) \cdot (\cosh (kx)) - \lambda a (y)\, e^{\frac{3}{2} kx} \cosh (kx) = 0.$$

Hence $a (y) = c^* e^{\frac{kb}{2} yi}$ for some constant c^*. Since ι_1 is $\frac{4\pi}{kb}$ periodic in y, ϕ is also $\frac{4\pi}{kb}$ periodic. The proof is finished. □

We also need to solve the linearized Bäcklund transformation between ι_0 and ι_1. A parallel result of Lemma 38 is

Lemma 39. *Suppose ϕ is a function solving the linearized bilinear KP-I equation at ι_1 obtained from Lemma 38. Then the system*

$$\begin{cases} \left(D_x^2+\mu D_x+\frac{1}{\sqrt{3}}iD_y\right)\xi\cdot\iota_1-\lambda\xi\iota_1=\mathcal{G}_1\phi, \\ \left(-D_x+3\lambda D_x-\sqrt{3}\mu iD_y+D_x^3-\sqrt{3}iD_xD_y-\frac{3k^2\mu}{4}\right)\xi\cdot\iota_1=\mathcal{N}_1\phi, \end{cases} \tag{4.21}$$

has a solution ξ, continuous in the region

$$\Omega_{**}:=\mathbb{R}^2\backslash\left\{(x,nt_k):x\le -x^*,n\in\mathbb{N}\right\},$$

$\frac{2\pi}{kb}$-periodic in y, with

$$\left|\xi\left(x,y\right)\right|\le C\left(1+x^2\right)e^{k|x|}, \tag{4.22}$$

and decaying exponentially fast as $x\to-\infty$:

$$\left|\xi\left(x,y\right)\right|\le Ce^{kx},\ \textit{for } x\le 0. \tag{4.23}$$

Moreover, there exists a constant c such that

$$\lim_{y\to 0^+}\xi\left(x,y\right)-\lim_{y\to 0^-}\xi\left(x,y\right)=c\chi_2\left(x\right)\ \textit{for } x\le x^*, \tag{4.24}$$

and ξ satisfies the linearized KP-I equation at ι_0:

$$-D_x^2\xi\cdot\iota_0+D_x^4\xi\cdot\iota_0=D_y^2\xi\cdot\iota_0\ \textit{in } \Omega_{**}.$$

Proof. This follows from the same arguments as in Lemma 38, using variation of parameter formula. We sketch the proof below.

After inserting the first equation of (4.21) into the second one, we get a third order ODE in x for the unknown function ξ :

$$\begin{aligned} &4\partial_x^3\xi+\left(6\mu-12\frac{\partial_x\iota_1}{\iota_1}\right)\partial_x^2\xi+\left(3\mu^2-1-12\mu\frac{\partial_x\iota_1}{\iota_1}+12\frac{\left(\partial_x\iota_1\right)^2}{\iota_1^2}\right)\partial_x\xi \\ &=\frac{1}{\iota_1}\left(3\partial_x\left(\mathcal{G}_1\eta\right)-\frac{6\partial_x\iota_1}{\iota_1}\mathcal{G}_1\eta+\mathcal{N}_1\eta+3\mu\mathcal{G}_1\eta\right):=A. \end{aligned}$$

Let $W=\partial_xq_1q_2-q_1\partial_xq_2$. Then the above equation has a solution

$$\tilde{\xi}_0:=\int_{-\infty}^x\left(q_2\int_{+\infty}^x\frac{q_1A}{4W}-q_1\int_{-\infty}^x\frac{q_2A}{4W}\right). \tag{4.25}$$

Let $\chi_j=\int_0^x q_j, j=1,2$. Note that although ϕ is only $\frac{4\pi}{kb}$ periodic, $\tilde{\xi}_0$ is actually $\frac{2\pi}{kb}$ periodic, due to the fact that the denominator in A is ι_1. Similarly as before, we can find functions ρ_j, with $\rho_j\left(0\right)=0, j=0,1,2,$ such that the function

$$\xi=\tilde{\xi}_0+\rho_0\left(y\right)+\rho_1\left(y\right)\chi_1+\rho_2\left(y\right)\chi_2$$

solve the system (4.21). Using the asymptotic behavior of χ_2 for $x\to+\infty$, we conclude

$$\rho_2\left(y\right)=0.$$

On the other hand, using the asymptotic behavior of χ_1 for $x\to-\infty$, we have $\rho_1=0$. Then by considering again the asymptotic behavior of ξ for $x\to-\infty$, we get $\rho_0=0$. Hence $\xi=\tilde{\xi}_0$. Now from (4.25), we get (4.22). The estimate (4.23) follows from the fact that when $k\in\left(0,\frac{1}{2}\right),-\frac{k+3\mu}{2}>k$.

Let us consider the function $a(y) := \int_{-\infty}^{+\infty} \frac{q_2 A}{4W}, y \in (0, t_k)$. As $x \to +\infty$, the main order of ξ is $a(y)\chi_1$. Hence

$$\left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right)\left(a(y) e^{kx}\right) \cdot \left(e^{\frac{k(x-py)}{2}}\right) - \lambda a(y) e^{kx} e^{\frac{k(x-py)}{2}} = 0.$$

It follows that $a(y) = Ce^{-kbyi}$ and $q_1 \int_{-\infty}^{x} \frac{q_2 A}{4W}$ can be extended continuously to the region Ω_{**}. Note that the function $q_2 \int_{+\infty}^{x} \frac{q_1 A}{4W}$ may have a jump on those half lines $\{(x, nt_k) : x \leq x^*, n \in \mathbb{N}\}$, this leads to (4.24). □

In the next step, for each given function ξ, we would like to solve the reversed linearized Backlund transformation system

$$\begin{cases} \left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right) \iota_0 \cdot \eta - \lambda \iota_0 \eta = -P\xi, \\ \left(-D_x + 3\lambda D_x - \sqrt{3}\mu i D_y + D_x^3 - \sqrt{3} i D_x D_y - \frac{3k^2\mu}{4}\right) \iota_0 \cdot \eta = -S\xi. \end{cases} \tag{4.26}$$

where

$$\begin{cases} P\xi := \left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right) \xi \cdot \iota_1 + \lambda \xi \iota_1, \\ S\xi := \left((3\lambda - 1) D_x - \sqrt{3}\mu i D_y + D_x^3 - \sqrt{3} i D_x D_y - \frac{3k^2\mu}{4}\right) \xi \cdot \iota_1. \end{cases} \tag{4.27}$$

The first equation of (4.26) can be written as

$$\partial_x^2 \eta - \mu \partial_x \eta + \frac{i}{\sqrt{3}}(-\partial_y \eta) - \lambda \eta = -P\xi.$$

Inserting this into the second equation, we get

$$-4\partial_x^3 \eta + 6\mu \partial_x^2 \eta + k^2 \partial_x \eta - 3\mu\lambda\eta - \frac{3k^2\mu}{4}\eta + 3\mu P\xi - 3\partial_x(P\xi) + S\xi = 0.$$

The characteristic equation of the ODE

$$-4\partial_x^3 \eta + 6\mu \partial_x^2 \eta + k^2 \partial_x \eta - 3\mu\lambda\eta - \frac{3k^2\mu}{4}\eta = 0 \tag{4.28}$$

has the form

$$-4t^3 - 2\sqrt{3}\sqrt{1-k^2}t^2 + k^2 t + \frac{\sqrt{3}k^2\sqrt{1-k^2}}{2} = 0.$$

Hence the indicial roots of the equation (4.28) are $-\frac{k}{2}, \frac{k}{2}, \frac{3\mu}{2}$.

Lemma 40. *Let $P\xi, S\xi$ be defined in (4.27). If $\xi = e^{k(x-py)}$. Then*

$$\begin{aligned} &3\mu P\xi - 3\partial_x(P\xi) + S\xi \\ &= -\frac{k^2}{8}\left(3ke^{\frac{3}{2}(k(x-py))} + (9k - 96\mu) e^{\frac{1}{2}(k(x-py))}\right). \end{aligned}$$

On the other hand, if $\xi = e^{k(x+py)}$, then

$$\begin{aligned} &3\mu P\xi - 3\partial_x(P\xi) + S\xi \\ &= -\frac{k^2}{8}\left((3k + 96\mu) e^{\frac{k}{2}(3x+py)} + 9ke^{\frac{k}{2}(x+3py)}\right). \end{aligned}$$

Proof. Taking derivatives in x for $P\xi$, we get

$$\begin{aligned}\partial_x(P\xi) &= \iota_1\partial_x^3\xi + (-\partial_x\iota_1 + \mu\iota_1)\partial_x^2\xi + \frac{i}{\sqrt{3}}\iota_1\partial_x\partial_y\xi + \frac{i}{\sqrt{3}}\partial_x\iota_1\partial_y\xi \\ &+ \left(-\partial_x^2\iota_1 - \frac{i}{\sqrt{3}}\partial_y\iota_1 - \lambda\iota_1\right)\partial_x\xi \\ &+ \left(\partial_x^3\iota_1 - \mu\partial_x^2\iota_1 - \frac{i}{\sqrt{3}}\partial_x\partial_y\iota_1 - \lambda\partial_x\iota_1\right)\xi.\end{aligned}$$

Therefore, $-3\mu P\xi + 3\partial_x(P\xi) - S\xi$ is equal to

$$\begin{aligned}&2\iota_1\partial_x^3\xi + \left(2\sqrt{3}i\iota_1\right)\partial_x\partial_y\xi + \left(-6\partial_x^2\iota_1 + 6\mu\partial_x\iota_1 + \lambda\iota_1 - 2\sqrt{3}i\partial_y\iota_1\right)\partial_x\xi \\ &+ \left(4\partial_x^3\iota_1 - 6\mu\partial_x^2\iota_1 - \frac{7k^2}{4}\partial_x\iota_1 + \frac{3k^2}{2}\mu\iota_1\right)\xi.\end{aligned}$$

Direct substitution of explicit formula of ξ into this expression gives us the desired results. □

Theorem 41. *The periodic solution $T_2 := 2\partial_x^2\iota_2$ of the KP-I equation is nondegenerate in the following sense: Suppose φ is smooth and $\frac{2\pi}{kb}$-periodic in the y variable. Assume*

$$\varphi(x,y) \to 0, \ \text{as } |x| \to +\infty,$$

and

$$\partial_x^2\left(\partial_x^2\varphi - \varphi + 6T_2\varphi\right) - \partial_y^2\varphi = 0.$$

Then $\varphi = c_1\partial_x T_2 + c_2\partial_y T_2$.

Proof. Let φ be the function satisfying the assumption of this theorem. Then by Lemma 37, $|\varphi| \leq Ce^{-k|x|}$. Let $K := \iota_2\partial_x^{-2}\frac{\varphi}{2}$. Then K can be written as

$$\mathbf{c}_1\iota_2 + \mathbf{c}_2\iota_2 x + K^*, \tag{4.29}$$

with $|K^*| \leq C$. Here $\mathbf{c}_1, \mathbf{c}_2$ are two constants independent of y.

By Lemma 38 and Lemma 39, we can solve the linearized Bäcklund transformation between $\iota_0, \iota_1, \iota_2$, and get a kernel ξ, given by Lemma 39, for the linearized bilinear KP-I operator at ι_0, with suitable growth and decay estimate. ξ satisfies

$$\partial_x^4\xi - \partial_x^2\xi - \partial_y^2\xi = 0 \text{ in } \Omega_{**}. \tag{4.30}$$

Now we define $\tilde{\xi}(x,y) = e^{-kx}\xi(x,y)$. Then

$$\partial_x^4\left(e^{kx}\tilde{\xi}\right) - \partial_x^2\left(e^{kx}\tilde{\xi}\right) - \partial_y^2\left(e^{kx}\tilde{\xi}\right) = 0 \text{ in } \Omega_{**}.$$

In view of the estimates (4.22), (4.23), we can take Fourier transform for $\tilde{\xi}$ in the x variable and solve the corresponding ODE of y for each frequency. Using the jump condition (4.24) and periodicity in Ω_{**}, we conclude that actually ξ is smooth and $\xi = (c\cos(kby) + d\sin(kby))e^{kx}$ for some constants c, d. To deal with this possibility, we now apply Lemma 40 to infer that the system (4.26) has a solution η with main order of the form

$$\eta(x,y) \sim a(y)e^{\frac{3}{2}kx}, \text{ as } x \to +\infty, \tag{4.31}$$

where $a(y) \neq 0$. Then after we solve the reverse linearized Bäcklund transformation from ι_1 to ι_2, we can see that (4.31) contradicts with the asymptotic behavior (4.29) of the kernel K of the linearized bilinear KP-I at ι_2. This finishes the proof. □

5. Morse index and orbital stability of the lump solution

We observe that for any $k \in \left(0, \frac{1}{2}\right)$, ι_2 is positive. It is also periodic in y, with minimal period $t_k := \frac{2\pi}{k\sqrt{1-k^2}}$. Let us define the region

$$\Omega_k := \left\{ (x, y) \in \mathbb{R}^2 : |y| < \frac{\pi}{k\sqrt{1-k^2}} \right\}.$$

Lemma 42. *As $k \to 0$, there holds $t_k \to +\infty$ and*

$$T_2 \to Q, \text{ in } C^2 (\Omega_k).$$

On the other hand, as $k \to \frac{1}{2}$, we have $t_k \to \frac{8\sqrt{3}\pi}{3}$ and

$$T_2 \to \frac{1}{2} \cosh^{-2} \left(\frac{x}{2}\right), \text{ in } C^2 \left(\mathbb{R}^2\right).$$

Proof. In each fixed bounded domain, when k is small, we can expand $\frac{\iota_2}{2}$ as

$$\begin{aligned} &1 + \frac{k^2 x^2}{2} - \left(1 - \frac{3}{2}k^2\right)\left(1 - \frac{k^2 y^2}{2}\right) + O\left(k^4\right) \\ &= \frac{k^2}{2}\left(x^2 + y^2 + 3\right) + O\left(k^4\right). \end{aligned}$$

Hence as $k \to 0$,

$$T_2 = 2\partial_x^2 \ln \iota_2 \to Q, \text{in } C^2_{loc}\left(\mathbb{R}^2\right). \tag{5.1}$$

Next, we want to show

$$\left|2\partial_x^2 \ln \iota_2 - Q\right| \leq \frac{C}{1 + x^2 + y^2}, \text{ in } \Omega_k, \tag{5.2}$$

where C is a constant independent of k. To do this, it will be sufficient to prove

$$\left|\partial_x^2 \ln \iota_2\right| \leq \frac{C}{1 + x^2 + y^2}, \text{ in } \Omega_k. \tag{5.3}$$

If both $|kx|$ and $|ky|$ are small, say $|kx| < \varepsilon$ and $|ky| < \varepsilon$, where ε is a small constant, then by the previous argument(expanding in terms of kx, ky), $2\partial_x^2 \ln \iota_2$ is close to Q. Hence (5.3) is true in this case. Now suppose $|kx| \geq \varepsilon$, then we compute

$$\begin{aligned} \left|2\partial_x^2 \ln \iota_2\right| &= \left|\frac{2k^2 \left(1 - A\cosh(kx)\cos(kby)\right)}{\left(\cosh(kx) - A\cos(kby)\right)^2}\right| \\ &\leq \frac{2k^2 x^2 \left(\cosh(kx) + 1\right)}{x^2 \left(\cosh(kx) - 1\right)^2} \\ &\leq \frac{C}{x^2}. \end{aligned}$$

On the other hand, if $|ky| > \varepsilon$ and $|kx| \leq \varepsilon$, then

$$\left|2\partial_x^2 \ln \iota_2\right| \leq \frac{k^2 C}{1 - \cos\varepsilon}.$$

Note that in Ω_k, $k\sqrt{1-k^2}\,|y| \leq 2\pi$. Thus $\left|2\partial_x^2 \ln \iota_2\right| \leq \frac{C}{y^2}$.

Now with (5.1) been proved, we can use (5.2) to deduce that

$$T_2 \to Q, \text{ in } C^2(\Omega_k).$$

The proof of convergence of T_2 to $\frac{1}{2}\cosh^{-2}\left(\frac{x}{2}\right)$ as $k \to \frac{1}{2}$ is easier, following from the fact that in this case $A \to 0$. ∎

Let us now consider the linearized operator around the one dimensional solution $\frac{1}{2}\cosh^{-2}\left(\frac{x}{2}\right)$.

Lemma 43. *The operator*

$$\mathbb{L}\eta := -\partial_x^2\eta + \eta - 3\cosh^{-2}\left(\frac{x}{2}\right)\eta + \partial_x^{-2}\partial_y^2\eta$$

has exactly one negative eigenvalue in the space $L^2\left(\mathbb{R}\times\mathbb{R}/\frac{8\sqrt{3}\pi}{3}\mathbb{Z}\right)$. *Moreover, the kernel of* $\mathbb{L}$ *in this space is spanned by* $\mathcal{Z}_i$, $i=0,1,2$, *where*

$$\mathcal{Z}_0 := \tanh\left(\frac{x}{2}\right)\cosh^{-2}\left(\frac{x}{2}\right),$$

$$\mathcal{Z}_1 := \left(\cosh^{-1}\frac{x}{2}\right)''\cos\left(\frac{\sqrt{3}}{4}y\right),$$

$$\mathcal{Z}_2 := \left(\cosh^{-1}\frac{x}{2}\right)''\sin\left(\frac{\sqrt{3}}{4}y\right).$$

Proof. This essentially follows from Lemma 2.3 of [60].

Let λ be a negative eigenvalue of $\mathbb{L}$, with η being an eigenfunction:

$$-\partial_x^2\eta + \eta - 3\cosh^{-2}\left(\frac{x}{2}\right)\eta + \partial_x^{-2}\partial_y^2\eta = \lambda\eta.$$

Since η is $\frac{8\sqrt{3}\pi}{3}$-period in y, we can write η as Fourier series:

$$\eta(x,y) = \sum_n\left(a_n(x)\cos\left(\frac{\sqrt{3}}{4}ny\right) + b_n(x)\sin\left(\frac{\sqrt{3}}{4}ny\right)\right).$$

For the $n=0$ component, the operator reduces to

$$a_0 \to -a_0'' + a_0 - 3\cosh^{-2}\left(\frac{x}{2}\right)a_0.$$

It is well known that this operator has a unique negative eigenvalue in $L^2(\mathbb{R})$. Moreover, 0 is an eigenvalue, with $\tanh\left(\frac{x}{2}\right)\cosh^{-2}\left(\frac{x}{2}\right)$ spanning the corresponding eigenspace. See, for instance, Lemma 2.2 of [43].

For general $n\in\mathbb{N}$, we get an equation of the form

$$B_n a := -a'' + a - 3\cosh^{-2}\left(\frac{x}{2}\right)a - \frac{3}{16}n^2\partial_x^{-2}a = \lambda a, \tag{5.4}$$

B_n can be seen as a self-adjoint operator in the space with norm

$$\sqrt{\int_{\mathbb{R}}(a')^2 + a^2 + \left(\partial_x^{-1}a\right)^2},$$

where in terms of Fourier transform, $\partial_x^{-1}a = \left(-i\xi^{-1}\hat{a}\right)^{\vee}$. In this context, if a is a solution of (5.4), then the antiderivative ∂_x^{-2} actually can be defined by

$$\partial_x^{-2}a = \int_{-\infty}^{x}\int_{-\infty}^{t}a(s)\,dsdt.$$

Note that if a satisfies equation (5.4), then automatically $\partial_x^{-2}a(x)\to 0$, as $x\to+\infty$. $\partial_x^{-2}a$ can also be defined in terms of Fourier transform:

$$\partial_x^{-2}a := \left(-\xi^{-2}\hat{a}(\xi)\right)^{\vee}.$$

For $n=1$, differentiating the family of periodic solution at $k=0$ with respect to k yields an eigenfunction for the eigenvalue $\lambda=0$. Consider the operator

$$\mathbb{T}a := -a^{(4)} + \left(\left(1-3\cosh^{-2}\left(\frac{x}{2}\right)\right)a'\right)' - \frac{3}{16}a.$$

Observe that $\mathbb{T}a = \partial_x B_1 \partial_x a$. From Lemma 2.3 of [60](see equation 2.5 there, see also [5]), we know that the operator $\mathbb{T}$ has no positive eigenvalue. Moreover, the kernel of $\mathbb{T}$ is one dimensional and spanned by the function $\left(\cosh\frac{x}{2}\right)'$. Hence B_1 has no negative eigenvalue and the kernel of B_1 is spanned by the function $\left(\cosh\frac{x}{2}\right)''$. As a consequence, for any $n>1$, the quadratic form associated to B_n is positive definite. This finishes the proof. □

We define the Morse index of the lump Q to be the number of negative eigenvalues(counted with multiplicity) of the operator $\mathcal{L}$:

$$\eta \to -\partial_x^2\eta + \eta - 6Q\eta + \partial_x^{-2}\partial_y^2\eta.$$

We also define the Morse index of the periodic solutions T_2(note that these solutions depend on k) to be the number of negative eigenvalues of the operator $\mathcal{L}_k$:

$$\eta \to -\partial_x^2\eta + \eta - 6T_2\eta + \partial_x^{-2}\partial_y^2\eta,$$

in the space of functions which are t_k-periodic in y. We remark that eigenfunction η corresponding to a native eigenvalue may not satisfy the condition $\int_{-\infty}^{+\infty}\eta = 0$.

Let E be the natural energy space associated to the KP-I lump solution, with the norm $\|\cdot\|$. More precisely, we define

$$E := \left\{ f : \|f\|^2 = \int_{\mathbb{R}^2} \left(|\partial_x f|^2 + f^2 + \left(\partial_x^{-1}\partial_y f\right)^2 \right) < +\infty \right\}.$$

Let E_k be the space obtained from the completion of the space consisting of those smooth solutions which are t_k periodic in y, under the norm

$$\|f\|_{*,k}^2 = \int_{\Omega_k} \left(|\partial_x f|^2 + f^2 + \left(\partial_x^{-1}\partial_y f\right)^2 \right) < +\infty.$$

Proposition 44. *The Morse index of Q is equal to* 1.

Proof. The negative eigenvalues of $\mathcal{L}_k$ have minimax characterization. If we list these negative eigenvalues in increasing order as $\lambda_{k,1}, \lambda_{k,2}, \dots$, then

$$\lambda_{k,i} = \max_{\phi_1,\dots,\phi_{i-1}\in E_k} \min_{f\in E_k} \left\{ \frac{\|f\|_{*,k}^2}{\int_{\Omega_k} f^2} : \int_{\Omega_k} f\phi_j = 0, j \le i-1 \right\}.$$

Observe that T_2 depends continuously on k and decaying exponentially fast to zero as $|x| \to +\infty$. Hence for fixed i, $\lambda_{k,i}$ depends continuously on k.

Now we would like to show that the Morse index of $\mathcal{L}_k$ is equal to one for k sufficient close to $1/2$. To see this, we assume to the contrary that there was a sequence k_n tending to $1/2$ such that the corresponding operator $\mathcal{L}_{k_n}$ has at least two negative eigenvalues $\lambda_{k_n,1}, \lambda_{k_n,2}$. By Lemma 43, the limiting operator $\mathbb{L}$ has exactly one negative eigenvalue. Thus $\lambda_{k_n,2}$ must tend to 0. We use $\eta_{k_n,2}$ to denote the corresponding eigenfunction, with L^∞ norm being equal to $\|\mathcal{Z}_1\|_{L^\infty}$. Note that

$$\int_{\Omega_k} \eta_{k_n,2}\partial_x T_2 = 0, \int_{\Omega_k} \eta_{k_n,2}\partial_y T_2 = 0;$$

while $\partial_x T_2$ tends to $c_0 \mathcal{Z}_0$ and $\frac{1}{\sqrt{\frac{1}{2}-k}} \partial_y T_2$ tends to $c_2 \mathcal{Z}_2$, for some nonzero constants c_0 and c_2. Hence up to a sign, $\eta_{k_n,2}$ converges to $\mathcal{Z}_1$. This argument also proves that there were exactly two negative eigenvalues, $\lambda_{k_n,1}$ and $\lambda_{k_n,2}$. We remark that differentiating T_2 with respect k yields another kernel of $\mathcal{L}_k$, but this kernel is linearly growing in the y direction.

Now by Theorem 1.1 of [66], for k sufficiently close to $\frac{1}{2}$, the Zaitsev soliton T_2 is orbitally stable in the energy space E_k. This contradicts with the instability theorem(See Page 309 of [24]). We then conclude that the Morse index of $\mathcal{L}_k$ is equal to one for k sufficient close to $1/2$.

Now Theorem 41 tells us that T_2 is nondegenerate, in the sense that the kernels of $\mathcal{L}_k$ are generated by space translation in the x and y directions. Therefore, a brach of negative eigenvalues cann't touch the zero eigenvalue and the Morse index of T_2 is invariant with respect to k. It follows that the Morse index of $\mathcal{L}_k$ is equal to one for all $k \in \left(0, \frac{1}{2}\right)$.

To prove the proposition, now we assume to the contrary that there were to negative eigenvalues for $\mathcal{L}$. The negative eigenvalues of the operator $\mathcal{L}$ also have minimax characterization. Lemma 42 tells us that T_2 converges to Q in $C^2(\Omega_k)$ as $k \to 0$. Hence from the decaying property of Q, we infer that L_k has at least two negative eigenvalues for k close to 0. This is a contradiction and hence the Morse index of the lump solution is equal to one. □

Theorem 45. *The lump is orbitally stable in the following sense: For any $\varepsilon > 0$, there exists $\delta > 0$, such that, if $u(t; x, y)$ is solution of KP-I with $\|u(0) - Q\| < \delta, u(0; \cdot) \in E$, then for all $t \in (0, +\infty)$,*

$$\inf_{\gamma_1, \gamma_2 \in \mathbb{R}} \|u(t) - Q(\cdot + \gamma_1, \cdot + \gamma_2)\| < \varepsilon.$$

Proof. With the spectral property of the lump solution understood, its orbital stability essentially follows from a result of Grillakis-Shatah-Strauss [23](See section 5 of [10], also see [60] for the case of line solitons). We briefly sketch the proof below.

Let u_c be the family of lumps with speed c. Define the function

$$d(c) := \int_{\mathbb{R}^2} \left(\frac{1}{2} (\partial_x u_c)^2 - u_c^3 + \frac{1}{2} \left(\partial_y \partial_x^{-1} u_c\right)^2 + \frac{1}{2} c u_c^2 \right).$$

Then $d'(c) = \frac{1}{2}\sqrt{c} \int_{\mathbb{R}^2} u_1^2$. Hence

$$d''(c) > 0. \tag{5.5}$$

Now using (5.5) and arguing similarly as Lemma 5.1 of [10], we can prove that if ρ is a function orthogonal in L^2 to Q and $\partial_x Q, \partial_y Q$, then,

$$\int_{\mathbb{R}^2} \rho(\mathcal{L}\rho) \geq c \|\rho\|^2,$$

for some constant $c > 0$. We define

$$Q_{a,b} := Q(x - a, y - b).$$

Let $u(t; x, y)$ be a solution of the KP-I flow, which is initially close to the lump:

$$\|u(0) - Q_{0,0}\| < \delta.$$

Without loss of generality, we assume $\|u(0)\|_{L^2} = \|Q_{0,0}\|_{L^2}$(In the general case, note that there is a family travelling wave lump type solutions u_s with speed s, defined in the first section. Hence there exists a c^* close to 1 such that $\|u(0)\|_{L^2} =$

$\|u_{c^*}\|_{L^2}$. Then one can follow the same arguments below to show that $u(t)$ is orbitally close to u_{c^*}. In view of the fact $\|u_{c^*}\|$ is close to $\|Q\|$, the result of the theorem then follows). The existence of this solution is guaranteed by the result of [29]. For t sufficiently small, we can find γ_1, γ_2, depending on t, such that the function

$$w(t) := u(t) - Q_{\gamma_1,\gamma_2}$$

satisfies the orthogonality condition

$$\int_{\mathbb{R}^2} (w\partial_x Q_{\gamma_1,\gamma_2}) = \int_{\mathbb{R}^2} (w\partial_y Q_{\gamma_1,\gamma_2}) = 0. \tag{5.6}$$

On the other hand, w can be written as

$$w = \alpha Q_{\gamma_1,\gamma_2} + w_1,$$

for some small constant $\alpha(t)$ and a function w_1 satisfying $\int_{\mathbb{R}^2} (w_1 Q_{\gamma_1,\gamma_2}) = 0$. Note that by (5.6),

$$\int_{\mathbb{R}^2} (w_1\partial_x Q_{\gamma_1,\gamma_2}) = \int_{\mathbb{R}^2} (w_1\partial_y Q_{\gamma_1,\gamma_2}) = 0.$$

Moreover, by the conservation of L^2 norm, we have

$$\int_{\mathbb{R}^2} ((1+\alpha) Q_{\gamma_1,\gamma_2} + w_1)^2 = \int_{\mathbb{R}^2} Q_{0,0}^2.$$

It follows that

$$\left(2\alpha + \alpha^2\right) \int_{\mathbb{R}^2} Q_{0,0}^2 = -\int_{\mathbb{R}^2} w_1^2. \tag{5.7}$$

On the other hand, the Hamiltonian energy

$$H(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}(\partial_x u)^2 - u^3 + \frac{1}{2}\left(\partial_y \partial_x^{-1} u\right)^2 + \frac{1}{2}u^2\right)$$

is conserved by the KP-I flow. Hence for any t, $H(u) = H(u(0))$. Note that

$$H(u) = H(Q_{\gamma_1,\gamma_2}) + \int_{\mathbb{R}^2} w(\mathcal{L}w) - \int_{\mathbb{R}^2} w^3.$$

Anisotropic Sobolev inequality(see for instance equation (2.9) in [60]) tells us that

$$\int_{\mathbb{R}^2} w^3 \leq C\|w\|^3.$$

Applying the spectral property of $\mathcal{L}$ and the estimate (5.7) of α, we deduce

$$H(u) \geq H(Q_{\gamma_1,\gamma_2}) + C\|w\|^2.$$

In view of the conservation of Hamiltonian H along the KP-I flow, we get

$$\|w\|^2 \leq C(H(u(0)) - H(Q_{0,0})).$$

This completes the proof. □

Remark 46. *Now with the spectral property of the periodic solutions T_2 understood, it is also possible to get their orbitally stability in the class of $\frac{2\pi}{kb}$-periodic solutions(The case of line soliton and Zaitsev solitons near the line soliton is discussed in [58–60, 66]). We will not address this issue here.*

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