FINITE-TIME SINGULARITY FORMATION FOR THE HEAT FLOW OF THE H-SYSTEM

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Abstract. We construct the first example of finite time blow-up solutions for the heat flow of the H-system, describing the evolution of surfaces with constant mean curvature

$$\begin{cases} u_t = \Delta u - 2u_{x_1} \wedge u_{x_2} & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2, \end{cases}$$

where u: $\mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^3$. The singularity at finite time forms as a scaled least energy H-bubble, denoted as W, exhibiting type II blow-up speed. One key observation is that the linearized operators around W projected onto $W^\perp$ and in the W-direction are in fact decoupled. On $W^\perp$, the linearization is the linearized harmonic map heat flow, while in the W-direction, it is the linearized Liouville-type flow. Based on this, we also prove the non-degeneracy of the H-bubbles with any degree.

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1. Introduction and main results

A classical problem in geometric analysis is the following *Plateau* problem: for a given curve Γ , find a surface with boundary Γ, with mean curvature $H(x)$ for a point x on the surface, where H is some (smooth) function. In the case of the ambient space $\mathbb{R}^3$, a parametric surface with prescribed mean curvature, satisfies the following equation, also known as the H-surface system

$$\Delta u - 2H(u)u_{x_1} \wedge u_{x_2} = 0 \ \text{ on } D, \tag{1.1}$$

where u: $D \subset \mathbb{R}^2 \to \mathbb{R}^3$ with D being the unit disk, H is a given scalar, "$\wedge$" denotes the wedge product, and, for instance, $u_{x_1} = \frac{\partial}{\partial x_1}u$. The geometric significance of system (1.1) is that conformal solutions u, i.e. solutions satisfying additionally,

$$|u_{x_1}|^2 - |u_{x_2}|^2 = u_{x_1} \cdot u_{x_2} = 0 \ \text{ on } D,$$

parameterize immersed 2D disk-type surfaces of prescribed mean curvature H. Solutions of (1.1) may arise as "soap bubbles", that is, surfaces of least area enclosing a given volume. Concerning the existence and optimal estimates, the Dirichlet problem of the H-system was studied intensively in many seminal works, such as Heinz [26], Hildebrandt [28, 29], Gulliver-Spruck [24, 25], Steffen [46, 47] and Wente [56]. Struwe [50] considered the Plateau problem of the H-system and proved its existence; see also Duzaar-Steffen [22]. For more geometric motivations and backgrounds, we refer to the comprehensive monographs of Struwe [51, 52], Duzaar-Steffen [21], Steffen [49], Bethuel-Caldiroli-Guida [2] and their references.

For the Dirichlet problem in a smooth bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} \Delta u = H(x, u, \nabla u)u_{x_1} \wedge u_{x_2} & \text{in } \ \Omega, \\ u(\cdot) = \tilde{g} & \text{on } \ \partial\Omega \end{cases} \tag{1.2}$$

with the scalar H being smooth, it was proved by Wente [57] and Chanillo-Malchiodi [13] that there is no nonzero solution in simply connected domain when $\tilde{g} = 0$. For $H(x, u, \nabla u) = 2$ and $\|\tilde{g}\|_{L^\infty} < 1$, a solution with minimal energy was constructed by Hildebrandt in [29], while in Brezis-Coron [6], Steffen [48], Struwe [51], the authors considered large energy solutions, and it was proved by Heinz [27] that the condition $\|\tilde{g}\|_{L^\infty} < 1$ is sharp. For general $H(x, u, \nabla u)$, the existence of solutions was proved in a series of works by Caldiroli-Musina [7, 8, 9, 39] via the variational perturbative method introduced by Ambrosetti and Badiale in [1]. Furthermore, bubbling and multi-bubble solutions have been constructed in Caldiroli-Musina [10] and Chanillo-Malchiodi [13]. Regularity of weak solutions was studied by Musina [40]. The asymptotic behavior of the solutions for (1.2) was studied, for instance, in [11, 12, 31, 32, 33, 44] and the references therein.

Rivière proved in the important work [42] that two-dimensional conformally invariant nonlinear elliptic PDEs, including the prescribed mean curvature equation and harmonic map equation, can be written in terms of suitable conservation laws. Based on this special compensated-compactness structure, he proved that the solutions of the prescribed bounded mean curvature equation and the harmonic map equation in any manifolds are continuous, and that critical points of two-dimensional continuously differentiable conformally invariant elliptic Lagrangians of the form

$$E[u] = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx_1 \wedge dx_2 + \int_{\mathbb{R}^2} \omega(u)(u_{x_1}, u_{x_2})dx_1 \wedge \ dx_2 \tag{1.3}$$

are continuous. Here ω is a C^1 differential two-form on a C^2-submanifolds N^k of $\mathbb{R}^m$, and k and m are arbitrary integers satisfying $1 \le k \le m$.

For the H-system in $\mathbb{R}^2$ with constant mean curvature $H = 1$, finite-energy entire solutions were classified in the classical paper [5] by Brezis-Coron as

$$u(z) = \Pi\left(\frac{P(z)}{Q(z)}\right) + C, \quad z = (x, y) = x + iy$$

where $\Pi : \mathbb{C} \to \mathbb{S}^2$ is the inverse stereographic projection defined by

$$\Pi(z) = \frac{1}{1 + |z|^2}\begin{bmatrix} 2z \\ |z|^2 - 1 \end{bmatrix},$$

P, Q are polynomials and C is a constant vector in $\mathbb{R}^3$. From this result we know that a typical class of solutions to

$$\Delta u - 2u_{x_1} \wedge u_{x_2} = 0 \ \text{ in } \ \mathbb{R}^2 \tag{1.4}$$

are $W^{(m)}(z) = \Pi(z^m)$ for $m \in \mathbb{Z}^+$, where m is the degree of the map. Here we consider the so called non-degenerate property of *the degree m bubble* $W^{(m)}$, which concerns the bounded kernel functions of the linearized equation around $W^{(m)}$ as follows

$$\Delta\phi = 2\left(W^{(m)}_{x_1} \wedge \phi_{x_2} + \phi_{x_1} \wedge W^{(m)}_{x_2}\right). \tag{1.5}$$

The non-degeneracy plays an important role in the construction of H-bubbles; see Caldiroli-Musina [10] for instance. In the polar coordinates, $W^{(m)}$ can be written as

$$W^{(m)}(x) = W^{(m)}(r,\theta) = \begin{bmatrix} e^{im\theta}\sin(w_m) \\ \cos(w_m) \end{bmatrix} = \begin{bmatrix} \frac{2r^m\cos(m\theta)}{r^{2m}+1} \\ \frac{2r^m\sin(m\theta)}{r^{2m}+1} \\ \frac{r^{2m}-1}{r^{2m}+1} \end{bmatrix}, \quad x = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2, \quad m \in \mathbb{Z}_+,$$

and

$$w_m = \pi - 2\arctan(r^m).$$

The linearized equation (1.5) then becomes

$$\Delta\phi = \frac{2}{r}\left(W^{(m)}_r \wedge \phi_\theta + \phi_r \wedge W^{(m)}_\theta\right) \tag{1.6}$$

for $\phi = \phi(r,\theta)$. For $m = \pm 1$, Chanillo and Malchiodi [13] proved the non-degeneracy result; see also [10, 32, 33, 39, 44] and the references therein. Chanillo and Malchiodi further conjectured such non-degeneracy holds true for *the degree m bubble* ($|m| \geq 2$). Our first result confirms this:

Theorem 1.1 (Non-degeneracy of H-bubbles)**.** *The solution to* (1.4)

$$W^{(m)}(x) = W^{(m)}(r,\theta) = \begin{bmatrix} \frac{2r^m\cos(m\theta)}{r^{2m}+1} \\ \frac{2r^m\sin(m\theta)}{r^{2m}+1} \\ \frac{r^{2m}-1}{r^{2m}+1} \end{bmatrix}, \quad x = (r\cos\theta,\ r\sin\theta) \in \mathbb{R}^2, \quad m \in \mathbb{Z}^+$$

is non-degenerate in the sense that all bounded solutions of the linearized equation (1.6) are linear combinations of $4m+5$ functions defined as follows,

$$\begin{aligned}
&\frac{r^{2m}-1}{r^{2m}+1}W^{(m)}, \quad \frac{r^m}{1+r^{2m}}\cos(m\theta)W^{(m)}, \quad \frac{r^m}{(1+r^{2m})}\sin(m\theta)W^{(m)},\\
&\frac{r^{m-k}}{1+r^{2m}}\left(\cos(k\theta)E^{(m)}_1 + \sin(k\theta)E^{(m)}_2\right), \quad \frac{r^{m-k}}{1+r^{2m}}\left(\sin(k\theta)E^{(m)}_1 - \cos(k\theta)E^{(m)}_2\right),\\
&\frac{r^{m+l}}{1+r^{2m}}\left(\cos(l\theta)E^{(m)}_1 - \sin(l\theta)E^{(m)}_2\right), \quad \frac{r^{m+l}}{1+r^{2m}}\left(\sin(l\theta)E^{(m)}_1 + \cos(l\theta)E^{(m)}_2\right)
\end{aligned} \tag{1.7}$$

for $k = 0, 1, \cdots, m$ and $l = 1, \cdots, m$. Here

$$E^{(m)}_1 = E^{(m)}_1(r,\theta) = \begin{bmatrix} \frac{r^{2m}-1}{r^{2m}+1}\cos(m\theta) \\ \frac{r^{2m}-1}{r^{2m}+1}\sin(m\theta) \\ \frac{-2r^m}{r^{2m}+1} \end{bmatrix}, \quad E^{(m)}_2 = E^{(m)}_2(r,\theta) = \begin{bmatrix} -\sin(m\theta) \\ \cos(m\theta) \\ 0 \end{bmatrix}.$$

Remark 1.1.1. *For $m \leq -1$, $W^{(m)}(z) = \Pi'\left(\bar{z}^{|m|}\right)$ with*

$$\Pi'(z) = \frac{1}{1+|z|^2}\begin{bmatrix} 2z \\ 1-|z|^2 \end{bmatrix},$$

and the same non-degeneracy result holds with m replaced by $|m|$.

Note that the method in [13] for the case $m = \pm 1$ depends on the spectrum of $\Delta_{\mathbb{S}^2}$ and the spherical decomposition on $L^2(\mathbb{S}^2)$. In [39], Musina further studied the role of the spectrum of the Laplace operator on $\mathbb{S}^2$ in the H-bubble problem. Here our proof of Theorem 1.1 is based on a decoupling property of the linearized operator and an ODE argument.

A key observation we make is that the linearized equations on $[W^{(m)}]^\perp$ and in the $W^{(m)}$-direction are decoupled, see the decompositions in Appendix 3 and Lemma 2.1 (for $m = 1$). On $[W^{(m)}]^\perp$, the linearized problem can be solved in a similar manner as the one for harmonic maps, see [14, Corollary 1]. Indeed, the kernel functions (1.7) in Theorem 1.1 are the same as those for the linearized harmonic map equation except for the first three functions parallel to $W^{(m)}$ at each point. Due to the critical growth in the nonlinearity, the H-system (1.4) shares some similarities with the harmonic map equation with target manifold being $\mathbb{S}^2$

$$\Delta u + |\nabla u|^2 u = 0 \ \text{ in } \ \mathbb{R}^2.$$

In fact, (1.4) is the Euler-Lagrange equation of the energy functional

$$E_H[u] = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \frac{2}{3}\int_{\mathbb{R}^2} u \cdot (u_{x_1} \wedge u_{x_2}) := E_D[u] + E_V[u], \tag{1.8}$$

that is scaling invariant. Here, $E_D[u]$ is the usual Dirichlet energy that appears in the context of harmonic maps, and $E_V[u]$ may be referred to as the H-volume functional of the surface parametrized by u.

Non-degeneracy of harmonic maps proved in [14] is a rather important property in the study of singularity formation for the harmonic map heat flow and the quantitative stability of harmonic maps, see [17, 18] and the references therein. In a related context, we refer the interested readers to [19, 38, 45] for the non-degeneracy of half harmonic maps. Inspired by the works [14, 18] on harmonic maps, it might be natural to expect the non-degeneracy of general bubble $u(z) = \Pi\left(\frac{P(z)}{Q(z)}\right) + C$ and the locally quantitative stability; see also [43].

The previous discussion, though important for our purposes, concerns the stationnary problem. In this paper, we focus on the geometric flow that describes the evolution of parametric surfaces with constant mean curvature. It is the associated heat flow of the H-system

$$\begin{cases} u_t = \Delta u - 2u_{x_1} \wedge u_{x_2} & \text{in } \ \mathbb{R}^2 \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0 & \text{in } \ \mathbb{R}^2, \end{cases} \tag{1.9}$$

where $u(x,t) = u(x_1, x_2, t) : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^3$, and $u_0 : \mathbb{R}^2 \to \mathbb{R}^3$ is a given smooth map. The system (1.9) is the negative L^2-gradient flow of the energy (1.8), and its stationary equation, namely H-system, is the equation satisfied by surfaces of mean curvature $H = 1$ in conformal representation. Global existence and regularity for weak solutions of the initial-boundary value problem to the heat flow of the H-system were established by Rey in [41]. Partial regularity of weak solution was studied in the works of Wang [53, 54]. Existence and uniqueness of short time regular solution were proved by Chen-Levine [15], where they also analyzed the bubbling phenomenon at the first singular time. In [3, 4], Bögelein-Duzaar-Scheven showed short-time regularity, and the global existence of weak solution which is regular except from finitely many singular times; see Duzaar-Scheven [20] for the global existence for a Plateau problem.

We are interested in the formation of singularity at finite time for the heat flow (1.9). Even though it is known that the heat flow for $H-$system shares many similarities with harmonic map flows (and there are now plenty examples of finite time singularities for harmonic map flows, e.g. [17] and references therein), there is no example of solutions to (1.9) which exhibit singularity formation at finite time, even in the equivariant case. See [30] for related existence results as well as blow-up and regularity criteria for (1.9) with Dirichlet boundary. The aim of this paper is to investigate the blow-up mechanism of the system (1.9) and its precise asymptotics, and we will prove the first instance of finite time blowup for (1.9). Though the elliptic theory for such systems share many similarities with the one for harmonic maps, the geometric flow exhibits behaviour which are surprising and make the analysis substantially harder. As alluded before the splitting of the linearization between directions behaving like the linearization of the harmonic map heat flow around a bubble and the one sharing strong similarities with a Liouville flow (i.e. an exponential nonlinearity) introduces major adjustments to the general strategy due to interaction issues. This behaviour at the linear level, though critical, is nevertheless decoupled which allows to close the argument.

The building block of our construction is the following least energy entire, nontrivial H-bubble ($m = 1$) given by

$$W(x) = \frac{1}{1+|x|^2}\begin{bmatrix} 2x \\ |x|^2 - 1 \end{bmatrix}, \quad x \in \mathbb{R}^2, \tag{1.10}$$

where we write $W = W^{(1)}$, $E_1 = E_1^{(1)}$ and $E_2 = E_2^{(1)}$ for simplicity. W satisfies

$$\int_{\mathbb{R}^2} |\nabla W|^2 = 8\pi, \quad W(\infty) = (0,0,1)^{\mathsf{T}}.$$

Note that $W(x)$ can be expressed as the 1-equivariant form

$$W(x) = \begin{bmatrix} e^{i\theta}\sin w(r) \\ \cos w(r) \end{bmatrix}, \ w(r) = \pi - 2\arctan(r), \ x = re^{i\theta}. \tag{1.11}$$

Define γ-rotation matrix around z-axis as

$$Q_\gamma := \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.12}$$

Our main result is stated as follows.

Theorem 1.2 (Type II finite-time blow-up)**.** *For any point $q \in \mathbb{R}^2$ and sufficiently small $T > 0$, there exists u_0 such that $\nabla_x u(x,t)$ with $u(x,t)$ solving (1.9) blows up at q as $t \to T$. More precisely, there exist $\kappa \in \mathbb{R}_+$, $\gamma_* \in \mathbb{R}$, and a map $u_* \in H^1_{\rm loc}(\mathbb{R}^2;\mathbb{R}^3) \cap L^\infty(\mathbb{R}^2;\mathbb{R}^3)$ such that*

$$u(x,t) - u_*(x) - Q_{\gamma_*}\left[W\left(\frac{x-\xi(t)}{\lambda(t)}\right) - W(\infty)\right] \to 0 \quad as \quad t \to T$$

in $H^1_{\rm loc}(\mathbb{R}^2;\mathbb{R}^3) \cap L^\infty(\mathbb{R}^2;\mathbb{R}^3)$ with

$$\lambda(t) = \kappa \frac{T-t}{|\log(T-t)|^2}(1+o(1)), \quad \xi(t) = q + o(1),$$

where $o(1) \to 0$ as $t \to T$. In particular, we have

$$|\nabla u(\cdot,t)|^2 dx \overset{*}{\rightharpoonup} |\nabla u_*|^2 dx + 8\pi\delta_q \quad as \quad t \to T$$

in the sense of Radon measures.

Remark 1.2.1.

- *The fine asymptotics of u is obtained in the construction:*

$$u(x,t) = Q_{\gamma(t)} W\left(\frac{x-\xi(t)}{\lambda(t)}\right) + \Phi_{\rm per},$$

  where the perturbation $\Phi_{\rm per}$ is small in the sense that $|\Phi_{\rm per}| \lesssim T^{\epsilon_1}$, $|\nabla_x \Phi_{\rm per}| \lesssim \lambda^{\epsilon_2}(t)$ for some $\epsilon_1 > 0$ and $-1 < \epsilon_2 < 0$. More precise asymptotic behavior can be found in Section 8.
- *The same construction works for the Cauchy-Dirichlet problem in a smooth bounded domain $\Omega \subset \mathbb{R}^2$:*

$$\begin{cases} u_t = \Delta u - 2u_{x_1} \wedge u_{x_2} & \text{in } \Omega \times \mathbb{R}_+, \\ u = (0,0,1)^\intercal & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(\cdot,0) = u_0 & \text{in } \Omega \end{cases}$$

  for given smooth map u_0 with $u_0|_{\partial\Omega} = (0,0,1)^\intercal$. It is worth mentioning that with such Dirichlet boundary, the solutions to (1.9) *are in fact different from those discussed in [30], where zero boundary condition was imposed.*
- *Motivated by [17] concerning the blow-up for the harmonic map heat flow, it is reasonable to expect that the finite time singularities might also happen in the gradient flow for the general energy functional* (1.3)*, and different blow-up mechanism may arise depending on the 2-form ω presumably.*
- *Bubbling solutions taking higher degree profile might be possible by virtue of the non-degeneracy in Theorem 1.1. However, the construction is more involved due to the presence of more bounded kernel functions for the linearized operator.*

The general framework of the construction is based on the inner-outer gluing method, first developed by Cortázar-del Pino-Musso [16] and Dávila-del Pino-Wei [17], and it is a versatile approach designed for the singularity formation for evolution PDEs.

The heat flow of the H-system is intimately related to the harmonic map heat flow

$$u_t = \Delta u + |\nabla u|^2 u \ \text{ in } \ \mathbb{R}^2 \times \mathbb{R}_+$$

due to their similar quadratic nonlinearities, and our construction is primarily inspired by Dávila-del Pino-Wei [17] concerning the blow-up for the latter geometric flow. Despite similar criticality and structure shared with harmonic map heat flow, the blow-up mechanism of (1.9) turns out to be more delicate due to the system in the W-direction. In fact, as a consequence of Lemma 2.1, the elliptic linearization decouples as the linearized harmonic map equation on $W^\perp$ and the linearized Liouville equation in the W-direction. Compared to the case of the harmonic map heat flow, the treatment required in the W-direction for (1.9) results in several technical novelties and a somewhat more unstable blow-up beyond the equivariant symmetry class. We shall describe these in what follows.

We now describe informally our construction. Recall we are interested in constructing solutions without symmetry assumptions. The major difficulties in the construction arise from the non-radial symmetry and the non-local/global effects, triggered by the slow spatial decay, in *both* W-direction and on $W^\perp$. Indeed, the ansatz for the desired blow-up solution with equivariant-type symmetry (1.13) turns out to be relatively simpler without the introduction of extra modulation parameters c_1 and c_2 corresponding to the parabolic linearized Liouville equation in the W-direction. Due to the non-radial symmetry, the ansatz has to be modified to further improve the slowly decaying errors in non-radial modes, yielding an extra non-local system governing the modes ± 1 parameters c_1 and c_2 in the W-direction (cf. the c_1-c_2 system in Section 9). Moreover, the c_1-c_2 system is in fact coupled with another λ-γ system, governing the scaling parameter λ and the rotational parameter γ on $W^\perp$ in a non-local way, and this is a consequence of slowly decaying error in mode 0 on $W^\perp$. The λ-γ system was first observed and derived in the context of the harmonic map heat flow [17]. For (1.9), the non-locality of the c_1-c_2 system and its coupling with the λ-γ system seem to be a new feature beyond the symmetry class.

Another aspect is the use of the distorted Fourier transform for the spectral analysis of mode 0 in the W-direction (i.e., the linearized Liouville equation, cf. Appendix B). The corresponding kernel is of order 1 at space infinity which makes the analysis more subtle, and the techniques for linear theories in all the other modes seem not to be sufficient to ensure a solution with sufficient decay. Instead, we use the techniques of the distorted Fourier transform for this specific mode and carry out a spectral analysis for the associated half-line Schrödinger operator, see Appendix B. The motivation is from a series important works of Krieger-Miao-Schlag-Tataru [34, 35, 36, 37] in the hyperbolic settings and from a recent one [55] in a dissipative-dispersive setting. These techniques might be of use in the study of semilinear elliptic and parabolic equations with exponential-type nonlinearity.

A roadmap to the construction. We elaborate in a bit more detailed fashion below. The first step of our proof is to find a good approximate solution. Let us begin with a simple motivational ansatz to solution u of (1.9), taking the co-rotational form similar to (1.11):

$$u(x,t) = \psi(r,t)\begin{bmatrix} e^{i\theta}\sin\varphi(r,t) \\ \cos\varphi(r,t)\end{bmatrix}. \tag{1.13}$$

Then system (1.9) becomes the following system of 1D evolution equations

$$\begin{cases} \varphi_t = \varphi_{rr} + \dfrac{\varphi_r}{r} + \dfrac{(-\sin\varphi + 2r\varphi_r)\psi_r}{r\psi} - \dfrac{\sin(2\varphi)}{2r^2}, \\ \psi_t = \psi_{rr} + \dfrac{\psi_r}{r} - \dfrac{2\psi^2\varphi_r\sin\varphi}{r} + \dfrac{-1+\cos(2\varphi) - 2r^2\varphi_r^2}{2r^2}\psi. \end{cases} \tag{1.14}$$

One can check directly that $(\pi - 2\arctan(r), 1)$ is a stationary solution. Then a natural approximation solution is

$$\varphi_0(r,t) = \pi - 2\arctan\left(\frac{r}{\lambda(t)}\right) \text{ and } \psi_0(r,t) = 1.$$

The linearized system of (1.14) around $(\varphi_0,\ \psi_0)$ is given by

$$\begin{cases} (\phi_1)_t = (\phi_1)_{rr} + \dfrac{(\phi_1)_r}{r} - \dfrac{r^4 - 6r^2\lambda^2 + \lambda^4}{r^2(r^2+\lambda^2)^2}\phi_1 - \dfrac{6\lambda}{r^2+\lambda^2}(\phi_2)_r, \\ (\phi_2)_t = (\phi_2)_{rr} + \dfrac{(\phi_2)_r}{r} + \dfrac{8\lambda^2}{(r^2+\lambda^2)^2}\phi_2 \end{cases}$$

for perturbations ϕ_1 and ϕ_2 of φ_0 and ψ_0, respectively. Note the second equation is the linearization for the 2D Liouville equation

$$\Delta u + e^u = 0$$

around the bubble $\log\frac{8\lambda^2}{(r^2+\lambda^2)^2}$.

Let us emphasize that the ansatz in the motivational example above enjoys the equivariant symmetry, but beyond such class, the ansatz has to be modified significantly due to the non-radial modes. Inspired by the simple ansatz (1.13) and the non-degeneracy in Theorem 1.1, we will find a suitable profile that approximates well the real solution to (1.9) and then further improve it by adding several corrections. First we define the error of u as

$$S[u] := -u_t + \Delta u - 2u_{x_1} \wedge u_{x_2}.$$

We take the approximate solution as

$$\begin{aligned} U_{\lambda,\gamma,\xi,c_1,c_2} &:= Q_{\gamma(t)}\left[W\left(\frac{x-\xi(t)}{\lambda(t)}\right) + c_1(t)\frac{2\rho}{\rho^2+1}\cos\theta W\left(\frac{x-\xi(t)}{\lambda(t)}\right) + c_2(t)\frac{2\rho}{\rho^2+1}\sin\theta W\left(\frac{x-\xi(t)}{\lambda(t)}\right)\right] \\ &:= Q_\gamma(W + c_1\mathcal{Z}_{1,1} + c_2\mathcal{Z}_{1,2}), \end{aligned}$$

where Q_γ is given in (1.12), and the modulation parameters $\lambda(t),\ \xi(t),\ \gamma(t),\ c_1(t),\ c_2(t)$ are to be adjusted. The most notable difference in the ansatz compared to the radial case is the introduction of the new parameters $c_1,\ c_2$, which in fact correspond to the modes ± 1 in the linearized Liouville problem in the $Q_\gamma W$-direction. Indeed, $\mathcal{Z}_{1,1}$ and $\mathcal{Z}_{1,2}$ are the corresponding kernels given in (2.2).

As we will see in Section 4, the non-radial ansatz $U_{\lambda,\gamma,\xi,c_1,c_2}$ generates slowly decaying errors in both W-direction and on $W^\perp$, and thus needs to be improved by adding several non-local corrections. We choose the corrected approximate solution as

$$U_* := U_{\lambda,\gamma,\xi,c_1,c_2} + \eta_1\left(\Phi^{(0)} + \Phi^{(1)} + \Phi_{U^\perp}^{(2)} + \Phi_{U^\perp}^{(-2)} + \Phi_U^{(2)}\right),$$

where the purpose of the cut-off function $\eta_1 := \eta(x-\xi(t))$ is to avoid potential slow spatial decay in the remote region, and the five corrections all have their own role. Here the corrections $\Phi^{(0)}$ and $\Phi^{(1)}$, in non-local form, are to improve slow spatial decay in mode 0 on $[Q_\gamma W]^\perp$ and in modes ± 1 in the $Q_\gamma W$-direction. The other three corrections $\Phi_{U^\perp}^{(\pm 2)}$, $\Phi_U^{(2)}$ are to improve slow time decay in modes ± 2 on $[Q_\gamma W]^\perp$ and in mode 2 in the $Q_\gamma W$-direction, respectively. Such bulky ansatz is necessary when designing weighted spaces for the gluing process, especially in the choice of constants measuring these spaces that the desired solutions reside in (see (8.29) in Section 8). In other words, without these corrections, weighted spaces for the perturbation cannot be found in the current set-up. This is one of the most important parts in the construction, and we do not know if there are simpler ansatzes for the approximate solution.

We next look for a solution with the following form

$$u = U_* + \Phi,$$

so $S[u] = 0$ yields

$$\partial_t\Phi = \Delta\Phi - 2\partial_{x_1}U_* \wedge \partial_{x_2}\Phi - 2\partial_{x_1}\Phi \wedge \partial_{x_2}U_* - 2\partial_{x_1}\Phi \wedge \partial_{x_2}\Phi + S[U_*].$$

To implement the inner-outer gluing procedure, we decompose Φ into

$$\Phi(x,t) = \eta_R Q_\gamma\Big[\Phi_W(y,t) + \Phi_{W^\perp}(y,t)\Big] + \Psi(x,t), \quad y = \frac{x-\xi(t)}{\lambda(t)},$$

where Φ_W is in the W-direction, $\Phi_{W^\perp}$ is on $W^\perp$, $\eta_R := \eta\left(\frac{x-\xi(t)}{R(t)\lambda(t)}\right)$, and $R(t) = \lambda^{-\beta}(t)$ with $\beta > 0$ to be chosen later. Then it is sufficient to find a desired solution u to (1.9) if the triple $(\Phi_W,\ \Phi_{W^\perp},\ \Psi)$ satisfies the coupled gluing system:

$$\lambda^2\partial_t\Phi_W = \Delta_y\Phi_W - 2\partial_{y_1}W \wedge \partial_{y_2}\Phi_W - 2\partial_{y_1}\Phi_W \wedge \partial_{y_2}W + \mathrm{RHS}_{\mathrm{in},W} \quad \text{in } \mathcal{D}_{2R}, \tag{1.15}$$

$$\lambda^2\partial_t\Phi_{W^\perp} = \Delta_y\Phi_{W^\perp} - 2\partial_{y_1}W \wedge \partial_{y_2}\Phi_{W^\perp} - 2\partial_{y_1}\Phi_{W^\perp} \wedge \partial_{y_2}W + \mathrm{RHS}_{\mathrm{in},W^\perp} \quad \text{in } \mathcal{D}_{2R}, \tag{1.16}$$

$$\partial_t\Psi = \Delta_x\Psi + \mathrm{RHS}_{\mathrm{out}} \text{ in } \mathbb{R}^2 \times (0,T), \tag{1.17}$$

i.e., Φ_W and $\Phi_{W^\perp}$ satisfy inner problem in the W-direction and on $W^\perp$, respectively, and Ψ solves the outer problem. The detailed form of the right hand sides $\mathrm{RHS}_{\mathrm{in},W}$, $\mathrm{RHS}_{\mathrm{in},W^\perp}$ and $\mathrm{RHS}_{\mathrm{out}}$ with couplings will be given in Section 5.

To attack the inner-outer gluing system (1.15)-(1.17), a key property derived in Lemma 2.1 is that the equation (1.16) for $\Phi_{W^\perp}$ is essentially a perturbation of the linearized harmonic map heat flow around the bubble W, while the equation (1.15) for Φ_W is a perturbation of the linearized Liouville-type flow, i.e., linearization of

$$u_t = \Delta u + e^u \text{ in } \mathbb{R}^2 \times \mathbb{R}_+$$

around the canonical bubble $\log \frac{8\lambda^{-2}}{(1+|y|^2)^2}$, and these two linearized operators are in fact decoupled. This remarkable structure allows us to develop the linear theories separately in the W-direction and on $W^\perp$, and the full nonlinear systems for $\Phi_{W^\perp}$ and Φ_W are weakly coupled provided their weighted spaces are properly designed. We will give the full linear theory in Section 7, and the gluing system will be solved in Section 8.

As in the ansatz $U_{\lambda,\gamma,\xi,c_1,c_2}$, we introduce two modulation parameters c_1 and c_2 that correspond to the slowly decaying kernels $\mathcal{Z}_{1,1}$, $\mathcal{Z}_{1,2}$ (Fourier modes ± 1) for the linearized Liouville operator. The non-local corrections $\Phi^{(0)}$ and $\Phi^{(1)}$, taking care of the slowly decaying errors respectively on $[Q_\gamma W]^\perp$ and in the $Q_\gamma W$-direction, in turn yield two non-local systems that govern the blow-up dynamics of λ-γ and c_1-c_2 through the orthogonality conditions at corresponding modes. For the λ-γ system, the influence of c_1, c_2 turns out to be a perturbation. However, in the c_1-c_2 system, the coupling from λ and γ in fact serves as one of the leading parts, and these parameters obey

$$\begin{gathered}\int_{-T}^{t-\lambda^2(t)} \frac{p_1(s)}{t-s} ds + \frac{2}{3}\left(\int_{-T}^{t-\lambda^2(t)} \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s} ds\right) \mathbf{c} = f(t), \\ p_1(t) = -2(\lambda \mathbf{c})', \quad \mathbf{c}(t) = c_1(t) + i c_2(t), \quad p_0(t) = -2(\dot{\lambda} + i\lambda\dot{\gamma})e^{i\gamma}\end{gathered} \tag{1.18}$$

for some $f(t) \to 0$ as $t \to T$. Our strategy is to approximate (1.18) by the local dynamics (6.15), where the second term involving $\mathbf{Z}$ can be regarded as the main contribution from the λ-γ system, and the role of the initial data can be seen; see Section 6. We also observe from here that, compared to the one for the heat flow of harmonic map, the blow-up for (1.9) seems to be somewhat more unstable beyond equivariant symmetry class due to the restrictive assumptions on the initial data. The solvability of the full problem (1.18) is more subtle due to the double integro-differential operators, and its resolution will be done by linearization in Section 9.

The rest of this paper is organized as follows. In Section 2, notations and necessary formulas for the linearized operators will be given. In Section 3, we prove Theorem 1.1. This part is of independent interest. In Section 4, we discuss the approximate solution for the desired blow-up and its improvement by introducing a couple of corrections, where some technical analysis is postponed to Appendix A. In Section 5, we make the final ansatz and formulate the gluing system. Asymptotics of the modulation parameters will be derived in Section 6. In Section 7, linear theories for the outer problem and the inner problems in the W-direction and on $W^\perp$ will be developed, where the spectral analysis and the pointwise control for the mode 0 in the W-direction via distorted Fourier transform will be carried out in Appendix B. The full gluing system will be solved in Section 8. In Section 9, the linear theory for the c_1-c_2 system will be established.

2. Notations and preliminaries

We first list in this section notations, useful properties and formulas for the linearized operator of the H-system around the H-bubble. Our building block is $W\left(\frac{x-\xi}{\lambda}\right)$ given in (1.10) with $y = \frac{x-\xi}{\lambda}$, and

$$E_1(y) = \begin{bmatrix} e^{i\theta}\cos w(\rho) \\ -\sin w(\rho) \end{bmatrix}, \quad E_2(y) = \begin{bmatrix} ie^{i\theta} \\ 0 \end{bmatrix}$$

form a Frenet basis associated to W. Here

$$w(\rho) = \pi - 2\arctan(\rho), \quad y = \rho e^{i\theta},$$

so

$$w_\rho = -\frac{2}{\rho^2+1}, \quad \sin w = -\rho w_\rho = \frac{2\rho}{\rho^2+1}, \quad \cos w = \frac{\rho^2-1}{\rho^2+1}.$$

- **Notations.**
  - A map f is said to be in the W-direction if there exists a scalar c such that $f = cW$. Similarly, f is said to be on $W^\perp$ (or $f \in W^\perp$) if $f \cdot W = 0$.
  - The complex form of a map $f = f_1E_1 + f_2E_2$ on $W^\perp$ is defined by
  $$(f)_{\mathbb{C}} = f_1 + if_2.$$
  - We write
  $$f := \Pi_W[f] + \Pi_{W^\perp}[f] \ \text{ for } \ \Pi_{W^\perp}[f] := f - (f \cdot W)W.$$

- A map f is said to be in mode k on $W^\perp$ if $\Pi_{W^\perp}[f]$ can be written as
$$\Pi_{W^\perp}[f] = \mathrm{Re}(f_k(r)e^{ik\theta})E_1 + \mathrm{Im}(f_k(r)e^{ik\theta})E_2, \quad x = re^{i\theta}.$$
- The notation $(\Pi_{W^\perp}[f])_{\mathbb{C},j}$ denotes the projection of the complex form of $\Pi_{W^\perp}[f]$ onto mode j.
- For $x \in \mathbb{R}^2$, $t \le T$ and admissiable functions $g(x), h(x,t)$, denote
$$\left(\Gamma_{\mathbb{R}^2} \circ g\right)(x,t) := (4\pi t)^{-1} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4t}} g(y)dy,$$
$$\left(\Gamma_{\mathbb{R}^2} \bullet h\right)(x,t) := \int_0^t \int_{\mathbb{R}^2} [4\pi(t-s)]^{-1} e^{-\frac{|x-y|^2}{4(t-s)}} h(y,s)dyds.$$
- Denote $\langle y \rangle = \sqrt{1+|y|^2}$ for any $y \in \mathbb{R}^2$.
- The symbol " $\lesssim$ " means " $\le C$" for a positive constant C independent of t and T. Here C might be different from line to line.

• **Linearized operator.**

The linearization around W reads
$$L_W[\varphi] := \Delta_y \varphi - 2W_{y_1} \wedge \varphi_{y_2} - 2\varphi_{y_1} \wedge W_{y_2}, \tag{2.1}$$
and as a consequence of the non-degeneracy of degree 1 map W proved by Chanillo-Malchiodi [13], all bounded kernel functions of the operator L_W must be linear combinations of
$$\begin{cases} Z_{0,1}(y) = \rho w_\rho E_1(y), \\ Z_{0,2}(y) = \rho w_\rho E_2(y), \\ Z_{1,1}(y) = w_\rho[\cos\theta E_1(y) + \sin\theta E_2(y)], \\ Z_{1,2}(y) = w_\rho[\sin\theta E_1(y) - \cos\theta E_2(y)], \\ Z_{-1,1}(y) = \rho^2 w_\rho[\cos\theta E_1(y) - \sin\theta E_2(y)], \\ Z_{-1,2}(y) = \rho^2 w_\rho[\sin\theta E_1(y) + \cos\theta E_2(y)], \\ \mathcal{Z}_0 = \cos w W, \\ \mathcal{Z}_{1,1} = \cos\theta \sin w W, \\ \mathcal{Z}_{1,2} = \sin\theta \sin w W. \end{cases} \tag{2.2}$$
For
$$U := Q_\gamma W,$$
we also define
$$L_U[\phi] := \Delta_x \phi - 2U_{x_1} \wedge \phi_{x_2} - 2\phi_{x_1} \wedge U_{x_2}.$$
Here, Q_γ is defined in (1.12). Clearly, one has
$$L_U[Q_\gamma \varphi] = \lambda^{-2} Q_\gamma L_W[\varphi].$$
We now give several useful formulations of L_W acting on φ in different forms.

Lemma 2.1. *If we set*
$$\Phi(y) = \phi_1(\rho,\theta)E_1 + \phi_2(\rho,\theta)E_2 + \phi_3(\rho,\theta)W$$
and suppose that $L_W[\Phi] = 0$, then the scalars ϕ_1, ϕ_2 and ϕ_3 should satisfy the following equations
$$\partial_{\rho\rho}\phi_1 + \frac{1}{\rho}\partial_\rho\phi_1 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_1 - \frac{1}{\rho^2}\phi_1 + \frac{8}{(1+\rho^2)^2}\phi_1 - \frac{2(\rho^2-1)}{\rho^2(\rho^2+1)}\partial_\theta\phi_2 = 0, \tag{2.3}$$
$$\partial_{\rho\rho}\phi_2 + \frac{1}{\rho}\partial_\rho\phi_2 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_2 - \frac{1}{\rho^2}\phi_2 + \frac{8}{(1+\rho^2)^2}\phi_2 + \frac{2(\rho^2-1)}{\rho^2(\rho^2+1)}\partial_\theta\phi_1 = 0, \tag{2.4}$$
$$\partial_{\rho\rho}\phi_3 + \frac{1}{\rho}\partial_\rho\phi_3 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_3 + \frac{8}{(1+\rho^2)^2}\phi_3 = 0. \tag{2.5}$$

Remark 2.1.1.

- *We observe from above Lemma that the the components in W-direction and on $W^\perp$ are in fact decoupled under linearization.*

- *Notice that the scalar equation* (2.5) *can be regarded as the linearization of the Liouville equation*
$$\Delta u + e^u = 0 \quad in \quad \mathbb{R}^2$$
around $\log \frac{8}{(1+\rho^2)^2}$, *which is non-degenerate in the sense that the linearization only has following bounded kernels*
$$\frac{\rho^2-1}{\rho^2+1}, \quad \frac{2\rho e^{i\theta}}{\rho^2+1}.$$
See also (2.2).

The next three lemmas concern the expansion of
$$\tilde{L}_U[\Phi] := -2U_{x_1} \wedge \Phi_{x_2} - 2\Phi_{x_1} \wedge U_{x_2}$$
in the linearization $L_U[\Phi]$, and these will be useful in analyzing the couplings in the gluing system.

Lemma 2.2. *In the polar coordinate system*
$$\Phi(x) = \Phi(r,\theta), \quad x = re^{i\theta}, \quad \rho = \frac{r}{\lambda},$$
the term $\tilde{L}_U[\Phi]$ *can be expressed as*
$$\begin{aligned}\tilde{L}_U[\Phi] = &-\frac{2}{\lambda} w_\rho \left(\Phi_r \cdot (Q_\gamma W)\right) Q_\gamma E_1 + \frac{2}{\lambda r} w_\rho (\Phi_\theta \cdot (Q_\gamma W)) Q_\gamma E_2 \\ &+ \frac{2}{\lambda r} w_\rho \left[r(\Phi_r \cdot (Q_\gamma E_1)) - (\Phi_\theta \cdot (Q_\gamma E_2))\right] Q_\gamma W.\end{aligned}$$

We consider a C^1 function $\Phi : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}$, that we express in the form
$$\Phi(x) = \begin{bmatrix} \varphi_1(x) + i\varphi_2(x) \\ \varphi_3(x) \end{bmatrix}.$$
We also denote
$$\varphi = \varphi_1 + i\varphi_2, \quad \bar{\varphi} = \varphi_1 - i\varphi_2$$
and define the operators
$$\operatorname{div} \varphi = \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2, \quad \operatorname{curl} \varphi = \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1.$$

Lemma 2.3. *In the polar coordinate system*
$$\Phi(x) = \Phi(r,\theta) = (\varphi_1, \ \varphi_2, \ \varphi_3)^T, \quad x = re^{i\theta}, \quad \rho = \frac{r}{\lambda},$$
the term $\tilde{L}_U[\Phi]$ *can be decomposed as follows*
$$\tilde{L}_U[\Phi] = \tilde{L}_U[\Phi]_0 + \tilde{L}_U[\Phi]_1 + \tilde{L}_U[\Phi]_2$$
with
$$\begin{aligned}\tilde{L}_U[\Phi]_0 &= \frac{1}{\lambda}\rho w_\rho^2 [\operatorname{div}(e^{-i\gamma}\varphi) Q_\gamma E_1 + \operatorname{curl}(e^{-i\gamma}\varphi) Q_\gamma E_2] + \frac{1}{\lambda} w_\rho^2 \operatorname{div}(e^{-i\gamma}\varphi)) Q_\gamma W, \\ \tilde{L}_U[\Phi]_1 &= -\frac{2}{\lambda} w_\rho \cos w \Big[[\partial_{x_1}\varphi_3 \cos\theta + \partial_{x_2}\varphi_3 \sin\theta] Q_\gamma E_1 + [\partial_{x_1}\varphi_3 \sin\theta - \partial_{x_2}\varphi_3 \cos\theta] Q_\gamma E_2 \Big] \\ &\quad - \frac{2}{\lambda} w_\rho \sin w [\partial_{x_1}\varphi_3 \cos\theta + \partial_{x_2}\varphi_3 \sin\theta] Q_\gamma W, \\ \tilde{L}_U[\Phi]_2 &= \frac{1}{\lambda}\rho w_\rho^2 \Big[[\operatorname{div}(e^{i\gamma}\bar{\varphi})\cos(2\theta) - \operatorname{curl}(e^{i\gamma}\bar{\varphi})\sin(2\theta)] Q_\gamma E_1 + [\operatorname{div}(e^{i\gamma}\bar{\varphi})\sin(2\theta) + \operatorname{curl}(e^{i\gamma}\bar{\varphi})\cos(2\theta)] Q_\gamma E_2 \Big] \\ &\quad + \frac{2}{\lambda} w_\rho \left[-\frac{1}{2}\rho^2 w_\rho \operatorname{div}(e^{i\gamma}\bar{\varphi})\cos(2\theta) + \frac{1}{2}\rho^2 w_\rho \operatorname{curl}(e^{i\gamma}\bar{\varphi})\sin(2\theta) \right] Q_\gamma W.\end{aligned}$$

Lemma 2.4. *For*

$$\Phi(x) = \begin{bmatrix} \phi(r)e^{i\theta} \\ 0 \end{bmatrix}$$

with ϕ complex-valued, the term $\tilde{L}_U[\Phi]$ can be expressed as

$$\tilde{L}_U[\Phi] = \frac{2}{\lambda}\rho w_\rho^2 \left[\mathrm{Re}(e^{-i\gamma}\phi_r(r))Q_\gamma E_1 + \frac{1}{r}\mathrm{Im}(e^{-i\gamma}\phi(r))Q_\gamma E_2 \right] + \frac{2}{\lambda} w_\rho \left(\mathrm{Re}(\phi_r e^{-i\gamma})\cos w - \frac{1}{r}\mathrm{Re}(\phi e^{-i\gamma}) \right) Q_\gamma W.$$

Assume ψ is real-valued and

$$\Psi(x) = \begin{bmatrix} 0 \\ 0 \\ \psi(r) \end{bmatrix}.$$

Then

$$\tilde{L}_U[\Psi] = -\frac{2}{\lambda} w_\rho \cos w \left(\psi_r Q_\gamma E_1 - \frac{1}{r}\psi_\theta Q_\gamma E_2 \right) + \frac{2}{\lambda}\rho w_\rho^2 \psi_r Q_\gamma W.$$

The proof of all the lemmas in this section will be postponed to Appendix A.1.

3. Proof of Theorem 1.1: Non-degeneracy of the degree m bubble

For notational simplicity, we write throughout this section

$$W^{(m)} = W, \quad E_1^{(m)} = E_1, \quad E_2^{(m)} = E_2.$$

Proof of Theorem 1.1. We divide the proof into the following steps.
Step 1. In the polar coordinate system, we write the solution $\phi = \phi(r,\theta)$ of (1.6) as

$$\phi(r,\theta) = \xi(r,\theta)E_1 + \eta(r,\theta)E_2 + \zeta(r,\theta)W$$

Then by direct computation, we know the linearized operator

$$L[\phi] = \Delta\phi - \frac{2}{r}\phi_r \wedge W_\theta - \frac{2}{r}W_r \wedge \phi_\theta := (L^{(1)}[\phi], L^{(2)}[\phi], L^{(3)}[\phi])$$

can be expressed as follows,

$$\begin{aligned}
&L^{(1)}[\phi] \\
&= \frac{1}{r^2\left(r^{2m}+1\right)^3}\cos(m\theta)\big[-m^2(-1+7r^{2m}-7r^{4m}+r^{6m})\xi + 16m^2r^{3m}\zeta \\
&\quad + (-2m+2mr^{2m}+2mr^{4m}-2mr^{6m})\eta_\theta \\
&\quad + (-1-r^{2m}+r^{4m}+r^{6m})\xi_{\theta\theta} + (-r-r^{1+2m}+r^{1+4m}+r^{1+6m})\xi_r \\
&\quad + (-r^2-r^{2+2m}+r^{2+4m}+r^{2+6m})\xi_{rr} + (2r^{2+m}+4r^{2+3m}+2r^{2+5m})\zeta_{rr} \\
&\quad +(2r^{1+m}+4r^{1+3m}+2r^{1+5m})\zeta_r + (2r^m+4r^{3m}+2r^{5m})\zeta_{\theta\theta}\big] \\
&\quad + \frac{(r^{2m}+1)}{r^2\left(r^{2m}+1\right)^3}\sin(m\theta)\big[(2m-2mr^{4m})\xi_\theta + m^2(1-6r^{2m}+r^{4m})\eta \\
&\qquad + (-1-2r^{2m}-r^{4m})\eta_{\theta\theta} + (-r-2r^{1+2m}-r^{1+4m})\eta_r \\
&\qquad +(-r-2r^{2+2m}-r^{2+4m})\eta_{rr})\big],
\end{aligned}$$

$$\begin{aligned}
L^{(2)}[\phi] &= \frac{-1-r^{2m}}{r^2\left(r^{2m}+1\right)^3}\cos(m\theta)\big[(2m-2mr^{4m})\xi_\theta + m^2(1-6r^{2m}+r^{4m})\eta \\
&\quad + (-1-2r^{2m}-r^{4m})\eta_{\theta\theta} + (-r-2r^{1+2m}-r^{1+4m})\eta_r \\
&\quad +(-r^2-2r^{2+2m}-r^{2+4m})\eta_{rr}\big] \\
&\quad + \frac{1}{r^2\left(r^{2m}+1\right)^3}\sin(m\theta)\big[-m^2(-1+7r^{2m}-7r^{4m}+r^{6m})\xi + 16m^2r^{3m}\zeta \\
&\quad + (-2m+2mr^{2m}+2mr^{4m}-2mr^{6m})\eta_\theta + (-1-r^{2m}+r^{4m}+r^{6m})\xi_{\theta\theta}
\end{aligned}$$

$$
\begin{aligned}
&+(-r-r^{1+2m}+r^{1+4m}+r^{1+6m})\xi_r+(-r^2-r^{2+2m}+r^{2+4m}+r^{2+6m})\xi_{rr}\\
&+(2r^m+4r^{3m}+2r^{5m})\zeta_{\theta\theta}+(2r^{1+m}+4r^{1+3m}+2r^{1+5m})\zeta_r\\
&+(2r^{2+m}+4r^{2+3m}+2r^{2+5m})\zeta_{rr}\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&L^{(3)}[\phi]\\
&=\frac{1}{r^2\left(r^{2m}+1\right)^3}\big[2m^2(r^m-6r^{3m}+r^{5m})\xi+8m^2r^{2m}\zeta+(-4mr^m+4mr^{5m})\eta_\theta\\
&+(-2r^m-4r^{3m}-2r^{5m})\xi_{\theta\theta}+(-1-r^{2m}+r^{4m}+r^{6m})\zeta_{\theta\theta}\\
&+(-2r^{1+m}-4r^{1+3m}-2r^{1+5m})\xi_r+(-r-r^{1+2m}+r^{1+4m}+r^{1+6m})\zeta_r\\
&+(-2r^{2+m}-4r^{2+3m}-2r^{2+5m})\xi_{rr}+(-r^2-r^{2+2m}+r^{2+4m}+r^{2+6m})\xi_{rr}\big].
\end{aligned}
$$

Step 2. Using Fourier expansion, we set $\xi(r,\theta)=a(r)\cos(k\theta)+b(r)\sin(k\theta)$, $\eta(r,\theta)=c(r)\cos(k\theta)+d(r)\sin(k\theta)$ and $\zeta(r,\theta)=e(r)\cos(k\theta)+f(r)\sin(k\theta)$, k is an integer, and we write

$$
L_\perp(\phi):=(L_\perp^{(1)}[\phi],L_\perp^{(1)}[\phi],L_\perp^{(1)}[\phi]):=(L^{(1)}[\phi],L^{(2)}[\phi],L^{(3)}[\phi])-((L^{(1)}[\phi],L^{(2)}[\phi],L^{(3)}[\phi])\cdot W)W.
$$

Then we have

$$
L_\perp^{(1)}[\phi]=\frac{1}{r^2\left(r^{2m}+1\right)^3}((1+r^{2m})\sin(m\theta)A_1+(-1+r^{2m})\cos(m\theta)A_2)
$$

with

$$
\begin{aligned}
A_1=&\cos(k\theta)(-2km(r^{4m}-1)b(r)+(k^2(1+r^{2m})^2+m^2(1-6r^{2m}+r^{4m}))c(r))\\
&-\cos(k\theta)(r(1+r^{2m})^2(c'(r)+rc''(r)))\\
&+\sin(k\theta)(2km(r^{4m}-1)a(r)+(k^2(1+r^{2m})^2+m^2(1-6r^{2m}+r^{4m}))d(r))\\
&-\sin(k\theta)(r(1+r^{2m})^2(d'(r)+rd''(r))),
\end{aligned}
$$

$$
\begin{aligned}
A_2=&\cos(k\theta)((-k^2(1+r^{2m})^2-m^2(1-6r^{2m}+r^{4m}))a(r)-2km(r^{4m}-1)d(r))\\
&+\cos(k\theta)(r(1+r^{2m})^2(a'(r)+ra''(r))\\
&+\sin(k\theta)(-k^2(1+r^{2m})^2-m^2(1-6r^{2m}+r^{4m}))b(r))\\
&+\sin(k\theta)(2km(r^{2m}-1)(r^{2m}+1)c(r)+r(1+r^{2m})^2(b'(r)+rb''(r)),
\end{aligned}
$$

$$
L_\perp^{(2)}[\phi]=\frac{1}{r^2\left(r^{2m}+1\right)^3}((1+r^{2m})\cos(m\theta)B_1+(-1+r^{2m})\sin(m\theta)B_2)
$$

with

$$
\begin{aligned}
B_1=&\cos(k\theta)(2km(r^{4m}-1)b(r)-(k^2(1+r^{2m})^2+m^2(1-6r^{2m}+r^{4m}))c(r))\\
&+\cos(k\theta)(r(1+r^{2m})^2(c'(r)+rc''(r)))\\
&-\sin(k\theta)(2km(r^{4m}-1)a(r)+(k^2(1+r^{2m})^2+m^2(1-6r^{2m}+r^{4m}))d(r))\\
&+\sin(k\theta)(r(1+r^{2m})^2(d'(r)+rd''(r))),
\end{aligned}
$$

$$
\begin{aligned}
B_2=&\cos(k\theta)(-(k^2(1+r^{2m})^2+m^2(1-6r^{2m}+r^{4m}))a(r)-2km(r^{4m}-1)d(r)\\
&+r(1+r^{2m})^2(a'(r)+ra''(r))\\
&+\sin(k\theta)(-(k^2(1+r^{2m})^2+m^2(1-6r^{2m}+r^{4m}))b(r)+2km(r^{4m}-1)c(r)\\
&+r(1+r^{2m})^2(b'(r)+rb''(r)),
\end{aligned}
$$

$$
L_\perp^{(3)}[\phi]=\frac{2r^{m-2}}{(1+r^{2m})^3}\cos(k\theta)C_1+\frac{2r^{m-2}}{(1+r^{2m})^3}\sin(k\theta)C_2,
$$

with

$$\begin{aligned}C_1 &= (k^2(1+r^{2m})^2 + m^2(1-6r^{2m}+r^{4m}))a(r) + (1+r^{2m})2km(r^{2m}-1)d(r)\\&\quad - r(1+r^{2m})(a'(r)+ra''(r)),\\C_2 &= (k^2(1+r^{2m})^2 + m^2(1-6r^{2m}+r^{4m}))b(r) - (1+r^{2m})2km(r^{2m}-1)c(r)\\&\quad + r(1+r^{2m})(b'(r)+rb''(r)).\end{aligned}$$

In the direction of W, we have

$$(L^{(1)}[\phi], L^{(2)}[\phi], L^{(3)}[\phi])\cdot W = \frac{1}{r^2(1+r^{2m})^2}\cos(k\theta)D_1 + \frac{2r^{m-2}}{(1+r^{2m})^3}\sin(k\theta)D_2$$

with

$$\begin{aligned}D_1 &= -(-8m^2r^{2m} + k^2(1+r^{2m})^2)e(r) + r(1+r^{2m})^2(e'(r)+re''(r)),\\D_2 &= -(-8m^2r^{2m} + k^2(1+r^{2m})^2)f(r) + r(1+r^{2m})^2(f'(r)+rf''(r)).\end{aligned}$$

We also write $L_\perp(\phi)$ as

$$L_\perp(\phi) = (L_\perp(\phi)\cdot E_1)E_1 + (L_\perp(\phi)\cdot E_2)E_2$$

with

$$\begin{aligned}&L_\perp(\phi)\cdot E_1\\&= ((-k^2(r^{2m}+1)^2 - m^2(1-6r^{2m}+r^{4m}))a(r) - 2km(r^{4m}-1)d(r))\cos(k\theta)\\&\quad + r(1+r^{2m})^2(a'(r)+ra''(r))\cos(k\theta)\\&\quad + ((-k^2(r^{2m}+1)^2 - m^2(1-6r^{2m}+r^{4m}))b(r) + 2km(r^{4m}-1)c(r))\sin(k\theta)\\&\quad + r(1+r^{2m})^2(b'(r)+rb''(r))\sin(k\theta)\end{aligned}$$

and

$$\begin{aligned}&L_\perp(\phi)\cdot E_2\\&= ((-k^2(r^{2m}+1)^2 - m^2(1-6r^{2m}+r^{4m}))c(r) + 2km(r^{4m}-1)b(r))\cos(k\theta)\\&\quad + r(1+r^{2m})^2(c'(r)+rc''(r))\cos(k\theta)\\&\quad + ((-k^2(r^{2m}+1)^2 - m^2(1-6r^{2m}+r^{4m}))d(r) - 2km(r^{4m}-1)a(r))\sin(k\theta)\\&\quad + r(1+r^{2m})^2(d'(r)+rd''(r))\sin(k\theta).\end{aligned}$$

Therefore, to solve the linearized equation (1.6), we need to solve the following system of ODEs,

$$\begin{cases} a''(r) + \frac{1}{r}a'(r) - \frac{k^2 + \frac{m^2(1-6r^{2m}+r^{4m})}{(r^{2m}+1)^2}}{r^2}a(r) - \frac{2km(r^{2m}-1)}{r^2(1+r^{2m})}d(r) = 0,\\ d''(r) + \frac{1}{r}d'(r) - \frac{k^2 + \frac{m^2(1-6r^{2m}+r^{4m})}{(r^{2m}+1)^2}}{r^2}d(r) - \frac{2km(r^{2m}-1)}{r^2(1+r^{2m})}a(r) = 0,\end{cases} \tag{3.1}$$

$$\begin{cases} b''(r) + \frac{1}{r}b'(r) - \frac{k^2 + \frac{m^2(1-6r^{2m}+r^{4m})}{(r^{2m}+1)^2}}{r^2}b(r) + \frac{2km(r^{2m}-1)}{r^2(1+r^{2m})}c(r) = 0,\\ c''(r) + \frac{1}{r}c'(r) - \frac{k^2 + \frac{m^2(1-6r^{2m}+r^{4m})}{(r^{2m}+1)^2}}{r^2}c(r) + \frac{2km(r^{2m}-1)}{r^2(1+r^{2m})}b(r) = 0,\end{cases} \tag{3.2}$$

and

$$\begin{cases} e''(r) + \frac{1}{r}e'(r) + \frac{8m^2r^{2m} - k^2(1+r^{2m})^2}{r^2(r^{2m}+1)^2}e(r) = 0,\\ f''(r) + \frac{1}{r}f'(r) + \frac{8m^2r^{2m} - k^2(1+r^{2m})^2}{r^2(r^{2m}+1)^2}f(r) = 0.\end{cases} \tag{3.3}$$

Step 3. We solve the system (3.1)-(3.3) in the following four cases.

Case 1: If $k=0$, then the solutions of (3.1)-(3.3) are

$$\begin{aligned}a(r) &= \frac{(c_1+c_3)}{2}\frac{r^m}{1+r^{2m}} + \frac{(c_2+c_4)}{2}\frac{2r^m\log(r^{2m}) + r^{3m} - r^{-m}}{1+r^{2m}},\\d(r) &= \frac{(c_1-c_3)}{2}\frac{r^m}{1+r^{2m}} + \frac{(c_2-c_4)}{2}\frac{2r^m\log(r^{2m}) + r^{3m} - r^{-m}}{1+r^{2m}},\end{aligned}$$

$$
\begin{aligned}
b(r) &= \frac{(c_5+c_7)}{2}\frac{r^m}{1+r^{2m}} + \frac{(c_6+c_8)}{2}\frac{2r^m\log(r^{2m})+r^{3m}-r^{-m}}{1+r^{2m}},\\
c(r) &= \frac{(c_5-c_7)}{2}\frac{r^m}{1+r^{2m}} + \frac{(c_6-c_8)}{2}\frac{2r^m\log(r^{2m})+r^{3m}-r^{-m}}{1+r^{2m}},\\
e(r) &= \frac{c_9(r^{2m}-1)}{1+r^{2m}} + \frac{c_{10}r^m((r^m-r^{-m})\log(r^{2m})-4r^{-m})}{1+r^{2m}},\\
f(r) &= \frac{c_{11}(r^{2m}-1)}{1+r^{2m}} + \frac{c_{12}r^m((r^m-r^{-m})\log(r^{2m})-4r^{-m})}{1+r^{2m}}.
\end{aligned}
$$

Therefore the bounded kernel functions take the form

$$
\begin{aligned}
\phi(r,\theta) = {} & \frac{(c_1+c_3)}{2}\frac{r^m}{1+r^{2m}}\cos(k\theta)E_1 + \frac{(c_5-c_7)}{2}\frac{r^m}{1+r^{2m}}\cos(k\theta)E_2\\
& + \frac{c_9(r^{2m}-1)}{1+r^{2m}}\cos(k\theta)W.
\end{aligned}
$$

Case 2: If $k=1$, then the solutions of (3.1)-(3.3) are

$$
\begin{aligned}
a(r) = {} & \frac{c_{13}r^{m-1}}{2(1+r^{2m})} + \frac{c_{14}r((-1-m)r^{-m}+2m^2r^m+mr^{3m}-2r^m-r^{3m})}{2(1+r^{2m})}\\
& + \frac{c_{15}r^{m+1}}{2(1+r^{2m})} + \frac{c_{16}r^{-1}((-1+m)r^{-m}+2m^2r^m-mr^{3m}-2r^m-r^{3m})}{2(1+r^{2m})},\\
d(r) = {} & \frac{c_{13}r^{m-1}}{2(1+r^{2m})} + \frac{c_{14}r((-1-m)r^{-m}+2m^2r^m+mr^{3m}-2r^m-r^{3m})}{2(1+r^{2m})}\\
& - \frac{c_{15}r^{m+1}}{2(1+r^{2m})} - \frac{c_{16}r^{-1}((-1+m)r^{-m}+2m^2r^m-mr^{3m}-2r^m-r^{3m})}{2(1+r^{2m})},\\
b(r) = {} & \frac{c_{17}r^{m-1}}{2(1+r^{2m})} + \frac{c_{18}r((-1-m)r^{-m}+2m^2r^m+mr^{3m}-2r^m-r^{3m})}{2(1+r^{2m})}\\
& + \frac{c_{19}r^{m+1}}{2(1+r^{2m})} + \frac{c_{20}r^{-1}((-1+m)r^{-m}+2m^2r^m-mr^{3m}-2r^m-r^{3m})}{2(1+r^{2m})},\\
c(r) = {} & -\frac{c_{17}r^{m-1}}{2(1+r^{2m})} - \frac{c_{18}r((-1-m)r^{-m}+2m^2r^m+mr^{3m}-2r^m-r^{3m})}{2(1+r^{2m})}\\
& + \frac{c_{19}r^{m+1}}{2(1+r^{2m})} + \frac{c_{20}r^{-1}((-1+m)r^{-m}+2m^2r^m-mr^{3m}-2r^m-r^{3m})}{2(1+r^{2m})},\\
e(r) = {} & \frac{c_{21}}{1+r^{2m}}((1-m)r^{-m}+(1+m)r^m)r^{m-1}\\
& + \frac{c_{22}}{1+r^{2m}}((1+m)r^{-m}+(1-m)r^m)r^{m+1},\\
f(r) = {} & \frac{c_{23}}{1+r^{2m}}((1-m)r^{-m}+(1+m)r^m)r^{m-1}\\
& + \frac{c_{24}}{1+r^{2m}}((1+m)r^{-m}+(1-m)r^m)r^{m+1}.
\end{aligned}
$$

If $m=1$, the bounded kernel functions take the form

$$
\begin{aligned}
&\phi(r,\theta)\\
&= \left(\left(\frac{c_{13}}{2(1+r^2)}+\frac{c_{15}r^2}{2(1+r^2)}\right)\cos(\theta)+\left(\frac{c_{17}}{2(1+r^2)}+\frac{c_{19}r^2}{2(1+r^2)}\right)\sin(\theta)\right)E_1\\
&+ \left(\left(-\frac{c_{17}}{2(1+r^2)}+\frac{c_{19}r^2}{2(1+r^2)}\right)\cos(\theta)+\left(\frac{c_{13}}{2(1+r^2)}-\frac{c_{15}r^2}{2(1+r^2)}\right)\sin(\theta)\right)E_2\\
&+ \left(\frac{(c_{21}+c_{22})2r}{1+r^2}\cos(\theta)+\frac{(c_{23}+c_{24})2r}{1+r^2}\sin(\theta)\right)W.
\end{aligned}
$$

If $m>1$, the bounded kernel functions take the form

$$\begin{aligned}&\phi(r,\theta)=\\&\left(\left(\frac{c_{13}r^{m-1}}{2(1+r^{2m})}+\frac{c_{15}r^{m+1}}{2(1+r^{2m})}\right)\cos(\theta)+\left(\frac{c_{17}r^{m-1}}{2(1+r^{2m})}+\frac{c_{19}r^{m+1}}{2(1+r^{2m})}\right)\sin(\theta)\right)E_1\\&+\left(\left(-\frac{c_{17}r^{m-1}}{2(1+r^{2m})}+\frac{c_{19}r^{m+1}}{2(1+r^{2m})}\right)\cos(\theta)+\left(\frac{c_{13}r^{m-1}}{2(1+r^{2m})}-\frac{c_{15}r^{m+1}}{2(1+r^{2m})}\right)\sin(\theta)\right)E_2.\end{aligned}$$

Case 3: If $k=m$, then the solutions of (3.1)-(3.3) are

$$\begin{aligned}a(r)&=\frac{c_{25}}{2(1+r^{2m})}+\frac{c_{26}r^m(r^{3m}+2r^{-m}\log(r^{2m})+4r^m)}{2(1+r^{2m})}\\&\quad+\frac{c_{27}r^{2m}}{2(1+r^{2m})}+\frac{c_{28}r^{-m}(-2r^{3m}\log(r^{2m})+4r^m+r^{-m})}{2(1+r^{2m})},\\d(r)&=\frac{c_{25}}{2(1+r^{2m})}+\frac{c_{26}r^m(r^{3m}+2r^{-m}\log(r^{2m})+4r^m)}{2(1+r^{2m})}\\&\quad-\frac{c_{27}r^{2m}}{2(1+r^{2m})}-\frac{c_{28}r^{-m}(-2r^{3m}\log(r^{2m})+4r^m+r^{-m})}{2(1+r^{2m})},\\b(r)&=\frac{c_{29}r^{2m}}{2(1+r^{2m})}+\frac{c_{30}r^{-m}(-2r^{3m}\log(r^{2m})+4r^m+r^{-m})}{2(1+r^{2m})}\\&\quad+\frac{c_{31}}{2(1+r^{2m})}+\frac{c_{32}r^m(r^{3m}+2r^{-m}\log(r^{2m})+4r^m)}{2(1+r^{2m})},\\c(r)&=\frac{c_{29}r^{2m}}{2(1+r^{2m})}+\frac{c_{30}r^{-m}(-2r^{3m}\log(r^{2m})+4r^m+r^{-m})}{2(1+r^{2m})}\\&\quad-\frac{c_{31}}{2(1+r^{2m})}-\frac{c_{32}r^m(r^{3m}+2r^{-m}\log(r^{2m})+4r^m)}{2(1+r^{2m})},\\e(r)&=\frac{c_{33}r^m}{(1+r^{2m})}+\frac{c_{34}(2r^m\log(r^{2m})+r^{3m}-r^{-m})}{1+r^{2m}},\\f(r)&=\frac{c_{35}r^m}{(1+r^{2m})}+\frac{c_{36}(2r^m\log(r^{2m})+r^{3m}-r^{-m})}{1+r^{2m}}.\end{aligned}$$

Therefore the bounded kernel functions take the form

$$\begin{aligned}&\phi(r,\theta)\\&=\left(\left(\frac{c_{25}}{2(1+r^{2m})}+\frac{c_{27}r^{2m}}{2(1+r^{2m})}\right)\cos(k\theta)+\left(\frac{c_{29}r^{2m}}{2(1+r^{2m})}+\frac{c_{31}}{2(1+r^{2m})}\right)\sin(k\theta)\right)E_1\\&+\left(\left(\frac{c_{29}r^{2m}}{2(1+r^{2m})}-\frac{c_{31}}{2(1+r^{2m})}\right)\cos(k\theta)+\left(\frac{c_{25}}{2(1+r^{2m})}-\frac{c_{27}r^{2m}}{2(1+r^{2m})}\right)\sin(k\theta)\right)E_2\\&+(\frac{c_{33}r^m}{(1+r^{2m})}\cos(k\theta)+\frac{c_{34}r^m}{(1+r^{2m})}\sin(k\theta))W.\end{aligned}$$

Case 4: The solutions of (3.1)-(3.3) in the case $k\neq 0$, $k\neq 1$ and $k\neq m$ are

$$\begin{aligned}a(r)&=c_{37}\frac{r^{m-k}}{2(1+r^{2m})}+c_{38}\frac{r^k(\frac{k(k+m)}{2}r^{-m}+(k-m)(kr^{3m}/2+(k+m)r^m))}{2(1+r^{2m})}\\&\quad+c_{39}\frac{r^{m+k}}{2(1+r^{2m})}+c_{40}\frac{r^{-k}(\frac{k(k-m)}{2}r^{-m}+(k+m)(kr^{3m}/2+(k-m)r^m))}{2(1+r^{2m})},\\d(r)&=c_{37}\frac{r^{m-k}}{2(1+r^{2m})}+c_{38}\frac{r^k(\frac{k(k+m)}{2}r^{-m}+(k-m)(kr^{3m}/2+(k+m)r^m))}{2(1+r^{2m})}\\&\quad-c_{39}\frac{r^{m+k}}{2(1+r^{2m})}-c_{40}\frac{r^{-k}(\frac{k(k-m)}{2}r^{-m}+(k+m)(kr^{3m}/2+(k-m)r^m))}{2(1+r^{2m})},\end{aligned}$$

$$
\begin{aligned}
b(r) = \frac{r^{-m-k}}{1+r^{2m}} \Bigg(& c_{41}(1+(k+m)r^{2m}(\frac{2}{k}+\frac{r^{2m}}{k-m})) + c_{42}r^{2(k+m)} \\
& + c_{43}\frac{r^{2k}(k(k+m)+2(k^2-m^2)r^{2m}+k(k-m)r^{4m})}{k(k+m)} + c_{44}r^{2m} \Bigg), \\
c(r) = \frac{r^{-m-k}}{1+r^{2m}} \Bigg(& c_{41}(1+(k+m)r^{2m}(\frac{2}{k}+\frac{r^{2m}}{k-m})) + c_{42}r^{2(k+m)} \\
& - c_{43}\frac{r^{2k}(k(k+m)+2(k^2-m^2)r^{2m}+k(k-m)r^{4m})}{k(k+m)} - c_{44}r^{2m} \Bigg), \\
e(r) = & \frac{c_{45}}{1+r^{2m}}((k-m)r^{-m}+(k+m)r^m)r^{m-k} \\
& + \frac{c_{46}}{1+r^{2m}}((k+m)r^{-m}+(k-m)r^m)r^{m+k}, \\
f(r) = & \frac{c_{47}}{1+r^{2m}}((k-m)r^{-m}+(k+m)r^m)r^{m-k} \\
& + \frac{c_{48}}{1+r^{2m}}((k+m)r^{-m}+(k-m)r^m)r^{m+k}.
\end{aligned}
$$

If $k > m$, there are no bounded kernel functions. If $k = 2, \cdots, m-1$, the bounded kernel functions take the form

$$
\begin{aligned}
\phi(r,\theta) = & \left(c_{37}\frac{r^{m-k}}{2(1+r^{2m})} + c_{39}\frac{r^{m+k}}{2(1+r^{2m})} \right) \cos(k\theta) E_1 \\
& + \frac{r^{-m-k}}{1+r^{2m}} \left(c_{42}r^{2(k+m)} + c_{44}r^{2m} \right) \sin(k\theta) E_1 \\
& + \frac{r^{-m-k}}{1+r^{2m}} \left(c_{42}r^{2(k+m)} - c_{44}r^{2m} \right) \cos(k\theta) E_2 \\
& + \left(c_{37}\frac{r^{m-k}}{2(1+r^{2m})} - c_{39}\frac{r^{m+k}}{2(1+r^{2m})} \right) \sin(k\theta) E_2.
\end{aligned}
$$

From the above computations, we conclude the validity of Theorem 1.1.

□

4. Approximation and improvement

In this section, we find a suitable profile that approximates well the real solution to (1.9) and then further improve it by adding several corrections. We first define the error of u as

$$
S[u] := -u_t + \Delta u - 2u_{x_1} \wedge u_{x_2}. \tag{4.1}
$$

Recall (2.2). We take the approximate solution as

$$
\begin{aligned}
& U_{\lambda,\gamma,\xi,c_1,c_2} \\
:= & Q_{\gamma(t)} \left[W\left(\frac{x-\xi(t)}{\lambda(t)}\right) + c_1(t)\frac{2\rho}{\rho^2+1}\cos\theta W\left(\frac{x-\xi(t)}{\lambda(t)}\right) + c_2(t)\frac{2\rho}{\rho^2+1}\sin\theta W\left(\frac{x-\xi(t)}{\lambda(t)}\right) \right] \\
= & Q_\gamma(W + c_1\mathcal{Z}_{1,1} + c_2\mathcal{Z}_{1,2}),
\end{aligned} \tag{4.2}
$$

where Q_γ in (1.12) can be written as

$$
Q_\gamma = e^{\gamma J_z}, \quad J_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

and the modulation parameters $\lambda(t),\ \xi(t),\ \gamma(t),\ c_1(t),\ c_2(t)$ are to be determined.

Remark 4.0.1. *By the non-degeneracy of degree 1 H-bubble, there are nine kernels for the associated linearized operator. In the above ansatz, we only introduce six modulation parameters. The other three correspond to the infinitesimal generators of rigid motions: rotations around x- and y-axes, and another translation for the map in $\mathbb{R}^3$, and the kernels are all of order one at space infinity. For technical reasons, we take advantage of regularity properties to control these modes in the absence of modulation.*

We have

$$\begin{aligned} \partial_\rho \mathcal{Z}_{1,1} &= \cos\theta w_\rho(\cos wW + \sin wE_1), \\ \partial_\theta \mathcal{Z}_{1,1} &= -\sin w \sin\theta W + \sin^2 w\cos\theta E_2, \\ \partial_\rho \mathcal{Z}_{1,2} &= \sin\theta w_\rho(\cos wW + \sin wE_1), \\ \partial_\theta \mathcal{Z}_{1,2} &= \sin w\cos\theta W + \sin^2 w\sin\theta E_2. \end{aligned} \tag{4.3}$$

Then the error of the approximation (4.2) is given by

$$\begin{aligned} & S[U_{\lambda,\gamma,\xi,c_1,c_2}] \\ = & -\partial_t(Q_\gamma W) - \sum_{j=1}^{2}[\dot c_j Q_\gamma \mathcal{Z}_{1,j} + c_j\partial_t(Q_\gamma \mathcal{Z}_{1,j})] \\ & - 2c_1c_2Q_\gamma \partial_{x_1}\mathcal{Z}_{1,1}\wedge\partial_{x_2}\mathcal{Z}_{1,2} - 2c_1c_2Q_\gamma\partial_{x_1}\mathcal{Z}_{1,2}\wedge\partial_{x_2}\mathcal{Z}_{1,1} \\ = & -\partial_t(Q_\gamma W) - \sum_{j=1}^{2}\dot c_jQ_\gamma\mathcal{Z}_{1,j} - \sum_{j=1}^{2}c_j\dot\gamma J_z e^{\gamma J_z}\mathcal{Z}_{1,j} - \sum_{j=1}^{2}c_jQ_\gamma(\partial_\rho\mathcal{Z}_{1,j}\rho_t + \partial_\theta\mathcal{Z}_{1,j}\theta_t) \\ & - \frac{2\lambda^{-2}c_1c_2}{\rho}Q_\gamma\left(\partial_\rho\mathcal{Z}_{1,1}\wedge\partial_\theta\mathcal{Z}_{1,2} + \partial_\rho\mathcal{Z}_{1,2}\wedge\partial_\theta\mathcal{Z}_{1,1}\right) \\ = & \lambda^{-1}\dot\lambda\rho w_\rho Q_\gamma E_1 + \dot\gamma\rho w_\rho Q_\gamma E_2 \\ & + \lambda^{-1}\dot\xi_1 w_\rho Q_\gamma\left[\cos\theta E_1 + \sin\theta E_2\right] + \lambda^{-1}\dot\xi_2 w_\rho Q_\gamma\left[\sin\theta E_1 - \cos\theta E_2\right] \\ & - \frac{2\rho}{1+\rho^2}(\dot c_1\cos\theta + \dot c_2\sin\theta)Q_\gamma W - \frac{2\dot\gamma\rho\sin w}{1+\rho^2}\left(c_1\cos\theta + c_2\sin\theta\right)Q_\gamma E_2 \\ & - \Big[\lambda^{-1}\rho^{-1}\sin w(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)(c_2\cos\theta - c_1\sin\theta) \\ & \quad - \lambda^{-1}w_\rho\cos w(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)(c_1\cos\theta + c_2\sin\theta)\Big]Q_\gamma W \\ & + \Big[\lambda^{-1}w_\rho\sin w(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)(c_1\cos\theta + c_2\sin\theta)\Big]Q_\gamma E_1 \\ & - \Big[\lambda^{-1}\rho^{-1}\sin^2 w(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)(c_1\cos\theta + c_2\sin\theta)\Big]Q_\gamma E_2 \\ & - \frac{2\lambda^{-2}c_1c_2}{\rho}Q_\gamma\left(-\sin 2\theta w_\rho\sin^2 w\cos wE_1 - \cos 2\theta w_\rho \sin^2 wE_2 + \sin 2\theta w_\rho\sin^3 wW\right), \end{aligned}$$

where we have used (4.3). We then arrange terms and write

$$\begin{aligned} & S[U_{\lambda,\gamma,\xi,c_1,c_2}] \\ = & \underbrace{\lambda^{-1}\dot\lambda\rho w_\rho Q_\gamma E_1 + \dot\gamma\rho w_\rho Q_\gamma E_2}_{:=\mathcal{E}^{(0)}_{U^\perp}} + \mathcal{R}_{U^\perp} \\ & + \underbrace{(-2)\lambda^{-2}c_1c_2w_\rho^2\sin w(\sin 2\theta\cos wQ_\gamma E_1 + \cos 2\theta Q_\gamma E_2)}_{:=\mathcal{E}^{(\pm 2)}_{U^\perp}} \\ & + \underbrace{\frac{\dot\xi_1}{\lambda}w_\rho Q_\gamma\left[\cos\theta E_1 + \sin\theta E_2\right] + \frac{\dot\xi_2}{\lambda}w_\rho Q_\gamma\left[\sin\theta E_1 - \cos\theta E_2\right]}_{:=\mathcal{E}^{(1)}_{U^\perp}} \\ & + \underbrace{\rho w_\rho\left[\cos\theta(\dot c_1 + \lambda^{-1}\dot\lambda c_1) + \sin\theta(\dot c_2 + \lambda^{-1}\dot\lambda c_2)\right]Q_\gamma W}_{:=\mathcal{E}^{(\pm 1)}_{U}} \\ & + \underbrace{2\lambda^{-2}c_1c_2\sin 2\theta\rho w_\rho^2\sin^2 wQ_\gamma W}_{:=\mathcal{E}^{(\pm 2)}_{U}} + \mathcal{R}_U, \end{aligned} \tag{4.4}$$

where

$$
\begin{aligned}
\mathcal{R}_{U^\perp} := &\ \dot\gamma\rho w_\rho \sin w\,(c_1\cos\theta + c_2\sin\theta)\,Q_\gamma E_2\\
&+\left[\lambda^{-1} w_\rho \sin w(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta+\dot\lambda\rho)(c_1\cos\theta+c_2\sin\theta)\right]Q_\gamma E_1\\
&-\left[\lambda^{-1}\rho^{-1}\sin^2 w(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)(c_1\cos\theta+c_2\sin\theta)\right]Q_\gamma E_2\\
=&\left[\lambda^{-1} w_\rho \sin w(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta+\dot\lambda\rho)(c_1\cos\theta+c_2\sin\theta)\right]Q_\gamma E_1\\
&+\left[\Big(\dot\gamma\rho w_\rho\sin w-\lambda^{-1}\rho^{-1}\sin^2 w(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)\Big)(c_1\cos\theta+c_2\sin\theta)\right]Q_\gamma E_2,\\
\mathcal{R}_U := &-\Big[\lambda^{-1}\rho^{-1}\sin w(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)(c_2\cos\theta-c_1\sin\theta)\\
&-\lambda^{-1}w_\rho[\cos w(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta)+\dot\lambda\rho w_\rho](c_1\cos\theta+c_2\sin\theta)\Big]Q_\gamma W.
\end{aligned}
$$

We observe that, compared to $\mathcal{R}_{U^\perp}$ and $\mathcal{R}_U$, the error terms $\mathcal{E}^{(0)}_{U^\perp}$ and $\mathcal{E}^{(\pm1)}_U$ decay slower in space, and $\mathcal{E}^{(\pm2)}_{U^\perp}$ and $\mathcal{E}^{(\pm2)}_U$ decay slower in time (in view of the blow-up dynamics that we search for). So we add several global corrections to improve the errors $\mathcal{E}^{(0)}_{U^\perp}$, $\mathcal{E}^{(\pm1)}_U$, $\mathcal{E}^{(\pm2)}_{U^\perp}$, and $\mathcal{E}^{(\pm2)}_U$.

To deal with $\mathcal{E}^{(0)}_{U^\perp}$ and $\mathcal{E}^{(\pm1)}_U$, we add two global/non-local corrections to improve the spatial decay at leading order. We aim to find $\Phi^{(0)}$ and $\Phi^{(1)}$ solving approximately

$$
\begin{aligned}
\partial_t\Phi^{(0)} &\approx \Delta\Phi^{(0)} + \mathcal{E}^{(0)}_{U^\perp},\\
\partial_t\Phi^{(1)} &\approx \Delta\Phi^{(1)} + \mathcal{E}^{(\pm1)}_{U}.
\end{aligned}
$$

Here $\Phi^{(0)}=\Phi^{(0)}[\lambda,\gamma,\xi,c_1,c_2]$ and $\Phi^{(1)}=\Phi^{(1)}[\lambda,\gamma,\xi,c_1,c_2]$ are non-local corrections, depending on the choice of parameters, to be specified below. To handle $\mathcal{E}^{(\pm2)}_{U^\perp}$ and $\mathcal{E}^{(\pm2)}_U$, we solve the following two linearized problems:

$$
\begin{aligned}
\partial_t\Phi^{(j)}_{U^\perp} &= L_U[\Phi^{(j)}_{U^\perp}] + \mathcal{E}^{(\pm2)}_{U^\perp},\quad j=\pm2,\\
\partial_t\Phi^{(2)}_{U} &= L_U[\Phi^{(2)}_{U}] + \mathcal{E}^{(\pm2)}_{U},
\end{aligned}
$$

where $\Phi^{(j)}_{U^\perp}=\Phi^{(j)}_{U^\perp}(\rho,t)$ is on $U^\perp$, and $\Phi^{(2)}_U=\Phi^{(2)}_U(\rho,t)$ is in the U-direction.

• **Approximate form of $\Phi^{(0)}$.**

We notice that

$$
\begin{aligned}
\mathcal{E}^{(0)}_{U^\perp} &= \lambda^{-1}\dot\lambda\rho w_\rho Q_\gamma E_1 + \dot\gamma\rho w_\rho Q_\gamma E_2\\
&\approx -\frac{2r(\dot\lambda+i\lambda\dot\gamma)}{\lambda^2+r^2}\begin{bmatrix} e^{i(\theta+\gamma)}\\ 0\end{bmatrix}\quad \text{for}\ \ r=|x-\xi|\gg\lambda.
\end{aligned}
$$

So we assume that

$$
\Phi^{(0)}(x,t)=\begin{bmatrix}\varphi^0(x,t)\\ 0\end{bmatrix}
$$

and φ^0 is an approximation of

$$
\varphi^0_t = \Delta\varphi^0 - \frac{2r}{r^2+\lambda^2}(\dot\lambda+i\lambda\dot\gamma)e^{i(\theta+\gamma)}.
$$

Let

$$
\varphi^0(x,t)=re^{i\theta}\psi^0(z(r),t),\quad z(r)=\sqrt{r^2+\lambda^2(t)},
$$

for $\psi^0(z,t)$ satisfying the equation

$$
\psi^0_t=\psi^0_{zz}+\frac{3\psi^0_z}{z}+\frac{p_0(t)}{z^2},
$$

where we define

$$
p_0(t) := -2(\dot\lambda+i\lambda\dot\gamma)e^{i\gamma}. \tag{4.5}
$$

From the Duhamel's principle, we know that

$$\psi^0(z,t) = \int_{-T}^t p_0(s)k(z,t-s)ds, \quad k(z,t) = \frac{1-e^{-\frac{z^2}{4t}}}{z^2} \tag{4.6}$$

is a weak solution. Notice $k_t = k_{zz} + \frac{3}{z}k_z$. Then we have

$$-\partial_t \Phi^{(0)} + \Delta_x \Phi^{(0)} = \tilde{\mathcal{R}}_0^0 + \tilde{\mathcal{R}}_1^0 := \tilde{\mathcal{R}}^0, \quad \tilde{\mathcal{R}}_0^0 = \begin{bmatrix} \mathcal{R}_0^0 \\ 0 \end{bmatrix}, \quad \tilde{\mathcal{R}}_1^0 = \begin{bmatrix} \mathcal{R}_1^0 \\ 0 \end{bmatrix}$$

where

$$\mathcal{R}_0^0 = -re^{i\theta}\frac{p_0(t)}{z^2} + re^{i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t p_0(s)(zk_z - z^2k_{zz})(z(r), t-s)ds$$

and

$$\mathcal{R}_1^0 = e^{i\theta}\mathrm{Re}[(\dot{\xi}e^{-i\theta})]\int_{-T}^t p_0(s)k(z(r),t-s)ds - \frac{r}{z^2}e^{i\theta}\left(\lambda\dot{\lambda} - \mathrm{Re}(re^{-i\theta}\dot{\xi}(t))\right)\int_{-T}^t p_0(s)zk_z(z(r),t-s)ds.$$

- **Approximate form of $\Phi^{(1)}$.**

Since

$$\begin{aligned}\mathcal{E}_U^{(\pm 1)} &= \rho w_\rho \left[\cos\theta(\dot{c}_1 + \lambda^{-1}\dot{\lambda}c_1) + \sin\theta(\dot{c}_2 + \lambda^{-1}\dot{\lambda}c_2)\right] Q_\gamma W \\ &\approx -\frac{2r}{r^2+\lambda^2}\begin{bmatrix} 0 \\ 0 \\ \cos\theta(\lambda c_1)' + \sin\theta(\lambda c_2)' \end{bmatrix} \quad \text{for} \quad r \gg \lambda,\end{aligned}$$

we assume that

$$\Phi^{(1)}(x,t) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{Re}(\varphi^1(x,t)) \end{bmatrix}$$

and φ^1 solves approximately

$$\varphi_t^1 = \Delta\varphi^1 - \frac{2r}{r^2+\lambda^2}e^{-i\theta}(\lambda\mathbf{c})',$$

where we use the notation

$$\mathbf{c}(t) := c_1(t) + ic_2(t). \tag{4.7}$$

We look for

$$\varphi^1(x,t) = re^{-i\theta}\psi^1(z(r),t), \quad z(r) = \sqrt{r^2+\lambda^2},$$

where $\psi^1(z,t)$ satisfies the equation

$$\psi_t^1 = \psi_{zz}^1 + \frac{3\psi_z^1}{z} + \frac{p_1(t)}{z^2},$$

and here we define

$$p_1(t) := -2(\lambda\mathbf{c})'. \tag{4.8}$$

Similar to (4.6), we see that

$$\psi^1(z,t) = \int_{-T}^t p_1(s)k(z,t-s)ds. \tag{4.9}$$

So we add a correction

$$\Phi^{(1)}(x,t) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{Re}(\varphi^1(r,t)) \end{bmatrix}$$

with

$$\varphi^1(r,t) = re^{-i\theta}\int_{-T}^t p_1(s)k(z(r),t-s)ds.$$

We next compute the new error produced by $\Phi^{(1)}$:

$$-\partial_t\Phi^{(1)} + \Delta_x\Phi^{(1)} = \mathrm{Re}(\tilde{\mathcal{R}}_0^1 + \tilde{\mathcal{R}}_1^1) := \tilde{\mathcal{R}}^1, \quad \tilde{\mathcal{R}}_0^1 = \begin{bmatrix} 0 \\ 0 \\ \mathcal{R}_0^1 \end{bmatrix}, \quad \tilde{\mathcal{R}}_1^1 = \begin{bmatrix} 0 \\ 0 \\ \mathcal{R}_1^1 \end{bmatrix}$$

where

$$\mathcal{R}_0^1 = -re^{-i\theta}\frac{p_1(t)}{z^2} + re^{-i\theta}\frac{\lambda^2}{z^4}\int_{-T}^{t} p_1(s)(zk_z - z^2 k_{zz})(z(r), t-s)ds$$

and

$$\mathcal{R}_1^1 = e^{-i\theta}\mathrm{Re}[(\dot{\xi}e^{-i\theta})]\int_{-T}^{t} p_1(s)k(z(r), t-s)ds - \frac{re^{-i\theta}}{z^2}\left(\lambda\dot{\lambda} - \mathrm{Re}(re^{-i\theta}\dot{\xi}(t))\right)\int_{-T}^{t} p_1(s)zk_z(z(r), t-s)ds.$$

• **Equations for $\Phi_{U^\perp}^{(\pm 2)}$.**

To handle $\mathcal{E}_{U^\perp}^{(\pm 2)}$ in (4.4), we first write it in complex form:

$$\begin{aligned}\left(Q_{-\gamma}\mathcal{E}_{U^\perp}^{(\pm 2)}\right)_{\mathbb{C}} &= -2\lambda^{-2}c_1c_2w_\rho^2\sin w(\sin 2\theta\cos w + i\cos 2\theta)\\ &= -i\lambda^{-2}c_1c_2w_\rho^2\sin w\left[e^{2i\theta}(1-\cos w) + e^{-2i\theta}(1+\cos w)\right]\\ &:= \left(Q_{-\gamma}\mathcal{E}_{U^\perp}^{(2)}\right)_{\mathbb{C}} + \left(Q_{-\gamma}\mathcal{E}_{U^\perp}^{(-2)}\right)_{\mathbb{C}}.\end{aligned}$$

We try to add two corrections, expressed in (ρ, t) coordinates

$$\begin{aligned}&\Phi_{U^\perp}^{(-2)}(\rho,t) = \phi_1^{(-2)}Q_\gamma E_1 + \phi_2^{(-2)}Q_\gamma E_2, \quad (\Phi_{U^\perp}^{(-2)})_{\mathbb{C}} = \phi_1^{(-2)} + i\phi_2^{(-2)} := \varphi_{-2}(\rho,t)e^{-2i\theta}\\ &\Phi_{U^\perp}^{(2)}(\rho,t) = \phi_1^{(2)}Q_\gamma E_1 + \phi_2^{(2)}Q_\gamma E_2, \quad (\Phi_{U^\perp}^{(2)})_{\mathbb{C}} = \phi_1^{(2)} + i\phi_2^{(2)} := \varphi_2(\rho,t)e^{2i\theta},\end{aligned} \tag{4.10}$$

where the complex-valued φ_{-2} and φ_2 solve

$$\lambda^2\partial_t\varphi_j = \partial_{\rho\rho}\varphi_j + \frac{1}{\rho}\partial_\rho\varphi_j + \left[\frac{8}{(1+\rho^2)^2} - \frac{1+j^2}{\rho^2} - \frac{2j(\rho^2-1)}{\rho^2(\rho^2+1)}\right]\varphi_j + \lambda^2 e^{-ji\theta}\left(Q_{-\gamma}\mathcal{E}_{U^\perp}^{(j)}\right)_{\mathbb{C}}, \quad j = \pm 2. \tag{4.11}$$

In other words, we solve the linearized problem

$$\partial_t\Phi_{U^\perp}^{(j)} = L_U[\Phi_{U^\perp}^{(j)}] + \mathcal{E}_{U^\perp}^{(\pm 2)}, \quad j = \pm 2,$$

where $\Phi_{U^\perp}^{(j)} = \Phi_{U^\perp}^{(j)}(\rho, t)$ is on $U^\perp$. A solution to (4.11) with zero initial data will be ensured by a linear theory developed later in Section 7.2.

• **Equation for $\Phi_U^{(2)}$.**

To deal with $\mathcal{E}_U^{(\pm 2)}$ given in (4.4), we need a correction $\Phi_U^{(2)}(\rho, t)$ in the form

$$\Phi_U^{(2)}(\rho,t) = \sin 2\theta\,\psi_2(\rho,t)Q_\gamma W \tag{4.12}$$

with ψ_2 solving

$$\lambda^2\partial_t\psi_2 = \partial_{\rho\rho}\psi_2 + \frac{1}{\rho}\partial_\rho\psi_2 - \frac{4}{\rho^2}\psi_2 + \frac{8}{(1+\rho^2)^2}\psi_2 + 2c_1c_2\rho w_\rho^2\sin^2 w \tag{4.13}$$

since it is exactly mode ± 2 of linearization in the U-direction (cf. (2.5)). Equation (4.13) with zero Cauchy data will be solved by the linear theory in Section 7.2.

We now compute the new error $S[U_*]$ of the corrected approximation

$$\begin{aligned}U_* &:= U_{\lambda,\gamma,\xi,c_1,c_2} + \eta_1\left(\Phi^{(0)} + \Phi^{(1)} + \Phi_{U^\perp}^{(2)} + \Phi_{U^\perp}^{(-2)} + \Phi_U^{(2)}\right)\\ &:= Q_\gamma(W + c_1\mathcal{Z}_{1,1} + c_2\mathcal{Z}_{1,2}) + \eta_1\Phi_*,\end{aligned} \tag{4.14}$$

where the purpose of the cut-off function

$$\eta_1 := \eta(x - \xi(t))$$

is to avoid potential slow spatial decay in the remote region. Here $\eta(s)$ is a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ for $s > 2$. We first analyze the leading terms

$$\mathcal{R}_{U^\perp} + \mathcal{R}_U + \mathcal{E}_{U^\perp}^{(1)} + \eta_1(\mathcal{E}_{U^\perp}^{(0)} + \tilde{\mathcal{R}}^0) + \eta_1(\mathcal{E}_U^{(\pm 1)} + \tilde{\mathcal{R}}^1) + \eta_1\tilde{L}_U[\Phi^{(0)} + \Phi^{(1)}]$$

in the corrected error $S[U_*]$. By definition, we have

$$\begin{aligned}
&\mathcal{R}_{U^\perp}+\mathcal{R}_U+\mathcal{E}^{(1)}_{U^\perp}\\
=&\Big[\lambda^{-1}w_\rho\sin w(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta+\dot\lambda\rho)(c_1\cos\theta+c_2\sin\theta)\Big]Q_\gamma E_1\\
&+\Big[\Big(\dot\gamma\rho w_\rho\sin w-\lambda^{-1}\rho^{-1}\sin^2 w(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)\Big)(c_1\cos\theta+c_2\sin\theta)\Big]Q_\gamma E_2\\
&-\Big[\lambda^{-1}\rho^{-1}\sin w(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)(c_2\cos\theta-c_1\sin\theta)\\
&\quad-\lambda^{-1}w_\rho[\cos w(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta)+\dot\lambda\rho w_\rho](c_1\cos\theta+c_2\sin\theta)\Big]Q_\gamma W\\
&+\frac{\dot\xi_1}{\lambda}w_\rho Q_\gamma\left[\cos\theta E_1+\sin\theta E_2\right]+\frac{\dot\xi_2}{\lambda}w_\rho Q_\gamma\left[\sin\theta E_1-\cos\theta E_2\right],
\end{aligned}\tag{4.15}$$

whose complex form on $W^\perp$ is

$$\begin{aligned}
&\Big(\mathcal{R}_{U^\perp}+\mathcal{R}_U+\mathcal{E}^{(1)}_{U^\perp}\Big)_{\mathbb{C}}\\
=&\ \rho w_\rho\sin w(\lambda^{-1}\dot\lambda+i\dot\gamma)(c_1\cos\theta+c_2\sin\theta)\\
&+\lambda^{-1}w_\rho\sin w(\dot\xi_1-i\dot\xi_2)e^{i\theta}(c_1\cos\theta+c_2\sin\theta)\\
&+\lambda^{-1}w_\rho(\dot\xi_1-i\dot\xi_2)e^{i\theta}.
\end{aligned}\tag{4.16}$$

Next, by using

$$\begin{aligned}
\begin{bmatrix} f(\rho)e^{i\theta}\\ g(\rho,\theta)\end{bmatrix}=&\Big[\cos w\mathrm{Re}(fe^{-i\gamma})-g\sin w\Big]Q_\gamma E_1+\mathrm{Im}(fe^{-i\gamma})Q_\gamma E_2\\
&+\Big[\sin w\mathrm{Re}(fe^{-i\gamma})+g\cos w\Big]Q_\gamma W,
\end{aligned}\tag{4.17}$$

one has

$$\begin{aligned}
&\mathcal{E}^{(0)}_{U^\perp}+\tilde{\mathcal{R}}^0\\
=&\ \rho w_\rho\begin{bmatrix}(\lambda^{-1}\dot\lambda+i\dot\gamma)e^{i(\theta+\gamma)}\\ 0\end{bmatrix}+\begin{bmatrix}\mathcal{R}^0_0\\ 0\end{bmatrix}+\begin{bmatrix}\mathcal{R}^0_1\\ 0\end{bmatrix}+\rho w_\rho\lambda^{-1}\dot\lambda\begin{bmatrix}e^{i(\theta+\gamma)}w_\rho\\ -\sin w\end{bmatrix}\\
=&\begin{bmatrix}(r\frac{\lambda^2}{z^4}\int_{-T}^t p_0(s)(zk_z-z^2k_{zz})(z(r),t-s)ds)e^{i\theta}\\ 0\end{bmatrix}+\begin{bmatrix}(\lambda^{-1}\dot\lambda e^{i\gamma}\rho w_\rho^2)e^{i\theta}\\ \lambda^{-1}\dot\lambda\rho^2w_\rho^2\end{bmatrix}+\begin{bmatrix}\mathcal{R}^0_1\\ 0\end{bmatrix}\\
=&\Big[\cos w\mathrm{Re}[(f^{(0)}_{U^\perp}+f^{(1)}_{U^\perp})e^{-i\gamma}]-\lambda^{-1}\dot\lambda\rho^2w_\rho^2\sin w\Big]Q_\gamma E_1+\mathrm{Im}[(f^{(0)}_{U^\perp}+f^{(1)}_{U^\perp})e^{-i\gamma}]Q_\gamma E_2\\
&+\Big[\sin w\mathrm{Re}[(f^{(0)}_{U^\perp}+f^{(1)}_{U^\perp})e^{-i\gamma}]+\lambda^{-1}\dot\lambda\rho^2w_\rho^2\cos w\Big]Q_\gamma W,
\end{aligned}\tag{4.18}$$

where

$$\begin{aligned}
&f^{(0)}_{U^\perp}:=r\frac{\lambda^2}{z^4}\int_{-T}^t p_0(s)(zk_z-z^2k_{zz})(z,t-s)ds+\lambda^{-1}\dot\lambda e^{i\gamma}\rho w_\rho^2-\frac{r}{z^2}\lambda\dot\lambda\int_{-T}^t p_0(s)zk_z(z,t-s)ds,\\
&f^{(1)}_{U^\perp}:=\mathrm{Re}[(\dot\xi e^{-i\theta})]\int_{-T}^t p_0(s)\Big(k(z,t-s)+\frac{r^2}{z}k_z(z,t-s)\Big)ds.
\end{aligned}\tag{4.19}$$

Similarly,

$$\begin{aligned}
&\mathcal{E}^{(\pm1)}_U+\tilde{\mathcal{R}}^1\\
=&\ \rho w_\rho\left[\cos\theta(\dot c_1+\lambda^{-1}\dot\lambda c_1)+\sin\theta(\dot c_2+\lambda^{-1}\dot\lambda c_2)\right]\begin{bmatrix}e^{i(\theta+\gamma)}\sin w\\ w_\rho\end{bmatrix}\\
&+\begin{bmatrix}0\\0\\ \mathrm{Re}\left[re^{-i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t p_1(s)(zk_z-z^2k_{zz})(z(r),t-s)ds\right]\end{bmatrix}+\begin{bmatrix}0\\0\\ \mathrm{Re}[\mathcal{R}^1_1]\end{bmatrix},
\end{aligned}$$

and from (4.17), it follows that

$$
\begin{aligned}
\left(\mathcal{E}_U^{(\pm1)}+\tilde{\mathcal{R}}^1\right)\cdot Q_\gamma W =&\ -\sin w\left(\rho^2 w_\rho^2\left[\cos\theta(\dot c_1+\lambda^{-1}\dot\lambda c_1)+\sin\theta(\dot c_2+\lambda^{-1}\dot\lambda c_2)\right]\right)\\
&+\left(\rho w_\rho^2\left[\cos\theta(\dot c_1+\lambda^{-1}\dot\lambda c_1)+\sin\theta(\dot c_2+\lambda^{-1}\dot\lambda c_2)\right]\right)\cos w\\
&+\mathrm{Re}\left[re^{-i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t p_1(s)(zk_z-z^2k_{zz})(z(r),t-s)ds+\mathcal{R}_1^1\right]\cos w\\
=&\ -\rho w_\rho^2\left[\cos\theta(\dot c_1+\lambda^{-1}\dot\lambda c_1)+\sin\theta(\dot c_2+\lambda^{-1}\dot\lambda c_2)\right]\\
&+\mathrm{Re}\left[re^{-i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t p_1(s)(zk_z-z^2k_{zz})(z(r),t-s)ds+\mathcal{R}_1^1\right]\cos w,\\
\left(\mathcal{E}_U^{(\pm1)}+\tilde{\mathcal{R}}^1\right)\cdot Q_\gamma E_1 =&\ -\cos w\left(\rho^2 w_\rho^2\left[\cos\theta(\dot c_1+\lambda^{-1}\dot\lambda c_1)+\sin\theta(\dot c_2+\lambda^{-1}\dot\lambda c_2)\right]\right)\\
&-\left(\rho w_\rho^2\left[\cos\theta(\dot c_1+\lambda^{-1}\dot\lambda c_1)+\sin\theta(\dot c_2+\lambda^{-1}\dot\lambda c_2)\right]\right)\sin w\\
&-\mathrm{Re}\left[re^{-i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t p_1(s)(zk_z-z^2k_{zz})(z(r),t-s)ds+\mathcal{R}_1^1\right]\sin w\\
=&\ -\rho^2 w_\rho^2\left[\cos\theta(\dot c_1+\lambda^{-1}\dot\lambda c_1)+\sin\theta(\dot c_2+\lambda^{-1}\dot\lambda c_2)\right]\\
&-\mathrm{Re}\left[re^{-i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t p_1(s)(zk_z-z^2k_{zz})(z(r),t-s)ds+\mathcal{R}_1^1\right]\sin w,\\
\left(\mathcal{E}_U^{(\pm1)}+\tilde{\mathcal{R}}^1\right)\cdot Q_\gamma E_2 =&\ 0.
\end{aligned}
\tag{4.20}
$$

Also, from Lemma 2.4, we have

$$
\begin{aligned}
&\tilde L_U[\Phi^{(0)}+\Phi^{(1)}]\\
=&\ \frac{2}{\lambda}\rho w_\rho^2\left[\mathrm{Re}(e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0))Q_\gamma E_1+\frac{1}{r}\mathrm{Im}(e^{-i\gamma}r\psi^0)Q_\gamma E_2\right]\\
&+\frac{2}{\lambda}w_\rho\left(\mathrm{Re}(e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0))\cos w-\frac{1}{r}\mathrm{Re}(e^{-i\gamma}r\psi^0)\right)Q_\gamma W\\
&-\frac{2}{\lambda}w_\rho\cos w\left(\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\psi^1_z)]Q_\gamma E_1-\frac{1}{r}\mathrm{Im}[e^{-i\theta}r\psi^1]Q_\gamma E_2\right)\\
&+\frac{2}{\lambda}\rho w_\rho^2\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\psi^1_z)]Q_\gamma W\\
=&\ \frac{2}{\lambda}w_\rho\left(\mathrm{Re}(e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0))\cos w-\frac{1}{r}\mathrm{Re}(e^{-i\gamma}r\psi^0)+\rho w_\rho\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\psi^1_z)]\right)Q_\gamma W\\
&+\frac{2}{\lambda}w_\rho\left(\rho w_\rho\mathrm{Re}(e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0))-\cos w\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\psi^1_z)]\right)Q_\gamma E_1\\
&+\frac{2}{\lambda r}w_\rho\left(\rho w_\rho\mathrm{Im}(e^{-i\gamma}r\psi^0)+\cos w\mathrm{Im}[e^{-i\theta}r\psi^1]\right)Q_\gamma E_2.
\end{aligned}
\tag{4.21}
$$

We denote the remaining terms in the error $S[U_*]$ by

$$
\mathcal{R}_*:=S[U_*]-\left[\mathcal{R}_{U^\perp}+\mathcal{R}_U+\mathcal{E}_{U^\perp}^{(1)}+\eta_1(\mathcal{E}_{U^\perp}^{(0)}+\tilde{\mathcal{R}}^0)+\eta_1(\mathcal{E}_U^{(\pm1)}+\tilde{\mathcal{R}}^1)+\eta_1\tilde L_U[\Phi^{(0)}+\Phi^{(1)}]\right]-E_{\eta_1},
\tag{4.22}
$$

where E_{η_1} is defined in (A.2), and we claim:

Lemma 4.1. *The remainder $\mathcal{R}_*$ in the corrected error $S[U_*]$ projected in each direction $Q_\gamma W$, $Q_\gamma E_1$ and $Q_\gamma E_2$ is given by*

$$
\begin{aligned}
&\mathcal{R}_*\cdot Q_\gamma W\\
=&-2\eta_1\lambda^{-1}w_\rho(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta+\dot\lambda\rho)(\phi_1^{(2)}+\phi_1^{(-2)})\\
&+\eta_1\sin w[\dot\gamma+\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)](\phi_2^{(2)}+\phi_2^{(-2)})
\end{aligned}
$$

$$
\begin{aligned}
&+\eta_1\lambda^{-1}(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta+\dot\lambda\rho)\sin 2\theta\partial_\rho\psi_2\\
&-2\eta_1\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)\cos 2\theta\psi_2\\
&-\frac{2\eta_1^2}{r}\Bigg[\left(\cos w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]-\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\sin w\right)\\
&\qquad+\lambda^{-1}\left(\sin 2\theta w_\rho\psi_2+\partial_\rho\phi_1^{(2)}+\partial_\rho\phi_1^{(-2)}\right)\Bigg]\\
&\quad\times\left(\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]+(\partial_\theta\phi_2^{(2)}+\partial_\theta\phi_2^{(-2)})+\cos w(\phi_1^{(2)}+\phi_1^{(-2)})+\sin 2\theta\sin w\psi_2\right)\\
&+\frac{2\eta_1^2}{r}\left(\mathrm{Im}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\lambda^{-1}(\partial_\rho\phi_2^{(2)}+\partial_\rho\phi_2^{(-2)})\right)\\
&\quad\times\left(\cos w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]-\mathrm{Im}(re^{-i\theta}\psi^1)\sin w+(\partial_\theta\phi_1^{(2)}+\partial_\theta\phi_1^{(-2)})-\cos w(\phi_2^{(2)}+\phi_2^{(-2)})\right)\\
&-\frac{2\eta_1}{r}\left(\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]+(\partial_\theta\phi_2^{(2)}+\partial_\theta\phi_2^{(-2)})+\cos w(\phi_1^{(2)}+\phi_1^{(-2)})+\sin 2\theta\sin w\psi_2\right)\\
&\quad\times\lambda^{-1}w_\rho\sin w(c_1\cos\theta+c_2\sin\theta)\\
&-\frac{2\eta_1}{r}\Bigg[\left(\cos w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]-\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\sin w\right)\\
&\qquad+\lambda^{-1}\left(\sin 2\theta w_\rho\psi_2+\partial_\rho\phi_1^{(2)}+\partial_\rho\phi_1^{(-2)}\right)\Bigg]\times\sin^2 w(c_1\cos\theta+c_2\sin\theta),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{R}_*\cdot Q_\gamma E_1\\
=\;&\eta_1\cos w[\dot\gamma+\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)](\phi_2^{(2)}+\phi_2^{(-2)})\\
&+2\eta_1\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)[\phi_2^{(2)}-\phi_2^{(-2)}]\\
&+\eta_1\lambda^{-1}(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta+\dot\lambda\rho)(\partial_\rho\phi_1^{(2)}+\partial_\rho\phi_1^{(-2)})\\
&+\eta_1\lambda^{-1}w_\rho(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta+\dot\lambda\rho)\sin 2\theta\psi_2\\
&-\frac{2\eta_1^2}{r}\left(\mathrm{Im}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\lambda^{-1}(\partial_\rho\phi_2^{(2)}+\partial_\rho\phi_2^{(-2)})\right)\\
&\quad\times\left(\sin w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]+\mathrm{Im}(re^{-i\theta}\psi^1)\cos w-\sin w(\phi_2^{(2)}+\phi_2^{(-2)})+2\psi_2\cos 2\theta\right)\\
&+\frac{2\eta_1^2}{r}\left(\sin w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\cos w\right)\\
&\quad\times\left(\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]+(\partial_\theta\phi_2^{(2)}+\partial_\theta\phi_2^{(-2)})+\cos w(\phi_1^{(2)}+\phi_1^{(-2)})+\sin 2\theta\sin w\psi_2\right)\\
&+\frac{2\eta_1}{r}\left(\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]+(\partial_\theta\phi_2^{(2)}+\partial_\theta\phi_2^{(-2)})+\cos w(\phi_1^{(2)}+\phi_1^{(-2)})+\sin 2\theta\sin w\psi_2\right)\\
&\quad\times\lambda^{-1}w_\rho\cos w(c_1\cos\theta+c_2\sin\theta)\\
&+\frac{2\eta_1}{r}\Bigg[\left(\sin w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\cos w\right)\\
&\qquad+\lambda^{-1}\left[\sin 2\theta\partial_\rho\psi_2-w_\rho(\phi_1^{(2)}+\phi_1^{(-2)})\right]\Bigg]\times\sin^2 w(c_1\cos\theta+c_2\sin\theta)\\
&-\frac{2\eta_1}{r}\left(\mathrm{Im}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\lambda^{-1}(\partial_\rho\phi_2^{(2)}+\partial_\rho\phi_2^{(-2)})\right)\times\sin w(c_2\cos\theta-c_1\sin\theta),
\end{aligned}
$$

$\mathcal{R}_*\cdot Q_\gamma E_2$

$$
\begin{aligned}
= & -\eta_1 \cos w[\dot\gamma+\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)](\phi_1^{(2)}+\phi_1^{(-2)}) \\
& +2\eta_1\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)[\phi_1^{(-2)}-\phi_1^{(2)}] \\
& +\eta_1\lambda^{-1}(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta+\dot\lambda\rho)(\partial_\rho\phi_2^{(2)}+\partial_\rho\phi_2^{(-2)}) \\
& -\eta_1\sin w[\dot\gamma+\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)]\sin 2\theta\psi_2 \\
& -\frac{2\eta_1^2}{r}\Bigg[\left(\sin w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\cos w\right) \\
& \qquad +\lambda^{-1}\left[\sin 2\theta\partial_\rho\psi_2-w_\rho(\phi_1^{(2)}+\phi_1^{(-2)})\right]\Bigg] \\
& \quad\times\left(\cos w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]-\mathrm{Im}(re^{-i\theta}\psi^1)\sin w+(\partial_\theta\phi_1^{(2)}+\partial_\theta\phi_1^{(-2)})-\cos w(\phi_2^{(2)}+\phi_2^{(-2)})\right) \\
& +\frac{2\eta_1^2}{r}\Bigg[\left(\cos w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]-\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\sin w\right) \\
& \qquad +\lambda^{-1}\left(\sin 2\theta w_\rho\psi_2+\partial_\rho\phi_1^{(2)}+\partial_\rho\phi_1^{(-2)}\right)\Bigg] \\
& \quad\times\left(\sin w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]+\mathrm{Im}(re^{-i\theta}\psi^1)\cos w-\sin w(\phi_2^{(2)}+\phi_2^{(-2)})+2\psi_2\cos 2\theta\right) \\
& -\frac{2\eta_1}{r}\left(\cos w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]-\mathrm{Im}(re^{-i\theta}\psi^1)\sin w+(\partial_\theta\phi_1^{(2)}+\partial_\theta\phi_1^{(-2)})-\cos w(\phi_2^{(2)}+\phi_2^{(-2)})\right) \\
& \quad\times\lambda^{-1}w_\rho\cos w(c_1\cos\theta+c_2\sin\theta) \\
& +\frac{2\eta_1}{r}\left(\sin w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]+\mathrm{Im}(re^{-i\theta}\psi^1)\cos w-\sin w(\phi_2^{(2)}+\phi_2^{(-2)})+2\psi_2\cos 2\theta\right) \\
& \quad\times\lambda^{-1}w_\rho\sin w(c_1\cos\theta+c_2\sin\theta) \\
& +\frac{2\eta_1}{r}\Bigg[\left(\cos w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]-\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\sin w\right) \\
& \qquad +\lambda^{-1}\left(\sin 2\theta w_\rho\psi_2+\partial_\rho\phi_1^{(2)}+\partial_\rho\phi_1^{(-2)}\right)\Bigg]\times\sin w(c_2\cos\theta-c_1\sin\theta).
\end{aligned}
$$

The derivation of above Lemma is rather lengthy, and we postpone it in Appendix A.2.

5. Formulating the gluing system

We aim to find a real solution

$$u=U_*+\Phi,$$

so $S[u]=0$ yields

$$\partial_t\Phi=\Delta\Phi-2\partial_{x_1}U_*\wedge\partial_{x_2}\Phi-2\partial_{x_1}\Phi\wedge\partial_{x_2}U_*-2\partial_{x_1}\Phi\wedge\partial_{x_2}\Phi+S[U_*].$$

We decompose Φ into

$$\Phi(x,t)=\eta_R Q_\gamma\Big[\Phi_W\,(y,t)+\Phi_{W^\perp}\,(y,t)\Big]+\Psi(x,t),\quad y=\frac{x-\xi(t)}{\lambda(t)},$$

where Φ_W is in the W-direction, $\Phi_{W^\perp}$ is on $W^\perp$,

$$\eta_R:=\eta\left(\frac{x-\xi(t)}{R(t)\lambda(t)}\right),$$

and $R(t)=\lambda^{-\beta}(t)$ with $\beta>0$ to be chosen later.

Then it is sufficient to find a desired solution u to (1.9) if the triple $(\Phi_W,\ \Phi_{W^\perp},\ \Psi)$ satisfies the gluing system:

$$
\begin{aligned}
\lambda^2\partial_t\Phi_W &= \Delta_y\Phi_W - 2\partial_{y_1}W\wedge\partial_{y_2}\Phi_W - 2\partial_{y_1}\Phi_W\wedge\partial_{y_2}W\\
&\quad + \lambda^2\Pi_W\Big[Q_{-\gamma}\tilde L_U[\Psi]\Big] + \lambda^2\Pi_W\Big[Q_{-\gamma}(S[U_*])\Big] + \lambda^2\mathcal{H}^W_{\rm in} \quad \text{in } \ \mathcal{D}_{2R},\\
\lambda^2\partial_t\Phi_{W^\perp} &= \Delta_y\Phi_{W^\perp} - 2\partial_{y_1}W\wedge\partial_{y_2}\Phi_{W^\perp} - 2\partial_{y_1}\Phi_{W^\perp}\wedge\partial_{y_2}W\\
&\quad + \lambda^2\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde L_U[\Psi]\Big] + \lambda^2\Pi_{W^\perp}\Big[Q_{-\gamma}(S[U_*])\Big] + \lambda^2\mathcal{H}^{W^\perp}_{\rm in} \quad \text{in } \ \mathcal{D}_{2R},\\
\partial_t\Psi &= \Delta_x\Psi + (1-\eta_R)\tilde L_U[\Psi] + (1-\eta_R)S[U_*]\\
&\quad + \Big[Q_\gamma(\Phi_W+\Phi_{W^\perp})\Delta_x\eta_R + 2\nabla_x\eta_R\cdot\nabla_x(Q_\gamma\Phi_W + Q_\gamma\Phi_{W^\perp}) - Q_\gamma(\Phi_W+\Phi_{W^\perp})\partial_t\eta_R\Big]\\
&\quad - 2(1-\eta_R)\partial_{x_1}(U_*-U)\wedge\partial_{x_2}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\\
&\quad - 2(1-\eta_R)\partial_{x_1}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\wedge\partial_{x_2}(U_*-U)\\
&\quad + (1-\eta_R)\Big[-\eta_R\dot\gamma J_z Q_\gamma(\Phi_W+\Phi_{W^\perp}) + \eta_R(\lambda^{-1}\dot\xi + \lambda^{-1}\dot\lambda y)\cdot\nabla_y(Q_\gamma\Phi_W + Q_\gamma\Phi_{W^\perp})\Big]\\
&\quad - 2(1-\eta_R)\partial_{x_1}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\wedge\partial_{x_2}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big) \ \text{in } \ \mathbb{R}^2\times(0,T),
\end{aligned}
\tag{5.1}
$$

i.e., Φ_W and $\Phi_{W^\perp}$ satisfy inner problem in the W-direction and on $W^\perp$, respectively, and Ψ solves the outer problem. Here

$$\mathcal{D}_{2R} := \Big\{(y,t)\in\mathbb{R}^2\times\mathbb{R}_+ : \ |y|\le 2R(t), \ t\in(0,T)\Big\},$$

$\widetilde{\mathcal{R}}_*$, E_{η_1} are defined in (6.2), (A.2), and

$$
\begin{aligned}
\mathcal{H}^W_{\rm in} &= \Pi_W\Bigg\{Q_{-\gamma}\Bigg[-2\partial_{x_1}(U_*-U)\wedge\partial_{x_2}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\\
&\qquad - 2\partial_{x_1}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\wedge\partial_{x_2}(U_*-U)\\
&\qquad - \eta_R\dot\gamma J_z Q_\gamma(\Phi_W+\Phi_{W^\perp}) + \eta_R(\lambda^{-1}\dot\xi+\lambda^{-1}\dot\lambda y)\cdot\nabla_y(Q_\gamma\Phi_W+Q_\gamma\Phi_{W^\perp})\\
&\qquad - 2\partial_{x_1}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\wedge\partial_{x_2}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\Bigg]\Bigg\},\\
\mathcal{H}^{W^\perp}_{\rm in} &= \Pi_{W^\perp}\Bigg\{Q_{-\gamma}\Bigg[-2\partial_{x_1}(U_*-U)\wedge\partial_{x_2}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\\
&\qquad - 2\partial_{x_1}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\wedge\partial_{x_2}(U_*-U)\\
&\qquad - \eta_R\dot\gamma J_z Q_\gamma(\Phi_W+\Phi_{W^\perp}) + \eta_R(\lambda^{-1}\dot\xi+\lambda^{-1}\dot\lambda y)\cdot\nabla_y(Q_\gamma\Phi_W+Q_\gamma\Phi_{W^\perp})\\
&\qquad - 2\partial_{x_1}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\wedge\partial_{x_2}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\Bigg]\Bigg\}.
\end{aligned}
\tag{5.2}
$$

We will solve equations $(5.1)_1$ and $(5.1)_2$ with zero initial data and $(5.1)_3$ with non-trivial initial data

$$\Psi(x,0) = Z^* = (z_1^*, z_2^*, z_3^*), \quad x\in\mathbb{R}^2.$$

Problems $(5.1)_1$ and $(5.1)_2$ are the inner problem which captures the blow-up property, while $(5.1)_3$ describes the main character with singularities removed. We will solve these problems by developing corresponding linear theories and the fixed point argument in suitable weighted spaces. We notice that the inner problem $(5.1)_2$ is essentially a perturbation of the linearized harmonic map heat flow around the bubble W, and the inner problem $(5.1)_1$ is a perturbation of the parabolic linearized Liouville equation (cf. Lemma 2.1 and Remark 2.1.1).

6. Leading dynamics for the parameters

In this section, we capture the leading dynamics of the parameters $\lambda(t)$, $\xi(t)$, $\gamma(t)$, $c_1(t)$, $c_2(t)$, and these are in fact determined by orthogonality conditions in corresponding modes that are needed to ensure the existence of desired solutions with fast space-time decay (in the linear theories in Section 7.2). We will choose $\lambda R \ll 1$, so $\eta_1(x) = \eta_1^2(x) = 1$ and $E_{\eta_1} = 0$ (cf. (A.2)) for $|x - \xi(t)| \le 2\lambda R$. Our aim is to adjust the parameters such that the orthogonalities hold

$$
\begin{aligned}
&\int_{B_{2R}} \left(\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U[\Psi]\right] + \Pi_{W^\perp}\left[Q_{-\gamma}(S[U_*])\right] + \mathcal{H}_{\rm in}^{W^\perp}\right) \cdot Z_{i,j}(y)dy = 0, \quad i = 0,1,\ j = 1,2,\\
&\int_{B_{2R}} \left(\Pi_{W}\left[Q_{-\gamma}\tilde{L}_U[\Psi]\right] + \Pi_{W}\left[Q_{-\gamma}(S[U_*])\right] + \mathcal{H}_{\rm in}^{W}\right) \cdot \mathcal{Z}_{1,j}(y)dy = 0, \quad j = 1,2
\end{aligned}
\tag{6.1}
$$

for all $t \in (0,T)$, where the kernels $Z_{i,j}$ and $\mathcal{Z}_{1,j}$ are defined in (2.2). The reason for not requiring orthogonalities with the other three kernel functions $Z_{-1,1}$, $Z_{-1,2}$ and $\mathcal{Z}_0$ is to avoid further complications due to the introduction of new modulation parameters, and we shall use linear theories without orthogonality conditions in Section 7.2 together with regularity estimates to control these modes.

The goal of this section is to derive the leading dynamics governing these parameters by approximating (6.1). To do this, we decompose the remainder $\mathcal{R}_*$ in $S[U_*]$ as

$$
\mathcal{R}_* := \widetilde{\mathcal{R}}_* + \mathcal{R}_{*,1}, \tag{6.2}
$$

where $\mathcal{R}_{*,1}$ is defined as

$$
\begin{aligned}
&\mathcal{R}_{*,1}\\
:=\ & -\frac{2\eta_1}{r}\sin w(c_1\cos\theta + c_2\sin\theta)\left(\lambda^{-1}w_\rho {\rm Im}\left[e^{-i\gamma}ir\psi^0\right] + \sin w\cos w {\rm Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right]\right)Q_\gamma W\\
&+\frac{2\eta_1}{r}(c_1\cos\theta + c_2\sin\theta)\left(\lambda^{-1}w_\rho\cos w {\rm Im}\left[e^{-i\gamma}ir\psi^0\right] + \sin^3 w {\rm Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right]\right)Q_\gamma E_1\\
&-\frac{2\eta_1}{r}\sin w(c_2\cos\theta - c_1\sin\theta){\rm Im}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right]Q_\gamma E_1\\
&-\frac{2\eta_1}{r}\lambda^{-1}w_\rho\cos 2w(c_1\cos\theta + c_2\sin\theta){\rm Re}\left[e^{-i\gamma}ir\psi^0\right]Q_\gamma E_2\\
&+\frac{2\eta_1}{r}\sin w\cos w(c_2\cos\theta - c_1\sin\theta){\rm Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right]Q_\gamma E_2.
\end{aligned}
\tag{6.3}
$$

So the error reads

$$
S[U_*] = \left[\mathcal{R}_{U^\perp} + \mathcal{R}_U + \mathcal{E}_{U^\perp}^{(1)} + \eta_1(\mathcal{E}_{U^\perp}^{(0)} + \tilde{\mathcal{R}}^0) + \eta_1(\mathcal{E}_U^{(\pm1)} + \tilde{\mathcal{R}}^1) + \eta_1\tilde{L}_U[\Phi^{(0)} + \Phi^{(1)}]\right] + \widetilde{\mathcal{R}}_* + \mathcal{R}_{*,1} + E_{\eta_1}. \tag{6.4}
$$

In what follows, $S[U_*]$ in the orthogonality conditions will be approximated by its main terms, and the analysis of the remainders will be done in Section 8.3.

6.1. λ-γ system and translation parameter ξ.

We start from the RHS on $W^\perp$, and this turns out to yield the dynamics for λ and γ at mode 0 and for $\xi := (\xi_1,\ \xi_2)$ at mode 1.

- **Mode 0 on $W^\perp$: λ-γ system.**

The orthogonality condition $(6.1)_1$ with $i=0$ and $j=1,2$ in the complex form implies

$$
\int_0^{2\pi}\int_0^{2R}\left(\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U[\Psi]\right] + \Pi_{W^\perp}\left[Q_{-\gamma}(S[U_*])\right] + \mathcal{H}_{\rm in}^{W^\perp}\right)_{\mathbb{C}}\rho^2 w_\rho d\rho d\theta = 0, \tag{6.5}
$$

which is a complex system for λ and γ. We single out the main terms at mode 0:

$$
\left(\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U[Z_0^*(q)]\right] + \Pi_{W^\perp}\left[Q_{-\gamma}\left((\mathcal{E}_{U^\perp}^{(0)} + \tilde{\mathcal{R}}^0) + \tilde{L}_U[\Phi^{(0)}]\right)\right]\right)_{\mathbb{C},0}.
$$

We will deal with the remainder terms, turn out to be faster vanishing in time, when we solve the full problem via fixed point argument. From Lemma 2.3, one has

$$\begin{aligned}&\Big(\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde{L}_U[Z_0^*(q)]\Big]\Big)_{\mathbb{C},0}\\&=\lambda^{-1}\rho w_\rho^2[\mathrm{div}(e^{-i\gamma}Z^*(q))+i\mathrm{curl}(e^{-i\gamma}Z^*(q))].\end{aligned}$$

Also, from (4.18), (4.19), (4.21), and (4.6), we obtain

$$\begin{aligned}&\Big(\Pi_{W^\perp}\Big[Q_{-\gamma}\Big((\mathcal{E}_{U^\perp}^{(0)}+\tilde{\mathcal{R}}^0)+\tilde{L}_U[\Phi^{(0)}]\Big)\Big]\Big)_{\mathbb{C},0}\\
=&\Big(\cos w\mathrm{Re}[(f_{U^\perp}^{(0)})e^{-i\gamma}]-\lambda^{-1}\dot{\lambda}\rho^2w_\rho^2\sin w\Big)+i\mathrm{Im}[(f_{U^\perp}^{(0)})e^{-i\gamma}]\\
&+2\lambda^{-1}\rho w_\rho^2\mathrm{Re}\Big[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\Big]+i2\lambda^{-1}\rho w_\rho^2\mathrm{Im}(e^{-i\gamma}\psi^0)\\
=&\cos w\mathrm{Re}\left[\frac{\lambda^{-1}\rho}{(1+\rho^2)^2}\int_{-T}^t p_0(s)e^{-i\gamma(t)}\big(zk_z(z,t-s)-z^2k_{zz}(z,t-s)\big)ds\right]\\
&-\cos w\mathrm{Re}\left[\frac{\dot{\lambda}\rho}{1+\rho^2}\int_{-T}^t p_0(s)e^{-i\gamma(t)}zk_z(z,t-s)ds\right]-\lambda^{-1}\dot{\lambda}\rho w_\rho^2\\
&+2\lambda^{-1}\rho w_\rho^2\mathrm{Re}\left[\int_{-T}^t p_0(s)e^{-i\gamma(t)}\Big(k(z,t-s)+\frac{r^2}{z}k_z(z,t-s)\Big)ds\right]\\
&+i\mathrm{Im}\left[\frac{\lambda^{-1}\rho}{(1+\rho^2)^2}\int_{-T}^t p_0(s)e^{-i\gamma(t)}\big(zk_z(z,t-s)-z^2k_{zz}(z,t-s)\big)ds\right]\\
&-i\mathrm{Im}\left[\frac{\dot{\lambda}\rho}{1+\rho^2}\int_{-T}^t p_0(s)e^{-i\gamma(t)}zk_z(z,t-s)ds\right]\\
&+i2\lambda^{-1}\rho w_\rho^2\mathrm{Im}\Big[\int_{-T}^t p_0(s)e^{-i\gamma(t)}k(z,t-s)ds\Big]\\
=&-\cos w\mathrm{Re}\left[\lambda^{-1}\rho w_\rho^2\int_{-T}^t\frac{p_0(s)e^{-i\gamma(t)}}{t-s}\zeta^2K_{\zeta\zeta}ds\right]\\
&+\cos w\mathrm{Re}\left[\dot{\lambda}\rho w_\rho\int_{-T}^t\frac{p_0(s)e^{-i\gamma(t)}}{t-s}\zeta K_\zeta ds\right]-\lambda^{-1}\dot{\lambda}\rho w_\rho^2\\
&+2\lambda^{-1}\rho w_\rho^2\mathrm{Re}\left[\int_{-T}^t\frac{p_0(s)e^{-i\gamma(t)}}{t-s}\Big(K+\frac{2\rho^2}{1+\rho^2}\zeta K_\zeta\Big)ds\right]\\
&+i\mathrm{Im}\left[\lambda^{-1}\rho w_\rho^2\int_{-T}^t\frac{p_0(s)e^{-i\gamma(t)}}{t-s}\zeta^2K_{\zeta\zeta}ds\right]\\
&+i\mathrm{Im}\left[\dot{\lambda}\rho w_\rho\int_{-T}^t\frac{p_0(s)e^{-i\gamma(t)}}{t-s}\zeta K_\zeta ds\right]+i2\lambda^{-1}\rho w_\rho^2\mathrm{Im}\left[\int_{-T}^t\frac{p_0(s)e^{-i\gamma(t)}}{t-s}Kds\right],\end{aligned}$$

where

$$k(z,t)=\frac{1-e^{-\frac{z^2}{4t}}}{z^2},\quad \zeta=\frac{z^2}{t-s}=\frac{\lambda^2(1+\rho^2)}{t-s},\quad K(\zeta)=\frac{1-e^{-\frac{\zeta}{4}}}{\zeta}. \tag{6.6}$$

Therefore, (6.5) at leading order gives the following system in complex form:

$$\begin{aligned}&\int_{-T}^t\frac{\mathrm{Re}(p_0(s)e^{-i\gamma(t)})}{t-s}\Gamma_1\left(\frac{\lambda(t)^2}{t-s}\right)ds+2\dot{\lambda}(t)+i\int_{-T}^t\frac{\mathrm{Im}(p_0(s)e^{-i\gamma(t)})}{t-s}\Gamma_2\left(\frac{\lambda(t)^2}{t-s}\right)ds\\
&=2\Big[\mathrm{div}(e^{-i\gamma(t)}Z^*(q))+i\mathrm{curl}(e^{-i\gamma(t)}Z^*(q))\Big],\end{aligned}\tag{6.7}$$

where

$$\Gamma_1(\tau) = \int_0^\infty \left(\rho^3 w_\rho^3 \left[\left(2K(\zeta) + 4\zeta K_\zeta(\zeta)\frac{\rho^2}{1+\rho^2}\right) - \cos w\zeta^2 K_{\zeta\zeta}(\zeta)\right] + \cos w\lambda\dot\lambda\rho^3 w_\rho^2 \zeta K_\zeta\right)_{\zeta=\tau(1+\rho^2)} d\rho,$$

$$\Gamma_2(\tau) = \int_0^\infty \left(\rho^3 w_\rho^3 \left[2K(\zeta) + \zeta^2 K_{\zeta\zeta}(\zeta)\right] + \lambda\dot\lambda\rho^3 w_\rho^2\zeta K_\zeta\right)_{\zeta=\tau(1+\rho^2)} d\rho,$$

and we have used $\int_0^{+\infty}\rho^3 w_\rho^3 d\rho = -2$. Clearly,

$$\Gamma_1(\tau) = \begin{cases} -1 + O(\tau), & \tau \le 1, \\ O\left(\frac{1}{\tau}\right), & \tau > 1, \end{cases} \qquad \Gamma_2(\tau) = \begin{cases} -1 + O(\tau), & \tau \le 1, \\ O\left(\frac{1}{\tau}\right), & \tau > 1, \end{cases}$$

so it follows from (6.7) that

$$\int_{-T}^{t-\lambda^2} \frac{p_0(s)}{t-s} ds = -2\Big[\operatorname{div} Z^*(q) + i\operatorname{curl} Z^*(q)\Big] + O(p_0(t)) + o(1), \tag{6.8}$$

as $t \to T$. Recall $p_0(t) := -2(\dot\lambda + i\lambda\dot\gamma)e^{i\gamma}$. We then proceed as [17, Section 5, p. 372] to obtain

$$\lambda(t) \sim \frac{T-t}{|\log(T-t)|^2}, \quad \gamma(t) = \gamma_* = \arctan\frac{\operatorname{curl} Z^*(q)}{\operatorname{div} Z^*(q)}$$

as $t \to T$ under the assumption that $\operatorname{div} Z^*(q) < 0$. So the leading part of $\lambda(t)$ is given by

$$\lambda_*(t) = \frac{(T-t)|\log T|}{|\log(T-t)|^2},$$

where the $|\log T|$ is a normalization factor. As we will see later, the dynamics of λ-γ system and c_1-c_2 system are in fact coupled, and this results in an extra restriction:

$$\cos\gamma_* \operatorname{div} Z^*(q) + \sin\gamma_* \operatorname{curl} Z^*(q) \le 0, \tag{6.9}$$

roughly speaking, yielding more instability.

Remark 6.0.1. *Both restrictions* $\operatorname{div} Z^*(q) < 0$ *and* (6.9) *can be achieved at the same time by choosing initial data* Z^* *such that*

$$\operatorname{div} Z^*(q) < 0, \quad \left|\frac{\operatorname{curl} Z^*(q)}{\operatorname{div} Z^*(q)}\right| \ll 1$$

since once Z^* *is fixed the rotation angle* γ_* *is determined automatically by*

$$\tan\gamma_* = \frac{\operatorname{curl} Z^*(q)}{\operatorname{div} Z^*(q)}.$$

In such case, $\gamma_* \ll 1$ *and thus one has* (6.9).

• **Mode 1 on $W^\perp$: ξ.**

Similarly, the orthogonality condition $(6.1)_1$ with $i = 1$ and $j = 1, 2$ in the complex form gives

$$\int_0^{2\pi}\int_0^{2R} \left(\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde L_U[\Psi]\Big] + \Pi_{W^\perp}\Big[Q_{-\gamma}(S[U_*])\Big] + \mathcal{H}_{\rm in}^{W^\perp}\right)_{\mathbb{C}} w_\rho e^{-i\theta}\rho d\rho d\theta = 0. \tag{6.10}$$

From (4.15), (4.16), (4.18), (4.19), (4.20), (4.21), (4.22) and (6.2), the main terms that have contribution in B_{2R} in above integration are given by

$$\begin{aligned} &\left(\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde L_U[\Psi]\Big] + \Pi_{W^\perp}\Big[Q_{-\gamma}(S[U_*] - \widetilde{\mathcal{R}}_*)\Big]\right)_{\mathbb{C},1} \\ = &\left(\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde L_U[Z_0^*(q)]\Big] + \Pi_{W^\perp}\Big[Q_{-\gamma}(S[U_*] - \widetilde{\mathcal{R}}_*)\Big]\right)_{\mathbb{C},1} + \left(\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde L_U[\Psi - Z_0^*(q)]\Big]\right)_{\mathbb{C},1} \\ = &-2\lambda^{-1}w_\rho\cos w e^{i\theta}\Big(\partial_{x_1}z_3^*(q) - i\partial_{x_2}z_3^*(q)\Big) + \left(\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde L_U[\Psi - Z_0^*(q)]\Big]\right)_{\mathbb{C},1} \\ &+ \lambda^{-1}w_\rho(\dot\xi_1 - i\dot\xi_2)e^{i\theta} + \rho w_\rho \sin w(\lambda^{-1}\dot\lambda + i\dot\gamma)\frac{e^{i\theta}}{2}(c_1 - ic_2) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}e^{i(\theta-\gamma)}(\dot\xi_1-i\dot\xi_2)\int_{-T}^{t}p_0(s)\Big(k+\frac{r^2}{z}k_z\Big)(z,t-s)ds\\
&+\frac{w_\rho}{4}e^{i\theta}(\dot\xi_1-i\dot\xi_2)\left[\int_{-T}^{t}\Big(p_0(s)e^{-i\gamma(t)}+\bar p_0(s)e^{i\gamma(t)}\Big)\Big(k+\frac{r^2}{z}k_z\Big)(z,t-s)ds\right]\\
&+\frac{\lambda^{-1}}{4}\rho^2w_\rho^2e^{i\theta}\bar p_1(t)+\frac{\lambda^{-1}}{8}\rho^2w_\rho^3e^{i\theta}\int_{-T}^{t}\bar p_1(s)(zk_z-z^2k_{zz})(z,t-s)ds\\
&+\frac{1}{4}\rho^2w_\rho^2e^{i\theta}\dot\lambda\int_{-T}^{t}\bar p_1(s)zk_z(z,t-s)ds-2\lambda^{-1}w_\rho\cos we^{i\theta}\int_{-T}^{t}\bar p_1(s)k(z,t-s)ds\\
&-\lambda^{-1}w_\rho\cos w\frac{r^2}{z}e^{i\theta}\int_{-T}^{t}\bar p_1(s)k_z(z,t-s)ds\\
&+\frac{2}{r}(c_1\cos\theta+c_2\sin\theta)\left(\lambda^{-1}w_\rho\cos w\,\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]+\sin^3w\,\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]\right)\\
&-\frac{2}{r}\sin w(c_2\cos\theta-c_1\sin\theta)\mathrm{Im}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]\\
&-i\frac{2}{r}\lambda^{-1}w_\rho\cos 2w(c_1\cos\theta+c_2\sin\theta)\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]\\
&+i\frac{2}{r}\sin w\cos w(c_2\cos\theta-c_1\sin\theta)\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right].
\end{aligned}
$$

Then the dynamics given by (6.10) is approximately the following ODE:

$$
\begin{aligned}
\dot\xi_1-i\dot\xi_2\sim{}& CR^{-2}\Big(\partial_{x_1}z_3^*(q)-i\partial_{x_2}z_3^*(q)\Big)+O(|\mathbf{c}|)\\
&-\frac{\lambda}{4\pi}\int_0^{2\pi}\int_0^{2R}\Big(\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde L_U[\Psi-Z_0^*(q)]\Big]+\Pi_{W^\perp}\Big[Q_{-\gamma}\widetilde{\mathcal R}_*\Big]+\mathcal H_{\rm in}^{W^\perp}\Big)_{\mathbb C,1}w_\rho e^{-i\theta}\rho d\rho d\theta
\end{aligned}
\tag{6.11}
$$

for some constant C, where we have used the fact that $\int_0^{+\infty}\rho w_\rho^2\cos w d\rho=0$. For later purpose (when dealing with c_1-c_2 system), we will choose initial data such that

$$\partial_{x_1}z_3^*(q)=\partial_{x_2}z_3^*(q)=0,$$

and

$$\lambda\Big|Q_{-\gamma}\tilde L_U[\Psi-Z_0^*(q)]\Big|\lesssim\lambda^\Theta,\quad |\mathbf{c}|\lesssim\lambda^\Theta(t)$$

for some $0<\Theta<1$. For the gluing procedure to work, we will eventually choose Θ to be slightly bigger than $1/3$ (cf. the final choice (8.29)). Because of this, also by estimates (8.11) and (8.24) that will be carried out later, one has

$$\lambda\Big|\Pi_{W^\perp}\Big[Q_{-\gamma}\widetilde{\mathcal R}_*\Big]+\mathcal H_{\rm in}^{W^\perp}\Big|\lesssim\lambda^{3\Theta-1}.$$

Therefore, there exists a solution

$$\xi(t)=q+o((T-t)^{3\Theta})$$

for above ODE, where we recall $\lambda(t)\sim\frac{T-t}{|\log(T-t)|^2}$.

6.2. **c_1-c_2 system.** Finally, we derive the asymptotic behavior of the translation parameters in the W-direction. From the linear theory of the inner problem in the W-direction, we need:

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^{2R}\Big[Q_{-\gamma}\tilde L_U[\Psi]\cdot W+\Big(Q_{-\gamma}(S[U_*])+\mathcal H_{\rm in}^W\Big)\cdot W\Big]\sin w\cos\theta\rho d\rho d\theta=0,\\
&\int_0^{2\pi}\int_0^{2R}\Big[Q_{-\gamma}\tilde L_U[\Psi]\cdot W+\Big(Q_{-\gamma}(S[U_*])+\mathcal H_{\rm in}^W\Big)\cdot W\Big]\sin w\sin\theta\rho d\rho d\theta=0.
\end{aligned}
\tag{6.12}
$$

By (4.15), (4.18), (4.20), (4.21), (4.22) and (6.2), we have

$$
\begin{aligned}
&Q_{-\gamma}(S[U_*]-\widetilde{\mathcal R}_*)\cdot W\\
={}&Q_{-\gamma}\Big(\mathcal R_{U^\perp}+\mathcal R_U+\mathcal E_{U^\perp}^{(1)}+(\mathcal E_{U^\perp}^{(0)}+\tilde{\mathcal R}^0)+(\mathcal E_U^{(\pm1)}+\tilde{\mathcal R}^1)+\tilde L_U[\Phi^{(0)}+\Phi^{(1)}]\Big)\cdot W
\end{aligned}
$$

$$
\begin{aligned}
&+Q_{-\gamma}(\mathcal{R}_*-\widetilde{\mathcal{R}}_*)\cdot W\\
=\;&-\lambda^{-1}\rho^{-1}\sin w(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)(c_2\cos\theta-c_1\sin\theta)\\
&+\lambda^{-1}w_\rho[\cos w(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta)+\dot\lambda\rho w_\rho](c_1\cos\theta+c_2\sin\theta)\\
&+\sin w\mathrm{Re}[(f^{(0)}_{U^\perp}+f^{(1)}_{U^\perp})e^{-i\gamma}]+\lambda^{-1}\dot\lambda\rho^2w_\rho^2\cos w\\
&-\rho w_\rho^2\left[\cos\theta(\dot c_1+\lambda^{-1}\dot\lambda c_1)+\sin\theta(\dot c_2+\lambda^{-1}\dot\lambda c_2)\right]\\
&+\mathrm{Re}\left[re^{-i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t p_1(s)(zk_z-z^2k_{zz})(z(r),t-s)ds+\mathcal{R}_1^1\right]\cos w\\
&+\frac{2}{\lambda}w_\rho\left(\mathrm{Re}(e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0))\cos w-\frac{1}{r}\mathrm{Re}(e^{-i\gamma}r\psi^0)+\rho w_\rho\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\psi^1_z)]\right),\\
&-\frac{2}{r}\sin w(c_1\cos\theta+c_2\sin\theta)\left(\lambda^{-1}w_\rho\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]+\sin w\cos w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]\right),
\end{aligned}
$$

and by Lemma 2.3,

$$
\begin{aligned}
&Q_{-\gamma}\tilde L_U[\Psi]\cdot W\approx Q_{-\gamma}\tilde L_U[Z^*(q)]\cdot W\\
=\;&\frac{1}{\lambda}w_\rho^2\,\mathrm{div}(e^{-i\gamma}Z^*(q)))-\frac{2}{\lambda}w_\rho\sin w[\partial_{x_1}z_3^*(q)\cos\theta+\partial_{x_2}z_3^*(q)\sin\theta]\\
&+\frac{2}{\lambda}w_\rho\left[-\frac12\rho^2w_\rho\,\mathrm{div}(e^{i\gamma}\bar Z^*(q))\cos(2\theta)+\frac12\rho^2w_\rho\mathrm{curl}(e^{i\gamma}\bar Z^*(q))\sin(2\theta)\right].
\end{aligned}
$$

So the terms in $\left(Q_{-\gamma}\tilde L_U[\Psi]\cdot W+Q_{-\gamma}(S[U_*]-\widetilde{\mathcal{R}}_*)\cdot W\right)$ that might contribute in (6.12) are given by

$$
\begin{aligned}
&-2\lambda^{-1}w_\rho\sin w[\partial_{x_1}z_3^*(q)\cos\theta+\partial_{x_2}z_3^*(q)\sin\theta]+Q_{-\gamma}\tilde L_U[\Psi-Z^*(q)]\cdot W\\
&-\lambda^{-1}\rho^{-1}\sin w(\dot\xi_1\sin\theta-\dot\xi_2\cos\theta)(c_2\cos\theta-c_1\sin\theta)\\
&+\lambda^{-1}w_\rho[\cos w(\dot\xi_1\cos\theta+\dot\xi_2\sin\theta)+\dot\lambda\rho w_\rho](c_1\cos\theta+c_2\sin\theta)\\
&+\sin w\mathrm{Re}[(f^{(1)}_{U^\perp})e^{-i\gamma}]-\rho w_\rho^2\left[\cos\theta(\dot c_1+\lambda^{-1}\dot\lambda c_1)+\sin\theta(\dot c_2+\lambda^{-1}\dot\lambda c_2)\right]\\
&+\mathrm{Re}\left[re^{-i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t p_1(s)(zk_z-z^2k_{zz})(z(r),t-s)ds+\mathcal{R}_1^1\right]\cos w\\
&+\frac{2}{\lambda}w_\rho\left(\rho w_\rho\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\psi^1_z)]\right)\\
&-\frac{2}{r}\sin w(c_1\cos\theta+c_2\sin\theta)\lambda^{-1}w_\rho\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]\\
&-\frac{2}{r}\sin w(c_1\cos\theta+c_2\sin\theta)\sin w\cos w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]\\
=\;&-2\lambda^{-1}w_\rho\sin w[\partial_{x_1}z_3^*(q)\cos\theta+\partial_{x_2}z_3^*(q)\sin\theta]+Q_{-\gamma}\tilde L_U[\Psi-Z^*(q)]\cdot W\\
&+\lambda^{-1}w_\rho^2\left[\dot\xi_1c_1\cos^2\theta+\dot\xi_2c_2\sin^2\theta+\dot\xi_1c_2\sin\theta\cos\theta+\dot\xi_2c_1\sin\theta\cos\theta\right]\\
&+\lambda^{-1}w_\rho\left[\sin2\theta(\dot\xi_1c_2+\dot\xi_2c_1)+\cos2\theta(\dot\xi_1c_1-\dot\xi_2c_2)\right]-\rho w_\rho^2\left[\cos\theta\dot c_1+\sin\theta\dot c_2\right]\\
&+\sin w\mathrm{Re}[(\dot\xi e^{-i\theta})]\mathrm{Re}\left[\int_{-T}^t p_0(s)e^{-i\gamma(t)}\left(k+\frac{r^2}{z}k_z\right)(z,t-s)ds\right]\\
&+\cos w\mathrm{Re}\left[re^{-i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t p_1(s)(zk_z-z^2k_{zz})(z(r),t-s)ds\right]\\
&+\cos w\mathrm{Re}\Bigg[e^{-i\theta}\mathrm{Re}[(\dot\xi e^{-i\theta})]\int_{-T}^t p_1(s)k(z(r),t-s)ds\\
&\qquad-\frac{re^{-i\theta}}{z^2}\left(\lambda\dot\lambda-\mathrm{Re}(re^{-i\theta}\dot\xi(t))\right)\int_{-T}^t p_1(s)zk_z(z(r),t-s)ds\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\lambda^{-1}\rho w_\rho^2 \mathrm{Re}\left[e^{-i\theta}\int_{-T}^{t} p_1(s)\Big(k+\frac{r^2}{z}k_z\Big)(z,t-s)ds\right]\\
&+ 2\lambda^{-1}w_\rho \sin w(c_1\cos\theta + c_2\sin\theta)\mathrm{Re}\left[\int_{-T}^{t} p_0(s)e^{-i\gamma(t)}\frac{r^2}{z}k_z(z,t-s)ds\right]\\
&+ 2\lambda^{-1}w_\rho^2 \sin w(c_1\cos\theta + c_2\sin\theta)\mathrm{Re}\left[\int_{-T}^{t} p_0(s)e^{-i\gamma(t)}\Big(k+\frac{r^2}{z}k_z\Big)(z,t-s)ds\right].
\end{aligned}
$$

So orthogonality condition $(6.12)_1$ implies

$$
\begin{aligned}
&-2\pi\partial_{x_1}z_3^*(q)\lambda^{-1}\int_0^{+\infty}\rho w_\rho \sin^2 w d\rho - \pi\dot{c}_1\int_0^{+\infty}\rho^2 w_\rho^2\sin w d\rho\\
&+\int_0^{2\pi}\int_0^{+\infty}\Big(Q_{-\gamma}\tilde{L}_U[\Psi - Z^*(q)]\cdot W\Big)\sin w\cos\theta\rho d\rho d\theta\\
&+\pi\dot{\xi}_1\int_0^{+\infty}\rho\sin^2 w\mathrm{Re}\left[\int_{-T}^{t} p_0(s)e^{-i\gamma(t)}\Big(k+\frac{r^2}{z}k_z\Big)(z,t-s)ds\right]d\rho\\
&+\frac{\pi}{4}\lambda^{-1}\int_0^{+\infty}\rho^2 w_\rho^2\sin w\cos w\int_{-T}^{t}\mathrm{Re}[p_1](s)(zk_z - z^2k_{zz})(z,t-s)dsd\rho\\
&+\frac{\pi}{2}\dot{\lambda}\int_0^{+\infty}\rho^2 w_\rho \sin w\cos w\int_{-T}^{t}\mathrm{Re}[p_1](s)zk_z(z,t-s)dsd\rho\\
&+2\pi\lambda^{-1}\int_0^{+\infty}\rho^2 w_\rho^2\sin w\int_{-T}^{t}\mathrm{Re}[p_1](s)\Big(k+\frac{r^2}{z}k_z\Big)(z,t-s)dsd\rho\\
&+2\pi\lambda^{-1}c_1\int_0^{+\infty}\rho w_\rho\sin^2 w\mathrm{Re}\left[\int_{-T}^{t} p_0(s)e^{-i\gamma(t)}\frac{r^2}{z}k_z(z,t-s)ds\right]d\rho\\
&+2\pi\lambda^{-1}c_1\int_0^{+\infty}\rho w_\rho^2\sin^2 w\mathrm{Re}\left[\int_{-T}^{t} p_0(s)e^{-i\gamma(t)}\Big(k+\frac{r^2}{z}k_z\Big)(z,t-s)ds\right]d\rho\\
&=0,
\end{aligned}
$$

and $(6.12)_2$ implies

$$
\begin{aligned}
&-2\pi\partial_{x_2}z_3^*(q)\lambda^{-1}\int_0^{+\infty}\rho w_\rho \sin^2 w d\rho - \pi\dot{c}_2\int_0^{+\infty}\rho^2 w_\rho^2\sin w d\rho\\
&+\int_0^{2\pi}\int_0^{+\infty}\Big(Q_{-\gamma}\tilde{L}_U[\Psi - Z^*(q)]\cdot W\Big)\sin w\sin\theta\rho d\rho d\theta\\
&+\pi\dot{\xi}_2\int_0^{+\infty}\rho\sin^2 w\mathrm{Re}\left[\int_{-T}^{t} p_0(s)e^{-i\gamma(t)}\Big(k(z,t-s)+\frac{r^2}{z}k_z(z,t-s)\Big)ds\right]d\rho\\
&+\frac{\pi}{4}\lambda^{-1}\int_0^{+\infty}\rho^2 w_\rho^2\sin w\cos w\int_{-T}^{t}\mathrm{Im}[p_1](s)(zk_z - z^2k_{zz})(z(r),t-s)dsd\rho\\
&+\frac{\pi}{2}\dot{\lambda}\int_0^{+\infty}\rho^2 w_\rho \sin w\cos w\int_{-T}^{t}\mathrm{Im}[p_1](s)zk_z(z,t-s)dsd\rho\\
&+2\pi\lambda^{-1}\int_0^{+\infty}\rho^2 w_\rho^2\sin w\int_{-T}^{t}\mathrm{Im}[p_1](s)\Big(k+\frac{r^2}{z}k_z\Big)(z,t-s)dsd\rho\\
&+2\pi\lambda^{-1}c_2\int_0^{+\infty}\rho w_\rho\sin^2 w\mathrm{Re}\left[\int_{-T}^{t} p_0(s)e^{-i\gamma(t)}\frac{r^2}{z}k_z(z,t-s)ds\right]d\rho\\
&+2\pi\lambda^{-1}c_2\int_0^{+\infty}\rho w_\rho^2\sin^2 w\mathrm{Re}\left[\int_{-T}^{t} p_0(s)e^{-i\gamma(t)}\Big(k+\frac{r^2}{z}k_z\Big)(z,t-s)ds\right]d\rho\\
&=0.
\end{aligned}
$$

Neglecting the terms that have faster vanishing in time, we arrive at

$$
\begin{aligned}
&4\partial_{x_1} z_3^*(q) - 2\lambda \dot{c}_1 + \frac{\lambda}{\pi}\int_0^{2\pi}\int_0^{+\infty} \Big(Q_{-\gamma}\tilde{L}_U[\Psi - Z^*(q)]\cdot W\Big)\sin w\cos\theta\rho d\rho d\theta \\
&- \int_0^{+\infty}\rho^2 w_\rho^2 \sin w\cos w\int_{-T}^t \frac{\mathrm{Re}[p_1](s)}{t-s}\zeta^2 K_{\zeta\zeta} ds d\rho \\
&+ 2\int_0^{+\infty}\rho^2 w_\rho^2 \sin w\int_{-T}^t \frac{\mathrm{Re}[p_1](s)}{t-s}\Big(K(\zeta) + \frac{2\rho^2}{1+\rho^2}\zeta K_\zeta\Big) ds d\rho \\
&+ 2c_1\int_0^{+\infty}\rho w_\rho \sin^2 w\int_{-T}^t \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\frac{2\rho^2}{1+\rho^2}\zeta K_\zeta ds d\rho \\
&+ 2c_1\int_0^{+\infty}\rho w_\rho^2 \sin^2 w\int_{-T}^t \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\Big(K(\zeta) + \frac{2\rho^2}{1+\rho^2}\zeta K_\zeta\Big) ds d\rho \\
\approx\ & 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&4\partial_{x_2} z_3^*(q) - 2\lambda \dot{c}_2 + \frac{\lambda}{\pi}\int_0^{2\pi}\int_0^{+\infty} \Big(Q_{-\gamma}\tilde{L}_U[\Psi - Z^*(q)]\cdot W\Big)\sin w\sin\theta\rho d\rho d\theta \\
&- \int_0^{+\infty}\rho^2 w_\rho^2 \sin w\cos w\int_{-T}^t \frac{\mathrm{Im}[p_1](s)}{t-s}\zeta^2 K_{\zeta\zeta} ds d\rho \\
&+ 2\int_0^{+\infty}\rho^2 w_\rho^2 \sin w\int_{-T}^t \frac{\mathrm{Im}[p_1](s)}{t-s}\Big(K(\zeta) + \frac{2\rho^2}{1+\rho^2}\zeta K_\zeta\Big) ds d\rho \\
&+ 2c_2\int_0^{+\infty}\rho w_\rho \sin^2 w\int_{-T}^t \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\frac{2\rho^2}{1+\rho^2}\zeta K_\zeta ds d\rho \\
&+ 2c_2\int_0^{+\infty}\rho w_\rho^2 \sin^2 w\int_{-T}^t \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\Big(K(\zeta) + \frac{2\rho^2}{1+\rho^2}\zeta K_\zeta\Big) ds d\rho \\
\approx\ & 0,
\end{aligned}
$$

where ζ and $K(\zeta)$ are defined in (6.6). Therefore, in complex form, the orthogonality condition in the W-direction reads

$$
\begin{aligned}
&\int_{-T}^t \frac{p_1(s)}{t-s}\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right) ds + 2\lambda\dot{\mathbf{c}} + \Gamma_4[p_0]\mathbf{c} \\
\approx\ & 4\big(\partial_{x_1} z_3^*(q) + i\partial_{x_2} z_3^*(q)\big) + \frac{\lambda}{\pi}\int_0^{2\pi}\int_0^{+\infty}\Big(Q_{-\gamma}\tilde{L}_U[\Psi - Z^*(q)]\cdot W\Big)\sin w e^{i\theta}\rho d\rho d\theta
\end{aligned}
$$

where

$$
\begin{aligned}
&p_1(t) = -2(\lambda\mathbf{c})', \quad \mathbf{c}(t) = c_1(t) + ic_2(t), \\
&\Gamma_3(\tau) = \int_0^{+\infty}\rho^2 w_\rho^2 \sin w\left[\cos w\zeta^2 K_{\zeta\zeta}(\zeta) - 2K(\zeta) - \frac{4\rho^2}{1+\rho^2}\zeta K_\zeta(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho,
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_4[p_0] =\ & -2\int_0^{+\infty}\rho w_\rho \sin^2 w\int_{-T}^t \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\frac{2\rho^2}{1+\rho^2}\zeta K_\zeta ds d\rho \\
& -2\int_0^{+\infty}\rho w_\rho^2 \sin^2 w\int_{-T}^t \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\Big(K(\zeta) + \frac{2\rho^2}{1+\rho^2}\zeta K_\zeta\Big) ds d\rho \\
=\ & \int_{-T}^t \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\Gamma_5\left(\frac{\lambda^2(t)}{t-s}\right) ds
\end{aligned}
$$

with

$$
\Gamma_5(\tau) = -2\int_0^{+\infty}\rho w_\rho \sin^2 w\left[(1+w_\rho)\frac{2\rho^2}{1+\rho^2}\zeta K_\zeta(\zeta) + w_\rho K(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho.
$$

It is direct to see that

$$\Gamma_3(\tau)=\begin{cases}-1+O(\tau), & \tau\le 1,\\ O\left(\frac{1}{\tau}\right), & \tau>1,\end{cases}\qquad \Gamma_5(\tau)=\begin{cases}-\frac{2}{3}+O(\tau), & \tau\le 1,\\ O\left(\frac{1}{\tau}\right), & \tau>1,\end{cases}$$

and thus above reduced problem, at main order, can be written in the complex form as

$$\int_{-T}^{t-\lambda^2(t)}\frac{p_1(s)}{t-s}ds+\frac{2}{3}\left(\int_{-T}^{t-\lambda^2(t)}\frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}ds\right)\mathbf{c}=-4\big(\partial_{x_1}z_3^*(q)+i\partial_{x_2}z_3^*(q)\big)+f(t),$$

where

$$f(t):=-\frac{\lambda}{\pi}\int_0^{2\pi}\int_0^{+\infty}\left(Q_{-\gamma}\tilde{L}_U[\Psi-Z^*(q)]\cdot W\right)\sin we^{i\theta}\rho d\rho d\theta.$$

In order for $p_1(t)$ to vanish as $t\to T$, we require

$$\partial_{x_1}z_3^*(q)=\partial_{x_2}z_3^*(q)=0.$$

Then the new balancing condition becomes

$$\int_{-T}^{t-\lambda^2(t)}\frac{p_1(s)}{t-s}ds+\frac{2}{3}\left(\int_{-T}^{t-\lambda^2(t)}\frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}ds\right)\mathbf{c}=f(t). \tag{6.13}$$

Here, $f(t)$ is essentially the contribution from the outer profile Ψ, and we will later see from the weighted space chosen for Ψ that

$$|f(t)|\lesssim\lambda_*^{\Theta}(t),\quad \lambda_*(t)=\frac{(T-t)|\log T|}{|\log(T-t)|^2}$$

for some $0<\Theta<1$.

We now derive the leading asymptotic behavior of (6.13) by approximating it by an ODE, and the non-local parts remained will be solved by a linear theory in later section together with the fixed point argument.

Notice, by the leading order of $\lambda\sim\lambda_*$, that

$$\begin{aligned}\int_{-T}^{t-\lambda^2(t)}\frac{p_1(s)}{t-s}ds&=\int_{-T}^{t-(T-t)}\frac{p_1(s)}{t-s}ds+p_1(t)\log(t-s)\Big|_{s=t-\lambda^2(t)}^{t-(T-t)}+\int_{t-(T-t)}^{t-\lambda^2(t)}\frac{p_1(s)-p_1(t)}{t-s}ds\\&=\int_{-T}^{t-(T-t)}\frac{p_1(s)}{t-s}ds+p_1(t)\left[\log(T-t)-2\log\lambda(t)\right]+\int_{t-(T-t)}^{t-\lambda^2(t)}\frac{p_1(s)-p_1(t)}{t-s}ds\\&\approx\int_{-T}^{t}\frac{p_1(s)}{T-s}ds-p_1(t)\log(T-t)\\&:=\mathcal{P}_1(t),\end{aligned}$$

and exactly the same argument works for

$$\begin{aligned}\int_{-T}^{t-\lambda^2(t)}\frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}ds&\approx\int_{-T}^{t}\frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{T-s}ds-\mathrm{Re}[p_0(t)e^{-i\gamma(t)}]\log(T-t)\\&:=\mathcal{P}_0(t).\end{aligned}$$

We then have an approximation for equation (6.13):

$$\mathcal{P}_1(t)+\frac{2}{3}\mathcal{P}_0(t)\mathbf{c}=f(t),$$

and we want to solve for $p_1(t)$ from here. Notice that

$$\begin{aligned}-\log(T-t)\mathcal{P}_1'&=\left[\log^2(T-t)p_1(t)\right]',\\-\log(T-t)\mathcal{P}_0'&=\left[\log^2(T-t)\mathrm{Re}[p_0(t)e^{-i\gamma(t)}]\right]'.\end{aligned}$$

Recall that

$$p_1=-2(\lambda\mathbf{c})',\quad p_0=-2(\dot\lambda+i\lambda\dot\gamma)e^{i\gamma}.$$

Then we get

$$2\Big[\log^2(T-t)(\lambda\mathbf{c})'\Big]' + \frac{4}{3}\Big[\log^2(T-t)\dot\lambda\Big]'\mathbf{c} + \frac{2}{3}\log(T-t)\mathcal{P}_0\mathbf{c}' = \log(T-t)f'(t). \tag{6.14}$$

The third term above can in fact be approximated by

$$\begin{aligned}\mathcal{P}_0 &\approx 2\dot\lambda - 2\mathrm{Re}\Big[\operatorname{div}(e^{-i\gamma_*}Z^*(q)) + i\operatorname{curl}(e^{-i\gamma_*}Z^*(q))\Big]\\ &\approx\ -2\mathrm{Re}\Big[\operatorname{div}(e^{-i\gamma_*}Z^*(q)) + i\operatorname{curl}(e^{-i\gamma_*}Z^*(q))\Big]\end{aligned}$$

due to (6.8), where γ_* is the rotation angle. Let us now write

$$\mathbf{Z} := -2\mathrm{Re}\Big[\operatorname{div}(e^{-i\gamma_*}Z^*(q)) + i\operatorname{curl}(e^{-i\gamma_*}Z^*(q))\Big],$$

and thus equation (6.14) reads approximately

$$2\Big[\log^2(T-t)(\lambda\mathbf{c})'\Big]' + \frac{2}{3}\log(T-t)\mathbf{Z}\mathbf{c}' = \log(T-t)f'(t)$$

as $\frac{4}{3}\Big[\log^2(T-t)\dot\lambda\Big]'\mathbf{c}$ is relatively smaller in size. Above equation should be understood in the weak sense when solving rigorously $\mathbf{c}$. But for now, we only single out the main part, smooth for $t\in(0,T)$, of $\mathbf{c}$. The full solvability of $\mathbf{c}$ is given in Section 9.

Integrating above equation implies

$$-2\log^2(T-t)(\lambda\mathbf{c})' + \frac{2\mathbf{Z}}{3}\int_t^T \log(T-s)\mathbf{c}'(s)ds = -f(t)\log(T-t) + \int_t^T \frac{f(s)}{T-s}ds$$

which is again approximately

$$-2\log^2(T-t)(\lambda\mathbf{c})' - \frac{2\mathbf{Z}}{3}\log(T-t)\mathbf{c}(t) = -f(t)\log(T-t) + \int_t^T \frac{f(s)}{T-s}ds. \tag{6.15}$$

The equation for $\lambda\mathbf{c}$ is given by

$$\Big[(T-t)^{-\frac{\mathbf{Z}}{6}\frac{\log(T-t)}{|\log T|}}(\lambda\mathbf{c})\Big]' = (T-t)^{-\frac{\mathbf{Z}}{6}\frac{\log(T-t)}{|\log T|}}\left(\frac{f(t)}{2\log(T-t)} - \frac{1}{2\log^2(T-t)}\int_t^T\frac{f(s)}{T-s}ds\right).$$

Thanks to (6.9), i.e., $\mathbf{Z}\ge 0$, one can find a solution $\mathbf{c}$ which vanishes as $t\to T$. Indeed,

$$\begin{aligned}|\lambda\mathbf{c}| &= \left|(T-t)^{\frac{\mathbf{Z}}{6}\frac{\log(T-t)}{|\log T|}}\int_t^T (T-\tau)^{-\frac{\mathbf{Z}}{6}\frac{\log(T-\tau)}{|\log T|}}\left(\frac{f(\tau)}{2\log(T-\tau)} - \frac{1}{2\log^2(T-\tau)}\int_\tau^T\frac{f(s)}{T-s}ds\right)d\tau\right|\\ &\lesssim \frac{|f(t)|}{|\log(T-t)|}\left|(T-t)^{\frac{\mathbf{Z}}{6}\frac{\log(T-t)}{|\log T|}}\int_t^T (T-\tau)^{-\frac{\mathbf{Z}}{6}\frac{\log(T-\tau)}{|\log T|}}d\tau\right|\\ &\lesssim \begin{cases}\frac{3|\log T|(T-t)|f(t)|}{\mathbf{Z}|\log(T-t)|^2}, & \mathbf{Z}>0,\\ \frac{(T-t)|f(t)|}{|\log(T-t)|}, & \mathbf{Z}=0,\end{cases}\end{aligned}$$

and

$$|\mathbf{c}| \lesssim \begin{cases}\lambda_*^\Theta(t), & \mathbf{Z}>0,\\ \lambda_*^\Theta(t)\frac{|\log(T-t)|}{|\log T|}, & \mathbf{Z}=0\end{cases}$$

because of $|f(t)|\lesssim\lambda_*^\Theta(t)$. Throughout the rest of this paper, we consider the most representative case when $\mathbf{Z}>0$, so we have

$$|\mathbf{c}| \lesssim \lambda_*^\Theta(t). \tag{6.16}$$

Remark 6.0.2. *The case with $\mathbf{Z}=0$ is in fact more special as the leading dynamics of c_1-c_2 system in such case reduces to the one similar to that of λ-γ system. The coupling between λ-γ system and c_1-c_2 system only appears in the case $\mathbf{Z}>0$.*

7. Linear theories

In this section, we give the linear theories concerning the a priori estimates for the linear problem of outer problem $(5.1)_3$, and the inner problems in the W-direction $(5.1)_1$ and on $W^\perp$ $(5.1)_2$.

7.1. **Linear theory for the outer problem.**

For $q \in \mathbb{R}^2$ and $T > 0$ sufficiently small, we consider the problem

$$\begin{cases} \psi_t = \Delta_x \psi + g(x,t) & \text{in } \mathbb{R}^2 \times (0,T), \\ \psi(x,0) = Z^*(x) & \text{in } \mathbb{R}^2 \end{cases} \tag{7.1}$$

for smooth initial value Z^* with compact support. The RHS g of (7.1) is assumed to be bounded with respect to some weights that appear in the outer problem $(5.1)_3$. Thus we define the weights

$$\begin{cases} \varrho_1 := \lambda_*^{\Theta} (\lambda_* R)^{-1} \mathbf{1}_{\{r \le 3\lambda_* R\}}, \\ \varrho_2 := T^{-\sigma_0} \dfrac{\lambda_*^{1-\sigma_0}}{r^2} \mathbf{1}_{\{r \ge \lambda_* R\}}, \\ \varrho_3 := T^{-\sigma_0}, \end{cases} \tag{7.2}$$

where $r = |x - q|$, $\Theta > 0$ and $\sigma_0 > 0$ is small. For a function $g(x,t)$ we define the L^∞-weighted norm

$$\|g\|_{**} := \sup_{\gamma \times (0,T)} \Big(1 + \sum_{i=1}^{3} \varrho_i(x,t) \Big)^{-1} |g(x,t)|. \tag{7.3}$$

We define the L^∞-weighted norm for ψ

$$\begin{aligned} \|\psi\|_{\sharp,\Theta,\alpha} := {} & \lambda_*^{-\Theta}(0) \frac{1}{|\log T| \lambda_*(0) R(0)} \|\psi\|_{L^\infty(\mathbb{R}^2 \times (0,T))} + \lambda_*^{-\Theta}(0) \|\nabla_x \psi\|_{L^\infty(\mathbb{R}^2 \times (0,T))} \\ & + \sup_{\mathbb{R}^2 \times (0,T)} \lambda_*^{-\Theta-1}(t) R^{-1}(t) \frac{1}{|\log(T-t)|} |\psi(x,t) - \psi(x,T)| \\ & + \sup_{\mathbb{R}^2 \times (0,T)} \lambda_*^{-\Theta}(t) |\nabla_x \psi(x,t) - \nabla_x \psi(x,T)| \\ & + \sup \lambda_*^{-\Theta}(t) (\lambda_*(t) R(t))^{\alpha} \frac{|\nabla_x \psi(x,t) - \nabla_x \psi(x',t')|}{(|x-x'|^2 + |t-t'|)^{\alpha/2}}, \end{aligned} \tag{7.4}$$

where $\Theta \in (0,1)$, $\alpha \in (0, \frac{1}{2})$, and the last supremum is taken in the region

$$x,\, x' \in \mathbb{R}^2, \quad t,\, t' \in (0,T), \quad |x - x'| \le 2\lambda_*(t) R(t), \quad |t - t'| < \frac{1}{4}(T-t).$$

Also, we choose the initial data $Z^*(x)$ so that

$$\psi_1(q,T) = \psi_2(q,T) = \psi_3(q,T) = \partial_{x_1} \psi_3(q,T) = \partial_{x_2} \psi_3(q,T) = 0. \tag{7.5}$$

This can be achieved by the following: choose regular maps $\mathscr{Z}_k$ with compact support, $k = 1, \dots, 5$ satisfying

$$\mathscr{Z}_k(q) = e_k^{(5)}$$

where $\{e_k^{(5)}\}_{k=1}^5$ forms an orthonormal basis of $\mathbb{R}^5$. We choose $\mathscr{C}_k$ so that

$$\begin{aligned} & \Big(\Gamma_{\mathbb{R}^2} \bullet \ \mathcal{G},\ [\nabla_x (\Gamma_{\mathbb{R}^2} \bullet \ \mathcal{G})]_3 \Big)(q,T) + \Big(\Gamma_{\mathbb{R}^2} \circ \ Z^*,\ [\nabla_x (\Gamma_{\mathbb{R}^2} \circ Z^*)]_3 \Big)(q,T) \\ & + \sum_{k=1}^{5} \mathscr{C}_k (\Gamma_{\mathbb{R}^2} \circ \mathscr{Z}_k,\ [\nabla_x (\Gamma_{\mathbb{R}^2} \circ \mathscr{Z}_k)]_3)(q,T) = 0. \end{aligned} \tag{7.6}$$

The first three and the last two vanishing conditions in (7.5) are needed in the gluing process and in c_1-c_2 system (i.e. $p_1(t)$), respectively.

We shall measure the solution ψ to the problem (7.1) in the norm $\| \ \|_{\sharp,\Theta,\alpha}$ defined in (7.4). We invoke some useful estimates proved in [17, Appendix A] as follows.

Proposition 7.1 ([17]). *For $T > 0$ sufficiently small, there is a linear operator that maps a function $g : \mathbb{R}^2 \times (0,T) \to \mathbb{R}^3$ with $\|g\|_{**} < \infty$ into ψ which solves problem (7.1). Moreover, the following estimate holds*

$$\|\psi\|_{\sharp,\Theta,\alpha} \le C \|g\|_{**},$$

where $\alpha \in (0, \frac{1}{2})$.

7.2. **Linear theories for the inner problems.** For the inner problem, we consider the model equation for the inner problem as follows.

$$\begin{cases} \partial_\tau \phi = L_W[\phi] + h(y,\tau), & |y| \le 2R(t(\tau)),\ \tau \in (\tau_0, +\infty), \\ \phi(y,\tau_0) = 0, & |y| \le 2R(t(\tau_0)), \end{cases} \tag{7.7}$$

where

$$\tau = \tau(t) = \tau_0 + \int_0^t \frac{ds}{\lambda(s)^2},$$

for τ_0 satisfying $t(\tau_0) = 0$, and L_W is the linearized operator defined in (2.1). We write the solution $\phi = \phi(\rho, \theta, \tau)$, $y = \rho e^{i\theta}$ of (7.7) as

$$\phi(\rho,\theta,\tau) = \phi_1(\rho,\theta,\tau)E_1 + \phi_2(\rho,\theta,\tau)E_2 + \phi_0(\rho,\theta,\tau)W.$$

We will deal with $\Pi_{W^\perp}[\phi] = \phi_1(\rho,\theta,\tau)E_1 + \phi_2(\rho,\theta,\tau)E_2$ and $\Pi_W[\phi] = \phi_0(\rho,\theta,\tau)W$ separately, and consider

$$\begin{cases} \partial_\tau(\Pi_{W^\perp}[\phi]) = L_W[\Pi_{W^\perp}[\phi]] + \Pi_{W^\perp}[h](y,\tau), & |y| \le 2R(t(\tau)),\ \tau \in (\tau_0, +\infty), \\ \Pi_{W^\perp}[\phi](y,\tau_0) = 0, & |y| \le 2R(t(\tau_0)), \end{cases} \tag{7.8}$$

and

$$\begin{cases} \partial_\tau(\Pi_W[\phi]) = L_W[\Pi_W[\phi]] + \Pi_W[h](y,\tau), & |y| \le 2R(t(\tau)),\ \tau \in (\tau_0, +\infty), \\ \Pi_W[\phi](y,\tau_0) = 0, & |y| \le 2R(t(\tau_0)). \end{cases} \tag{7.9}$$

We will measure the RHS by the norm:

$$\|h\|_{\nu,a} := \sup_{|y| \le 2R(t),\ t \in (0,T)} \lambda_*^{-\nu}(t) \langle y \rangle^a |h(y,t)| \tag{7.10}$$

if we use (y,t) variables and by

$$\|h\|_{v,\ell}^{(\tau)} := \sup_{|y| \le 2R(t),\ t \in (0,T)} v^{-1}(\tau) \langle y \rangle^\ell |h(y,\tau)|$$

if we use (y,τ) variables. Here, $v(\tau)$ is some Hölder continuous function decaying in τ as $\tau \to \infty$.

7.2.1. *Linear theory for the inner problem on $W^\perp$.* If we use the complex notation $W^\perp$

$$(\Pi_{W^\perp}[\phi])_{\mathbb{C}} = \phi_1 + i\phi_2,$$

then the equation on $W^\perp$ reads as

$$\partial_\tau(\Pi_{W^\perp}[\phi])_{\mathbb{C}} = \Delta(\Pi_{W^\perp}[\phi])_{\mathbb{C}} - \frac{1}{\rho^2}(\Pi_{W^\perp}[\phi])_{\mathbb{C}} + \frac{8}{(1+\rho^2)^2}(\Pi_{W^\perp}[\phi])_{\mathbb{C}} + i\frac{2(\rho^2-1)}{\rho^2(\rho^2+1)}\partial_\theta(\Pi_{W^\perp}[\phi])_{\mathbb{C}} + (\Pi_{W^\perp}[h])_{\mathbb{C}}$$

with zero initial data. We further expand $(\Pi_{W^\perp}[\phi])_{\mathbb{C}}$ and $(\Pi_{W^\perp}[h])_{\mathbb{C}}$ in Fourier modes

$$(\Pi_{W^\perp}[\phi])_{\mathbb{C}} = \sum_{k \in \mathbb{Z}} \phi_k^{W^\perp}(\rho,\tau)e^{ik\theta}, \quad (\Pi_{W^\perp}[h])_{\mathbb{C}} = \sum_{k \in \mathbb{Z}} h_k^{W^\perp}(\rho,\tau)e^{ik\theta}.$$

Then the complex-valued scalar $\phi_k^{W^\perp}$ at mode k satisfies

$$\begin{aligned} \partial_\tau \phi_k^{W^\perp} &= \partial_{\rho\rho}\phi_k^{W^\perp} + \frac{1}{\rho}\partial_\rho \phi_k^{W^\perp} - \frac{1+k^2}{\rho^2}\phi_k^{W^\perp} + \frac{8}{(1+\rho^2)^2}\phi_k^{W^\perp} - \frac{2k(\rho^2-1)}{\rho^2(\rho^2+1)}\phi_k^{W^\perp} + h_k^{W^\perp} \\ &:= \mathcal{L}_k^{W^\perp}[\phi_k^{W^\perp}] + h_k^{W^\perp} \end{aligned} \tag{7.11}$$

which is precisely the linearization of harmonic map equation around degree 1 harmonic map dealt with in [17, Section 7]. So we have:

Proposition 7.2. ([17, Proposition 7.1]) *Assume that $a \in (2,3)$, $\nu > 0$, and $\|\Pi_{W^\perp}[h]\|_{\nu,a} < +\infty$. Let us write*

$$\Pi_{W^\perp}[h] = h_0^{W^\perp} + h_1^{W^\perp} + h_{-1}^{W^\perp} + h_\perp^{W^\perp} \quad \textit{with} \quad h_\perp^{W^\perp} = \sum_{k \ne 0, \pm 1} h_k^{W^\perp}.$$

Then there exists a solution $\Pi_{W^\perp}[\phi](h)$ of problem (7.8), which defines a linear operator of $\Pi_{W^\perp}[h]$, and satisfies the following estimate for $|y| \le 2R(t)$ with $t \in (0,T)$:

$$|\Pi_{W^\perp}[\phi](y,t)| + (1+|y|)\,|\nabla_y \Pi_{W^\perp}[\phi](y,t)|$$

$$\begin{aligned}
&\lesssim \lambda_*^\nu(t)\,\min\left\{\frac{R^{5-a}(t)}{1+|y|^3},\ \frac{R^{\frac{5-a}{2}}(t)}{1+|y|}\right\}\|h_0^{W^\perp}-\bar h_0^{W^\perp}\|_{\nu,a}+\frac{\lambda_*^\nu(t)R^2(t)}{1+|y|}\|\bar h_0^{W^\perp}\|_{\nu,a}\\
&+\frac{\lambda_*^\nu(t)}{1+|y|^{a-2}}\left\|h_1^{W^\perp}-\bar h_1^{W^\perp}\right\|_{\nu,a}+\frac{\lambda_*^\nu(t)R^4(t)}{1+|y|^2}\left\|\bar h_1^{W^\perp}\right\|_{\nu,a}\\
&+\lambda_*^\nu(t)\,\min\left\{\frac{R^{4-a}(t)}{1+|y|^2},\ \log R\right\}\|h_{-1}^{W^\perp}-\bar h_{-1}^{W^\perp}\|_{\nu,a}+\lambda_*^\nu(t)\log R(t)\,\|\bar h_{-1}^{W^\perp}\|_{\nu,a}\\
&+\frac{\lambda_*^\nu(t)}{1+|y|^{a-2}}\|h_\perp^{W^\perp}\|_{\nu,a}.
\end{aligned}$$

Here

$$\bar h_k^{W^\perp}(y,t):=\sum_{j=1}^{2}\frac{\chi Z_{k,j}(y)}{\int_{\mathbb{R}^2}\chi|Z_{k,j}|^2}\int_{\mathbb{R}^2}\Pi_{W^\perp}[h](z,t)\cdot Z_{k,j}(z)\,dz,\ \ k=0,\pm1,\ \ j=1,2, \tag{7.12}$$

with $Z_{k,j}$ defined in (2.2), where

$$\chi(y,\tau)=\begin{cases} w_\rho^2(|y|) & \text{if } |y|<2R(t),\\ 0 & \text{if } |y|\ge 2R(t).\end{cases} \tag{7.13}$$

Via a re-gluing process, better estimates can be gained at mode 0 with a slight modification on the orthogonality condition. Consider the linear problem at mode 0 on $W^\perp$:

$$\begin{cases} \lambda^2\partial_t\phi=L_W\phi+h(\rho,t)+\sum_{j=1,2}\tilde c_{0j}Z_{0,j}w_\rho^2, & |y|\le 2R(t),\ t\in(0,T),\\ \phi\perp W,\quad \phi=0\ \text{ on }\ \partial B_{2R}\times(0,T),\quad \phi(\cdot,0)=0\ \text{ in }\ B_{2R(0)}. \end{cases} \tag{7.14}$$

Proposition 7.3. ([17, Proposition 7.2]) *Let $\sigma_*\in(0,1)$. Under the assumptions of Proposition 7.2, there exists $(\phi,\ \tilde c_{0j})$, linear in h, solving (7.14) such that*

$$|\phi|+(1+\rho)|\partial_\rho\phi|\lesssim\lambda_*^\nu(t)\|h\|_{\nu,a}\min\left\{\frac{R^{\sigma_*(5-a)}(t)}{1+|y|^3},\frac{1}{1+|y|^{a-2}}\right\}$$

and such that

$$\tilde c_{0j}=\tilde c_{0j}[h]=-\frac{\int_{\mathbb{R}^2}h\cdot Z_{0,j}}{\int_{\mathbb{R}^2}w_\rho^2Z_{0,j}^2}-G[h],\quad j=1,2$$

for some operator G linear in h with

$$|G[h]|\lesssim\lambda_*^\nu R^{-\sigma_*\sigma'}\|h\|_{\nu,a},\quad \sigma'\in(0,\ a-2).$$

More refined versions of above linear theory on $W^\perp$ are obtained in [55, Section 9] (in a special case of $a=1$, $b=0$).

7.2.2. *Linear theory for inner problem in the W-direction.* For the equation (7.9) in the W-direction, the use of

$$\Pi_W[\phi]=\sum_{k\in\mathbb{Z}}\phi_k^W(\rho,\tau)e^{ik\theta},\quad \Pi_W[h]=\sum_{k\in\mathbb{Z}}h_k^W(\rho,\tau)e^{ik\theta}$$

gives equation in mode k:

$$\begin{cases}\partial_\tau\phi_k^W(\rho,\tau)=\mathcal{L}_k^W[\phi_k^W]+h_k^W(\rho,\tau), & (\rho,\tau)\in(0,\infty)\times(\tau_0,\infty),\\ \phi_k^W(\rho,\tau_0)=0, & \rho\in(0,\infty),\end{cases} \tag{7.15}$$

where

$$\mathcal{L}_k^W:=\partial_{\rho\rho}+\frac{1}{\rho}\partial_\rho-\frac{k^2}{\rho^2}+\frac{8}{(1+\rho^2)^2}.$$

- **Mode 0.**

In mode $k=0$, we establish a linear theory via the distorted Fourier transform (DFT).

Proposition 7.4. *In (7.15) with $k=0$, if $\|h_0^W\|_{v,\ell}^{(\tau)}<\infty$ with $\ell>\frac{3}{2}$, then there exists a solution ϕ_0^W that satisfies the following estimate*

$$|\phi_0^W(\rho,\tau)|\lesssim \|h_0^W\|_{v,\ell}^{(\tau)}\mathbf{1}_{\{\rho\le\tau^{\frac12}\}}\begin{cases} v(\tau)\tau^{1-\frac{\ell}{2}}+\tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell<2\\ v(\tau)(\log\tau)^2+\tau^{-1}\log\tau\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell=2\\ v(\tau)\log\tau+\tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell>2\end{cases}$$
$$+\|h_0^W\|_{v,\ell}^{(\tau)}\mathbf{1}_{\{\rho>\tau^{\frac12}\}}\rho^{-1/2}\begin{cases} v(\tau)\tau^{\frac54-\frac{\ell}{2}}+\tau^{\frac14-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell<2\\ v(\tau)\tau^{\frac14}\langle\log\tau\rangle+\tau^{-\frac34}\langle\log\tau\rangle\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell=2\\ v(\tau)\tau^{\frac14}+\tau^{-\frac34}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \ell>2\end{cases}.$$

If, in addition, $2<\ell<\frac52$ and the orthogonality condition

$$\int_0^\infty h_0^W(\rho,\tau)\frac{\rho^2-1}{\rho^2+1}\rho d\rho=0 \quad \text{for all} \quad \tau>\tau_0 \tag{7.16}$$

is satisfied, then the following estimate holds

$$|\phi_0^W(\rho,\tau)|\lesssim\|h_0^W\|_{v,\ell}^{(\tau)}\begin{cases} v(\tau)\langle\rho\rangle^{2-\ell}+\tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds & \text{if } \rho\le\tau^{\frac12}\\ \rho^{-1/2}\left(v(\tau)\tau^{\frac54-\frac{\ell}{2}}+\tau^{\frac14-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}}v(s)ds\right) & \text{if } \rho>\tau^{\frac12}\end{cases}.$$

We postpone the proof of Proposition 7.4 in Appendix B.2.

• **Modes $k=\pm1$.**

We notice that $\mathcal{L}_{\pm1}^W\equiv\mathcal{L}_0^{W^\perp}$, where $\mathcal{L}_k^{W^\perp}$ is defined in (7.11) (on $W^\perp$), meaning that the linear theory for modes ±1 in the W-direction is exactly the same as the one for mode 0 on $W^\perp$. So for modes $|k|=1$ in the W-direction, we have

Proposition 7.5. ([17, Lemma 7.1, Lemma 7.3]) *For modes $k=\pm1$ in (7.15), there exists a solution ϕ_k^W that satisfies the following estimate*

$$|\phi_k^W|\lesssim\lambda_*^\nu(t)\min\left\{\frac{R^{5-a}(t)}{1+|y|^3},\ \frac{R^{\frac{5-a}{2}}(t)}{1+|y|}\right\}\|h_k^W-\bar h_k^W\|_{\nu,a}+\frac{\lambda_*^\nu(t)R^2(t)}{1+|y|}\|\bar h_k^W\|_{\nu,a}.$$

Here $a\in(2,3)$, $\nu>0$, and

$$\bar h_1^W:=\frac{\chi\mathcal{Z}_{1,1}(y)}{\int_{\mathbb{R}^2}\chi|\mathcal{Z}_{1,1}|^2}\int_{\mathbb{R}^2}\Pi_W[h](z,t)\cdot\mathcal{Z}_{1,1}(z)\,dz,$$
$$\bar h_{-1}^W:=\frac{\chi\mathcal{Z}_{1,2}(y)}{\int_{\mathbb{R}^2}\chi|\mathcal{Z}_{1,2}|^2}\int_{\mathbb{R}^2}\Pi_W[h](z,t)\cdot\mathcal{Z}_{1,2}(z)\,dz,$$

with χ, $\mathcal{Z}_{1,k}$ defined respectively in (7.13) and (2.2).

Remark 7.5.1. *For the non-orthogonal parts in Proposition 7.2 and Proposition 7.5, one can in fact relax the assumption on the spatial decay to $0<a<3$ with estimates modified accordingly; see [17, p. 394, Lemma 7.1]. Later we will use this version.*

An analogue of Proposition 7.3 also holds true for modes $k=\pm1$. Consider the linear problem in modes ±1 in the W-direction:

$$\begin{cases}\lambda^2\partial_t\phi=L_W\phi+h(\rho,t)+\sum_{j=1,2}\hat c_{1j}\mathcal{Z}_{1,j}w_\rho^2, & |y|\le 2R(t),\ t\in(0,T),\\ \phi\perp W^\perp,\quad \phi=0 \ \text{ on } \ \partial B_{2R}\times(0,T), & \phi(\cdot,0)=0 \ \text{ in } \ B_{2R(0)}.\end{cases} \tag{7.17}$$

We have the following refined estimates:

Proposition 7.6. *Let $\sigma_*\in(0,1)$. Under the assumptions of Proposition 7.5, there exists $(\phi,\ \hat c_{1j})$, linear in h, solving (7.17) such that*

$$|\phi|+(1+\rho)|\partial_\rho\phi|\lesssim\lambda_*^\nu(t)\|h\|_{\nu,a}\min\left\{\frac{R^{\sigma_*(5-a)}(t)}{1+|y|^3},\frac{1}{1+|y|^{a-2}}\right\}$$

and such that

$$\hat{c}_{1j} = \hat{c}_{1j}[h] = -\frac{\int_{\mathbb{R}^2} h \cdot \mathcal{Z}_{1,j}}{\int_{\mathbb{R}^2} w_\rho^2 \mathcal{Z}_{1,j}^2} - \hat{G}[h], \quad j = 1, 2$$

for some operator $\hat{G}$ linear in h with

$$|\hat{G}[h]| \lesssim \lambda_*^\nu R^{-\sigma_* \sigma'} \|h\|_{\nu,a}, \quad \sigma' \in (0,\ a-2).$$

• **Higher modes $|k| \geq 2$.**

Proposition 7.7. *For modes $|k| \geq 2$ in (7.15), there exists a solution ϕ_k^W that satisfies*

$$|\phi_k^W| \lesssim \frac{1}{k^2} \lambda_*^\nu(t) \langle \rho \rangle^{2-a} \|h_k^W\|_{\nu,a},$$

where $a \in (2,3)$, $\nu > 0$.

Proof. We start with $k = 2$ (the same for $k = -2$). In such case, $\mathcal{L}_2^W$ has a (non sign-changing) kernel function, denoted by Z_2, whose asymptotic behavior is ρ^2 both near origin and at infinity. More generally, for $|k| \geq 2$, $\mathcal{L}_k^W$ has kernel functions

$$\frac{\rho^{-k}(k\rho^2 + k + \rho^2 - 1)}{\rho^2 + 1}, \quad \frac{\rho^k(k\rho^2 + k - \rho^2 + 1)}{\rho^2 + 1}.$$

We choose barrier function

$$\bar{\varphi}_2 = 2\|h_2^W\|_{\nu,a} \lambda_*^\nu \varphi_2$$

with φ_2 solving

$$\mathcal{L}_2^W[\varphi_2] + \langle \rho \rangle^{-a} = 0.$$

With Dirichlet boundary $\varphi_2(2R) = 0$, above equation admits a unique solution given by

$$\varphi_2(\rho) = Z_2(\rho) \int_\rho^{2R} \frac{dr}{r Z_2^2(r)} \int_0^r \langle s \rangle^{-a} Z_2(s) s ds$$

where $a \in (2,3)$. Then one has

$$|\varphi_2| \leq C \langle \rho \rangle^{2-a},$$

and thus

$$\begin{aligned} -\lambda^2 \partial_t \bar{\varphi}_2 + \mathcal{L}_2^W[\bar{\varphi}_2] + h_2^W &\leq 2\nu \|h_2^W\|_{\nu,a} |\dot{\lambda}_*| \lambda_*^{\nu+1} |\varphi_2| - \|h_2^W\|_{\nu,a} \lambda_*^\nu \langle \rho \rangle^{-a} \\ &\leq \|h_2^W\|_{\nu,a} \lambda_*^\nu \langle \rho \rangle^{-a} \Big(2\nu C |\dot{\lambda}_*| \lambda_* \langle \rho \rangle^2 - 1 \Big) \\ &< 0 \end{aligned}$$

since $|\dot{\lambda}_*| \lambda_* R^2 \ll 1$ (we choose $\beta < \frac{1}{2}$ in $R = \lambda_*^{-\beta}$). Therefore, the pointwise estimate for mode $k = 2$ follows. For the higher modes $|k| \geq 3$, similar argument works. Indeed, We choose barrier function

$$\bar{\varphi}_2 = 2\|h_k^W\|_{\nu,a} \lambda_*^\nu \varphi_k$$

with φ_k solving

$$\mathcal{L}_k^W[\varphi_k] + \langle \rho \rangle^{-a} = 0,$$

where in this case,

$$|\varphi_k| \leq \frac{C}{k^2} \langle \rho \rangle^{2-a}$$

by variation of parameters. The proof is thus completed. □

We then combine Proposition 7.4 (returning to (y,t) variables), Proposition 7.5 and Proposition 7.7 and use a scaling argument together with parabolic regularity estimates to get gradient estimate for (7.9). We thus conclude the following linear theory for the inner problem in the W-direction.

Proposition 7.8. *Assume that $a \in (2, \frac{5}{2})$, $\nu > 0$, and $\|\Pi_W[h]\|_{\nu,a} < +\infty$. Then there exists a solution $\Pi_W[\phi](h)$ of problem (7.9), which defines a linear operator of $\Pi_W[h]$, and satisfies the following estimate for $|y| \le 2R(t)$ with $t \in (0,T)$:*

$$\begin{aligned}
&|\Pi_W[\phi](y,t)| + (1+|y|)\,|\nabla_y \Pi_W[\phi](y,t)| \\
\lesssim\ & \frac{\lambda_*^\nu(t)}{1+|y|^{a-2}}\,\|h_0^W - \bar h_0^W\|_{\nu,a} + \lambda_*^\nu(t)\log R(t)\,\|\bar h_0^W\|_{\nu,a} \\
&+ \sum_{k=\pm1}\left(\lambda_*^\nu(t)\,\min\left\{\frac{R^{5-a}(t)}{1+|y|^3}, \frac{R^{\frac{5-a}{2}}(t)}{1+|y|}\right\}\|h_k^W - \bar h_k^W\|_{\nu,a} + \frac{\lambda_*^\nu(t)R^2(t)}{1+|y|}\|\bar h_k^W\|_{\nu,a}\right) \\
&+ \frac{\lambda_*^\nu(t)}{1+|y|^{a-2}}\left\|h - h_0^W - h_1^W - h_{-1}^W\right\|_{\nu,a}.
\end{aligned}$$

Remark 7.8.1. *In practice, we are going to impose orthogonality conditions on the inner problems for modes 0, 1 on $W^\perp$ and modes ±1 in the W-direction, and these orthogonalities yield the dynamics of the λ-γ system, translation parameter $\xi = (\xi_1, \xi_2)$ and c_1-c_2 system, respectively. Under these orthogonalities, we will solve both inner solutions Φ_W and $\Phi_{W^\perp}$ to $(5.1)_1$ and $(5.1)_2$ in the wighted space:*

$$\|\phi\|_{\mathrm{in},\sigma_*,\nu,a} := \sup_{|y|\le 2R(t),\ t\in(0,T)}\left[\lambda_*^\nu(t)\,\max\left\{\frac{R^{\sigma_*(5-a)}(t)}{1+|y|^3}, \frac{1}{1+|y|^{a-2}}\right\}\right]^{-1}\Big(|\phi(y,t)| + \langle y\rangle|\nabla_y\phi(y,t)|\Big) \tag{7.18}$$

for some $\sigma_*,\ \nu \in (0,1)$, $a \in (2,3)$.

8. Solving the gluing system

In this section, we solve the full gluing system. We are going to work in the following weighted topologies.

- RHS for both inner problems $(5.1)_1$ and $(5.1)_2$: $\|\cdot\|_{\nu,a}$-norm defined in (7.10).
- RHS for the outer problem $(5.1)_3$: $\|\cdot\|_{**}$-norm defined in (7.3).
- Inner solutions Φ_W and $\Phi_{W^\perp}$: $\|\cdot\|_{\mathrm{in},\sigma_*,\nu,a}$-norm defined in (7.18).
- Outer solution Ψ: $\|\cdot\|_{\sharp,\Theta,\alpha}$-norm defined in (7.4).

The constants measuring above weighted norms will be put into a specific range ensuring the implementation of the gluing process. We now start to estimate RHS in the gluing system (5.1) and reveal these restrictions on those constants. We are also going to use the leading asymptotics of the modulation parameters:

$$|\lambda| \sim \lambda_* = \frac{(T-t)|\log T|}{|\log(T-t)|^2}, \quad |\dot\xi| \lesssim \lambda_*^{3\Theta-1}, \quad |c_1| \lesssim \lambda_*^\Theta, \quad |c_2| \lesssim \lambda_*^\Theta,$$

and recall $R = R(t) = \lambda_*^{-\beta}$. Our first assumption on the constants is the following:

$$0<\nu<1, \quad 2<a<3, \quad 0<\beta<1/2, \quad 0<\Theta<1/2. \tag{8.1}$$

8.1. Non-local systems.

As discussed in Section 6, translation parameter ξ obeys essentially an ODE of first order, while the dynamics of λ-γ system and c_1-c_2 system are governed by integro-differential equations. The leading non-local operator might be regarded as the Abel's integral operator in the end-point case, so the solvability is rather involved.

For the λ-γ system, we recall (4.5). To introduce the space for the parameter function $p_0(t) = -2(\dot\lambda + i\lambda\dot\gamma)e^{i\gamma}$, we recall the non-local operator $\mathcal{B}_0$ appears at mode 0 on $W^\perp$ is of the approximate form

$$\mathcal{B}_0[p_0] = \int_{-T}^{t-\lambda_*^2}\frac{p_0(s)}{t-s}ds + O(p_0(t)).$$

For $\Theta \in (0,1)$, $\varpi \in \mathbb{R}$ and a continuous function $g : [-T,T] \to \mathbb{C}$, we define the norm

$$\|g\|_{\Theta,\varpi} = \sup_{t\in[-T,T]}(T-t)^{-\Theta}|\log(T-t)|^{\varpi}|g(t)|, \tag{8.2}$$

and for $\alpha \in (0,1)$, $m,\ \varpi \in \mathbb{R}$, we define the semi-norm

$$[g]_{\frac{\alpha}{2},m,\varpi} = \sup (T-t)^{-m} |\log(T-t)|^{\varpi} \frac{|g(t)-g(s)|}{(t-s)^{\frac{\alpha}{2}}},$$

where the supremum is taken over $s \le t$ in $[-T,T]$ such that $t-s \le \frac{1}{4}(T-t)$.

The following proposition gives an approximate inverse of the non-local operator $\mathcal{B}_0$ with a small remainder $\mathcal{R}_0$.

Proposition 8.1. ([17, Proposition 6.5, Proposition 6.6]) *Let $\alpha_0, \frac{\alpha}{2} \in (0, \frac{1}{2})$, $\varpi \in \mathbb{R}$. There is $\flat > 0$ such that if $\Theta \in (0,\flat)$ and $m \le \Theta - \frac{\alpha}{2}$, then for $h(t):[0,T] \to \mathbb{C}$ satisfying*

$$\begin{cases} |h(T)| > 0, \\ T^{\Theta} |\log T|^{1+\sigma-\varpi} \|h(\cdot) - h(T)\|_{\Theta,\varpi-1} + [h]_{\frac{\alpha}{2},m,\varpi-1} \le C_1, \end{cases} \tag{8.3}$$

for some $C_1 > 0$, $\sigma \in (0,1)$, then, for $T > 0$ sufficiently small there exist two operators $\mathcal{P}$ and $\mathcal{R}_0$ so that $p_0 = \mathcal{P}[h] : [-T,T] \to \mathbb{C}$ satisfies

$$\mathcal{B}_0[p_0](t) = h(t) + \mathcal{R}_0[h](t), \quad t \in [0,T]$$

with

$$|\mathcal{R}_0[h](t)| \le C\Big(T^{\sigma} + T^{\Theta} \frac{\log|\log T|}{|\log T|} \|h(\cdot)-h(T)\|_{\Theta,\varpi-1} + [h]_{\frac{\alpha}{2},m,\varpi-1}\Big) \frac{(T-t)^{m+\frac{(1+\alpha_0)\alpha}{2}}}{|\log(T-t)|^{\varpi}}.$$

Moreover,

$$\mathcal{P}[h] = p_{0,\kappa} + \mathcal{P}_1[h] + \mathcal{P}_2[h],$$

with

$$p_{0,\kappa}(t) = \frac{\kappa |\log T|}{|\log(T-t)|^2}, \quad t \le T,$$

where $\kappa = \kappa[h]$. Moreover, the following bounds hold:

$$\begin{aligned} &\kappa = 2h(T)\left(1 + O(|\log T|^{-1})\right), \quad |\partial_t \mathcal{P}_1[h](t)| \le C \frac{|\log T|^{1-\sigma} \left(\log(|\log T|)\right)^2}{|\log(T-t)|^{3-\sigma}}, \\ &|\partial_t^2 \mathcal{P}_1[h](t)| \le C \frac{|\log T|}{|\log(T-t)|^3 (T-t)}, \quad \|\partial_t \mathcal{P}_2[h]\|_{\Theta,\varpi} \le C\left(T^{\frac{1}{2}+\sigma-\Theta} + \|h(\cdot)-h(T)\|_{\Theta,\varpi-1}\right), \\ &[\partial_t \mathcal{P}_2[h]]_{\frac{\alpha}{2},m,\varpi} \le C\left(|\log T|^{\varpi-3} T^{\flat-m-\frac{\alpha}{2}} + T^{\Theta} \frac{\log|\log T|}{|\log T|} \|h(\cdot)-h(T)\|_{\Theta,\varpi-1} + [h]_{\frac{\alpha}{2},m,\varpi-1}\right). \end{aligned} \tag{8.4}$$

The dealing of c_1-c_2 system is similar as the one of λ-γ system as their leading non-local operator turn out to be the same. Indeed, from Section 9, we have:

Proposition 8.2. *For $|f(t)| \lesssim \lambda_*^{\Theta}(t)$, when $T > 0$ is sufficiently small, there exist two operators $\mathcal{P}_{\mathbf{c}}$ and $\mathcal{R}_{\mathbf{c}}$ such that $p_1 = \mathcal{P}_{\mathbf{c}}[f] : [-T,T] \to \mathbb{C}$ satisfies the nonlocal equation*

$$\int_{-T}^{t} \frac{p_1(s)}{t-s} \Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right) ds + 2\lambda \dot{\mathbf{c}} + \Gamma_4[p_0]\mathbf{c} = f(t) + \mathcal{R}_{\mathbf{c}}[f](t), \quad t \in [0,T]$$

with

$$|R_{\mathbf{c}}[f]| \le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^{\Theta}(T-t)^{\alpha_1}}{|\log(T-t)|^2}$$

and

$$\left|\frac{d}{dt} R_{\mathbf{c}}[f]\right| \le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^{\Theta}(T-t)^{\alpha_1-1}}{|\log(T-t)|^2}.$$

The function $\mathcal{P}_{\mathbf{c}}[f]$ can be decomposed as $\mathcal{P}_{\mathbf{c}}[f] = p_{1,0} + p_{1,1}$. Here $p_{0,1}$ and $p_{1,1}$ satisfy the following properties

$$|p_{1,0}(t)| \le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|},$$

$$|p_{1,1}(t)| \le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^{\Theta} |\log T|^{k-1}}{|\log(T-t)|^{k+1}},$$

$$|\dot{p}_{1,1}(t)| \le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^2(T-t)}$$

and

$$|\ddot{p}_{1,1}(t)| \le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^2(T-t)^2}.$$

Here $k\in(0,1)$, $\alpha_1\in(0,1/3)$ *and*

$$\|f\|_{*,\Theta,0} := \sup_{t\in[0,T]}\left|\lambda_*^{-\Theta}(t)f(t)\right|.$$

8.2. **The inner-outer gluing system.** Because of the refined linear theories for inner problems (Proposition 7.3, Proposition 7.6), λ-γ system (Proposition 8.1) and c_1-c_2 system (Proposition 8.2), we need to further decompose inner problems $(5.1)_1$-$(5.1)_2$ into three pieces. We search for

$$\Phi^*_W + \Phi^*_{W^\perp} + \Phi^\dagger = \Phi_W + \Phi_{W^\perp}$$

with

$$\begin{aligned}
\lambda^2\partial_t\Phi^*_W &= \Delta_y\Phi^*_W - 2\partial_{y_1}W\wedge\partial_{y_2}\Phi^*_W - 2\partial_{y_1}\Phi^*_W\wedge\partial_{y_2}W\\
&\quad + \lambda^2\Pi_W\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right] + \lambda^2\Pi_W\left[Q_{-\gamma}(S[U_*])-\mathcal{R}^\dagger\right] + \mathcal{H}^W_{\rm in}\\
&\quad + \mathbf{R_c}[\Psi] + \sum_{j=1,2}\hat{c}_{1j}\mathcal{Z}_{1,j}w_\rho^2 \quad \text{in } \mathcal{D}_{2R},\\
\lambda^2\partial_t\Phi^*_{W^\perp} &= \Delta_y\Phi^*_{W^\perp} - 2\partial_{y_1}W\wedge\partial_{y_2}\Phi^*_{W^\perp} - 2\partial_{y_1}\Phi^*_{W^\perp}\wedge\partial_{y_2}W\\
&\quad + \lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right] + \lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}(S[U_*])-\mathcal{R}^\dagger\right] + \mathcal{H}^{W^\perp}_{\rm in}\\
&\quad + \mathbf{R_0}[\Psi] + \lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}^{(1)}_U[\Psi]\right] + \sum_{j=1,2}\tilde{c}_{0j}Z_{0,j}w_\rho^2 + \sum_{j=1,2}c_{1j}Z_{1,j}w_\rho^2 \quad \text{in } \mathcal{D}_{2R},\\
\lambda^2\partial_t\Phi^\dagger &= \Delta_y\Phi^\dagger - 2\partial_{y_1}W\wedge\partial_{y_2}\Phi^\dagger - 2\partial_{y_1}\Phi^\dagger\wedge\partial_{y_2}W\\
&\quad + \lambda^2\left[Q_{-\gamma}\left(\tilde{L}_U[\Psi](\lambda y+\xi,t) - \tilde{L}_U[\Psi](q,t)\right)\right] - \lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}^{(1)}_U[\Psi]\right]\\
&\quad + \mathbf{R_c}[\Psi] + \mathbf{R_0}[\Psi] + \lambda^2\mathcal{R}^\dagger \quad \text{in } \mathcal{D}_{2R},
\end{aligned} \tag{8.5}$$

where

$$\begin{aligned}
\mathbf{R_c}[\Psi] &:= \lambda\mathcal{R}_{\mathbf{c}}\left[\int_{\mathbb{R}^2}\lambda\Pi_W\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right]\cdot\mathcal{Z}_{1,1}\right]\frac{w_\rho^2\mathcal{Z}_{1,1}}{\int_{\mathbb{R}^2}w_\rho^2\mathcal{Z}_{1,1}^2}\\
&\quad + \lambda\mathcal{R}_{\mathbf{c}}\left[\int_{\mathbb{R}^2}i\lambda\Pi_W\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right]\cdot\mathcal{Z}_{1,2}\right]\frac{w_\rho^2\mathcal{Z}_{1,2}}{\int_{\mathbb{R}^2}w_\rho^2\mathcal{Z}_{1,2}^2},\\
\mathbf{R_0}[\Psi] &:= \lambda\mathcal{R}_0\left[\int_{\mathbb{R}^2}\lambda\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right]\cdot Z_{0,1}\right]\frac{w_\rho^2 Z_{0,1}}{\int_{\mathbb{R}^2}w_\rho^2 Z_{0,1}^2}\\
&\quad + \lambda\mathcal{R}_0\left[\int_{\mathbb{R}^2}i\lambda\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right]\cdot Z_{0,2}\right]\frac{w_\rho^2 Z_{0,2}}{\int_{\mathbb{R}^2}w_\rho^2 Z_{0,2}^2}
\end{aligned} \tag{8.6}$$

with the operators $\mathcal{R}_0$ and $\mathcal{R}_{\mathbf{c}}$ given in Proposition 8.1 and Proposition 8.2, respectively; for $j=1,2$,

$$\begin{aligned}
\hat{c}_{1j} &= \hat{c}_{1j}\left[\lambda^2\Pi_W\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right] + \lambda^2\Pi_W\left[Q_{-\gamma}(S[U_*])-\mathcal{R}^\dagger\right] + \mathcal{H}^W_{\rm in} + \mathbf{R_c}[\Psi]\right]\\
\tilde{c}_{0j} &= \tilde{c}_{0j}\left[\lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right] + \lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}(S[U_*])-\mathcal{R}^\dagger\right] + \mathcal{H}^{W^\perp}_{\rm in} + \mathbf{R_0}[\Psi]\right]
\end{aligned} \tag{8.7}$$

are given in Proposition 7.6 and Proposition 7.3, respectively; the term $\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U^{(1)}[\Psi]\right]$ denotes the part of mode 1 in the projection of $Q_{-\gamma}\left(\tilde{L}_U[\Psi](\lambda y+\xi,t)-\tilde{L}_U[\Psi](q,t)\right)$ onto $W^\perp$,

$$c_{1j}=\frac{\displaystyle\int_{\mathbb{R}^2}\left(\lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right]+\lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}(S[U_*])-\mathcal{R}^\dagger\right]+\mathcal{H}_{\rm in}^{W^\perp}\right)\cdot Z_{1,j}}{\displaystyle\int_{\mathbb{R}^2}w_\rho^2Z_{1,j}^2},\tag{8.8}$$

and

$$\begin{aligned}\mathcal{R}^\dagger:=&-\frac{2\eta_1}{r}\left((\partial_\theta\phi_2^{(2)}+\partial_\theta\phi_2^{(-2)})+\cos w(\phi_1^{(2)}+\phi_1^{(-2)})+\sin2\theta\sin w\psi_2\right)\\&\times\lambda^{-1}w_\rho\sin w(c_1\cos\theta+c_2\sin\theta)W\\&-\frac{2\eta_1}{r}\left[\lambda^{-1}\left(\sin2\theta w_\rho\psi_2+\partial_\rho\phi_1^{(2)}+\partial_\rho\phi_1^{(-2)}\right)\right]\times\sin^2w(c_1\cos\theta+c_2\sin\theta)W.\end{aligned}$$

We will solve $(8.5)_1$ and $(8.5)_2$ with orthogonality conditions imposed at corresponding modes

$$\hat{c}_{1j}=0,\quad\tilde{c}_{0j}=0,\quad c_{1j}=0,$$

and solve $(8.5)_3$ (consisting of components both on $W^\perp$ and in the W-direction) without orthogonality.

- In the inner problems (supported in B_{2R}) with orthogonality condition imposed, by (A.7), one has

$$\begin{aligned}&\left|\lambda^2\Pi_W\left[Q_{-\gamma}(S[U_*])-\mathcal{R}^\dagger\right]\right|\\\lesssim&\ \lambda^2|Q_{-\gamma}\mathcal{R}_U|+\lambda^2\left|\Pi_W\left[Q_{-\gamma}(\mathcal{R}_*)-\mathcal{R}^\dagger\right]\right|\\&+\lambda^2\left|\Pi_W\left[Q_{-\gamma}\left(\eta_1(\mathcal{E}_{U^\perp}^{(0)}+\tilde{\mathcal{R}}^0)+\eta_1(\mathcal{E}_U^{(\pm1)}+\tilde{\mathcal{R}}^1)+\eta_1\tilde{L}_U[\Phi^{(0)}+\Phi^{(1)}]\right)\right]\right|\\\lesssim&\ \lambda_*^{4\Theta}\langle\rho\rangle^{-2}+\lambda_*^{1+\Theta}\langle\rho\rangle^{-3}+\lambda_*\langle\rho\rangle^{-2}+\lambda_*^{3\Theta+1}+\lambda_*^2|\dot\lambda_*|\langle\rho\rangle^{-1}\\&+\left[\lambda_*^2+\lambda_*^{2\Theta+1}\left(\langle\rho\rangle^{-1-\delta}+\langle\rho\rangle^{-\delta_1}\right)+\lambda_*^{5\Theta}\left(\langle\rho\rangle^{-2-\delta}+\langle\rho\rangle^{-1-\delta_1}\right)\right.\\&\left.\quad+\lambda_*^{4\Theta}\langle\rho\rangle^{-2-2\delta}+\lambda_*^{\Theta+1}\langle\rho\rangle^{-3}\right]\mathbf{1}_{\{\rho\lesssim\lambda_*^{-1}\}}\\\lesssim&\ \lambda_*^{4\Theta}\langle\rho\rangle^{-2}+\lambda_*\langle\rho\rangle^{-2}+\lambda_*^{3\Theta+1}+\lambda_*^2+\lambda_*^{2\Theta+1}\left(\langle\rho\rangle^{-1-\delta}+\langle\rho\rangle^{-\delta_1}\right),\end{aligned}$$

and

$$\begin{aligned}&\left|\lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}(S[U_*])-\mathcal{R}^\dagger\right]\right|\\\lesssim&\ \lambda^2\left|Q_{-\gamma}(\mathcal{R}_{U^\perp}+\mathcal{E}_{U^\perp}^{(1)})\right|+\lambda^2\left|\Pi_{W^\perp}\left[Q_{-\gamma}(\mathcal{R}_*)\right]\right|\\&+\lambda^2\left|\Pi_{W^\perp}\left[Q_{-\gamma}\left(\eta_1(\mathcal{E}_{U^\perp}^{(0)}+\tilde{\mathcal{R}}^0)+\eta_1(\mathcal{E}_U^{(\pm1)}+\tilde{\mathcal{R}}^1)+\eta_1\tilde{L}_U[\Phi^{(0)}+\Phi^{(1)}]\right)\right]\right|\\\lesssim&\ \lambda_*^{3\Theta}\langle\rho\rangle^{-2}+\lambda_*\langle\rho\rangle^{-2}+\lambda_*^{3\Theta+1}\\&+\left[\lambda_*^2+\lambda_*^{2\Theta+1}\langle\rho\rangle^{-\delta}+\lambda_*^{4\Theta}\left(\langle\rho\rangle^{-3-2\delta}+\langle\rho\rangle^{-2-\delta-\delta_1}\right)+\lambda_*^{5\Theta}\langle\rho\rangle^{-1-\delta}\right.\\&\left.\quad+\lambda_*^{\Theta+1}\langle\rho\rangle^{-2}+\lambda_*^{3\Theta}\langle\rho\rangle^{-3-\delta}\right]\mathbf{1}_{\{\rho\lesssim\lambda_*^{-1}\}}\\\lesssim&\ \lambda_*^{3\Theta}\langle\rho\rangle^{-2}+\lambda_*\langle\rho\rangle^{-2}+\lambda_*^{3\Theta+1}+\lambda_*^2+\lambda_*^{2\Theta+1}\langle\rho\rangle^{-\delta}+\lambda_*^{5\Theta}\langle\rho\rangle^{-1-\delta}.\end{aligned}$$

We thus need following restrictions on the parameters after simplification by (8.1)

$$\begin{aligned}&3\Theta-\nu-\beta(a-2)>0,\quad 1-\nu-\beta(a-2)>0,\\&2\Theta+1-\nu-\beta(a-\delta)>0,\quad 5\Theta-\nu-\beta(a-1-\delta)>0\end{aligned}\tag{8.9}$$

so that above terms have finite $\|\cdot\|_{\nu,a}$-norm. Next, we estimate by Lemma 2.3

$$\left|\lambda^2\Pi_W\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right]\right|,\ \left|\lambda^2\Pi_{W^\perp}\left[Q_{-\gamma}\tilde{L}_U[\Psi](q,t)\right]\right|\lesssim\lambda_*\langle y\rangle^{-2}.$$

So we need

$$1-\nu-\beta(a-2)>0.\tag{8.10}$$

Recall (5.2) and (4.14). We estimate

$$\begin{aligned}&\left|\mathcal{H}_{\rm in}^{W^\perp}\right|,\quad\left|\mathcal{H}_{\rm in}^{W}\right|\\
\lesssim&\left|\nabla_y\Big(\eta_RQ_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\right|\left|\nabla_y\big(c_1Q_\gamma\mathcal{Z}_{1,1}+c_2Q_\gamma\mathcal{Z}_{1,2}+\eta_1\Phi_*\big)\right|\\
&+\left|-\eta_R\lambda^2\dot\gamma J_zQ_\gamma(\Phi_W+\Phi_{W^\perp})+\eta_R(\lambda\dot\xi+\lambda\dot\lambda y)\cdot\nabla_y(Q_\gamma\Phi_W+Q_\gamma\Phi_{W^\perp})\right|\\
&+\left|\partial_{y_1}\Big(\eta_RQ_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\wedge\partial_{y_2}\Big(\eta_RQ_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\right|\\
\lesssim&\left[\lambda_*^\nu R^{\sigma_*(5-a)}\langle y\rangle^{-4}\Big(\|\Phi_W\|_{{\rm in},\sigma_*,\nu,a}+\|\Phi_{W^\perp}\|_{{\rm in},\sigma_*,\nu,a}\Big)+\lambda_*^\Theta(0)\lambda_*(t)\|\Psi\|_{\sharp,\Theta,\alpha}\right]\\
&\quad\times\Big(\lambda_*(t)+\lambda_*^\Theta\langle y\rangle^{-2}+\lambda_*^{2\Theta}\langle y\rangle^{-\delta-1}\Big)\\
&+(\lambda_*^{1+\nu}+\lambda_*^{3\Theta+\nu})R^{\sigma_*(5-a)}\langle y\rangle^{-3}\Big(\|\Phi_W\|_{{\rm in},\sigma_*,\nu,a}+\|\Phi_{W^\perp}\|_{{\rm in},\sigma_*,\nu,a}\Big)\\
&+\lambda_*^{2\nu}R^{2\sigma_*(5-a)}\langle y\rangle^{-8}\Big(\|\Phi_W\|_{{\rm in},\sigma_*,\nu,a}+\|\Phi_{W^\perp}\|_{{\rm in},\sigma_*,\nu,a}\Big)^2+\lambda_*^{2\Theta}(0)\lambda_*^2(t)\|\Psi\|^2_{\sharp,\Theta,\alpha},\end{aligned}\tag{8.11}$$

and thus need

$$\Theta-\beta\sigma_*(5-a)>0,\quad \nu-2\beta\sigma_*(5-a)>0,\tag{8.12}$$

where parts of the restrictions are in (8.9) already.

From Proposition 8.1 and Proposition 8.2, we have

$$\begin{aligned}&|\mathbf{R_0}[\Psi]|\lesssim(\lambda_*(t))^{1+m+\frac{(1+\alpha_0)\alpha}{2}}\langle\rho\rangle^{-5}\\&|\mathbf{R_c}[\Psi]|\lesssim(\lambda_*(t))^{1+\alpha_1+\Theta}\langle\rho\rangle^{-5}\end{aligned}\tag{8.13}$$

with sufficient decay in space-time.

- In $(8.5)_3$ without orthogonality imposed on the RHS, we notice that the term

$$\lambda^2\Big[Q_{-\gamma}\left(\tilde{L}_U[\Psi](\lambda y+\xi,t)-\tilde{L}_U[\Psi](q,t)\right)\Big]-\lambda^2\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde{L}_U^{(1)}[\Psi]\Big]$$

has no mode 1 on $W^\perp$. In order for it to be small such that $\Phi^\dagger$, solved from non-orthogonal versions of Proposition 7.4, Proposition 7.8 (in the W-direction), Proposition 7.2 (on $W^\perp$) and Remark 7.5.1, stays within the space with the $\|\cdot\|_{{\rm in},\sigma_*,\nu,a}$-topology, we need

$$1+\Theta+\alpha\beta-2\beta>\nu-\sigma_*\beta(5-a),\quad 0<\alpha<1.\tag{8.14}$$

Indeed, we have

$$\left|\lambda^2\Big[Q_{-\gamma}\left(\tilde{L}_U[\Psi](\lambda y+\xi,t)-\tilde{L}_U[\Psi](q,t)\right)\Big]-\lambda^2\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde{L}_U^{(1)}[\Psi]\Big]\right|\lesssim\lambda_*\langle y\rangle^{\alpha-2}\lambda_*^\Theta R^{-\alpha}\|\Psi\|_{\sharp,\Theta,\alpha}$$

where we have used Lemma 2.3 and

$$|\nabla_x\Psi(x,t)-\nabla_x\Psi(q,t)|\lesssim|x-q|^\alpha\lambda_*^\Theta(\lambda_*R)^{-\alpha}\|\Psi\|_{\sharp,\Theta,\alpha}.$$

The necessity of restriction (8.14) is clear for all the modes except for the mode 0 in the W-direction. For the mode 0 in the W-direction, we can estimate

$$\begin{aligned}&\left|\lambda^2\Big[Q_{-\gamma}\left(\tilde{L}_U[\Psi](\lambda y+\xi,t)-\tilde{L}_U[\Psi](q,t)\right)\Big]-\lambda^2\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde{L}_U^{(1)}[\Psi]\Big]\right|\\
\lesssim&\ \lambda_*\langle y\rangle^{\alpha-2}\lambda_*^\Theta R^{-\alpha}\|\Psi\|_{\sharp,\Theta,\alpha}\lesssim\lambda_*^{1+\Theta}\langle y\rangle^{-\ell}R^{\ell-2}\|\Psi\|_{\sharp,\Theta,\alpha}\end{aligned}$$

thanks to the support of the inner problem. Here $\ell > 3/2$ and is close to $3/2$, and $\alpha \in (0,1)$ and is close to 1. So we need

$$1 + \Theta - \beta(\ell - 2) > \nu - \sigma_* \beta(5 - a),$$

which is true if (8.14) holds.

For the other part $\lambda^2 \mathcal{R}^\dagger$, by definitions (4.10) and (4.12), we first observe that it is in modes ± 1, ± 3 (in the W-direction), and also

$$|\lambda^2 \mathcal{R}^\dagger| \lesssim \lambda_*^{3\Theta} \langle y \rangle^{-3-\delta}.$$

Then from Proposition 7.5 and Proposition 7.7, we require

$$3\Theta - 2\beta > \nu - \sigma_* \beta(5 - a) \tag{8.15}$$

so that $\|\Phi^\dagger\|_{\mathrm{in},\sigma_*,\nu,a} < +\infty$.

For $\mathbf{R_c}[\Psi]$ and $\mathbf{R}_0[\Psi]$, since they are respectively in mode ± 1 of the W-direction and in mode 0 on $W^\perp$, we apply the non-orthogonal part of Proposition 7.2 and Proposition 7.5. By (8.13), we need the restrictions (8.21) and (8.23) and so that $\Phi^\dagger$ survives in the desired topology.

• We now consider the outer problem

$$\begin{cases} \partial_t \Psi = \Delta_x \Psi + \mathcal{G} & \text{in } \mathbb{R}^2 \times (0,T), \\ \Psi(x,0) = Z^*(x) + \sum_{k=1}^{5} \mathscr{C}_k \mathscr{Z}_k(x) & \text{in } \mathbb{R}^2, \end{cases} \tag{8.16}$$

where $\mathcal{G} = \mathcal{G}[\Phi_W, \Phi_{W^\perp}, \Psi, p_0, p_1, \xi, Z^*]$ is the RHS in the outer problem $(5.1)_3$, and $\mathscr{C}_k$ will be chosen such that we have desired vanishing (7.5). The outer problem can be formulated as a fixed-point problem

$$\mathcal{T}_{\mathrm{out}}[\Psi] = \Gamma_{\mathbb{R}^2} \bullet \mathcal{G} + \Gamma_{\mathbb{R}^2} \circ \left(Z^* + \sum_{k=1}^{5} \mathscr{C}_k \mathscr{Z}_k \right).$$

For the RHS $\mathcal{G}$ given in $(5.1)_3$, by (A.7) and (A.8), we have

$$\begin{aligned} & |(1-\eta_R) S[U_*]| \\ \lesssim\ & (1-\eta_R) \left| \left[\mathcal{R}_{U^\perp} + \mathcal{R}_U + \mathcal{E}_{U^\perp}^{(1)} + \eta_1 (\mathcal{E}_{U^\perp}^{(0)} + \tilde{\mathcal{R}}^0) + \eta_1 (\mathcal{E}_U^{(\pm 1)} + \tilde{\mathcal{R}}^1) + \eta_1 \tilde{L}_U [\Phi^{(0)} + \Phi^{(1)}] \right] \right| \\ & + (1-\eta_R) |\mathcal{R}_*| + |E_{\eta_1}| \\ \lesssim\ & \left(\lambda_*^{\Theta-1} + \lambda_*^{3\Theta-2} \right) \langle \rho \rangle^{-2} \mathbf{1}_{\{\lambda_* R \lesssim r\}} + \left(\lambda_*^{-1} \langle \rho \rangle^{-2} + \lambda_*^{3\Theta-1} + |\dot{\lambda}_*| \langle \rho \rangle^{-1} \right) \mathbf{1}_{\{\lambda_* R \lesssim r \lesssim 1\}} \\ & + \Bigg[1 + \lambda_*^{2\Theta-1} \langle \rho \rangle^{-\delta} + \lambda_*^{4\Theta-2} \left(\langle \rho \rangle^{-3-2\delta} + \langle \rho \rangle^{-2-\delta-\delta_1} \right) + \lambda_*^{5\Theta-2} \langle \rho \rangle^{-1-\delta} \\ & \quad + \lambda_*^{\Theta-1} \langle \rho \rangle^{-2} + \lambda_*^{3\Theta-2} \langle \rho \rangle^{-3-\delta} \Bigg] \mathbf{1}_{\{\lambda_* R \lesssim r \lesssim 1\}} + 1. \end{aligned}$$

In order for above terms to have finite $\|\cdot\|_{**}$-norm, following restrictions should be imposed on parameters

$$\sigma_0 > 0, \quad 3\Theta - 1 \geq 0, \quad 2\Theta - 1 + \delta\beta > 0. \tag{8.17}$$

Also

$$\left| (1-\eta_R) \tilde{L}_U[\Psi] \right| \lesssim \frac{\lambda_*(t)}{r^2} \mathbf{1}_{\{r \geq \lambda_* R\}} \lesssim T^\epsilon \varrho_2$$

for some $\epsilon > 0$ provided $\sigma_0 > 0$. For the remaining terms, we estimate

$$\begin{aligned} & |Q_\gamma(\Phi_W + \Phi_{W^\perp}) \Delta_x \eta_R + 2\nabla_x \eta_R \cdot \nabla_x (Q_\gamma \Phi_W + Q_\gamma \Phi_{W^\perp}) - Q_\gamma(\Phi_W + \Phi_{W^\perp}) \partial_t \eta_R| \\ \lesssim\ & \left[\lambda_*^{\nu-2} R^{-a} + \lambda_*^{\nu} R^{2-a} (T-t)^{-1} \right] \left(\|\Phi_W\|_{\mathrm{in},\sigma_*,\nu,a} + \|\Phi_{W^\perp}\|_{\mathrm{in},\sigma_*,\nu,a} \right) \mathbf{1}_{\{r \sim \lambda_* R\}}, \\ & \left| (1-\eta_R) \left[-\eta_R \dot{\gamma} J_z Q_\gamma (\Phi_W + \Phi_{W^\perp}) + \eta_R (\lambda^{-1} \dot{\xi} + \lambda^{-1} \dot{\lambda} y) \cdot \nabla_y (Q_\gamma \Phi_W + Q_\gamma \Phi_{W^\perp}) \right] \right| \\ \lesssim\ & \left(\lambda_*^{\nu-1} R^{2-a} + \lambda_*^{3\Theta+\nu-2} R^{1-a} \right) \left(\|\Phi_W\|_{\mathrm{in},\sigma_*,\nu,a} + \|\Phi_{W^\perp}\|_{\mathrm{in},\sigma_*,\nu,a} \right) \mathbf{1}_{\{r \geq \lambda_* R\}}, \end{aligned}$$

and by (4.14), one has

$$
\begin{aligned}
&(1-\eta_R)\Big| -2\partial_{x_1}(U_*-U)\wedge\partial_{x_2}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\\
&\qquad -2\partial_{x_1}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\wedge\partial_{x_2}(U_*-U)\Big|\\
\lesssim\ & (1-\eta_R)\Big|\nabla_x\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\Big|\Big|\nabla_x\left(c_1 Q_\gamma \mathcal{Z}_{1,1}+c_2 Q_\gamma\mathcal{Z}_{1,2}+\eta_1\Phi_*\right)\Big|\\
\lesssim\ & \Big[\lambda_*^{\nu-1}R^{1-a}\mathbf{1}_{\{r\sim\lambda_* R\}}\Big(\|\Phi_W\|_{\mathrm{in},\sigma_*,\nu,a}+\|\Phi_{W^\perp}\|_{\mathrm{in},\sigma_*,\nu,a}\Big)+\lambda_*^{\Theta}(0)\mathbf{1}_{\{r\ge\lambda_* R\}}\|\Psi\|_{\sharp,\Theta,\alpha}\Big]\\
&\times\Big(1+\lambda_*^{\Theta-1}\langle y\rangle^{-2}+\lambda_*^{2\Theta-1}\langle y\rangle^{-\delta-1}\Big).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&(1-\eta_R)\left|-2\partial_{x_1}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\wedge\partial_{x_2}\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\right|\\
\lesssim\ & (1-\eta_R)\Big|\nabla_x\Big(\eta_R Q_\gamma(\Phi_W+\Phi_{W^\perp})+\Psi\Big)\Big|^2\\
\lesssim\ & \lambda_*^{2\nu-2}\langle y\rangle^{2-2a}\Big(\|\Phi_W\|_{\mathrm{in},\sigma_*,\nu,a}+\|\Phi_{W^\perp}\|_{\mathrm{in},\sigma_*,\nu,a}\Big)^2\mathbf{1}_{\{|y|\sim R\}}+\lambda_*^{2\Theta}(0)\|\Psi\|^2_{\sharp,\Theta,\alpha}.
\end{aligned}
$$

To bound above terms in the $\|\cdot\|_{**}$-topology, we need

$$
\begin{aligned}
&\nu-2+a\beta>\Theta-1+\beta,\quad \nu-1-\beta(2-a)>\Theta-1+\beta,\\
&3\Theta+\nu-2-\beta(1-a)>\Theta-1+\beta,\quad \nu-1-\beta(1-a)>\Theta-1+\beta,\\
&\nu+\Theta-2+\beta(1+a)>\Theta-1+\beta,\quad 2\nu-2-\beta(2-2a)>\Theta-1+\beta.
\end{aligned}
\tag{8.18}
$$

For the Cauchy integrals $\Gamma_{\mathbb{R}^2}\circ\Big(Z^*+\sum_{k=1}^5\mathscr{C}_k\mathscr{Z}_k\Big)$, we claim

$$
\left|\Gamma_{\mathbb{R}^2}\circ\left(Z^*+\sum_{k=1}^5\mathscr{C}_k\mathscr{Z}_k\right)\right|\lesssim 1
$$

since $\|Z^*+\sum_{k=1}^5\mathscr{C}_k\mathscr{Z}_k\|_{L^\infty}\lesssim 1$. Indeed, $\mathscr{C}_k$ is chosen such that the vanishing (7.6) holds, so the Cauchy integrals have the same control as the space-time convolutions estimated above. In fact, one has better estimate $\sum_{k=1}^5|\mathscr{C}_k|\lesssim\lambda_*^{-\Theta-1}(0)R^{-1}(0)|\log T|^{-1}$ that can be derived similar to [17, Proposition A.1].

8.3. **Orthogonal equations.** We first employ Proposition 8.1 and Proposition 8.2 to derive restrictions on constants, measuring weighted topologies, that ensure the implementation of the gluing process, and then analyze the remainder terms neglected in Section 6.

• For the orthogonal equation of p_0:

$$
\tilde c_{0j}=0,
$$

we apply Proposition 8.1 with

$$
h[\Psi]:=\left[\partial_{x_1}\Psi_1+\partial_{x_2}\Psi_2+i\left(\partial_{x_1}\Psi_2-\partial_{x_2}\Psi_1\right)\right](q,t).
$$

The vanishing and Hölder properties (8.3) are exactly the ones inherited from the weighted topology (7.4) for the outer problem, namely

$$
\begin{aligned}
&|h[\Psi](t)-h[\Psi](T)|\lesssim\lambda_*^{\Theta}(t),\\
&\frac{|h[\Psi](t)-h[\Psi](s)|}{|t-s|^{\alpha/2}}\lesssim\lambda_*(t)^{\Theta-\alpha(1-\beta)}\quad\text{for}\quad |t-s|<\frac{T-t}{4}.
\end{aligned}
$$

So it is then natural to choose in the $[\cdot]_{\frac{\alpha}{2},m,\varpi-1}$-seminorm

$$
m=\Theta-\alpha(1-\beta).
$$

Then in the last line of (8.4), $\flat-m-\frac{\alpha}{2}>0$. In order for both $\|h[\Psi](\cdot)-h[\Psi](T)\|_{\Theta,\varpi-1}$, $[h[\Psi]]_{\frac{\alpha}{2},m,\varpi-1}$ to be finite, we need

$$
\varpi-1-2\Theta<0,\quad \varpi-1-2m<0.
$$

Also the assumption $m \le \Theta - \frac{\alpha}{2}$ in Proposition 8.1 implies

$$\beta \le 1/2,$$

which is in the desired self-similar regime as we require before. Recall the estimate of $\mathcal{R}_0[h[\Psi]]$. We require

$$m + (1+\alpha_0)\frac{\alpha}{2} > \Theta,$$

namely,

$$0 < \alpha_0 < 1/2, \quad 2\beta - 1 + \alpha_0 > 0,$$

so that the vanishing order of $\mathcal{R}_0[h[\Psi]]$ as $t \to T$ is faster than the leading part $h[\Psi]$ itself.

Under the paramter assumption above, the equation for p_0 will be solve by Proposition 8.1 and the solution p_0 satisfies the estimate in (8.4). We can then find a solution ξ to the orthogonal equation

$$c_{1j} = 0$$

with the estimate

$$|\dot{\xi}| \lesssim \lambda_*^{3\Theta-1} \tag{8.19}$$

provided

$$2\nu - 2\sigma_*\beta(5-a) - 3\Theta > 0.$$

We then conclude that with above choices of $m,\ \alpha_0,\ \frac{\alpha}{2},\ \varpi$, the remainder gains smallness

$$|\mathcal{R}_0[h[\Psi]](t)| \lesssim \lambda_*^{\Theta+\sigma_1} \tag{8.20}$$

compare to the leading part $h[\Psi]$ itself, where

$$0 < \sigma_1 < m + \frac{(1+\alpha_0)\alpha}{2} - \Theta.$$

We put the remainder $\mathcal{R}_0[h[\Psi]]$, i.e., $\mathbf{R_0}[\Psi]$ defined in (8.6), in the non-orthogonal inner problem, where the extra smallness measured above by σ_1 is crucial to control the non-orthogonal part. Indeed, from the version of linear theory without orthogonality condition imposed at mode 0 on $W^\perp$ in Proposition 7.2, we need

$$1 + \Theta - \alpha(1-\beta) + \frac{(1+\alpha_0)\alpha}{2} - 2\beta > \nu - \sigma_*\beta(5-a). \tag{8.21}$$

In summary, the restrictions on the constants needed when dealing with the reduced problem are given by

$$\begin{aligned} &0 < \beta < \frac{1}{2}, \quad 0 < \alpha_0 < \frac{1}{2}, \quad 2\beta - 1 + \alpha_0 > 0, \quad 2\nu - 2\sigma_*\beta(5-a) - 3\Theta > 0, \\ &1 + \Theta - \alpha(1-\beta) + \frac{(1+\alpha_0)\alpha}{2} - 2\beta > \nu - \sigma_*\beta(5-a). \end{aligned} \tag{8.22}$$

• By the same reasoning as in λ-γ system, c_1-c_2 system will not be completely solved. Instead, the term $\mathbf{R_c}$ defined in (8.6) that produces the remainder $R_{\mathbf{c}}[f]$ will be put in the piece of inner problem with no orthogonality condition imposed. Recall the non-orthogonal linear theory Proposition 7.5 for modes ± 1 in the W-direction. Therefore, from its vanishing bound, one requires

$$1 + \Theta + \alpha_1 - 2\beta > \nu - \sigma_*\beta(5-a).$$

A more convenient way to achieve above restriction is to assume

$$\alpha_1 > \alpha\left(\frac{\alpha_0}{2} - \frac{1}{2} + \beta\right) \tag{8.23}$$

since it will be satisfied provided (8.22) and (8.23) hold.

We now analyze those terms that we neglect when deriving the leading dynamics of the modulation parameters. We recall (6.4) and consider the remainders for the mode 0 on $W^\perp$:

$$\begin{aligned} &\Pi_{W^\perp}\Big[Q_{-\gamma}\tilde{L}_U[\Psi]\Big] + \Pi_{W^\perp}\Big[Q_{-\gamma}(S[U_*])\Big] + \mathcal{H}_{\text{in}}^{W^\perp} \\ &- \Pi_{W^\perp}\Big[Q_{-\gamma}\tilde{L}_U[Z_0^*(q)]\Big] - \Pi_{W^\perp}\Big[Q_{-\gamma}\Big((\mathcal{E}_{U^\perp}^{(0)} + \tilde{\mathcal{R}}^0) + \tilde{L}_U[\Phi^{(0)}]\Big)\Big]. \end{aligned}$$

In $\mathcal{D}_{2R}$, one has

$$\left|\Big(\Pi_{W^\perp}\Big[Q_{-\gamma}(S[U_*])\Big] - \Pi_{W^\perp}\Big[Q_{-\gamma}\Big((\mathcal{E}_{U^\perp}^{(0)} + \tilde{\mathcal{R}}^0) + \tilde{L}_U[\Phi^{(0)}]\Big)\Big]\Big)_{\mathbb{C},0}\right|$$

$$
\begin{aligned}
&= \left|\left(\Pi_{W^\perp}\left[Q_{-\gamma}\left(\mathcal{R}_{U^\perp}+\tilde{\mathcal{R}}^1+\widetilde{\mathcal{R}}_*+\mathcal{R}_{*,1}\right)\right]\right)_{\mathbb{C},0}\right| \\
&\lesssim \lambda_*^{\Theta-1}\langle\rho\rangle^{-2}+\lambda_*^{3\Theta-1}\langle\rho\rangle^{-1}+1+\lambda_*^{2\Theta-1}\langle\rho\rangle^{-\delta}+\lambda_*^{4\Theta-2}\left(\langle\rho\rangle^{-3-2\delta}+\langle\rho\rangle^{-2-\delta-\delta_1}\right)+\lambda_*^{5\Theta-2}\langle\rho\rangle^{-1-\delta} \\
&\quad +\lambda_*^{\Theta-1}\langle\rho\rangle^{-2}+\lambda_*^{3\Theta-2}\langle\rho\rangle^{-3-\delta},
\end{aligned}
$$

where we have used (4.20), (6.2) and (A.7). Also, it follows from (7.4) that

$$
\left|Q_{-\gamma}\tilde{L}_U[\Psi-Z_0^*(q)]\right|\lesssim\lambda_*^{\Theta-1},
$$

and $\mathcal{H}_{\rm in}^{W^\perp}$ is estimated in (8.11).

We next consider the remainders in mode 1 on $W^\perp$. To estimate $\Pi_{W^\perp}\left[Q_{-\gamma}\widetilde{\mathcal{R}}_*\right]+\mathcal{H}_{\rm in}^{W^\perp}$, we recall (6.2). Similar to (A.7), one can verify that

$$
\begin{aligned}
\left|\widetilde{\mathcal{R}}_*\cdot Q_\gamma W\right| &\lesssim \Big[1+\lambda_*^{2\Theta-1}\left(\langle\rho\rangle^{-1-\delta}+\langle\rho\rangle^{-\delta_1}\right)+\lambda_*^{5\Theta-2}\left(\langle\rho\rangle^{-2-\delta}+\langle\rho\rangle^{-1-\delta_1}\right) \\
&\qquad +\lambda_*^{4\Theta-2}\langle\rho\rangle^{-2-2\delta}+\lambda_*^{3\Theta-2}\langle\rho\rangle^{-4-\delta}\Big]\mathbf{1}_{\{\rho\lesssim\lambda_*^{-1}\}}, \\
\left|\widetilde{\mathcal{R}}_*\cdot Q_\gamma E_1\right|,\ \left|\widetilde{\mathcal{R}}_*\cdot Q_\gamma E_2\right| &\lesssim \Big[1+\lambda_*^{2\Theta-1}\langle\rho\rangle^{-\delta}+\lambda_*^{4\Theta-2}\left(\langle\rho\rangle^{-3-2\delta}+\langle\rho\rangle^{-2-\delta-\delta_1}\right)+\lambda_*^{5\Theta-2}\langle\rho\rangle^{-1-\delta} \\
&\qquad +\lambda_*^{3\Theta-2}\langle\rho\rangle^{-3-\delta}\Big]\mathbf{1}_{\{\rho\lesssim\lambda_*^{-1}\}}.
\end{aligned}
\tag{8.24}
$$

The estimates for remainders in modes ±1 in the W-direction $\Pi_W\left[Q_{-\gamma}\widetilde{\mathcal{R}}_*\right]+\mathcal{H}_{\rm in}^W$ are similar (cf. (8.11) and (8.24)). Recall that the outer problem will be solved within the weighted space (7.4) with the vanishings (7.5). Thanks to this, in the reduced problem for c_1 and c_2, we have

$$
\begin{aligned}
&\left|\frac{\lambda}{\pi}\int_0^{2\pi}\int_0^{+\infty}\left(Q_{-\gamma}\tilde{L}_U[\Psi(q,t)]\cdot W\right)\sin w e^{i\theta}\rho d\rho d\theta\right| \\
=&\ \frac{2}{\pi}\left|\int_0^{2\pi}\int_0^{+\infty}w_\rho\sin^2 w\Big[\partial_{x_1}\Psi_3(q,t)\cos\theta+\partial_{x_2}\Psi_3(q,t)\sin\theta\Big]e^{i\theta}\rho d\rho d\theta\right| \\
=&\ \frac{2}{\pi}\left|\int_0^{2\pi}\int_0^{+\infty}w_\rho\sin^2 w\Big[\Big(\partial_{x_1}\Psi_3(q,t)-\partial_{x_1}\Psi_3(q,T)\Big)\cos\theta+\Big(\partial_{x_2}\Psi_3(q,t)-\partial_{x_2}\Psi_3(q,T)\Big)\sin\theta\Big]e^{i\theta}\rho d\rho d\theta\right| \\
\lesssim&\ \lambda_*^\Theta(t)\|\Psi\|_{\sharp,\Theta,\alpha},
\end{aligned}
$$

where we have used Lemma 2.3. This term appears as part of $f(t)$ in Proposition 8.2.

From the estimates above, we see that, under the final choice of constants (8.29), the remainders in the orthogonal equations $\tilde{c}_{0j}=c_{1j}=\hat{c}_{1j}=0$ are indeed of smaller order than the main order taken into account when deriving the leading dynamics in Section 6.

8.4. **Solving the full system: Proof of Theorem 1.2.** We now formulate the full gluing system (8.5) & (8.16) together with the orthogonal equations

$$
\tilde{c}_{0j}=c_{1j}=\hat{c}_{1j}=0 \tag{8.25}
$$

as a fixed-point problem. Here the definitions for $\hat{c}_{1j}$, $\tilde{c}_{0j}$, c_{1j} are given in (8.7) and (8.8).

We will solve the outer problem (8.16) in the space

$$
X_\Psi:=\Big\{\Psi=(\Psi_1,\Psi_2,\Psi_3):\|\Psi\|_{\sharp,\Theta,\alpha}<+\infty,\ \Psi_j(q,T)=\partial_{x_1}\Psi_3(q,T)=\partial_{x_2}\Psi_3(q,T)=0,\ j=1,2,3\Big\} \tag{8.26}
$$

and the inner problems $(8.5)_1$-$(8.5)_3$ for Φ^*_W, $\Phi^*_{W^\perp}$ and $\Phi^\dagger$ all in the space

$$\begin{aligned} X_{\Phi^*_W} &:= \Big\{\Phi^*_W,\ \nabla_y \Phi^*_W \in L^\infty(\mathcal{D}_{2R}) : \|\Phi^*_W\|_{\mathrm{in},\sigma_*,\nu,a} < +\infty\Big\}, \\ X_{\Phi^*_{W^\perp}} &:= \Big\{\Phi^*_{W^\perp},\ \nabla_y \Phi^*_{W^\perp} \in L^\infty(\mathcal{D}_{2R}) : \|\Phi^*_{W^\perp}\|_{\mathrm{in},\sigma_*,\nu,a} < +\infty\Big\}, \\ X_{\Phi^\dagger} &:= \Big\{\Phi^\dagger,\ \nabla_y \Phi^\dagger \in L^\infty(\mathcal{D}_{2R}) : \|\Phi^\dagger\|_{\mathrm{in},\sigma_*,\nu,a} < +\infty\Big\}. \end{aligned} \tag{8.27}$$

We consider the modulation problem (8.25). For $\tilde{c}_{0j} = 0$, Proposition 8.1 gives an approximate inverse $\mathcal{P}$ of the operator $\mathcal{B}_0$, so that for given $h(t)$ satisfying (8.3), $p_0 := \mathcal{P}[h]$ satisfies

$$\mathcal{B}_0[p_0] = h + \mathcal{R}_0[h] \quad \text{ in } \ [0, T]$$

for a small remainder $\mathcal{R}_0[h]$. Moreover, estimates (8.4) for $p_{0,1} := \mathcal{P}_1[h] + \mathcal{P}_2[h]$ in Proposition 8.1 lead us to define the space

$$X_{p_0} := \{p_{0,1} \in C([-T,T;\mathbb{C}]) \cap C^1([-T,T;\mathbb{C}]) \ : \ p_{0,1}(T) = 0,\ \|p_{0,1}\|_{*,3-\sigma_0} < +\infty\}$$

for some $\sigma_0 \in (0,1)$, where we represent p_0 by the pair $(\kappa, p_{0,1})$ in the form $p_0 = p_{0,\kappa} + p_{0,1}$, and the $\|\cdot\|_{*,3-\sigma_0}$-seminorm is defined by

$$\|f\|_{*,3-\sigma_0} := \sup_{t\in[-T,T]} |\log(T-t)|^{3-\sigma_0} |\dot f(t)|.$$

For $c_{1j} = 0$, we define the space for $\xi(t)$ as

$$X_\xi = \Big\{\xi \in C^1((0,T);\mathbb{R}^2) \ : \ \dot\xi(T) = 0,\ \|\xi\|_{X_\xi} < +\infty\Big\}$$

where

$$\|\xi\|_{X_\xi} = \|\xi\|_{L^\infty(0,T)} + \sup_{t\in[-T,T]} \lambda_*^{-\sigma_1}(t) |\dot\xi(t)|$$

for some $\sigma_1 \in (0, 3\Theta - 1)$.

Similarly, for $\hat{c}_{1j} = 0$, Proposition 8.2 concerns the solvability of c_1-c_2 system, i.e., $p_1 = -2\big[\lambda(c_1 + ic_2)\big]'$, up to a small remainder $\mathcal{R}_{\mathbf{c}}[f](t)$. The control of the resolution $p_1 = \mathcal{P}_{\mathbf{c}}[f] = p_{1,0} + p_{1,1}$ for the modified non-local problem motivates us to solve p_1 in the space

$$X_{p_1} := \{p_1 \in C([-T,T;\mathbb{C}]) \cap C^1([-T,T;\mathbb{C}]) \ : \ p_1(T) = 0,\ \|p_1\|_{\Theta,1} < +\infty\},$$

where the norm above is defined in (8.2).

We define $\mathfrak{X} := X_\Psi \times X_{\Phi^*_W} \times X_{\Phi^*_{W^\perp}} \times X_{\Phi^\dagger} \times \mathbb{C} \times X_{p_0} \times X_\xi \times X_{p_1}$ and take a closed subset $\mathfrak{B} \subset \mathfrak{X}$ for which $(\Psi, \Phi^*_W,\ \Phi^*_{W^\perp},\ \Phi^\dagger,\ \kappa,\ p_{0,1},\ \xi,\ p_1) \in \mathfrak{B}$ satisfies

$$\begin{aligned} &\|\Psi\|_{\sharp,\Theta,\alpha} + \|\Phi^*_W\|_{\mathrm{in},\sigma_*,\nu,a} + \|\Phi^*_{W^\perp}\|_{\mathrm{in},\sigma_*,\nu,a} + \|\Phi^\dagger\|_{\mathrm{in},\sigma_*,\nu,a} \le 1, \\ &|\kappa - \kappa_0| \le |\log T|^{-1}, \quad \kappa_0 = \mathrm{div}\, Z^*(q) + i \mathrm{curl} Z^*(q), \\ &\|p_{0,1}\|_{*,3-\sigma_0} \le C_0 |\log T|^{1-\sigma_0} (\log(|\log T|))^2, \quad \|\xi\|_{X_\xi} + \|p_1\|_{\Theta,\sigma_2} \le 1 \end{aligned}$$

for a sufficiently large constant C_0. Then we define an operator $\mathfrak{F}$ which returns the solution from $\mathfrak{B}$ to $\mathfrak{X}$

$$\begin{aligned} \mathfrak{F} : \mathfrak{B} \subset \mathfrak{X} &\to \mathfrak{X} \\ v &\mapsto \mathfrak{F}(v) := \Big(\mathfrak{F}_\Psi(v),\ \mathfrak{F}_{\Phi^*_W}(v),\ \mathfrak{F}_{\Phi^*_{W^\perp}}(v),\ \mathfrak{F}_{\Phi^\dagger}(v),\ \mathfrak{F}_\kappa(v),\ \mathfrak{F}_{p_{0,1}}(v),\ \mathfrak{F}_\xi(v),\ \mathfrak{F}_{p_1}(v)\Big). \end{aligned}$$

Here the operator $\mathfrak{F}_\Psi$ corresponds to the outer problem (8.16) with linear theory given by Proposition 7.1. The operators $\mathfrak{F}_{\Phi^*_W}$, $\mathfrak{F}_{\Phi^*_{W^\perp}}$, $\mathfrak{F}_{\Phi^\dagger}$ handle respectively three pieces of inner problems $(8.5)_1$, $(8.5)_2$, $(8.5)_3$, and their linear theories are given in Proposition 7.8 and Proposition 7.2. The operators $\mathfrak{F}_\kappa$, $\mathfrak{F}_{p_{0,1}}$ deal with the λ-γ system from $\tilde{c}_{0j} = 0$ whose linear theory is in Proposition 8.1. The operator $\mathfrak{F}_\xi$ concerns $c_{1j} = 0$ yielding a first order ODE for ξ. The operator $\mathfrak{F}_{p_1}$ is related to the c_1-c_2 system from $\hat{c}_{1j} = 0$ with linear theory given by Proposition 8.2.

The property that $\mathfrak{F} : \mathfrak{B} \to \mathfrak{B}$ follows from those estimates in Section 8.2 and Section 8.3 under all the restrictions (8.1), (8.9), (8.10), (8.12), (8.14), (8.15), (8.17), (8.18), (8.22) and (8.23) for the constants. These can be simplified as

$$\begin{aligned}
&0<\nu<1, \quad 2<a<3, \quad 1/3<\Theta<\beta<1/2,\\
&1-\nu-\beta(a-2)>0, \quad 0<\sigma_*<1,\\
&\Theta-\beta\sigma_*(5-a)>0, \quad \nu-2\beta\sigma_*(5-a)>0,\\
&3\Theta-2\beta>\nu-\sigma_*\beta(5-a), \quad \nu-2+a\beta>\Theta-1+\beta,\\
&0<\alpha<1, \quad \alpha_0\approx 1/2, \quad 0<\alpha_1<1/3, \quad \alpha_1>\alpha\left(\frac{\alpha_0}{2}-\frac{1}{2}+\beta\right),\\
&1+\Theta-\alpha(1-\beta)+\frac{(1+\alpha_0)\alpha}{2}-2\beta>\nu-\sigma_*\beta(5-a),
\end{aligned} \tag{8.28}$$

where we take $\delta=\frac{999}{1000}\approx 1$, $\alpha_0=\frac{49}{100}\approx 1/2$. With the aid of Mathematica, the system (8.28) admits a valid choice of constants:

$$\begin{aligned}
&\frac{37}{60}<\nu<\frac{5}{8}, \quad \frac{209-120\nu}{45}<a<3, \quad \beta=\frac{3}{8}, \quad \Theta=\frac{11}{30},\\
&\frac{8\nu-\frac{14}{5}}{15-3a}<\sigma_*<\frac{4\nu}{15-3a}, \quad \alpha=\frac{99}{100}, \quad \frac{297}{2500}<\alpha_1<\frac{1}{3}.
\end{aligned} \tag{8.29}$$

The compactness of the operator $\mathfrak{F}$ can be proved by suitable variants of (8.29). Indeed, one can vary slightly the constants such that all the restrictions 8.28 still hold, and get (8.29) with the weighted norms measured by the new constants with the closed ball $\mathfrak{B}$ remains the same. For instance, for fixed Θ', α' close to Θ,α, one can show that if $v\in\mathfrak{B}$, then

$$\|\mathfrak{F}_\Psi(v)\|_{\sharp,\Theta',\alpha'}\le CT^{\epsilon'}$$

for some constants C, $\epsilon'>0$. Moreover, one can show that for $\alpha'>\alpha$ and $\Theta'-\Theta>2(\alpha'-\alpha)$, one has a compact embedding in the sense that if a sequence $\{\Psi_k\}_k$ is bounded in the $\|\cdot\|_{\sharp,\Theta',\alpha'}$-norm, then there exists a subsequence that converges in the $\|\cdot\|_{\sharp,\Theta,\alpha}$-norm. The compactness thus follows directly from Arzelà–Ascoli's theorem by a standard diagonal argument. The compactness of the rest operators can be proved in a similar manner. Therefore, the Schauder fixed-point theorem implies the existence of a desired solution. The proof of Theorem 1.2 is complete.

9. Non-local c_1-c_2 system

In this section, we solve the equation for c_1 and c_2 defined in Section 6

$$\int_{-T}^{t}\frac{p_1(s)}{t-s}\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right)ds+2\lambda\dot{\mathbf{c}}+\Gamma_4[p_0]\mathbf{c}=f(t). \tag{9.1}$$

Recall from Section 6 that $p_1=-2(\lambda\mathbf{c})'$, $p_0=-2(\dot\lambda+i\lambda\dot\gamma)e^{i\gamma}$, $\mathbf{c}(t):=c_1(t)+ic_2(t)$ and $f(t)$ is a smooth function satisfying the condition $|f(t)|\lesssim\lambda_*^\Theta(t)$. For simplicity, we will use the notations in this following: $\mathbf{c}^{(0)}$, $\mathbf{c}^{(1)}$ and $p_{1,0}(t)=\left(-2\lambda\mathbf{c}^{(0)}\right)'$, $p_{1,1}(t)=\left(-2\lambda\mathbf{c}^{(1)}\right)'$.

9.1. **The construction of a solution.** According to the computations in Section 6, (9.1) can be approximated by the following equation

$$\mathcal{P}_1(t)+\frac{2}{3}\mathcal{P}_0(t)\mathbf{c}=f(t) \text{ in } [0,T),$$

so we need to construct an operator $\mathcal{P}_\mathbf{c}$ which assigns $p_1=\mathcal{P}_\mathbf{c}[f]$ to a function f in a suitable class such that

$$\mathcal{P}_1(t)+\frac{2}{3}\mathcal{P}_0(t)\mathbf{c}=f(t)+\mathcal{R}_\mathbf{c}[f](t) \text{ in } [0,T), \tag{9.2}$$

so that $\mathcal{R}_\mathbf{c}[f](t)$ is a suitable small remainder term. The first approximation is the following function

$$\lambda\mathbf{c}^{(0)}=(T-t)^{\frac{\mathbf{Z}}{6}\log(T-t)}\int_t^T(T-\tau)^{-\frac{\mathbf{Z}}{6}\log(T-\tau)}\left(\frac{f(\tau)}{2\log(T-\tau)}-\frac{1}{2\log^2(T-\tau)}\int_\tau^T\frac{f(s)}{T-s}ds\right)d\tau,$$

which is a solution of the equation

$$\mathcal{P}(t):=\int_{-T}^{t}\frac{p_{1,0}(s)}{T-s}ds-p_{1,0}(t)\log(T-t)-\frac{2}{3}\mathbf{Z}\mathbf{c}^{(0)}=f(t), \quad p_{1,0}(t)=(-2\lambda\mathbf{c}^{(0)})'$$

and it is an approximation to the equation (9.2). Observe that we have $|\lambda \mathbf{c}^{(0)}| \le \frac{\lambda_*^{\Theta}(t)|\log T|(T-t)}{|\log(T-t)|^2}$ and $|(\lambda \mathbf{c}^{(0)})'| \le \frac{\lambda_*^{\Theta}(t)}{|\log(T-t)|}$ since $f(t)$ satisfies the condition $|f(t)| \lesssim \lambda_*^{\Theta}(t)$.

We look for p_1 with the form $p_1 = p_{1,0} + p_{1,1}$, where $p_{1,0}(t) = (-2\lambda \mathbf{c}^{(0)})'$ and p_1 satisfies the following equation

$$\mathcal{I}[p_{1,0}] + \mathcal{I}[p_{1,1}] + \tilde{B}[p_{1,0} + p_{1,1}] - \frac{2}{3}\mathbf{Z}\left(\mathbf{c}^{(0)} + \mathbf{c}^{(1)}\right) = f(t) + \mathcal{R}_{\mathbf{c}}[f](t) \text{ for } t \in [0,T].$$

Here we define $\mathcal{I}[p] = \int_{-T}^{t-\lambda_*^2(t)} \frac{p(s)}{t-s} ds$, the term $\tilde{B}[p_{1,0} + p_{1,1}]$ are small terms defined as follows,

$$\tilde{\mathcal{B}}[p_1] = \tilde{\mathcal{B}}_1[p_1] + \tilde{\mathcal{B}}_2[p_1]\mathbf{c} + \tilde{\mathcal{B}}_3[p_1]\mathbf{c} - 2\lambda\dot{\mathbf{c}},$$

$$\tilde{\mathcal{B}}_1[p_1] = -\int_{-T}^{t-\lambda_*^2(t)} \frac{p_1(s)}{t-s}\left(\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right) + 1\right) ds - \int_{t-\lambda_*^2(t)}^{t} \frac{p_1(s)}{t-s}\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right) ds,$$

$$\tilde{\mathcal{B}}_2[p_1] = \int_{-T}^{t-\lambda_*^2(t)} \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\left(\Gamma_5\left(\frac{\lambda^2(t)}{t-s}\right) + \frac{2}{3}\right) ds + \int_{t-\lambda_*^2(t)}^{t} \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\Gamma_5\left(\frac{\lambda^2(t)}{t-s}\right) ds,$$

$$\tilde{\mathcal{B}}_3[p_1] = -\frac{2}{3}\int_{-T}^{t-\lambda_*^2(t)} \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s} ds + \frac{2}{3}\mathbf{Z}.$$

The idea is to decompose $\mathcal{I}[p_{1,1}]$ into $\mathcal{I}[p_{1,1}] = S_{\alpha_1}[p_{1,1}] + \mathcal{R}_{\alpha_1}[p_{1,1}]$, then replace the operator $\mathcal{I}[p_{1,1}]$ by $S_{\alpha_1}[p_{1,1}]$ and try to solve the corresponding equation. If $\alpha_1 > 0$ is small, then we can find $p_{1,1}$ such that

$$\mathcal{I}[p_{1,0}] + S_{\alpha_1}[p_{1,1}] + \tilde{B}[p_{1,0} + p_{1,1}] - \frac{2}{3}\mathbf{Z}\left(\mathbf{c}^{(0)} + \mathbf{c}^{(1)}\right) = f(t) \text{ for } t \in [0,T].$$

This means that we have

$$\mathcal{B}[p_{1,0} + p_{1,1}] - \frac{2}{3}\mathbf{Z}\left(\mathbf{c}^{(0)} + \mathbf{c}^{(1)}\right) = f(t) + \mathcal{R}_{\alpha_1}[p_{1,1}] \text{ for } t \in [0,T].$$

We will prove that

$$|\mathcal{R}_{\alpha_1}[p_{1,1}]| \le C(T-t)^{\Theta+\alpha_1} \text{ for } t \in [0,T].$$

Now we decompose

$$S_{\alpha_1}[g] = \tilde{L}_0[g] + \tilde{L}_1[g]$$

where

$$\tilde{L}_0[g](t) = (1-\alpha_1)|\log(T-t)|g(t)$$

and $\tilde{L}_1[g]$ contains all other terms, we have

$$\tilde{L}_1[g] = \int_{-T}^{t} \frac{g(s)}{T-s} ds + \int_{t-(T-t)}^{t-(T-t)^{1+\alpha_1}} \frac{g(s)}{t-s} ds - \int_{t-(T-t)}^{t} \frac{g(s)}{T-s} ds$$
$$+ \int_{-T}^{t-(T-t)} g(s)\left(\frac{1}{t-s} - \frac{1}{T-s}\right) ds + (4\log(|\log(T-t)|) - 2\log(|\log T|))g(t).$$

Now we look for a function g solving the following problem

$$S_{\alpha_1}[g] - \frac{1}{3}\frac{\mathbf{Z}}{\lambda} g = \tilde{f}(t) \text{ in } [-T,T].$$

We solve a modified version of this equation. Let η be a smooth cut-off function such that

$$\eta(s) = 1 \text{ for } s \ge 0, \ \eta(s) = 0 \text{ for } s \le -\frac{1}{4}.$$

We will find a function g such that

$$\tilde{L}_0[g] + \eta(\frac{t}{T})\tilde{L}_1[g] - \frac{1}{3}\frac{\mathbf{Z}}{\lambda} g = \tilde{f}(t) \text{ in } [-T,T]. \tag{9.3}$$

Lemma 9.1. *When $\alpha_1 \in (0, \frac{1}{3})$ and $T > 0$ are sufficiently small, there is a linear operator T_1 such that $g = T_1[\tilde{f}]$ satisfies (9.3) and the following estimate holds*

$$\|g\|_{*,\Theta,k+1} \le C\|\tilde{f}\|_{*,\Theta,k}.$$

Here the norms are defined by

$$\|h\|_{*,\Theta,k} = \sup_{t\in[-T,T]} (\lambda_*(t))^{-\Theta}|\log(T-t)|^k|h(t)|.$$

Here $k \in (0,1)$.

Let us start with the construction of the linear operator T_1. We will find an inverse for $\tilde{L}_0$, namely given a function $\tilde{f}$, find g such that $\tilde{L}_0[g] - \frac{1}{3}\frac{\mathbf{Z}}{\lambda} g = \tilde{f}$. Set $g = (-2\lambda \mathbf{c})'$, differentiating this equation we get

$$-2(\lambda\mathbf{c})'' - 2\frac{1}{(T-t)|\log(T-t)|}(\lambda\mathbf{c})' + \frac{2\mathbf{Z}}{3(1-\alpha_1)}\frac{\mathbf{c}'}{|\log(T-t)|} = \frac{1}{1-\alpha_1}\frac{\tilde{f}'(t)}{|\log(T-t)|}.$$

This equation can be rewritten as

$$-2\left(|\log(T-t)|(\lambda\mathbf{c})'\right)' + \frac{2\mathbf{Z}}{3(1-\alpha_1)}\mathbf{c}' = \frac{1}{1-\alpha_1}\tilde{f}'(t).$$

Integrating this equation gives us

$$2|\log(T-t)|(\lambda\mathbf{c})' + \frac{2\mathbf{Z}}{3(1-\alpha_1)}\int_t^T \mathbf{c}'(s)ds = -\frac{1}{1-\alpha_1}\tilde{f}(t),$$

which is equivalent to

$$(\lambda\mathbf{c})' - \frac{\mathbf{Z}}{3(1-\alpha_1)}|\log(T-t)|^{-1}\mathbf{c}(t) = \frac{1}{2(1-\alpha_1)}|\log(T-t)|^{-1}\tilde{f}(t). \tag{9.4}$$

Rewrite it as

$$\left[(T-t)^{-\frac{\mathbf{Z}}{6(1-\alpha_1)}\frac{\log(T-t)}{|\log T|}}(\lambda\mathbf{c})\right]' = (T-t)^{-\frac{\mathbf{Z}}{6(1-\alpha_1)}\frac{\log(T-t)}{|\log T|}}\frac{1}{2(1-\alpha_1)}|\log(T-t)|^{-1}\tilde{f}(t).$$

We finally get

$$\lambda\mathbf{c} = (T-t)^{\frac{\mathbf{Z}}{6(1-\alpha_1)}\frac{\log(T-t)}{|\log T|}}\int_t^T (T-\tau)^{-\frac{\mathbf{Z}}{6(1-\alpha_1)}\frac{\log(T-\tau)}{|\log T|}}\frac{1}{2(1-\alpha_1)}|\log(T-\tau)|^{-1}\tilde{f}(\tau)d\tau. \tag{9.5}$$

We denote the the operator which assigns $\tilde{f}(t)$ to $(-2\lambda\mathbf{c})'$ by the formulas (9.5) and (9.4) as T_0.

Lemma 9.2. *Set* $\alpha_1 \in (0,\frac{1}{3})$, $k \in (0,1)$ *and* $(-2\lambda\mathbf{c})' := T_0(\tilde{f})$ *given by (9.5). Then the following estimate holds*

$$\|T_0[\tilde{f}]\|_{*,\Theta,k+1} \le \frac{2}{1-\alpha_1}\|\tilde{f}\|_{*,\Theta,k}$$

for a constant $C > 0$ *which is independent of* k *and* T.

Proof. We recall that

$$\|\tilde{f}\|_{*,\Theta,k} = \sup_{t\in[-T,T]}(\lambda_*(t))^{-\Theta}|\log(T-t)|^k|\tilde{f}(t)|.$$

Then from the formula (9.5), we have the following estimate

$$\begin{aligned}
|\lambda\mathbf{c}| &= (T-t)^{\frac{\mathbf{Z}}{6(1-\alpha_1)}\frac{\log(T-t)}{|\log T|}}\int_t^T (T-\tau)^{-\frac{\mathbf{Z}}{6(1-\alpha_1)}\frac{\log(T-\tau)}{|\log T|}}\frac{1}{2(1-\alpha_1)}|\log(T-\tau)|^{-1}\tilde{f}(\tau)d\tau \\
&\le \frac{1}{2(1-\alpha_1)}\frac{(\lambda_*(t))^{\Theta}\|\tilde{f}\|_{**,\Theta,k}}{|\log(T-t)|^{k+1}}(T-t)^{\frac{\mathbf{Z}}{6(1-\alpha_1)}\frac{\log(T-t)}{|\log T|}}\int_t^T (T-\tau)^{-\frac{\mathbf{Z}}{6(1-\alpha_1)}\frac{\log(T-\tau)}{|\log T|}}d\tau \\
&\le \frac{3(\lambda_*(t))^{\Theta}|\log T|(T-t)}{2\mathbf{Z}|\log(T-t)|^{k+2}}\|\tilde{f}\|_{**,\Theta,k}.
\end{aligned}$$

Using equation (9.4), we know that

$$|(-2\lambda\mathbf{c})'| \le \frac{2(\lambda_*(t))^{\Theta}}{(1-\alpha_1)|\log(T-t)|^{k+1}}\|\tilde{f}\|_{**,\Theta,k}.$$

□

Proof of Lemma 9.1: We construct g as a solution of the fixed point problem

$$g = T_0\left[\tilde{f} - \eta(\frac{t}{T})\tilde{L}_1[g]\right]$$

where T_0 is the operator defined in (9.5) and η is the cut-off function. By Lemma 9.2, we have

$$\|T_0\left[\eta(\frac{t}{T})\tilde{L}_1[g]\right]\|_{*,\Theta,k+1} \le \frac{2}{1-\alpha_1}\|\tilde{L}_1[g]\|_{*,\Theta,k}.$$

Let us estimate the different terms in $\tilde{L}_1$, which are defined by

$$\tilde{L}_1[g] = \sum_{j=0}^{4} \tilde{L}_{1j}[g]$$

where

$$\tilde{L}_{10}[g] = \int_{-T}^{t} \frac{g(s)}{T-s} ds, \quad \tilde{L}_{11}[g] = \int_{t-(T-t)}^{t-(T-t)^{1+\alpha_1}} \frac{g(s)}{t-s} ds, \quad \tilde{L}_{12}[g] = \int_{t-(T-t)}^{t} \frac{g(s)}{T-s} ds$$

$$\tilde{L}_{13}[g] = \int_{-T}^{t-(T-t)} g(s) \left(\frac{1}{t-s} - \frac{1}{T-s} \right) ds, \quad \tilde{L}_{14}[g] = (4\log(|\log(T-t)|) - 2\log(|\log T|))g(t).$$

Then we have the following estimates. First,

$$|\tilde{L}_{10}[g]| \le \|g\|_{*,\Theta,k+1} \int_{-T}^{t} \frac{(\lambda_*(s))^{\Theta}}{(T-s)|\log(T-s)|^{k+1}} ds \le \|g\|_{*,\Theta,k+1} \frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^{k+1}},$$

and

$$\begin{aligned} |\tilde{L}_{11}[g]| &\le \|g\|_{*,\Theta,k+1} \int_{t-(T-t)}^{t-(T-t)^{1+\alpha_1}} \frac{(\lambda_*(s))^{\Theta}}{(t-s)|\log(T-s)|^{k+1}} ds \\ &\le \|g\|_{*,\Theta,k+1} \frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^{k+1}} \int_{t-(T-t)}^{t-(T-t)^{1+\alpha_1}} \frac{1}{t-s} ds \\ &\le \|g\|_{*,\Theta,k+1} \frac{\alpha_1(\lambda_*(t))^{\Theta}}{|\log(T-t)|^{k}}, \end{aligned}$$

therefore

$$\|\tilde{L}_{10}[g]\|_{*,\Theta,k} \le \frac{1}{|\log T|} \|g\|_{*,\Theta,k+1}, \quad \|\tilde{L}_{11}[g]\|_{*,\Theta,k} \le \alpha_0 \|g\|_{*,\Theta,k+1}.$$

Second,

$$\begin{aligned} |\tilde{L}_{12}[g]| &\le \|g\|_{*,\Theta,k+1} \int_{t-(T-t)}^{t} \frac{(\lambda_*(s))^{\Theta}}{(T-s)|\log(T-s)|^{k+1}} ds \\ &\le \|g\|_{*,\Theta,k+1} \frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^{k+1}} \int_{t-(T-t)}^{t} \frac{1}{T-s} ds \\ &\le \|g\|_{*,\Theta,k+1} \frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^{k+1}} \log 2 \end{aligned}$$

which implies that

$$\|\tilde{L}_{12}[g]\|_{*,\Theta,k} \le \frac{\log 2}{|\log T|} \|g\|_{*,\Theta,k+1}.$$

For $\tilde{L}_{13}$ we have

$$\begin{aligned} |\tilde{L}_{13}[g]| &\le \|g\|_{*,\Theta,k+1} \int_{-T}^{t-(T-t)} \frac{(\lambda_*(s))^{\Theta}}{|\log(T-s)|^{k+1}} \left(\frac{1}{t-s} - \frac{1}{T-s} \right) ds \\ &\le C\|g\|_{*,\Theta,k+1} (T-t) \int_{-T}^{t-(T-t)} \frac{(\lambda_*(s))^{\Theta}}{(T-s)^2|\log(T-s)|^{k+1}} ds \\ &\le C\|g\|_{*,\Theta,k+1} \frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^{k+1}} \end{aligned}$$

and this gives

$$\|\tilde{L}_{13}[g]\|_{*,\Theta,k} \le \frac{C}{|\log T|} \|g\|_{*,\Theta,k+1}.$$

Finally, we have

$$|\tilde{L}_{14}[g]| \le \|g\|_{*,\Theta,k+1} (\lambda_*(t))^{\Theta} \frac{\log(|\log(T-t)|) + \log(|\log T|)}{|\log(T-t)|^{k+1}}$$

and this gives us the estimate

$$\|\tilde{L}_{14}[g]\|_{*,\Theta,k} \le \frac{C\log(|\log T|)}{|\log T|} \|g\|_{*,\Theta,k+1}.$$

Combine all the estimates above we get

$$\|T_0[\eta(\frac{t}{T})\tilde{L}_1[g]]\|_{*,\Theta,k+1} \le \frac{2}{1-\alpha_1}\left(\alpha_1 + \frac{1}{|\log T|} + \frac{\log|\log T|}{|\log T|}\right)\|g\|_{*,\Theta,k+1}.$$

If $0<\alpha_1<\frac{1}{3}$ is fixed and $T>0$ is sufficiently small, then we get

$$\|g\|_{*,\Theta,k+1} \le C\|\tilde{f}\|_{*,\Theta,k}.$$

□

Let

$$E(t) = \mathcal{I}[p_{1,0}](t) + \frac{2}{3}\mathbf{Z}\mathbf{c}^{(0)} - f(t) \tag{9.6}$$

and we consider the fixed point problem

$$p_{1,1} = \mathcal{A}[p_{1,1}] \tag{9.7}$$

where

$$\mathcal{A}[p_{1,1}] = T_1[-\eta(\frac{t}{T})E - \tilde{\mathcal{B}}[p_{1,0}+p_{1,1}]].$$

Then we have

Proposition 9.3. *When $\alpha\in(0,\frac{1}{3})$ and $k\in(0,1)$, there is a function $p_{1,1}$ satisfying (9.7) satisfying $\|p_{1,1}\|_{*,k+1} \le C_0|\log T|^{k-1}$. Here C_0 is a sufficiently large but fixed constant.*

We divide the proof into the following lemmas.

Lemma 9.4. *For the term $E(t)$ defined in (9.6), we have*

$$|E(t)| \le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|}, \quad -\frac{T}{4}\le t\le T. \tag{9.8}$$

Proof. By definition (9.6) we have

$$E(t) = \int_{-T}^{t-\lambda_*(t)^2}\frac{p_{1,0}(s)}{t-s}ds + \frac{2}{3}\mathbf{Z}\mathbf{c}^{(0)} - f(t).$$

Let $t\in[-\frac{T}{4},T]$ and we write

$$\begin{aligned}
E(t) &= \int_{-T}^{t-(T-t)/5}\frac{p_{1,0}(s)}{t-s}ds + \int_{t-(T-t)/5}^{t-\lambda_*(t)^2}\frac{p_{1,0}(s)}{t-s}ds + \frac{2}{3}\mathbf{Z}\mathbf{c}^{(0)} - f(t)\\
&= \int_{-T}^{t}\frac{p_{1,0}(s)}{T-s}ds - \int_{t-(T-t)/5}^{t}\frac{p_{1,0}(s)}{T-s}ds\\
&\quad + \int_{-T}^{t-(T-t)/5}p_{1,0}(s)\left(\frac{1}{t-s}-\frac{1}{T-s}\right)ds + \int_{t-(T-t)/5}^{t-\lambda_*(t)^2}\frac{p_{1,0}(s)}{t-s}ds + \frac{2}{3}\mathbf{Z}\mathbf{c}^{(0)} - f(t).
\end{aligned}$$

Then we estimate as follows,

$$\begin{aligned}
\left|\int_{t-(T-t)/5}^{t}\frac{p_{1,0}(s)}{T-s}ds\right| &\lesssim \|f\|_{*,\Theta,0}\int_{t-(T-t)/5}^{t}\frac{(\lambda_*(s))^\Theta}{|\log(T-s)|}\frac{1}{T-s}ds\\
&\lesssim \|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|}\frac{1}{T-t}\int_{t-(T-t)/5}^{t}ds \lesssim \|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|}
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{-T}^{t-(T-t)/5}p_{1,0}(s)\left(\frac{1}{t-s}-\frac{1}{T-s}\right)ds\right| &\lesssim \|f\|_{*,\Theta,0}\int_{-T}^{t-(T-t)/5}\frac{(\lambda_*(s))^\Theta}{|\log(T-s)|}\frac{T-t}{(t-s)(T-s)}ds\\
&\lesssim \|f\|_{*,\Theta,0}\int_{-T}^{t-(T-t)/5}\frac{(\lambda_*(s))^\Theta}{|\log(T-s)|}\frac{T-t}{(T-s)^2}ds\\
&\lesssim \|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|}.
\end{aligned}$$

For the fourth term in E we have

$$\int_{t-(T-t)/5}^{t-\lambda_*(t)^2}\frac{p_{1,0}(s)}{t-s}ds=p_{1,0}(t)\int_{t-(T-t)/5}^{t-\lambda_*(t)^2}\frac{1}{t-s}ds-\int_{t-(T-t)/5}^{t-\lambda_*(t)^2}\frac{p_{1,0}(t)-p_{1,0}(s)}{t-s}ds$$
$$=p_{1,0}(t)\left(\log(T-t)/5-2\log(\lambda_*(t))\right)-\int_{t-(T-t)/5}^{t-\lambda_*(t)^2}\frac{p_{1,0}(t)-p_{1,0}(s)}{t-s}ds.$$

Furthermore,

$$\left|\int_{t-(T-t)/5}^{t-\lambda_*(t)^2}\frac{p_{1,0}(t)-p_{1,0}(s)}{t-s}ds\right|\le\sup_{-T\le s\le t}|\dot p_{1,0}(s)|(T-t)\le\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|}.$$

Since $p_{1,0}$ satisfies the equation $\int_{-T}^{t}\frac{p_{1,0}(s)}{T-s}ds-p_{1,0}(t)\log(T-t)-\frac{2}{3}\mathbf{Z}\mathbf{c}^{(0)}-f(t)=0$, we have

$$E(t)\le C\|f\|_{*,\Theta,0}\left(\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|}\right).$$

This is the desired estimate (9.8). □

Lemma 9.5. *We have the estimate*

$$\int_{-T}^{t-(\lambda_*(t))^2}\frac{(\lambda_*(s))^{\Theta}}{(t-s)^a|\log(T-s)|^b}ds\le C\frac{(\lambda_*(t))^{2(1-a)+\Theta}}{|\log(T-t)|^b},\quad t\in[0,T]$$

for $a>1$ and $b>0$. For $\mu\in(0,1)$, $l\in\mathbb{R}$, we have

$$\int_{-T}^{t-(\lambda_*(t))^2}\frac{(T-s)^{\mu}}{(t-s)^2|\log(T-s)|^l}ds\le C\frac{(T-t)^{\mu}}{(\lambda_*(t))^2|\log(T-t)|^l}.$$

Proof. Let us write

$$\int_{-T}^{t-(\lambda_*(t))^2}\frac{(\lambda_*(s))^{\Theta}}{(t-s)^a|\log(T-s)|^b}ds$$
$$=\int_{-T}^{t-(T-t)}\frac{(\lambda_*(s))^{\Theta}}{(t-s)^a|\log(T-s)|^b}ds+\int_{t-(T-t)}^{t-(\lambda_*(t))^2}\frac{(\lambda_*(s))^{\Theta}}{(t-s)^a|\log(T-s)|^b}ds.$$

Then, for $t\in[0,T]$, we estimate,

$$\int_{-T}^{t-(T-t)}\frac{(\lambda_*(s))^{\Theta}}{(t-s)^a|\log(T-s)|^b}ds\le C\int_{-T}^{t-(T-t)}\frac{1}{(T-s)^{a-\Theta}|\log(T-s)|^{b+2\Theta}}ds$$
$$\le C\frac{1}{(T-t)^{a-\Theta-1}|\log(T-t)|^{b+2\Theta}}\le C\frac{(\lambda_*(t))^{\Theta}}{(T-t)^{a-1}|\log(T-t)|^b}\le C\frac{(\lambda_*(t))^{2(1-a)+\Theta}}{|\log(T-t)|^b}$$

and

$$\int_{t-(T-t)}^{t-(\lambda_*(t))^2}\frac{(\lambda_*(s))^{\Theta}}{(t-s)^a|\log(T-s)|^b}ds$$
$$\le C\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^b}\int_{t-(T-t)}^{t-(\lambda_*(t))^2}\frac{1}{(t-s)^a}ds\le C\frac{(\lambda_*(t))^{\Theta}}{(\lambda_*(t))^{2a-2}|\log(T-t)|^b}.$$

The case for $t\in[-T,0]$ is similar. Indeed, we have

$$\left|\int_{-T}^{t-(\lambda_*(t))^2}\frac{(\lambda_*(s))^{\Theta}}{(t-s)^a|\log(T-s)|^b}ds\right|\le\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^b}\int_{-T}^{t-(\lambda^*(t))^2}\frac{1}{(t-s)^a}ds\le C\frac{(\lambda_*(t))^{\Theta+2(1-a)}}{|\log(T-t)|^b}.$$

Similarly, we have

$$\int_{-T}^{t-(\lambda_*(t))^2}\frac{(T-s)^{\mu}}{(t-s)^2|\log(T-s)|^l}ds\le C\frac{(T-t)^{\mu}}{(\lambda_*(t))^2|\log(T-t)|^l}.$$

□

Lemma 9.6. *Let $M=C|\log T|^{k-1}$, $k\in(0,1)$, then for $\|p_{1,1}\|_{*,\Theta,k+1}\le M$, we have*

$$\|\tilde B[p_{1,0}+p_{1,1}](\cdot)\|_{*,\Theta,k}\le C\|f\|_{*,\Theta,0}|\log T|^{k-1}.$$

Proof. Observe that with the choice of M, if $\|p_{1,1}\|_{*,\Theta,k+1} \leq M$, then we have

$$\left|\frac{p_{1,1}}{p_{1,0}}\right| \leq \frac{C_0}{|\log T|^k}$$

since $T > 0$ is sufficiently small. Let us recall that

$$\tilde{\mathcal{B}}[p_1] = \tilde{\mathcal{B}}_1[p_1] + \tilde{\mathcal{B}}_2[p_1]\mathbf{c} + \tilde{\mathcal{B}}_3[p_1]\mathbf{c} - 2\lambda\dot{\mathbf{c}},$$

$$\tilde{\mathcal{B}}_1[p_1] = -\int_{-T}^{t-\lambda_*^2(t)} \frac{p_1(s)}{t-s}\left(\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right)+1\right)ds - \int_{t-\lambda_*^2(t)}^{t} \frac{p_1(s)}{t-s}\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right)ds,$$

$$\tilde{\mathcal{B}}_2[p_1] = \int_{-T}^{t-\lambda_*^2(t)} \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\left(\Gamma_5\left(\frac{\lambda^2(t)}{t-s}\right)+\frac{2}{3}\right)ds + \int_{t-\lambda_*^2(t)}^{t} \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\Gamma_5\left(\frac{\lambda^2(t)}{t-s}\right)ds,$$

$$\tilde{\mathcal{B}}_3[p_1] = -\frac{2}{3}\int_{-T}^{t-\lambda_*^2(t)} \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}ds + \frac{2}{3}\mathbf{Z}.$$

From the definition, we have

$$\|2\lambda\dot{\mathbf{c}}\|_{*,\Theta,k} \leq C\|f\|_{*,\Theta,0}|\log T|^{k-1}.$$

For the other terms, we write

$$\tilde{B}_{1,a}[p_{1,0}+p_{1,1}](t) = \int_{-T}^{t-\lambda_*^2(t)} \frac{p_1(s)}{t-s}\left(\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right)+1\right)ds,$$

$$\tilde{B}_{2,a}[p_{1,0}+p_{1,1}](t) = \int_{-T}^{t-\lambda_*^2(t)} \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\left(\Gamma_5\left(\frac{\lambda^2(t)}{t-s}\right)+\frac{2}{3}\right)ds$$

and

$$\tilde{B}_{1,b}[p_{1,0}+p_{1,1}](t) = \int_{t-\lambda_*^2(t)}^{t} \frac{p_1(s)}{t-s}\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right)ds,$$

$$\tilde{B}_{2,b}[p_{1,0}+p_{1,1}](t) = \int_{t-\lambda_*^2(t)}^{t} \frac{\mathrm{Re}[p_0(s)e^{-i\gamma(t)}]}{t-s}\Gamma_5\left(\frac{\lambda^2(t)}{t-s}\right)ds.$$

From Lemma 9.5 and the asymptotic estimates for Γ_3, Γ_5, we have the following estimates,

$$\begin{aligned}
|\tilde{B}_{1,a}[p_{1,0}+p_{1,1}](t)| &\leq C(\lambda_*(t))^{2\sigma}\left|\int_{-T}^{t-(\lambda_*(t))^2} \frac{(p_{1,0}+p_{1,1})(s)}{(t-s)^{1+\sigma}}ds\right| \\
&\leq C\|f\|_{*,\Theta,0}(\lambda_*(t))^{2\sigma}\int_{-T}^{t-(\lambda_*(t))^2} \frac{(\lambda_*(s))^{\Theta}}{(t-s)^{1+\sigma}|\log(T-s)|}ds \\
&\leq C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|}, \\
|\tilde{B}_{2,a}[p_{1,0}+p_{1,1}](t)\mathbf{c}(t)| &\leq C(\lambda_*(t))^{2\sigma}\left|\int_{-T}^{t-(\lambda_*(t))^2} \frac{p_0(s)}{(t-s)^{1+\sigma}}ds\mathbf{c}(t)\right| \\
&\leq C\|f\|_{*,\Theta,0}(\lambda_*(t))^{2\sigma+\Theta}\int_{-T}^{t-(\lambda_*(t))^2} \frac{1}{(t-s)^{1+\sigma}|\log(T-s)|^2}ds \\
&\leq C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^2}
\end{aligned}$$

and

$$|\tilde{B}_{1,b}[p_{1,0}+p_{1,1}](t)| \leq C\frac{1}{(\lambda_*(t))^2}\int_{t-(\lambda_*(t))^2}^{t} |(p_{1,0}+p_{1,1})(s)|\,ds \leq C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|},$$

$$|\tilde{B}_{2,b}[p_{1,0}+p_{1,1}](t)\mathbf{c}(t)| \leq C\frac{\|f\|_{*,\Theta,0}}{(\lambda_*(t))^2}\int_{t-(\lambda_*(t))^2}^{t} |p_0(s)|\,ds(\lambda_*(t))^{\Theta} \leq C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^2},$$

$$
\begin{aligned}
&|\tilde{B}_3[p_{1,0}+p_{1,1}](t)\mathbf{c}(t)| \\
&\leq C\|f\|_{*,\Theta,0}(\lambda_*(t))^{\Theta}|\dot{\lambda}_*| + C\|f\|_{*,\Theta,0}(\lambda_*(t))^{\Theta}\left|\int_{t-(T-t)}^{t-\lambda^2(t)} \frac{p_0(s)-p_0(t)}{t-s}ds\right| \\
&\leq C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^2}.
\end{aligned}
$$

Combine all the above estimates, we obtain

$$
\|\tilde{B}[p_{1,0}+p_{1,1}](\cdot)\|_{*,\Theta,k} \leq C\|f\|_{*,\Theta,0}|\log T|^{k-1}.
$$

□

Lemma 9.7. *Let $M = C|\log T|^{k-1}$, then for $\|p_{1,i}\|_{*,\Theta,k} \leq M$, $i = 1, 2$, we have the following estimate,*

$$
\|\tilde{B}[p_{1,0}+p_{1,1}](\cdot) - \tilde{B}[p_{1,0}+p_{1,2}](\cdot)\|_{*,\Theta,k} \leq \frac{C}{|\log T|}\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}.
$$

Proof. Let us use the notations in Lemma 9.6. The estimate for $2\lambda\dot{\mathbf{c}}$ is from definition directly. For the other terms, we write

$$
\begin{aligned}
D_{1,a} &:= \tilde{B}_{1,a}[p_{1,0}+p_{1,1}](t) - \tilde{B}_{1,a}[p_{1,0}+p_{1,2}](t), \quad D_{2,a} := \tilde{B}_{2,a}[p_1](\mathbf{c}^{(0)}+\mathbf{c}^{(1,1)})(t) - \tilde{B}_{2,a}[p_1](\mathbf{c}^{(0)}+\mathbf{c}^{(1,2)})(t) \\
D_{1,b} &:= \tilde{B}_{1,b}[p_{1,0}+p_{1,1}](t) - \tilde{B}_{1,b}[p_{1,0}+p_{1,2}](t), \quad D_{2,b} := \tilde{B}_{2,b}[p_1](\mathbf{c}^{(0)}+\mathbf{c}^{(1,1)})(t) - \tilde{B}_{2,b}[p_1](\mathbf{c}^{(0)}+\mathbf{c}^{(1,2)})(t), \\
D_3 &:= \tilde{B}_3[p_1](\mathbf{c}^{(0)}+\mathbf{c}^{(1,1)})(t) - \tilde{B}_3[p_1](\mathbf{c}^{(0)}+\mathbf{c}^{(1,2)})(t).
\end{aligned}
$$

Let us consider the term

$$
\frac{d}{d\zeta}\tilde{B}_{1,a}[p_{1,0}+p_{1,\zeta}](t) = \int_{-T}^{t-(\lambda_*(t))^2} \frac{p_{1,1}(s)-p_{1,2}(s)}{t-s}\left(\Gamma_3\left(\frac{|(p_0+p_1)(t)|^2}{t-s}\right)+1\right)ds
$$

with $p_{1,\zeta}(t) = \zeta p_{1,1}(t) + (1-\zeta)p_{1,2}(t)$. Using the decaying estimates for Γ_3 and Lemma 9.5, we have

$$
\begin{aligned}
&\left|\int_{-T}^{t-(\lambda_*(t))^2} \frac{p_{1,1}(s)-p_{1,2}(s)}{t-s}\left(\Gamma_3\left(\frac{|(p_0+p_1)(t)|^2}{t-s}\right)+1\right)ds\right| \\
&\leq C\int_{-T}^{t-(\lambda_*(t))^2} \frac{|p_{1,1}(s)-p_{1,2}(s)|}{t-s}\left(\frac{|(p_0+p_\zeta)(t)|^2}{t-s}\right)^{\sigma}ds \\
&\leq C\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}(\lambda_*(t))^{2\sigma}\int_{-T}^{t-(\lambda_*(t))^2} \frac{(\lambda_*(s))^{\Theta}}{(t-s)^{1+\sigma}|\log(T-s)|^{k+1}}ds \\
&\leq C\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}(\lambda_*(t))^{2\sigma}\frac{(\lambda_*(t))^{\Theta}}{(\lambda_*(t))^{2\sigma}|\log(T-t)|^{k+1}} \\
&= C\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^{k+1}}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
|D_{1,a}(t)| &= \left|\int_0^1 \frac{d}{d\zeta}\tilde{B}_{1,a}[p_{1,0}+p_{1,\zeta}](t)\right| \\
&\leq \left|\int_{-T}^{t-(\lambda_*(t))^2} \frac{p_{1,1}(s)-p_{1,2}(s)}{t-s}\left(\Gamma_3\left(\frac{|(p_0+p_1)(t)|^2}{t-s}\right)-1\right)ds\right| \\
&\leq C\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^{k+1}}.
\end{aligned}
$$

Similarly, we estimate $D_{1,b}$ as follows.

$$
\frac{d}{d\zeta}\tilde{B}_{1,b}[p_{1,0}+p_{1,\zeta}](t) = \int_{t-(\lambda_*(t))^2}^{t} \frac{p_{1,1}(s)-p_{1,2}(s)}{t-s}\Gamma_3\left(\frac{|(p_0+p_1)(t)|^2}{t-s}\right)ds.
$$

For this term we have

$$
\begin{aligned}
|D_{1,b}(t)| &= \left| \int_0^1 \frac{d}{d\zeta} \tilde{B}_{1,a}[p_{1,0} + p_{1,\zeta}](t) \right| \\
&\le \left| \int_{t-(\lambda_*(t))^2}^{t} \frac{p_{1,1}(s) - p_{1,2}(s)}{t-s} \Gamma_3 \left(\frac{|(p_0+p_1)(t)|^2}{t-s} \right) ds \right| \\
&\le \frac{C}{|(p_0+p_\zeta)(t)|^2} \int_{t-(\lambda_*(t))^2}^{t} |p_{1,1}(s) - p_{1,2}(s)| ds \\
&\le C \|p_{1,1} - p_{1,2}\|_{*,\Theta,k+1} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}}.
\end{aligned}
$$

Now we estimate $D_{2,a}$. We have

$$
\begin{aligned}
|D_{2,a}| &= |\tilde{B}_{2,a}[p_1](\mathbf{c}^{(0)} + \mathbf{c}^{(1,1)})(t) - \tilde{B}_{2,a}[p_1](\mathbf{c}^{(0)} + \mathbf{c}^{(1,2)})(t)| \\
&\le C(\lambda_*(t))^{2\sigma} \left| \int_{-T}^{t-(\lambda_*(t))^2} \frac{p_0(s)}{(t-s)^{1+\sigma}} ds (\mathbf{c}^{(1,1)} - \mathbf{c}^{(1,2)})(t) \right| \\
&\le C(\lambda_*(t))^{2\sigma-1} \left| \int_{-T}^{t-(\lambda_*(t))^2} \frac{p_0(s)}{(t-s)^{1+\sigma}} ds \int_t^T (p_{1,1}(s) - p_{1,2}(s)) ds \right| \\
&\le C \|p_{1,1} - p_{1,2}\|_{*,\Theta,k+1} (\lambda_*(t))^{2\sigma-1} \left| \int_{-T}^{t-(\lambda_*(t))^2} \frac{p_0(s)}{(t-s)^{1+\sigma}} ds \int_t^T \frac{(\lambda_*(s))^\Theta}{|\log(T-s)|^{k+1}} ds \right| \\
&\le C \|p_{1,1} - p_{1,2}\|_{*,\Theta,k+1} (\lambda_*(t))^{2\sigma-1} \left| \int_{-T}^{t-(\lambda_*(t))^2} \frac{|\log T|}{(t-s)^{1+\sigma} |\log(T-s)|^2} ds \int_t^T \frac{(\lambda_*(s))^\Theta}{|\log(T-s)|^{k+1}} ds \right| \\
&\le C \|p_{1,1} - p_{1,2}\|_{*,\Theta,k+1} (\lambda_*(t))^{2\sigma-1} \frac{|\log T| (\lambda_*(t))^{-2\sigma}}{|\log(T-t)|^2} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}} (T-t) \\
&\le C \|p_{1,1} - p_{1,2}\|_{*,\Theta,k+1} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}}.
\end{aligned}
$$

For $D_{2,b}$, we have

$$
\begin{aligned}
|D_{2,b}| &= |\tilde{B}_{2,b}[p_1](\mathbf{c}^{(0)} + \mathbf{c}^{(1,1)})(t) - \tilde{B}_{2,b}[p_1](\mathbf{c}^{(0)} + \mathbf{c}^{(1,2)})(t)| \\
&\le C(\lambda_*(t))^{-2} \left| \int_{t-(\lambda_*(t))^2}^{t} |p_0(s)| ds (\mathbf{c}^{(1,1)} - \mathbf{c}^{(1,2)})(t) \right| \\
&\le C(\lambda_*(t))^{-2} \left| \int_{t-(\lambda_*(t))^2}^{t} |p_0(s)| ds \int_t^T (p_{1,1}(s) - p_{1,2}(s)) ds \right| \\
&\le C \|p_{1,1} - p_{1,2}\|_{*,\Theta,k+1} (\lambda_*(t))^{-2} \left| \int_{t-(\lambda_*(t))^2}^{t} |p_0(s)| ds \int_t^T \frac{(\lambda_*(s))^\Theta}{|\log(T-s)|^{k+1}} ds \right| \\
&\le C \|p_{1,1} - p_{1,2}\|_{*,\Theta,k+1} (\lambda_*(t))^{-2} \left| \frac{|\log T| (\lambda_*(t))^2}{|\log(T-t)|^2} \int_t^T \frac{(\lambda_*(s))^\Theta}{|\log(T-s)|^{k+1}} ds \right| \\
&\le C \|p_{1,1} - p_{1,2}\|_{*,\Theta,k+1} (\lambda_*(t))^{-2} \frac{|\log T| (\lambda_*(t))^2}{|\log(T-t)|^2} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}} (T-t) \\
&\le C \|p_{1,1} - p_{1,2}\|_{*,\Theta,k+1} \frac{(\lambda_*(t))^{\Theta+1}}{|\log(T-t)|^{k+1}}.
\end{aligned}
$$

For D_3, we have

$$
\begin{aligned}
|D_3| &= |\tilde{B}_3[p_1](\mathbf{c}^{(0)} + \mathbf{c}^{(1,1)})(t) - \tilde{B}_3[p_1](\mathbf{c}^{(0)} + \mathbf{c}^{(1,2)})(t)| \\
&\le C \left| \left(|\dot{\lambda}_*| + \left| \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{p_0(s) - p_0(t)}{t-s} ds \right| \right) (\mathbf{c}^{(1,1)} - \mathbf{c}^{(1,2)})(t) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\le C\frac{|\log T|}{|\log(T-t)|^2}\left|\int_t^T (p_{1,1}(s)-p_{1,2}(s))ds\right| \\
&\le C\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}\left|\frac{|\log T|}{|\log(T-t)|^2}\int_t^T \frac{(\lambda_*(s))^{\Theta}}{|\log(T-s)|^{k+1}}ds\right| \\
&\le C\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}\frac{|\log T|}{|\log(T-t)|^2}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^{k+1}}(T-t) \\
&\le C\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}\frac{(\lambda_*(t))^{\Theta+1}}{|\log(T-t)|^{k+1}}.
\end{aligned}
$$

Combine all the above estimates, we obtain

$$
\|\tilde{B}[p_{1,0}+p_{1,1}](\cdot)-\tilde{B}[p_{1,0}+p_{1,2}](\cdot)\|_{*,\Theta,k}\le \frac{C}{|\log T|}\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}.
$$

□

Proof of Proposition 9.3: Now we prove Proposition 9.3 based on the above estimates and the fixed point arguments. Indeed, we choose $k\in(0,1)$. Then from Lemma 9.1 we have

$$
\|\mathcal{A}[p_{1,1}]\|_{*,\Theta,k+1}\le C\left(\|\eta\tilde{E}\|_{*,\Theta,k}+\|\tilde{B}[p_{1,0}+p_{1,1}]\|_{*,\Theta,k}\right)
$$

and

$$
\|\eta\tilde{E}\|_{*,\Theta,k}\le C_E|\log T|^{k-1}
$$
$$
\|\tilde{B}[p_{1,0}+p_{1,1}]\|_{*,\Theta,k}\le C|\log T|^{k-1}.
$$

Therefore we have

$$
\|\mathcal{A}[p_{1,1}]\|_{*,\Theta,k+1}\le C\cdot C_E|\log T|^{k-1}+C|\log T|^{k-1}\le C_0|\log T|^{k-1}
$$

for fixing C_0 large. Hence the operator maps $\overline{B}_M(0)$ into itself for $M=C_0|\log T|^{k-1}$. Moreover, we have

$$
\|\mathcal{A}[p_{1,1}]-\mathcal{A}[p_{1,2}]\|_{*,\Theta,k+1}\le C\|\tilde{B}[p_{1,0}+p_{1,1}]-\tilde{B}[p_{1,0}+p_{1,2}]\|_{*,\Theta,k}\le\frac{C}{|\log T|}\|p_{1,1}-p_{1,2}\|_{*,\Theta,k+1}.
$$

Then we obtain a fixed point of problem (9.7) by the contraction mapping theorem. □

9.2. **Estimates of the derivatives.**

Lemma 9.8. *We have the following estimates for $p_{1,1}$,*

$$
|\dot{p}_{1,1}(t)|\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^2(T-t)}, \tag{9.9}
$$

$$
|\ddot{p}_{1,1}(t)|\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|^2(T-t)^2}. \tag{9.10}
$$

Proof. To prove (9.9), let us recall that $p_{1,1}$ satisfies the following equation

$$
\tilde{L}_0[p_{1,1}]+\eta(\frac{t}{T})\tilde{L}_1[p_{1,1}]+\eta(\frac{t}{T})E+\tilde{B}[p_{1,0}+p_{1,1}](t)-\frac{2\mathbf{Z}}{3}\mathbf{c}^{(1)}=0 \text{ in } [-T,T].
$$

Differentiate this equation with respect to t, we have

$$
\begin{aligned}
(1-\alpha_1)|\log(T-t)|\dot{p}_{1,1}+(1-\alpha_1)\frac{p_{1,1}}{T-t}+\eta(\frac{t}{T})\frac{d}{dt}\tilde{L}_1[p_{1,1}]+\frac{1}{T}\eta'\left(\frac{t}{T}\right)\tilde{L}_1[p_{1,1}]+\eta\left(\frac{t}{T}\right)\frac{d}{dt}E \\
+\frac{1}{T}\eta'\left(\frac{t}{T}\right)E+\frac{d}{dt}\tilde{B}[p_{1,0}+p_{1,1}](t)-\frac{d}{dt}\left(\frac{2\mathbf{Z}}{3}\mathbf{c}^{(1)}\right)=0.
\end{aligned}
$$

Rewrite the above equation as

$$
(1-\alpha_1)|\log(T-t)|\dot{p}_{1,1}+\eta(\frac{t}{T})\tilde{L}_1[\dot{p}_{1,1}]+\mathcal{U}[\dot{p}_{1,1}](t)=h(t)
$$

where h is a function satisfying

$$
|h(t)|\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|(T-t)}
$$

and $\mathcal{U}$ is the operator defined by

$$\mathcal{U}[\dot{p}_{1,1}] = -\int_{-T}^{t-(\lambda_*(t))^2} \frac{\dot{p}_{1,1}(s)}{t-s}\left(\Gamma_3\left(\frac{(\lambda(t))^2}{t-s}\right)+1\right)ds - \int_{t-(\lambda_*(t))^2}^{t} \frac{\dot{p}_{1,1}(s)}{t-s}\Gamma_3\left(\frac{(\lambda(t))^2}{t-s}\right)ds.$$

Observe that One of the terms in h is $\eta(\frac{t}{T})\frac{d}{dt}E$, from the proof of Lemma 9.4, we have

$$\left|\eta(\frac{t}{T})\frac{d}{dt}E\right| \le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)}.$$

Let us recall the proof of Lemma 9.1, we estimate the term $\frac{d}{dt}\tilde{L}_{10}[p_{1,1}]$ as follows

$$\left|\frac{d}{dt}\tilde{L}_{10}[p_{1,0}]\right| = \left|\frac{d}{dt}\int_{-T}^{t}\frac{p_{1,0}(s)}{T-s}ds\right| = \left|\frac{p_{1,0}(t)}{T-t}\right| \le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)}.$$

Next we compute $\frac{d}{dt}\tilde{L}_{11}[p_{1,1}]$,

$$\begin{aligned}\frac{d}{dt}\tilde{L}_{11}[p_{1,1}] &= \frac{d}{dt}\int_{(T-t)^{1+\alpha_1}}^{T-t}\frac{p_{1,1}(t-r)}{r}dr\\ &= \int_{(T-t)^{1+\alpha_1}}^{T-t}\frac{\dot{p}_{1,1}(t-r)}{r}dr - \frac{p_{1,1}(t-(T-t))}{T-t}\\ &\quad + (1+\alpha_1)\frac{p_{1,1}(t-(T-t)^{1+\alpha_0})}{T-t}\\ &= \tilde{L}_{11}[\dot{p}_{1,1}] - \frac{p_{1,1}(t-(T-t))}{T-t} + (1+\alpha_1)\frac{p_{1,1}(t-(T-t)^{1+\alpha_0})}{T-t}.\end{aligned}$$

The last two terms can be estimated as follows

$$\left|\frac{p_{1,1}(t-(T-t))}{T-t}\right| \le C\|f\|_{*,\Theta,0}\frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}(T-t)} \le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)},$$

$$\left|(1+\alpha_1)\frac{p_{1,1}(t-(T-t)^{1+\alpha_0})}{T-t}\right| \le C\|f\|_{*,\Theta,0}\frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}(T-t)} \le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)}.$$

Similarly, the term $\frac{d}{dt}\tilde{L}_{12}[p_{1,1}]$ can be estimated as

$$\begin{aligned}\frac{d}{dt}\tilde{L}_{12}[p_{1,1}] &= \frac{d}{dt}\int_0^{T-t}\frac{p_{1,1}(t-r)}{T-t+r}dr\\ &= \int_0^{T-t}\frac{\dot{p}_{1,1}(t-r)}{T-t+r}dr - \frac{p_{1,1}(t-(T-t))}{2(T-t)}\\ &\quad + \int_0^{T-t}\frac{p_{1,1}(t-r)}{(T-t+r)^2}dr\\ &= \tilde{L}_{12}[\dot{p}_{1,1}] - \frac{p_{1,1}(t-(T-t))}{2(T-t)} + \int_{t-(T-t)}^{t}\frac{p_{1,1}(s)}{(T-s)^2}ds\end{aligned}$$

and

$$\left|\frac{p_{1,1}(t-(T-t))}{2(T-t)}\right| \le C\|f\|_{*,\Theta,0}\frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}(T-t)} \le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)},$$

$$\left|\int_{t-(T-t)}^{t}\frac{p_{1,1}(s)}{(T-s)^2}ds\right| \le C\|f\|_{*,\Theta,0}\frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}(T-t)} \le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)}.$$

The term $\frac{d}{dt}\tilde{L}_{13}[p_{1,1}]$ can be estimated as

$$\begin{aligned}\frac{d}{dt}\tilde{L}_{13}[p_{1,1}] &= \frac{d}{dt}\int_{T-t}^{T+t}\frac{p_{1,1}(t-r)(T-t)}{r(T-t+r)}dr\\ &= \int_{T-t}^{T+t}\frac{\dot{p}_{1,1}(t-r)(T-t)}{r(T-t+r)}dr + \frac{p_{1,1}(t-(T-t))}{2(T-t)} + \frac{p_{1,1}(t-(T+t))(T-t)}{2T(T+t)}\\ &\quad - \int_{T-t}^{T+t}\frac{p_{1,1}(t-r)}{r(T-t+r)}dr + \int_{T-t}^{T+t}\frac{p_{1,1}(t-r)(T-t)}{r(T-t+r)^2}dr\end{aligned}$$

$$= \tilde{L}_{13}[\dot{p}_{1,1}] + \frac{p_{1,1}(t-(T-t))}{2(T-t)} + \frac{p_{1,1}(-T)(T-t)}{2T(T+t)}$$
$$- \int_{-T}^{t-(T-t)} \frac{p_{1,1}(s)}{(t-s)(T-s)} ds + \int_{-T}^{t-(T-t)} \frac{p_{1,1}(s)(T-t)}{(t-s)(T-s)^2} ds$$

and

$$\left| \frac{p_{1,1}(t-(T-t))}{2(T-t)} \right| \le C\|f\|_{*,\Theta,0} \frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}(T-t)} \le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)},$$
$$\left| \frac{p_{1,1}(-T)(T-t)}{2T(T+t)} \right| \le C \left| \frac{p_{1,1}(-T)}{2T} \right| \le C\|f\|_{*,\Theta,0} \frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}(T-t)} \le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)},$$
$$\left| \int_{-T}^{t-(T-t)} \frac{p_{1,1}(s)}{(t-s)(T-s)} ds \right| \le C\|f\|_{*,\Theta,0} \int_{-T}^{t-(T-t)} \frac{|\log T|^{k-1}(\lambda_*(s))^\Theta}{(T-s)^2|\log(T-s)|^{k+1}} ds$$
$$\le C\|f\|_{*,\Theta,0} \frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}(T-t)} \le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)},$$
$$\left| \int_{-T}^{t-(T-t)} \frac{p_{1,1}(s)(T-t)}{(t-s)(T-s)^2} ds \right| \le C\|f\|_{*,\Theta,0} \int_{-T}^{t-(T-t)} \frac{|\log T|^{k-1}(\lambda_*(s))^\Theta(T-t)}{(T-s)^3|\log(T-s)|^{k+1}} ds$$
$$\le C\|f\|_{*,\Theta,0} \frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+1}(T-t)} \le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)}.$$

The term $\frac{d}{dt}\tilde{L}_{14}[p_{1,1}]$ can be estimated as

$$\frac{d}{dt}\tilde{L}_{14}[p_{1,1}] = \frac{d}{dt}\left((4\log(|\log(T-t)|) - 2\log(|\log T|))p_{1,1}(t)\right)$$
$$= \tilde{L}_{14}[\dot{p}_{1,1}] + \frac{4}{(T-t)|\log(T-t)|} p_{1,1}$$

and

$$\left| \frac{4}{(T-t)|\log(T-t)|} p_{1,1} \right| \le C\|f\|_{*,\Theta,0} \frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^{k+2}(T-t)} \le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|^2(T-t)}.$$

For the term $\frac{1}{T}\eta'\left(\frac{t}{T}\right)\tilde{L}_1[p_{1,1}]$, we have

$$\left| \frac{1}{T}\eta'\left(\frac{t}{T}\right)\tilde{L}_1[p_{1,1}] \right| \le C\|f\|_{*,\Theta,0} \frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{T|\log(T-t)|^k} \le C\|f\|_{*,\Theta,0} \frac{|\log T|^{k-1}(\lambda_*(t))^\Theta}{|\log(T-t)|^k(T-t)}$$
$$\le C\|f\|_{*,\Theta,0} \frac{(\lambda_*(t))^\Theta}{|\log(T-t)|(T-t)}$$

since $|\log(T-t)| \sim |\log T|$ in the interval $t \in [-\frac{T}{4}, 0]$ where $\eta'\left(\frac{t}{T}\right)$ is not zero. Now we consider the term $\frac{d}{dt}\tilde{B}[p_{1,0}+p_{1,1}](t)$. Observe that one of these terms are

$$\frac{d}{dt}\int_{-T}^{t-\lambda_*^2(t)} \frac{p_1(s)}{t-s}\left(\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right)+1\right) ds,$$

we have

$$\frac{d}{dt}\int_{-T}^{t-\lambda_*^2(t)} \frac{p_1(s)}{t-s}\left(\Gamma_3\left(\frac{\lambda^2(t)}{t-s}\right)+1\right) ds = \frac{d}{dt}\int_{\lambda_*^2(t)}^{t+T} \frac{p_1(t-r)}{r}\left(\Gamma_3\left(\frac{\lambda^2(t)}{r}\right)+1\right) dr$$
$$= \int_{\lambda_*^2(t)}^{t+T} \frac{\dot{p}_1(t-r)}{r}\left(\Gamma_3\left(\frac{\lambda^2(t)}{r}\right)+1\right) dr + \int_{\lambda_*^2(t)}^{t+T} \frac{p_1(t-r)}{r}\left(\Gamma_3'\left(\frac{\lambda^2(t)}{r}\right)\right)\frac{2\lambda(t)\lambda'(t)}{r} dr$$
$$+ \frac{p_1(-T)}{t+T}\left(\Gamma_3\left(\frac{\lambda^2(t)}{t+T}\right)+1\right) + \frac{p_1(t-\lambda_*^2(t))}{\lambda_*^2(t)}\left(\Gamma_3\left(\frac{\lambda^2(t)}{\lambda_*^2(t)}\right)+1\right)2\lambda_*(t)\lambda_*'(t)$$

and from Lemma 9.5, it holds that

$$\left| \int_{\lambda_*^2(t)}^{t+T} \frac{p_1(t-r)}{r}\left(\Gamma_3'\left(\frac{\lambda^2(t)}{r}\right)\right)\frac{2\lambda(t)\lambda'(t)}{r} dr \right| = \left| \int_{-T}^{t-\lambda_*^2(t)} \frac{p_1(s)}{t-s}\left(\Gamma_3'\left(\frac{\lambda^2(t)}{t-s}\right)\right)\frac{2\lambda(t)\lambda'(t)}{t-s} ds \right|$$

$$\begin{aligned}
&\le C|\lambda(t)\lambda'(t)|\int_{-T}^{t-\lambda_*^2(t)}\frac{|p_1(s)|}{(t-s)^2}\left(\frac{t-s}{\lambda^2(t)}\right)^{\sigma}ds\\
&\le C\lambda(t)^{1-2\sigma}|\lambda'(t)|\|f\|_{*,\Theta,0}\int_{-T}^{t-\lambda_*^2(t)}\frac{1}{(t-s)^{2-\sigma}}\frac{(\lambda_*(s))^{\Theta}}{|\log(T-s)|}ds\\
&\le C\lambda(t)^{1-2\sigma}|\lambda'(t)|\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta+2(\sigma-1)}}{|\log(T-t)|}\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|(T-t)},
\end{aligned}$$

$$\left|\frac{p_1(-T)}{t+T}\left(\Gamma_3\left(\frac{\lambda^2(t)}{t+T}\right)+1\right)\right|\le\left|\frac{|p_1(-T)|}{T}\right|\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(2T))^{\Theta}}{|\log(2T)|T}\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|(T-t)},$$

$$\begin{aligned}
&\left|\frac{p_1(t-\lambda_*^2(t))}{\lambda_*^2(t)}\left(\Gamma_3\left(\frac{\lambda^2(t)}{\lambda_*^2(t)}\right)+1\right)2\lambda_*(t)\lambda_*'(t)\right|\\
&\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|}\frac{|\lambda_*'(t)|}{\lambda_*(t)}\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}}{|\log(T-t)|(T-t)}.
\end{aligned}$$

The other terms in h as well as the proof of (9.10) can be estimated similarly, we omit the details here. □

9.3. **Estimate of the remainder** $R_{\alpha_1}[p_{1,1}]$.

Lemma 9.9. *Let $p_{1,1}$ be the solution constructed in Proposition 9.3, then we have*

$$|R_{\alpha_1}[p_{1,1}]|\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}(T-t)^{\alpha_1}}{|\log(T-t)|^2}.$$

Furthermore, it holds that

$$\left|\frac{d}{dt}R_{\alpha_1}[p_{1,1}]\right|\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}(T-t)^{\alpha_1-1}}{|\log(T-t)|^2}.$$

Proof. From the definition $\mathcal{R}_{\alpha_1}[p_{1,1}]=\mathcal{I}[p_{1,1}]-S_{\alpha_1}[p_{1,1}]$ and the estimates in Lemma 9.8, we have

$$\begin{aligned}
|R_{\alpha_1}[p_{1,1}]|&\le\int_{t-(T-t)^{1+\alpha_1}}^{t-(\lambda_*(t))^2}\frac{|p_{1,1}(t)-p_{1,1}(s)|}{t-s}ds\\
&\le\sup_{r\in(t-(T-t)^{1+\alpha_1},t-(\lambda_*(t))^2)}|\dot p_{1,1}(r)|(T-t)^{1+\alpha_1}\\
&\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}(T-t)^{\alpha_1}}{|\log(T-t)|^2}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{d}{dt}R_{\alpha_1}[p_{1,1}]&=-\frac{d}{dt}\int_{t-(T-t)^{1+\alpha_1}}^{t-(\lambda_*(t))^2}\frac{p_{1,1}(t)-p_{1,1}(s)}{t-s}ds\\
&=-\frac{d}{dt}\int_{(\lambda_*(t))^2}^{(T-t)^{1+\alpha_1}}\frac{p_{1,1}(t)-p_{1,1}(t-r)}{r}dr\\
&=-\int_{(\lambda_*(t))^2}^{(T-t)^{1+\alpha_1}}\frac{\dot p_{1,1}(t)-\dot p_{1,1}(t-r)}{r}dr-2(1+\alpha_1)\frac{p_{1,1}(t-(\lambda_*(t))^2)}{\lambda_*(t)}\dot\lambda_*(t)\\
&\quad+(1+\alpha_1)\frac{p_{1,1}(t-(T-t)^{1+\alpha_0})}{T-t}.
\end{aligned}$$

From the estimates in Lemma 9.8, we obtain

$$\left|\frac{d}{dt}R_{\alpha_1}[p_{1,1}]\right|\le C\|f\|_{*,\Theta,0}\frac{(\lambda_*(t))^{\Theta}(T-t)^{\alpha_1-1}}{|\log(T-t)|^2}.$$

□

Combine the results in Proposition 9.3, Lemma 9.8 and Lemma 9.9, define $\mathcal{R}_{\mathbf{c}}[f](t):=R_{\alpha_1}[p_{1,1}](t)$, then we obtain the conclusion of Proposition 8.2.

Appendix A. Proof of technical lemmas

Several technical lemmas will be proved in this section.

A.1. **Decomposition of the linearization.**

Proof of Lemma 2.1. We compute

$$\begin{aligned}
\Delta(\phi_3 W) =& \left[\partial_{\rho\rho}\phi_3 + \frac{1}{\rho}\partial_\rho\phi_3 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_3 - \left(w_\rho^2 + \frac{\sin^2 w}{\rho^2}\right)\phi_3\right] W \\
&+ \left[\left(w_{\rho\rho} + \frac{w_\rho}{\rho} - \frac{\sin w\cos w}{\rho^2}\right)\phi_3 + 2w_\rho\partial_\rho\phi_3\right] E_1 + 2\frac{\sin w\partial_\theta\phi_3}{\rho^2}E_2, \\
\Delta(\phi_1 E_1) =& -\left[\left(w_{\rho\rho} + \frac{w_\rho}{\rho} + \frac{\sin w\cos w}{\rho^2}\right)\phi_1 + 2w_\rho\partial_\rho\phi_1\right] W \\
&+ \left[\partial_{\rho\rho}\phi_1 + \frac{1}{\rho}\partial_\rho\phi_1 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_1 - \left(w_\rho^2 + \frac{\cos^2 w}{\rho^2}\right)\phi_1\right] E_1 + 2\frac{\cos w\partial_\theta\phi_1}{\rho^2}E_2,
\end{aligned}$$

and

$$\begin{aligned}
\Delta(\phi_2 E_2) =& -\frac{2\sin w\partial_\theta\phi_2}{\rho^2}W - \frac{2\cos w\partial_\theta\phi_2}{\rho^2}E_1 \\
&+ \left[\partial_{\rho\rho}\phi_2 + \frac{1}{\rho}\partial_\rho\phi_2 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_2 - \frac{\phi_2}{\rho^2}\right] E_2.
\end{aligned}$$

So we get

$$\begin{aligned}
&\Delta\phi \\
=& \left[\partial_{\rho\rho}\phi_3 + \frac{1}{\rho}\partial_\rho\phi_3 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_3 - \left(w_\rho^2 + \frac{\sin^2 w}{\rho^2}\right)\phi_3 - \left(w_{\rho\rho} + \frac{w_\rho}{\rho} + \frac{\sin w\cos w}{\rho^2}\right)\phi_1 - 2w_\rho\partial_\rho\phi_1 - \frac{2\sin w\partial_\theta\phi_2}{\rho^2}\right] W \\
&+ \left[\left(w_{\rho\rho} + \frac{w_\rho}{\rho} - \frac{\sin w\cos w}{\rho^2}\right)\phi_3 + 2w_\rho\partial_\rho\phi_3 + \partial_{\rho\rho}\phi_1 + \frac{1}{\rho}\partial_\rho\phi_1 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_1 - \left(w_\rho^2 + \frac{\cos^2 w}{\rho^2}\right)\phi_1 - \frac{2\cos w\partial_\theta\phi_2}{\rho^2}\right] E_1 \\
&+ \left[\partial_{\rho\rho}\phi_2 + \frac{1}{\rho}\partial_\rho\phi_2 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_2 - \frac{\phi_2}{\rho^2} + 2\frac{\sin w\partial_\theta\phi_3}{\rho^2} + 2\frac{\cos w\partial_\theta\phi_1}{\rho^2}\right] E_2.
\end{aligned}$$

Also, one has

$$\begin{aligned}
\phi_{y_1} =& \left(\cos\theta\partial_\rho - \frac{\sin\theta}{\rho}\partial_\theta\right)(\phi_3 W + \phi_1 E_1 + \phi_2 E_2) \\
=& \left[\cos\theta\partial_\rho\phi_3 - \frac{\sin\theta}{\rho}\partial_\theta\phi_3 - \cos\theta w_\rho\phi_1 + \frac{\sin\theta\sin w}{\rho}\phi_2\right] W \\
&+ \left[\cos\theta\partial_\rho\phi_1 - \frac{\sin\theta}{\rho}\partial_\theta\phi_1 + \cos\theta w_\rho\phi_3 + \frac{\sin\theta\cos w}{\rho}\phi_2\right] E_1 \\
&+ \left[\cos\theta\partial_\rho\phi_2 - \frac{\sin\theta}{\rho}\partial_\theta\phi_2 - \frac{\sin\theta\sin w}{\rho}\phi_3 - \frac{\sin\theta\cos w}{\rho}\phi_1\right] E_2
\end{aligned}$$

$$\begin{aligned}
\phi_{y_2} =& \left(\sin\theta\partial_\rho + \frac{\cos\theta}{\rho}\partial_\theta\right)(\phi_3 W + \phi_1 E_1 + \phi_2 E_2) \\
=& \left[\sin\theta\partial_\rho\phi_3 + \frac{\cos\theta}{\rho}\partial_\theta\phi_3 - \sin\theta w_\rho\phi_1 - \frac{\cos\theta\sin w}{\rho}\phi_2\right] W \\
&+ \left[\sin\theta\partial_\rho\phi_1 + \frac{\cos\theta}{\rho}\partial_\theta\phi_1 + \sin\theta w_\rho\phi_3 - \frac{\cos\theta\cos w}{\rho}\phi_2\right] E_1 \\
&+ \left[\sin\theta\partial_\rho\phi_2 + \frac{\cos\theta}{\rho}\partial_\theta\phi_2 + \frac{\cos\theta\sin w}{\rho}\phi_3 + \frac{\cos\theta\cos w}{\rho}\phi_1\right] E_2
\end{aligned}$$

and thus

$$\begin{aligned} U_{y_1} \wedge \phi_{y_2} = &\cos\theta w_\rho \left[\sin\theta\partial_\rho\phi_2 + \frac{\cos\theta}{\rho}\partial_\theta\phi_2 + \frac{\cos\theta\cos w}{\rho}\phi_1\right] W \\ &+ \frac{\sin\theta\sin w}{\rho}\left[\sin\theta\partial_\rho\phi_1 + \frac{\cos\theta}{\rho}\partial_\theta\phi_1 - \frac{\cos\theta\cos w}{\rho}\phi_2\right] W + \frac{w_\rho\sin w}{\rho}\phi_3 W \\ &- \frac{\sin\theta\sin w}{\rho}\left[\sin\theta\partial_\rho\phi_3 + \frac{\cos\theta}{\rho}\partial_\theta\phi_3 - \sin\theta w_\rho\phi_1 - \frac{\cos\theta\sin w}{\rho}\phi_2\right] E_1 \\ &- \cos\theta w_\rho\left[\sin\theta\partial_\rho\phi_3 + \frac{\cos\theta}{\rho}\partial_\theta\phi_3 - \sin\theta w_\rho\phi_1 - \frac{\cos\theta\sin w}{\rho}\phi_2\right] E_2, \end{aligned}$$

$$\begin{aligned} \phi_{y_1} \wedge U_{y_2} = &\frac{\cos\theta\sin w}{\rho}\left[\cos\theta\partial_\rho\phi_1 - \frac{\sin\theta}{\rho}\partial_\theta\phi_1 + \frac{\sin\theta\cos w}{\rho}\phi_2\right] W \\ &- \sin\theta w_\rho\left[\cos\theta\partial_\rho\phi_2 - \frac{\sin\theta}{\rho}\partial_\theta\phi_2 - \frac{\sin\theta\cos w}{\rho}\phi_1\right] W + \frac{w_\rho\sin w}{\rho}\phi_3 W \\ &- \frac{\cos\theta\sin w}{\rho}\left[\cos\theta\partial_\rho\phi_3 - \frac{\sin\theta}{\rho}\partial_\theta\phi_3 - \cos\theta w_\rho\phi_1 + \frac{\sin\theta\sin w}{\rho}\phi_2\right] E_1 \\ &+ \sin\theta w_\rho\left[\cos\theta\partial_\rho\phi_3 - \frac{\sin\theta}{\rho}\partial_\theta\phi_3 - \cos\theta w_\rho\phi_1 + \frac{\sin\theta\sin w}{\rho}\phi_2\right] E_2. \end{aligned}$$

Therefore, linearization

$$\Delta_y\phi - 2U_{y_1}\wedge\phi_{y_2} - 2\phi_{y_1}\wedge U_{y_2} = 0$$

implies

$$\begin{aligned} &\left[\partial_{\rho\rho}\phi_3 + \frac{1}{\rho}\partial_\rho\phi_3 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_3 - \left(w_\rho^2 + \frac{\sin^2 w}{\rho^2}\right)\phi_3 - \left(w_{\rho\rho} + \frac{w_\rho}{\rho} + \frac{\sin w\cos w}{\rho^2}\right)\phi_1 - 2w_\rho\partial_\rho\phi_1 - \frac{2\sin w\partial_\theta\phi_2}{\rho^2}\right] W \\ &+ \left[\left(w_{\rho\rho} + \frac{w_\rho}{\rho} - \frac{\sin w\cos w}{\rho^2}\right)\phi_3 + 2w_\rho\partial_\rho\phi_3 + \partial_{\rho\rho}\phi_1 + \frac{1}{\rho}\partial_\rho\phi_1 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_1 - \left(w_\rho^2 + \frac{\cos^2 w}{\rho^2}\right)\phi_1 - \frac{2\cos w\partial_\theta\phi_2}{\rho^2}\right] E_1 \\ &+ \left[\partial_{\rho\rho}\phi_2 + \frac{1}{\rho}\partial_\rho\phi_2 + \frac{1}{\rho^2}\partial_{\theta\theta}\phi_2 - \frac{\phi_2}{\rho^2} + 2\frac{\sin w\partial_\theta\phi_3}{\rho^2} + 2\frac{\cos w\partial_\theta\phi_1}{\rho^2}\right] E_2 \\ = &\, 2\left[\frac{w_\rho}{\rho}\partial_\theta\phi_2 + \frac{w_\rho\cos w}{\rho}\phi_1 + \frac{\sin w}{\rho}\partial_\rho\phi_1 + 2\frac{w_\rho\sin w}{\rho}\phi_3\right] W \\ &+ 2\left[-\frac{\sin w}{\rho}\partial_\rho\phi_3 + \frac{\sin w}{\rho}w_\rho\phi_1\right] E_1 + 2\left[-\frac{w_\rho}{\rho}\partial_\theta\phi_3 + \frac{w_\rho\sin w}{\rho}\phi_2\right] E_2, \end{aligned}$$

and the proof is complete by collecting terms in the same direction.

□

Proof of Lemma 2.2. First, we notice the fact

$$U_r = \frac{1}{\lambda}w_\rho(\rho)Q_\gamma E_1, \quad \frac{1}{r}\partial_\theta U = -\frac{1}{\lambda}w_\rho(\rho)Q_\gamma E_2.$$

By direct computation, we have

$$\begin{aligned} \frac{1}{r}U_r \wedge \Phi_\theta &= \frac{1}{r}\left(\frac{1}{\lambda}w_\rho(\rho)Q_\gamma E_1\right)\wedge\Phi_\theta \\ &= \frac{1}{\lambda r}w_\rho(\rho)(Q_\gamma E_1 \wedge \Phi_\theta) \\ &= \frac{1}{\lambda r}w_\rho(\rho)\left[(-\Phi_\theta\cdot(Q_\gamma W))Q_\gamma E_2 + (\Phi_\theta\cdot(Q_\gamma E_2))Q_\gamma W\right] \end{aligned}$$

and

$$\begin{aligned}\frac{1}{r}\Phi_r\wedge U_\theta &= \Phi_r\wedge\left(-\frac{1}{\lambda}w_\rho(\rho)Q_\gamma E_2\right)\\ &= \left(-\frac{1}{\lambda}w_\rho(\rho)\right)\Phi_r\wedge(Q_\gamma E_2)\\ &= \left(-\frac{1}{\lambda}w_\rho(\rho)\right)\left[\left(-\Phi_r\cdot(Q_\gamma W)\right)Q_\gamma E_1+(\Phi_r\cdot(Q_\gamma E_1))Q_\gamma W\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}\tilde{L}_U[\Phi] =\ & -\frac{2}{\lambda r}w_\rho(\rho)\left[(-\Phi_\theta\cdot(Q_\gamma W))Q_\gamma E_2+(\Phi_\theta\cdot(Q_\gamma E_2))Q_\gamma W\right]\\ &+\frac{2}{\lambda}w_\rho(\rho)\left[\left(-\Phi_r\cdot(Q_\gamma W)\right)Q_\gamma E_1+(\Phi_r\cdot(Q_\gamma E_1))Q_\gamma W\right]\\ =\ & -\frac{2}{\lambda}w_\rho(\rho)\left(\Phi_r\cdot(Q_\gamma W)\right)Q_\gamma E_1+\frac{2}{\lambda r}w_\rho(\rho)(\Phi_\theta\cdot(Q_\gamma W))Q_\gamma E_2\\ &+\frac{2}{\lambda r}w_\rho(\rho)\left[r(\Phi_r\cdot(Q_\gamma E_1))-(\Phi_\theta\cdot(Q_\gamma E_2))\right]Q_\gamma W.\end{aligned}$$

□

Proof of Lemma 2.3. Clearly,

$$\Phi_r=\cos\theta\partial_{x_1}\Phi+\sin\theta\partial_{x_2}\Phi,\quad \frac{1}{r}\Phi_\theta=-\sin\theta\partial_{x_1}\Phi+\cos\theta\partial_{x_2}\Phi.$$

We have $(\Phi\cdot Q_\gamma E_j)=(Q_{-\gamma}\Phi\cdot E_j)$ and write $\phi=Q_{-\gamma}\Phi=(\phi_1,\ \phi_2,\ \phi_3)^T$ so that

$$\Phi=\begin{bmatrix}\varphi_1+i\varphi_2\\ \varphi_3\end{bmatrix}=\begin{bmatrix}e^{i\gamma}(\phi_1+i\phi_2)\\ \phi_3\end{bmatrix}. \tag{A.1}$$

Then, one has

$$\begin{aligned}\phi_r\cdot W &= \frac{1}{2}\sin w\Big[[\partial_{x_1}\phi_1+\partial_{x_2}\phi_2]+\cos(2\theta)[\partial_{x_1}\phi_1-\partial_{x_2}\phi_2]+\sin(2\theta)[\partial_{x_2}\phi_1+\partial_{x_1}\phi_2]\Big]\\ &\quad+\cos w[\partial_{x_1}\phi_3\cos\theta+\partial_{x_2}\phi_3\sin\theta]\\ \frac{1}{r}\phi_\theta\cdot W &= \frac{1}{2}\sin w\Big[[\partial_{x_2}\phi_1-\partial_{x_1}\phi_2]+\cos(2\theta)[\partial_{x_1}\phi_2+\partial_{x_2}\phi_1]+\sin(2\theta)[\partial_{x_2}\phi_2-\partial_{x_1}\phi_1]\Big]\\ &\quad+\cos w[-\partial_{x_1}\phi_3\sin\theta+\partial_{x_2}\phi_3\cos\theta]\end{aligned}$$

and

$$\begin{aligned}\phi_r\cdot E_1 &= \frac{1}{2}\cos w\Big[[\partial_{x_1}\phi_1+\partial_{x_2}\phi_2]+\cos(2\theta)[\partial_{x_1}\phi_1-\partial_{x_2}\phi_2]+\sin(2\theta)[\partial_{x_2}\phi_1+\partial_{x_1}\phi_2]\Big]\\ &\quad-\sin w[\partial_{x_1}\phi_3\cos\theta+\partial_{x_2}\phi_3\sin\theta]\\ \frac{1}{r}\phi_\theta\cdot E_2 &= [-\sin\theta\partial_{x_1}\phi_1+\cos\theta\partial_{x_2}\phi_1](-\sin\theta)+[-\sin\theta\partial_{x_1}\phi_2+\cos\theta\partial_{x_2}\phi_2](\cos\theta)\\ &= [\partial_{x_1}\phi_1\sin^2\theta+\partial_{x_2}\phi_2\cos^2\theta]-\frac{1}{2}\sin(2\theta)[\partial_{x_2}\phi_1+\partial_{x_1}\phi_2]\\ &= [\partial_{x_1}\phi_1\frac{1-\cos(2\theta)}{2}+\partial_{x_2}\phi_2\frac{\cos(2\theta)+1}{2}]-\frac{1}{2}\sin(2\theta)[\partial_{x_2}\phi_1+\partial_{x_1}\phi_2]\\ &= \frac{1}{2}[\partial_{x_1}\phi_1+\partial_{x_2}\phi_2]-\frac{1}{2}\sin(2\theta)[\partial_{x_2}\phi_1+\partial_{x_1}\phi_2]-\frac{1}{2}\cos(2\theta)[\partial_{x_1}\phi_1-\partial_{x_2}\phi_2]\\ \phi_r\cdot E_1-\frac{1}{r}\phi_\theta\cdot E_2 &= \frac{1}{2}\cos w\Big[[\partial_{x_1}\phi_1+\partial_{x_2}\phi_2]+\cos(2\theta)[\partial_{x_1}\phi_1-\partial_{x_2}\phi_2]+\sin(2\theta)[\partial_{x_2}\phi_1+\partial_{x_1}\phi_2]\Big]\\ &\quad-\sin w[\partial_{x_1}\phi_3\cos\theta+\partial_{x_2}\phi_3\sin\theta]\\ &\quad-\frac{1}{2}[\partial_{x_1}\phi_1+\partial_{x_2}\phi_2]+\frac{1}{2}\sin(2\theta)[\partial_{x_2}\phi_1+\partial_{x_1}\phi_2]+\frac{1}{2}\cos(2\theta)[\partial_{x_1}\phi_1-\partial_{x_2}\phi_2].\end{aligned}$$

Combining above terms with above Lemma and (A.1), we obtain the desired decomposition. □

A.2. **Analysis of the new error.**

Proof of Lemma 4.1. We have

$$
\begin{aligned}
&S[U_*]\\
&= \mathcal{R}_{U^\perp} + \mathcal{R}_U + \eta_1(\mathcal{E}^{(0)}_{U^\perp} - \partial_t\Phi^{(0)} + \Delta\Phi^{(0)}) + \eta_1(\mathcal{E}^{(\pm1)}_{U} - \partial_t\Phi^{(1)} + \Delta\Phi^{(1)}) + \mathcal{E}^{(1)}_{U^\perp}\\
&\quad + (1-\eta_1)\Big(\mathcal{E}^{(0)}_{U^\perp} + \mathcal{E}^{(\pm1)}_{U} + \mathcal{E}^{(\pm2)}_{U^\perp} + \mathcal{E}^{(\pm2)}_{U}\Big) + (\Delta\eta_1 - \partial_t\eta_1)\Phi_* + 2\nabla\Phi_*\cdot\nabla\eta_1\\
&\quad - \eta_1\left[\phi_1^{(2)}\partial_t(Q_\gamma E_1) + \phi_2^{(2)}\partial_t(Q_\gamma E_2) + 2\theta_t(\phi_1^{(2)}Q_\gamma E_2 - \phi_2^{(2)}Q_\gamma E_1) + \partial_\rho\Phi^{(2)}_{U^\perp}\rho_t\right]\\
&\quad - \eta_1\left[\phi_1^{(-2)}\partial_t(Q_\gamma E_1) + \phi_2^{(-2)}\partial_t(Q_\gamma E_2) - 2\theta_t(\phi_1^{(-2)}Q_\gamma E_2 - \phi_2^{(-2)}Q_\gamma E_1) + \partial_\rho\Phi^{(-2)}_{U^\perp}\rho_t\right]\\
&\quad - \eta_1\left[\sin 2\theta\psi_2\partial_t(Q_\gamma W) + 2\cos 2\theta\psi_2\theta_t Q_\gamma W + \sin 2\theta\partial_\rho\psi_2\rho_t Q_\gamma W\right]\\
&\quad + \eta_1\tilde{L}_U[\Phi^{(0)} + \Phi^{(1)}] - 2\partial_{x_1}(c_1Q_\gamma\mathcal{Z}_{1,1} + c_2Q_\gamma\mathcal{Z}_{1,2})\wedge\partial_{x_2}(\eta_1\Phi_*) - 2\partial_{x_1}(\eta_1\Phi_*)\wedge\partial_{x_2}(c_1Q_\gamma\mathcal{Z}_{1,1} + c_2Q_\gamma\mathcal{Z}_{1,2})\\
&\quad - 2\partial_{x_2}\eta_1\partial_{x_1}(Q_\gamma W)\wedge\Phi_* - 2\partial_{x_1}\eta_1\Phi_*\wedge\partial_{x_2}(Q_\gamma W) - 2\partial_{x_1}(\eta_1\Phi_*)\wedge\partial_{x_2}(\eta_1\Phi_*)\\
&= \mathcal{R}_{U^\perp} + \mathcal{R}_U + \eta_1(\mathcal{E}^{(0)}_{U^\perp} + \tilde{\mathcal{R}}^0) + \eta_1(\mathcal{E}^{(\pm1)}_{U} + \tilde{\mathcal{R}}^1) + \mathcal{E}^{(1)}_{U^\perp}\\
&\quad + (1-\eta_1)\Big(\mathcal{E}^{(0)}_{U^\perp} + \mathcal{E}^{(\pm1)}_{U} + \mathcal{E}^{(\pm2)}_{U^\perp} + \mathcal{E}^{(\pm2)}_{U}\Big) + (\Delta\eta_1 - \partial_t\eta_1)\Phi_* + 2\nabla\Phi_*\cdot\nabla\eta_1\\
&\quad - \eta_1\lambda^{-1}w_\rho(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)(\phi_1^{(2)} + \phi_1^{(-2)})Q_\gamma W\\
&\quad + \eta_1\sin w[\dot\gamma + \lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)](\phi_2^{(2)} + \phi_2^{(-2)})Q_\gamma W\\
&\quad - \eta_1\lambda^{-1}w_\rho(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)(\phi_1^{(2)} + \phi_1^{(-2)})Q_\gamma W\\
&\quad + \eta_1\lambda^{-1}(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)\sin 2\theta\partial_\rho\psi_2 Q_\gamma W\\
&\quad - 2\eta_1\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)\cos 2\theta\psi_2 Q_\gamma W\\
&\quad + \eta_1\cos w[\dot\gamma + \lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)](\phi_2^{(2)} + \phi_2^{(-2)})Q_\gamma E_1\\
&\quad + 2\eta_1\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)[\phi_2^{(2)} - \phi_2^{(-2)}]Q_\gamma E_1\\
&\quad + \eta_1\lambda^{-1}(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)(\partial_\rho\phi_1^{(2)} + \partial_\rho\phi_1^{(-2)})Q_\gamma E_1\\
&\quad + \eta_1\lambda^{-1}w_\rho(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)\sin 2\theta\psi_2 Q_\gamma E_1\\
&\quad - \eta_1\cos w[\dot\gamma + \lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)](\phi_1^{(2)} + \phi_1^{(-2)})Q_\gamma E_2\\
&\quad + 2\eta_1\lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)[\phi_1^{(-2)} - \phi_1^{(2)}]Q_\gamma E_2\\
&\quad + \eta_1\lambda^{-1}(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)(\partial_\rho\phi_2^{(2)} + \partial_\rho\phi_2^{(-2)})Q_\gamma E_2\\
&\quad - \eta_1\sin w[\dot\gamma + \lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)]\sin 2\theta\psi_2 Q_\gamma E_2\\
&\quad + \eta_1\tilde{L}_U[\Phi^{(0)} + \Phi^{(1)}] - \frac{2}{r}\partial_r(\eta_1\Phi_*)\wedge\partial_\theta(\eta_1\Phi_*)\\
&\quad - \frac{2}{r}\Big[\partial_r(c_1Q_\gamma\mathcal{Z}_{1,1} + c_2Q_\gamma\mathcal{Z}_{1,2})\wedge\partial_\theta(\eta_1\Phi_*) + \partial_r(\eta_1\Phi_*)\wedge\partial_\theta(c_1Q_\gamma\mathcal{Z}_{1,1} + c_2Q_\gamma\mathcal{Z}_{1,2})\Big]\\
&\quad - 2\partial_{x_2}\eta_1\partial_{x_1}(Q_\gamma W)\wedge\Phi_* - 2\partial_{x_1}\eta_1\Phi_*\wedge\partial_{x_2}(Q_\gamma W),
\end{aligned}
$$

where we have used

$$
\begin{aligned}
&\rho_t = -\lambda^{-1}(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho),\\
&\theta_t = \lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta),\\
&\partial_t(Q_\gamma W) = -\lambda^{-1}(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)w_\rho Q_\gamma E_1 + [\dot\gamma + \lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)]\sin w Q_\gamma E_2,\\
&\partial_t(Q_\gamma E_1) = \lambda^{-1}(\dot\xi_1\cos\theta + \dot\xi_2\sin\theta + \dot\lambda\rho)w_\rho Q_\gamma W + [\dot\gamma + \lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)]\cos w Q_\gamma E_2,\\
&\partial_t(Q_\gamma E_2) = -[\dot\gamma + \lambda^{-1}\rho^{-1}(\dot\xi_1\sin\theta - \dot\xi_2\cos\theta)](\cos w Q_\gamma E_1 + \sin w Q_\gamma W).
\end{aligned}
$$

We collect some extra terms produced by the cut-off η_1 and define

$$
\begin{aligned}
E_{\eta_1} &:= (1-\eta_1)\Big(\mathcal{E}^{(0)}_{U^\perp} + \mathcal{E}^{(\pm 1)}_{U} + \mathcal{E}^{(\pm 2)}_{U^\perp} + \mathcal{E}^{(\pm 2)}_{U}\Big) + (\Delta\eta_1 - \partial_t\eta_1)\Phi_* + 2\nabla\Phi_*\cdot\nabla\eta_1 \\
&\quad - 2\partial_{x_2}\eta_1\partial_{x_1}(Q_\gamma W)\wedge\Phi_* - 2\partial_{x_1}\eta_1\Phi_*\wedge\partial_{x_2}(Q_\gamma W) - \frac{2\eta_1\partial_r\eta_1}{r}\Phi_*\wedge\partial_\theta\Phi_* \\
&\quad - \frac{2\partial_r\eta_1}{r}\Phi_*\wedge\partial_\theta(c_1 Q_\gamma \mathcal{Z}_{1,1} + c_2 Q_\gamma\mathcal{Z}_{1,2}).
\end{aligned}
\tag{A.2}
$$

To further expand the error, we use (4.17) and first compute

$$
\partial_r\Phi^{(0)} = \begin{bmatrix} e^{i\theta}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0) \\ 0\end{bmatrix}, \quad \partial_\theta\Phi^{(0)} = \begin{bmatrix} ire^{i\theta}\psi^0 \\ 0\end{bmatrix},
$$

$$
\partial_r\Phi^{(1)} = \begin{bmatrix} 0 \\ 0 \\ \mathrm{Re}[e^{-i\theta}(\psi^1 + \frac{r^2}{z}\partial_z\psi^1)]\end{bmatrix}, \quad \partial_\theta\Phi^{(1)} = \begin{bmatrix} 0 \\ 0 \\ \mathrm{Im}(re^{-i\theta}\psi^1)\end{bmatrix},
$$

$$
\begin{aligned}
&\partial_r(\Phi^{(2)}_{U^\perp} + \Phi^{(-2)}_{U^\perp} + \Phi^{(2)}_{U}) \\
&= \lambda^{-1}\Big[\partial_\rho\phi_1^{(2)} Q_\gamma E_1 + \partial_\rho\phi_2^{(2)} Q_\gamma E_2 - w_\rho\phi_1^{(2)} Q_\gamma W\Big] \\
&\quad + \lambda^{-1}\Big[\partial_\rho\phi_1^{(-2)} Q_\gamma E_1 + \partial_\rho\phi_2^{(-2)} Q_\gamma E_2 - w_\rho\phi_1^{(-2)} Q_\gamma W\Big] \\
&\quad + \lambda^{-1}\sin 2\theta(\partial_\rho\psi_2 Q_\gamma W + w_\rho\psi_2 Q_\gamma E_1),
\end{aligned}
$$

$$
\begin{aligned}
&\partial_\theta(\Phi^{(2)}_{U^\perp} + \Phi^{(-2)}_{U^\perp} + \Phi^{(2)}_{U}) \\
= &-\phi_2^{(2)}\sin w Q_\gamma W + (\partial_\theta\phi_1^{(2)} - \cos w\phi_2^{(2)})Q_\gamma E_1 + (\partial_\theta\phi_2^{(2)} + \cos w\phi_1^{(2)})Q_\gamma E_2 \\
&-\phi_2^{(-2)}\sin w Q_\gamma W + (\partial_\theta\phi_1^{(-2)} - \cos w\phi_2^{(-2)})Q_\gamma E_1 + (\partial_\theta\phi_2^{(-2)} + \cos w\phi_1^{(-2)})Q_\gamma E_2 \\
&+\psi_2(2\cos 2\theta Q_\gamma W + \sin 2\theta\sin w Q_\gamma E_2).
\end{aligned}
$$

We then have

$$
\begin{aligned}
\partial_r\Phi_* = &\begin{bmatrix} e^{i\theta}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0) \\ 0\end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathrm{Re}[e^{-i\theta}(\psi^1 + \frac{r^2}{z}\partial_z\psi^1)]\end{bmatrix} \\
&+ \lambda^{-1}\Big[\partial_\rho\phi_1^{(2)} Q_\gamma E_1 + \partial_\rho\phi_2^{(2)} Q_\gamma E_2 - w_\rho\phi_1^{(2)} Q_\gamma W\Big] \\
&+ \lambda^{-1}\Big[\partial_\rho\phi_1^{(-2)} Q_\gamma E_1 + \partial_\rho\phi_2^{(-2)} Q_\gamma E_2 - w_\rho\phi_1^{(-2)} Q_\gamma W\Big] \\
&+ \lambda^{-1}\sin 2\theta(\partial_\rho\psi_2 Q_\gamma W + w_\rho\psi_2 Q_\gamma E_1) \\
= &\cos w\mathrm{Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right] Q_\gamma E_1 + \mathrm{Im}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right] Q_\gamma E_2 \\
&+ \sin w\mathrm{Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right] Q_\gamma W \\
&+ \mathrm{Re}[e^{-i\theta}(\psi^1 + \frac{r^2}{z}\partial_z\psi^1)]\cos w Q_\gamma W - \mathrm{Re}[e^{-i\theta}(\psi^1 + \frac{r^2}{z}\partial_z\psi^1)]\sin w Q_\gamma E_1 \\
&+ \lambda^{-1}\Big[\partial_\rho\phi_1^{(2)} Q_\gamma E_1 + \partial_\rho\phi_2^{(2)} Q_\gamma E_2 - w_\rho\phi_1^{(2)} Q_\gamma W\Big] \\
&+ \lambda^{-1}\Big[\partial_\rho\phi_1^{(-2)} Q_\gamma E_1 + \partial_\rho\phi_2^{(-2)} Q_\gamma E_2 - w_\rho\phi_1^{(-2)} Q_\gamma W\Big] \\
&+ \lambda^{-1}\sin 2\theta(\partial_\rho\psi_2 Q_\gamma W + w_\rho\psi_2 Q_\gamma E_1) \\
= &\left(\sin w\mathrm{Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right] + \mathrm{Re}[e^{-i\theta}(\psi^1 + \frac{r^2}{z}\partial_z\psi^1)]\cos w\right) Q_\gamma W \\
&+ \lambda^{-1}\left[\sin 2\theta\partial_\rho\psi_2 - w_\rho(\phi_1^{(2)} + \phi_1^{(-2)})\right] Q_\gamma W
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\cos w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]-\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\sin w\right)Q_\gamma E_1\\
&+\lambda^{-1}\left(\sin 2\theta w_\rho\psi_2+\partial_\rho\phi_1^{(2)}+\partial_\rho\phi_1^{(-2)}\right)Q_\gamma E_1\\
&+\left(\mathrm{Im}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\lambda^{-1}(\partial_\rho\phi_2^{(2)}+\partial_\rho\phi_2^{(-2)})\right)Q_\gamma E_2,
\end{aligned}
$$

$$
\begin{aligned}
\partial_\theta\Phi_* =\;& \begin{bmatrix} ire^{i\theta}\psi^0\\ 0\end{bmatrix}+\begin{bmatrix}0\\0\\ \mathrm{Im}(re^{-i\theta}\psi^1)\end{bmatrix}\\
&-\phi_2^{(2)}\sin wQ_\gamma W+(\partial_\theta\phi_1^{(2)}-\cos w\phi_2^{(2)})Q_\gamma E_1+(\partial_\theta\phi_2^{(2)}+\cos w\phi_1^{(2)})Q_\gamma E_2\\
&-\phi_2^{(-2)}\sin wQ_\gamma W+(\partial_\theta\phi_1^{(-2)}-\cos w\phi_2^{(-2)})Q_\gamma E_1+(\partial_\theta\phi_2^{(-2)}+\cos w\phi_1^{(-2)})Q_\gamma E_2\\
&+\psi_2(2\cos 2\theta Q_\gamma W+\sin 2\theta\sin wQ_\gamma E_2)\\
=\;&\cos w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]Q_\gamma E_1+\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]Q_\gamma E_2+\sin w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]Q_\gamma W\\
&+\mathrm{Im}(re^{-i\theta}\psi^1)\cos wQ_\gamma W-\mathrm{Im}(re^{-i\theta}\psi^1)\sin wQ_\gamma E_1\\
&-\phi_2^{(2)}\sin wQ_\gamma W+(\partial_\theta\phi_1^{(2)}-\cos w\phi_2^{(2)})Q_\gamma E_1+(\partial_\theta\phi_2^{(2)}+\cos w\phi_1^{(2)})Q_\gamma E_2\\
&-\phi_2^{(-2)}\sin wQ_\gamma W+(\partial_\theta\phi_1^{(-2)}-\cos w\phi_2^{(-2)})Q_\gamma E_1+(\partial_\theta\phi_2^{(-2)}+\cos w\phi_1^{(-2)})Q_\gamma E_2\\
&+\psi_2(2\cos 2\theta Q_\gamma W+\sin 2\theta\sin wQ_\gamma E_2)\\
=\;&\left(\sin w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]+\mathrm{Im}(re^{-i\theta}\psi^1)\cos w-\sin w(\phi_2^{(2)}+\phi_2^{(-2)})+2\psi_2\cos 2\theta\right)Q_\gamma W\\
&+\left(\cos w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]-\mathrm{Im}(re^{-i\theta}\psi^1)\sin w+(\partial_\theta\phi_1^{(2)}+\partial_\theta\phi_1^{(-2)})-\cos w(\phi_2^{(2)}+\phi_2^{(-2)})\right)Q_\gamma E_1\\
&+\left(\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]+(\partial_\theta\phi_2^{(2)}+\partial_\theta\phi_2^{(-2)})+\cos w(\phi_1^{(2)}+\phi_1^{(-2)})+\sin 2\theta\sin w\psi_2\right)Q_\gamma E_2.
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
&\partial_r\Phi_*\wedge\partial_\theta\Phi_*\\
=&\Bigg[\left(\cos w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]-\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\sin w\right)\\
&\qquad+\lambda^{-1}\left(\sin 2\theta w_\rho\psi_2+\partial_\rho\phi_1^{(2)}+\partial_\rho\phi_1^{(-2)}\right)\Bigg]\\
&\quad\times\left(\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]+(\partial_\theta\phi_2^{(2)}+\partial_\theta\phi_2^{(-2)})+\cos w(\phi_1^{(2)}+\phi_1^{(-2)})+\sin 2\theta\sin w\psi_2\right)Q_\gamma W\\
&-\left(\mathrm{Im}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\lambda^{-1}(\partial_\rho\phi_2^{(2)}+\partial_\rho\phi_2^{(-2)})\right)\\
&\quad\times\left(\cos w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]-\mathrm{Im}(re^{-i\theta}\psi^1)\sin w+(\partial_\theta\phi_1^{(2)}+\partial_\theta\phi_1^{(-2)})-\cos w(\phi_2^{(2)}+\phi_2^{(-2)})\right)Q_\gamma W\\
&+\left(\mathrm{Im}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\lambda^{-1}(\partial_\rho\phi_2^{(2)}+\partial_\rho\phi_2^{(-2)})\right)\\
&\quad\times\left(\sin w\mathrm{Re}\left[e^{-i\gamma}ir\psi^0\right]+\mathrm{Im}(re^{-i\theta}\psi^1)\cos w-\sin w(\phi_2^{(2)}+\phi_2^{(-2)})+2\psi_2\cos 2\theta\right)Q_\gamma E_1\\
&-\left(\sin w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\cos w\right)\\
&\quad\times\left(\mathrm{Im}\left[e^{-i\gamma}ir\psi^0\right]+(\partial_\theta\phi_2^{(2)}+\partial_\theta\phi_2^{(-2)})+\cos w(\phi_1^{(2)}+\phi_1^{(-2)})+\sin 2\theta\sin w\psi_2\right)Q_\gamma E_1\\
&+\Bigg[\left(\sin w\mathrm{Re}\left[e^{-i\gamma}(\psi^0+\frac{r^2}{z}\partial_z\psi^0)\right]+\mathrm{Re}[e^{-i\theta}(\psi^1+\frac{r^2}{z}\partial_z\psi^1)]\cos w\right)\\
&\qquad+\lambda^{-1}\left[\sin 2\theta\partial_\rho\psi_2-w_\rho(\phi_1^{(2)}+\phi_1^{(-2)})\right]\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(\cos w \mathrm{Re}\left[e^{-i\gamma} ir\psi^0\right] - \mathrm{Im}(re^{-i\theta}\psi^1)\sin w + (\partial_\theta \phi_1^{(2)} + \partial_\theta \phi_1^{(-2)}) - \cos w(\phi_2^{(2)} + \phi_2^{(-2)})\Big) Q_\gamma E_2 \\
&- \Bigg[\Big(\cos w \mathrm{Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right] - \mathrm{Re}[e^{-i\theta}(\psi^1 + \frac{r^2}{z}\partial_z\psi^1)]\sin w\Big) \\
&\quad + \lambda^{-1}\Big(\sin 2\theta w_\rho \psi_2 + \partial_\rho \phi_1^{(2)} + \partial_\rho\phi_1^{(-2)}\Big)\Bigg] \\
&\times \Big(\sin w \mathrm{Re}\left[e^{-i\gamma} ir\psi^0\right] + \mathrm{Im}(re^{-i\theta}\psi^1)\cos w - \sin w(\phi_2^{(2)} + \phi_2^{(-2)}) + 2\psi_2\cos 2\theta\Big) Q_\gamma E_2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_r(c_1 Q_\gamma \mathcal{Z}_{1,1} + c_2 Q_\gamma \mathcal{Z}_{1,2}) \wedge \partial_\theta \Phi_* + \partial_r\Phi_* \wedge \partial_\theta(c_1 Q_\gamma \mathcal{Z}_{1,1} + c_2 Q_\gamma \mathcal{Z}_{1,2}) \\
= &\Big(\mathrm{Im}\left[e^{-i\gamma} ir\psi^0\right] + (\partial_\theta\phi_2^{(2)} + \partial_\theta \phi_2^{(-2)}) + \cos w(\phi_1^{(2)} + \phi_1^{(-2)}) + \sin 2\theta \sin w\psi_2\Big) \\
&\quad \times \lambda^{-1} w_\rho \sin w(c_1\cos\theta + c_2\sin\theta) Q_\gamma W \\
&- \Big(\mathrm{Im}\left[e^{-i\gamma} ir\psi^0\right] + (\partial_\theta\phi_2^{(2)} + \partial_\theta \phi_2^{(-2)}) + \cos w(\phi_1^{(2)} + \phi_1^{(-2)}) + \sin 2\theta \sin w\psi_2\Big) \\
&\quad \times \lambda^{-1} w_\rho \cos w(c_1\cos\theta + c_2\sin\theta) Q_\gamma E_1 \\
&+ \Big(\cos w \mathrm{Re}\left[e^{-i\gamma} ir\psi^0\right] - \mathrm{Im}(re^{-i\theta}\psi^1)\sin w + (\partial_\theta \phi_1^{(2)} + \partial_\theta \phi_1^{(-2)}) - \cos w(\phi_2^{(2)} + \phi_2^{(-2)})\Big) \\
&\quad \times \lambda^{-1} w_\rho \cos w(c_1\cos\theta + c_2\sin\theta) Q_\gamma E_2 \\
&- \Big(\sin w \mathrm{Re}\left[e^{-i\gamma} ir\psi^0\right] + \mathrm{Im}(re^{-i\theta}\psi^1)\cos w - \sin w(\phi_2^{(2)} + \phi_2^{(-2)}) + 2\psi_2\cos 2\theta\Big) \\
&\quad \times \lambda^{-1} w_\rho \sin w(c_1\cos\theta + c_2\sin\theta) Q_\gamma E_2 \\
&+ \Bigg[\Big(\cos w \mathrm{Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right] - \mathrm{Re}[e^{-i\theta}(\psi^1 + \frac{r^2}{z}\partial_z\psi^1)]\sin w\Big) \\
&\quad + \lambda^{-1}\Big(\sin 2\theta w_\rho \psi_2 + \partial_\rho \phi_1^{(2)} + \partial_\rho\phi_1^{(-2)}\Big)\Bigg] \times \sin^2 w(c_1\cos\theta + c_2\sin\theta) Q_\gamma W \\
&- \Bigg[\Big(\sin w \mathrm{Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right] + \mathrm{Re}[e^{-i\theta}(\psi^1 + \frac{r^2}{z}\partial_z\psi^1)]\cos w\Big) \\
&\quad + \lambda^{-1}\left[\sin 2\theta \partial_\rho \psi_2 - w_\rho(\phi_1^{(2)} + \phi_1^{(-2)})\right]\Bigg] \times \sin^2 w(c_1\cos\theta + c_2\sin\theta) Q_\gamma E_1 \\
&+ \Big(\mathrm{Im}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right] + \lambda^{-1}(\partial_\rho\phi_2^{(2)} + \partial_\rho\phi_2^{(-2)})\Big) \times \sin w(c_2\cos\theta - c_1\sin\theta) Q_\gamma E_1 \\
&- \Bigg[\Big(\cos w \mathrm{Re}\left[e^{-i\gamma}(\psi^0 + \frac{r^2}{z}\partial_z\psi^0)\right] - \mathrm{Re}[e^{-i\theta}(\psi^1 + \frac{r^2}{z}\partial_z\psi^1)]\sin w\Big) \\
&\quad + \lambda^{-1}\Big(\sin 2\theta w_\rho \psi_2 + \partial_\rho \phi_1^{(2)} + \partial_\rho\phi_1^{(-2)}\Big)\Bigg] \times \sin w(c_2\cos\theta - c_1\sin\theta) Q_\gamma E_2
\end{aligned}
$$

since

$$
\begin{aligned}
\partial_r(c_1\mathcal{Z}_{1,1} + c_2\mathcal{Z}_{1,2}) &= \lambda^{-1} w_\rho (c_1\cos\theta + c_2\sin\theta)(\cos w W + \sin w E_1), \\
\partial_\theta(c_1\mathcal{Z}_{1,1} + c_2\mathcal{Z}_{1,2}) &= \sin w(c_2\cos\theta - c_1\sin\theta) W + \sin^2 w(c_1\cos\theta + c_2\sin\theta) E_2.
\end{aligned}
$$

Collecting all the identities above, we conclude the desired results. □

To estimate terms appearing in the remainder $\mathcal{R}_*$, we give now several estimates for the corrections ψ^0, ψ^1, $\phi_j^{(k)}$, $j = 1, 2$, $k = \pm 2$, and ψ_2, where their definitions can be found in (4.6), (4.9), (4.10), (4.11), (4.12) and (4.13).

By the definition of $k(z,t)$ in (4.6), we have

$$|\psi^j| \lesssim \int_{-T}^{t} \frac{|p_j(s)|}{t-s} \left(\mathbf{1}_{\{\zeta\le 1\}} + \zeta^{-1}\mathbf{1}_{\{\zeta>1\}}\right)\Big|_{\zeta=z^2(t-s)^{-1}} ds, \quad j=0,\ 1,$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A. For instance, we estimate

$$|\psi^1| \lesssim z^{-2}\int_{-T}^{t} |p_1(s)| ds \lesssim z^{-2}(t+T)T^{\Theta}|\log T|^{-1-2\Theta} \ \text{ for } \ z^2 \ge t,$$

where we have used

$$|p_1| \sim \frac{\lambda_*^{\Theta}}{|\log(T-s)|}, \quad \lambda_* = \frac{(T-t)|\log T|}{|\log(T-t)|^2}.$$

For $z^2 < t$, we have

$$|\psi^1| \lesssim \int_{-T}^{t-z^2} \frac{|p_1(s)|}{t-s} ds + z^{-2}\int_{t-z^2}^{t} |p_1(s)| ds,$$

and if $t \le \frac{T}{2}$,

$$|\psi^1| \lesssim T^{\Theta}|\log T|^{-1-2\Theta}\left\langle \log\left(\frac{t+T}{z^2}\right)\right\rangle.$$

If $z^2 < t$, $z^2 < T-t$ and $t > \frac{T}{2}$, one has

$$\begin{aligned} |\psi^1| &\lesssim \int_{-T}^{t-(T-t)} \frac{|p_1(s)|}{T-s} ds + \int_{t-(T-t)}^{t-z^2} \frac{|p_1(s)|}{t-s} ds + z^{-2}\int_{t-z^2}^{t} |p_1(s)| ds \\ &\lesssim |\log T|^{\Theta}\int_{-T}^{t-(T-t)} \frac{(T-s)^{\Theta-1}}{|\log(T-s)|^{1+2\Theta}} ds + |p_1(t)| \log\frac{T-t}{z^2} + \lambda_*^{\Theta}(t)|\log(T-t)|^{-1} \\ &\lesssim \frac{T^{\Theta}}{|\log T|^{1+\Theta}}. \end{aligned}$$

If $z^2 < t$, $z^2 \ge T-t$ and $t > \frac{T}{2}$, we estimate

$$\begin{aligned} |\psi^1| &\lesssim \frac{T^{\Theta}}{|\log T|^{1+\Theta}} + z^{-2}\int_{t-z^2}^{t} \frac{(T-s)^{\Theta}|\log T|^{\Theta}}{|\log(T-s)|^{1+2\Theta}} ds \\ &\lesssim \frac{T^{\Theta}}{|\log T|^{1+\Theta}} + \frac{(T-t+z^2)^{\Theta}}{|\log(T-t+z^2)|^{1+2\Theta}}. \end{aligned}$$

Above estimates then give

$$|\psi^1| \lesssim \mathbf{1}_{\{z^2<t\}} + z^{-2}(t+T)T^{\Theta}|\log T|^{-1-2\Theta}\mathbf{1}_{\{z^2\ge t\}},$$

and similarly,

$$|\psi^1| + z|\partial_z\psi^1| \lesssim \mathbf{1}_{\{z^2<t\}} + z^{-2}(t+T)T^{\Theta}|\log T|^{-1-2\Theta}\mathbf{1}_{\{z^2\ge t\}}. \tag{A.3}$$

ψ^0 is estimated in the same way. One directly checks that

$$|\psi^0| + z|\partial_z\psi^0| \lesssim \mathbf{1}_{\{z^2<t\}} + z^{-2}(t+T)|\log T|^{-1}\mathbf{1}_{\{z^2\ge t\}}. \tag{A.4}$$

Terms $\phi_j^{(k)}$, $j=1,2$, $k=\pm 2$ and ψ_2 are estimated via the linear theory in both W-direction and on $W^{\perp}$ in mode ± 2, namely Proposition 7.2 and Proposition 7.7 with $k=\pm 2$.

In the scalar equation (4.11), the right hand sides for modes $j=2$ and $j=-2$ satisfy

$$\begin{aligned} \left|\lambda^2\left(Q_{-\gamma}\mathcal{E}_{U^{\perp}}^{(2)}\right)_{\mathbb{C}}\right| &= \left| -ic_1c_2w_\rho^2 \sin w e^{2i\theta}(1-\cos w)\right| \lesssim \lambda_*^{2\Theta}(t)\rho\langle\rho\rangle^{-8}, \\ \left|\lambda^2\left(Q_{-\gamma}\mathcal{E}_{U^{\perp}}^{(-2)}\right)_{\mathbb{C}}\right| &= \left| -ic_1c_2w_\rho^2 \sin w e^{-2i\theta}(1+\cos w)\right| \lesssim \lambda_*^{2\Theta}(t)\rho^3\langle\rho\rangle^{-8}, \end{aligned}$$

where we have used (6.16). Then Proposition 7.2 implies

$$\begin{aligned} |\varphi_2| &\lesssim \lambda_*^{2\Theta}(t)\langle\rho\rangle^{-\delta}, \\ |\varphi_{-2}| &\lesssim \lambda_*^{2\Theta}(t)\langle\rho\rangle^{-\delta} \end{aligned} \tag{A.5}$$

for some $\delta \in (0,1)$ and close to 1. In fact, a more careful inspection on the proof of Proposition 7.2 enables one to obtain faster spatial decay for corrections in higher modes as well as vanishing properties near the origin.

For instance, δ can be taken close to 3 for mode 2 and close to 1 for mode -2. Similarly, in (4.13), since the RHS

$$|2c_1c_2\rho w_\rho^2 \sin^2 w| \lesssim \lambda_*^{2\Theta}(t)\rho^3\langle\rho\rangle^{-8},$$

one has

$$|\psi_2| \lesssim \lambda_*^{2\Theta}(t)\langle\rho\rangle^{-\delta_1} \tag{A.6}$$

by Proposition 7.7, where δ_1 can be taken close to 2.

One then uses the bounds (A.3), (A.4), (A.5) and (A.6) and directly checks that

$$\begin{aligned}\left|\mathcal{R}_*\cdot Q_\gamma W\right| \lesssim& \Bigg[1+\lambda_*^{2\Theta-1}\left(\langle\rho\rangle^{-1-\delta}+\langle\rho\rangle^{-\delta_1}\right)+\lambda_*^{5\Theta-2}\left(\langle\rho\rangle^{-2-\delta}+\langle\rho\rangle^{-1-\delta_1}\right)\\ &+\lambda_*^{4\Theta-2}\langle\rho\rangle^{-2-2\delta}+\lambda_*^{\Theta-1}\langle\rho\rangle^{-3}+\lambda_*^{3\Theta-2}\langle\rho\rangle^{-4-\delta}\Bigg]\mathbf{1}_{\{\rho\lesssim\lambda_*^{-1}\}},\\ \left|\mathcal{R}_*\cdot Q_\gamma E_1\right|,\ \left|\mathcal{R}_*\cdot Q_\gamma E_2\right| \lesssim& \Bigg[1+\lambda_*^{2\Theta-1}\langle\rho\rangle^{-\delta}+\lambda_*^{4\Theta-2}\left(\langle\rho\rangle^{-3-2\delta}+\langle\rho\rangle^{-2-\delta-\delta_1}\right)+\lambda_*^{5\Theta-2}\langle\rho\rangle^{-1-\delta}\\ &+\lambda_*^{\Theta-1}\langle\rho\rangle^{-2}+\lambda_*^{3\Theta-2}\langle\rho\rangle^{-3-\delta}\Bigg]\mathbf{1}_{\{\rho\lesssim\lambda_*^{-1}\}},\end{aligned} \tag{A.7}$$

where the expression of the remainder $\mathcal{R}_*$ is given in Lemma 4.1, and we have used

$$|\mathbf{c}| \lesssim \lambda_*^{\Theta}, \quad |\dot{\xi}| \lesssim \lambda_*^{3\Theta-1}.$$

For the remaining terms E_{η_1} (defined in (A.2)) supported outside of the inner region, we have

$$|E_{\eta_1}| \lesssim 1 \tag{A.8}$$

by (A.3), (A.4), (A.5), (A.6), and (4.4).

Appendix B. Spectral analysis of the linearized Liouville equation

B.1. **Generalized eigenfunctions and density of spectral measure.** We consider the operator

$$L_0 := \partial_{rr} + \frac{1}{r}\partial_r + \frac{8}{(1+r^2)^2},$$

which has kernels

$$K_1 = \frac{r^2-1}{r^2+1}, \quad K_2 = \frac{(r^2-1)\log r - 2}{r^2+1}.$$

The operator L_0 corresponds to the linearization at mode 0 of the Liouville equation. Let $u(\rho) = r^{-1/2}v(r)$. Then

$$L_0(u) = r^{-1/2}\mathcal{L}_0 v,$$

with

$$\mathcal{L}_0 v := \partial_{rr}v + \frac{v}{4r^2} + \frac{8v}{(1+r^2)^2}.$$

The new operator $\mathcal{L}_0$ has kernel

$$\Phi_0^0(r) = \frac{r^{1/2}(r^2-1)}{r^2+1},$$

and the other one is given by

$$\begin{aligned}\Theta_0^0(r) =& -\Phi_0^0(r)\int \frac{1}{(\Phi(r))^2}dr\\ =& -\frac{r^{1/2}(r^2-1)}{r^2+1}\left(\log r - \frac{2}{r^2-1} + C\right)\end{aligned}$$

for which

$$W[\Theta_0^0, \Phi_0^0] = 1.$$

Here for any C, $\Theta_0^0(1) = 0$, and we take $C = 0$.

Lemma B.1. *([37, Lemma 4.2], [23]) The spectrum of $\mathcal{L}_0$ equals*

$$spec(-\mathcal{L}_0) = \{\xi_d\} \cup [0, \infty),$$

where the only negative eigenvalue ξ_d is negative and simple, and its corresponding eigenfunction ϕ_d has exponential decay.

Remark B.1.1. *The operator $\mathcal{L}_0$ has a resonance at zero since $\mathcal{L}_0[\Phi_0^0] = 0$ and $\Theta_0^0 \notin L^2(dr)$.*

We now consider the fundamental system of solutions $\phi(r, z)$ and $\theta(r, z)$ to

$$-\mathcal{L}_0 y = zy$$

for all $z \in \mathbb{C}$ so that

$$W[\theta(\cdot, z), \phi(\cdot, z)] = 1.$$

Notice that these functions are entire in z, $\phi(r, 0) = c\Phi_0^0(r)$ for some normalization constant c. Let $\psi(r, z)$ be a Weyl-Titchmarsh solution. The generalized Weyl-Titchmarsh m function is defined as

$$C\psi(\cdot, z) = \theta(\cdot, z) + m(z)\phi(\cdot, z)$$

for some constant $C \neq 0$. Then

$$m(z) = \frac{W[\theta(\cdot, z), \psi(\cdot, z)]}{W[\psi(\cdot, z), \phi(\cdot, z)]}.$$

A spectral measure of $\mathcal{L}_0$ is obtained as

$$\rho((\lambda_1, \lambda_2]) = \frac{1}{\pi} \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \int_{\lambda_1+\delta}^{\lambda_2+\delta} \text{Im } m(\lambda + i\epsilon) d\lambda.$$

For the detailed definitions and properties, see [23].

Proposition B.2. *The m function is*

$$d\rho = \delta_{\xi_d} + \rho(\xi) d\xi, \quad \rho(\xi) = \pi^{-1} \text{Im } m(\xi + i0^+).$$

The distorted Fourier transform (DFT) defined as

$$\mathcal{F} : f \to \hat{f},$$

$$\hat{f}(\xi_d) = \int_0^\infty \phi_d(r) f(r) dr, \quad \hat{f}(\xi) = \lim_{b \to \infty} \int_0^b \phi(r, \xi) f(r) dr, \quad \xi \geq 0,$$

is a unitary operator from $L^2(\mathbb{R}^+)$ to $L^2(\{\xi_d\} \cup \mathbb{R}^+, \rho)$, and its inverse is given by

$$\mathcal{F}^{-1} : \hat{f} \to f(r) = \hat{f}(\xi_d)\phi_d(r) + \lim_{\mu \to \infty} \int_0^\mu \phi(r, \xi) \hat{f}(\xi) \rho(\xi) d\xi.$$

Next, we want to give the asymptotic expansion of distorted basis $\Phi^0(r, \xi)$ satisfying $-\mathcal{L}_0 \Phi^0(r, \xi) = \xi \Phi^0(r, \xi)$.

Proposition B.3. *For any $\xi \in \mathbb{C}$, $\Phi^0(r, \xi)$ admits the asymptotic expansion*

$$\Phi^0(r, \xi) = \Phi_0^0(r) + r^{1/2} \sum_{j=1}^\infty (-r^2 \xi)^j \Phi_j^0(r^2),$$

which converges absolutely for any fixed r, ξ and converges uniformly if $r^2\xi$ remains bounded with a fixed r. Here for $j \geq 1$, $\Phi_j^0(u)$ are smooth functions in $u \geq 0$ satisfying

$$\begin{aligned} \Phi_1^0(u) = \frac{u-1}{12u(u+1)^2} \Bigg[&(3u - 2\pi^2)(1+u) + 6(1+u)\log(1+u)[2 + \log(1+u) - 2\log u] \\ &+ 12(1+u)\text{Polylog}\left(2, \frac{1}{1+u}\right) \Bigg] + \frac{(u+3)\log(u+1) - 3u}{u(u+1)}, \end{aligned}$$

$$|\Phi_j^0(u)| \leq C, \quad j \geq 2$$

for a large constant $C > 0$.

Proof. We look for

$$\Phi^0(r,\xi) = r^{1/2}\sum_{j=0}^{\infty}(-\xi)^j f_j(r), \quad f_0(r) = \frac{r^2-1}{r^2+1}$$

with

$$\mathcal{L}_0(r^{1/2}f_j) = r^{1/2}f_{j-1}, \quad f_{-1}\equiv 0.$$

Then above ansatz yields the recurrence relation

$$\begin{aligned} f_j(r) &= \int_0^r r^{-1/2}\left[\Phi_0^0(r)\Theta_0^0(s) - \Theta_0^0(r)\Phi_0^0(s)\right] s^{1/2} f_{j-1}(s)ds \\ &= \int_0^r \left[\frac{(r^2-1)}{r^2+1}\frac{s\left[2-(s^2-1)\log s\right]}{s^2+1} + \frac{\left[(r^2-1)\log r - 2\right]}{r^2+1}\frac{s(s^2-1)}{s^2+1}\right] f_{j-1}(s)ds \\ &= \int_0^r \frac{2s(r^2-s^2)+s(s^2-1)(r^2-1)\log(r/s)}{(r^2+1)(s^2+1)} f_{j-1}(s)ds \end{aligned}$$

Now we change variable $v=s^2$, $u=r^2$ and define

$$\tilde{f}_j(v) = f_j(s).$$

In particular,

$$\tilde{f}_0(u) = \frac{u-1}{u+1}.$$

Then we get

$$\tilde{f}_j(u) = \int_0^u \frac{4(u-v)+(v-1)(u-1)\log(u/v)}{4(u+1)(v+1)}\tilde{f}_{j-1}(v)dv.$$

We first consider $\tilde{f}_1(u)$

$$\begin{aligned} \tilde{f}_1(u) &= \int_0^u \frac{4(u-z)(z-1)-(z-1)^2(u-1)\log(z/u)}{4(u+1)(z+1)^2}dz \\ &= \frac{1}{1+u}\int_0^u \frac{(u-z)(z-1)}{(z+1)^2}dz - \frac{u-1}{4(u+1)}\int_0^u \frac{(z-1)^2\log(z/u)}{(z+1)^2}dz, \end{aligned}$$

where we have

$$\int_0^u \frac{(u-z)(z-1)}{(z+1)^2}dz = (u+3)\log(u+1) - 3u,$$

and

$$\begin{aligned} &\int_0^u \frac{(z-1)^2\log(z/u)}{(z+1)^2}dz \\ =& -\frac{1}{3(1+u)}\Big[3u(1+u) - 2\pi^2(1+u) - 15u\log u - 3u^2\log u + 12(1+u)\log(1+u) \\ &+ 6(1+u)\log^2(1+u) + 3[u(5+u) - 4(1+u)\log(1+u)](\log u) \\ &+ 12(1+u)\mathrm{Polylog}\left(2,\frac{1}{1+u}\right)\Big] \end{aligned}$$

with

$$\mathrm{Polylog}\left(2,\frac{1}{1+u}\right) \sim \begin{cases} \frac{\pi^2}{6} + (\log u - 1)u - \frac{1}{4}(2\log u+1)u^2 + \frac{1}{18}(6\log u + 7)u^3 + O(u^4\log u) & \text{for } u \ll 1, \\ \frac{1}{u} + \frac{1}{4u^2} + \frac{1}{9u^3} + O\left(\frac{1}{u^4}\right) & \text{for } u \gg 1, \end{cases}$$

and for $u \ll 1$

$$\begin{aligned} \frac{u-1}{u+1}\mathrm{Polylog}\left(2,\frac{1}{1+u}\right) =& -\frac{\pi^2}{6} + \frac{1}{3}(3+\pi^2-3\log u)u + \frac{1}{12}(-21-4\pi^2+30\log u)u^2 \\ &+ \frac{1}{9}(10+3\pi^2-30\log u)u^3 + O(u^4\log u). \end{aligned}$$

So we obtain

$$\begin{aligned}\tilde{f}_1(u) = &\frac{(u+3)\log(u+1)-3u}{u+1}\\ &+\frac{u-1}{12(u+1)^2}\Bigg[(3u-2\pi^2)(1+u)+6(1+u)\log(1+u)[2+\log(1+u)-2\log u]\\ &\qquad+12(1+u)\mathrm{Polylog}\left(2,\frac{1}{1+u}\right)\Bigg],\end{aligned}$$

and

$$\tilde{f}_1(u)\sim\begin{cases}-\dfrac{u}{4}+\dfrac{u^2}{4}-\dfrac{2u^3}{9}+O(u^4) & \text{for } u\ll 1,\\ \dfrac{u}{4}-\dfrac{3\log^2 u-12\log u+21+\pi^2}{6}+\dfrac{6\log^2 u+39+2\pi^2}{6u}+O\left(\dfrac{\log^2 u}{u^2}\right) & \text{for } u\gg 1.\end{cases} \tag{B.1}$$

Next we estimate $\tilde{f}_j(u)$ inductively. We consider

$$\tilde{f}_j(u)=\int_0^u \frac{4(u-v)+(v-1)(u-1)\log(u/v)}{4(u+1)(v+1)}\tilde{f}_{j-1}(v)dv$$

and bound the kernel function as

$$\left|\frac{4(u-v)+(v-1)(u-1)\log(u/v)}{4(u+1)(v+1)}\right|\le C[\log(u/v)+1]$$

for some constant $C>0$ since $v\le u$. Then from (B.1) we have

$$|\tilde{f}_1(u)|\le Cu,$$

and thus

$$\begin{aligned}|\tilde{f}_2(u)|&\le C\int_0^u[\log(u/v)+1]|\tilde{f}_1(v)|dv\\ &\le C\int_0^u v[\log(u/v)+1]dv\\ &\le Cu^2.\end{aligned}$$

By induction, clearly one has

$$|\tilde{f}_j(u)|\le Cu^j.$$

In the original variable, we have $r^{2j}\Phi_j^0(r^2)=f_j(r)=\tilde{f}_j(r^2)=\tilde{f}_j(u)$, and thus

$$|\Phi_j^0(u)|\le C$$

as desired. □

Next we estimate the Weyl-Titchmarsh function $\Psi_0^+(r,\xi)$ with

$$-\mathcal{L}_0\Psi_0^+(r,\xi)=\xi\Psi_0^+(r,\xi),\quad \xi>0,$$

namely

$$\partial_{rr}\Psi_0^+(r,\xi)+\frac{\Psi_0^+(r,\xi)}{4r^2}+\xi\Psi_0^+(r,\xi)=-\frac{8\Psi_0^+(r,\xi)}{(1+r^2)^2}.$$

We write

$$\Psi_0^+(r,\xi)=g(r\xi^{1/2}).$$

Then one has

$$g''(r\xi^{1/2})+\frac{1}{4r^2\xi}g(r\xi^{1/2})+g(r\xi^{1/2})=-\frac{8}{\xi(1+r^2)^2}g(r\xi^{1/2}),$$

and

$$g''(q)+\frac{1}{4q^2}g(q)+g(q)=-\frac{8}{\xi(1+q^2/\xi)^2}g(q)$$

for

$$q:=r\xi^{1/2}.$$

Homogeneous solutions can be written in terms of special functions. Here we follow the argument in [34, Proposition 5.6] and [36, Section 4]. We now consider the case $r\xi^{1/2} \gtrsim 1$.

Lemma B.4. *A Weyl-Titchmarsh function is of the form*

$$\Psi_0^+(r,\xi) = \xi^{-1/4} e^{ir\xi^{1/2}} \sigma(r\xi^{1/2}, r) \quad \text{for} \quad r\xi^{1/2} \gtrsim 1$$

where σ admits the asymptotic series approximation about q with a fixed r,

$$\sigma(q,r) \sim \sum_{j=0}^{\infty} q^{-j} \psi_j^+(r), \quad \psi_0^+(r) = 1,$$

$$\psi_1^+(r) = -\frac{i}{8} + i\left(\frac{2r^2}{r^2+1} + 2r\arctan r - r\pi\right)$$

with

$$\sup_{r>0} |(r\partial_r)^k \psi_j^+| \le d_{jk},$$

and

$$|(r\partial_r)^\alpha (q\partial_q)^\beta (\sigma(q,r) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r))| \le e_{\alpha,\beta,j_0} q^{-j_0-1}.$$

Proof. From the form of $\Psi_0^+(r,\xi)$, we only need to consider

$$\left(\partial_{rr} + 2i\xi^{1/2}\partial_r + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\sigma(r\xi^{1/2}, r) = 0.$$

We look for a formal power series that solves above equation

$$\sigma(r\xi^{1/2}, r) = \sum_{j=0}^{\infty} \xi^{-j/2} f_j(r)$$

which implies the following recurrence relation

$$2i\partial_r f_j(r) = \left(-\partial_{rr} - \frac{1}{4r^2} - \frac{8}{(1+r^2)^2}\right) f_{j-1}(r), \quad j \ge 1,\ f_0(r) = 1.$$

A solution is given by

$$f_j(r) = \frac{i}{2}\partial_r f_{j-1}(r) + \frac{i}{2}\int_r^\infty \left(-\frac{1}{4s^2} - \frac{8}{(1+s^2)^2}\right) f_{j-1}(s)ds.$$

Then we get

$$|f_1(r)| \le c_{10} r^{-1}, \quad |(r\partial_r)^k f_1| \le c_{1k} r^{-1},\ k \in \mathbb{N},$$

and by induction

$$|(r\partial_r)^k f_j| \le c_{jk} r^{-j}, \quad j \ge 1,\ r > 0.$$

Now write

$$\psi_j^+(r) := r^j f_j(r),$$

and it follows from the control of f_j that

$$\sup_{r>0} |(r\partial_r)^k \psi_j^+| \le d_{jk}$$

for some constant d_{jk}. Write $q = r\xi^{1/2}$. Then

$$q^{-j}\psi_j^+(r) = \xi^{-j/2} f_j(r).$$

Define

$$\sigma_{ap}(q,r) := \sum_{j=0}^{\infty} q^{-j}\psi_j^+(r)(1 - \eta(q\delta_j)),$$

where $\delta_j \to 0^+$ sufficiently fast to ensure the convergence of the summation. Indeed, one has

$$\begin{aligned}
&\left|\sigma_{ap}(q,r) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)\right| \\
&= \left|\sum_{j=j_0+1}^{\infty} q^{-j}\psi_j^+(r)(1-\eta(q\delta_j)) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)\eta(q\delta_j)\right| \\
&\le e_{j_0} q^{-j_0-1}.
\end{aligned}$$

In the last step, we have used the following estimates. For $j \ge j_0+2$,

$$\left|q^{-j}\psi_j^+(r)(1-\eta(q\delta_j))\right| \le q^{-j}\mathbf{1}_{\{q\delta_j>1\}} d_{j0} \le q^{-j_0-1}2^{-j} \quad \text{if} \quad \delta_j d_{j0} \le 2^{-j}.$$

For $0 \le j \le j_0$,

$$|q^{-j}\psi_j^+\eta(q\delta_j)| = |q^{-j_0-1}q^{j_0+1-j}\psi_j^+\eta(q\delta_j)| \le q^{-j_0-1}(2\delta_j^{-1})^{j_0+1}d_{j0} \le q^{-j_0-1}\max_{0\le j\le j_0}(2\delta_j^{-1})^{j_0+1}d_{j0}.$$

Similarly, we have

$$\left|(r\partial_r)^\alpha(q\partial_q)^\beta(\sigma_{ap}(q,r) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r))\right| \le e_{\alpha,\beta,j_0} q^{-j_0-1}.$$

This implies that $\sigma_{ap}(r\xi^{1/2},r)$ is a good approximation at infinity, namely the error

$$\begin{aligned}
e(r\xi^{1/2},r) &= \left(\partial_{rr} + 2i\xi^{1/2}\partial_r + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\sigma_{ap} \\
&= \left(\partial_{rr} + 2i\xi^{1/2}\partial_r + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\left(\sigma_{ap} - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r) + \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)\right)
\end{aligned}$$

is small. Indeed, for the first term, we estimate

$$\left|\left(\partial_{rr} + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\left(\sigma_{ap} - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)\right)\right| \lesssim r^{-2}q^{-j_0},$$

and

$$\left|2i\xi^{1/2}\partial_r\left(\sigma_{ap} - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)\right)\right| \lesssim \xi^{1/2}r^{-1}q^{-j_0-1} = r^{-2}q^{-j_0}.$$

For the second term, we have

$$\begin{aligned}
&\left(\partial_{rr} + 2i\xi^{1/2}\partial_r + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\sum_{j=0}^{j_0} q^{-j}\psi_j^+(r) \\
&= \left(\partial_{rr} + 2i\xi^{1/2}\partial_r + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\sum_{j=0}^{j_0} \xi^{-j/2}f_j(r) \\
&= \left(\partial_{rr} + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\sum_{j=0}^{j_0} \xi^{-j/2}f_j(r) + 2i\left(\sum_{j=0}^{j_0-1} \xi^{-j/2}\partial_r f_{j+1}(r)\right) \\
&= \left(\partial_{rr} + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\xi^{-j_0/2}f_{j_0}(r),
\end{aligned}$$

so

$$\left|\left(\partial_{rr} + 2i\xi^{1/2}\partial_r + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)\right| \lesssim \xi^{-j_0/2}r^{-j_0-2} = r^{-2}q^{-j_0}.$$

Therefore, the error has the following bound

$$\left|e(r\xi^{1/2},r)\right| \le c_{j_0} r^{-2}q^{-j_0} \quad \text{for all} \quad j_0 \ge 0.$$

We then look for perturbation $\sigma_1 = -\sigma + \sigma_{ap}$ solved by

$$\left(\partial_{rr} + 2i\xi^{1/2}\partial_r + \frac{1}{4r^2} + \frac{8}{(1+r^2)^2}\right)\sigma_1 = e(r\xi^{1/2}, r),$$

which can be formulated, by writing $(v_1, v_2) = (\sigma, r\partial_r\sigma)$, as a system of first order

$$\begin{cases} \partial_r v_1 = r^{-1}v_2 \\ \partial_r v_2 = \left(-\frac{1}{4r} - \frac{8r}{(1+r^2)^2}\right)v_1 + (r^{-1} - 2i\xi^{1/2})v_2 + re(r\xi^{1/2}, r) \end{cases} \tag{B.2}$$

We then estimate

$$\begin{aligned} &\partial_r(|v_1|^2 + |v_2|^2) \\ =& 2\mathrm{Re}(\bar{v}_1\partial_r v_1) + 2\mathrm{Re}(\bar{v}_2\partial_r v_2) \\ =& 2r^{-1}\mathrm{Re}(\bar{v}_1 v_2) + 2\mathrm{Re}\Big[\bar{v}_2\Big(\Big(-\frac{1}{4r} - \frac{8r}{(1+r^2)^2}\Big)v_1 + (r^{-1} - 2i\xi^{1/2})v_2 + re(r\xi^{1/2}, r)\Big)\Big] \\ =& 2r^{-1}\mathrm{Re}(\bar{v}_1 v_2) - \left(\frac{1}{2r} + \frac{16r}{(1+r^2)^2}\right)\mathrm{Re}(\bar{v}_2 v_1) + 2r^{-1}|v_2|^2 + 2r\mathrm{Re}\big(\bar{v}_2 e(r\xi^{1/2}, r)\big) \\ \geq& -C(r^{-1}|v|^2 + r|v||e(r\xi^{1/2}, r)|) \end{aligned}$$

for some $C > 0$. So we have

$$\partial_r|v| \geq -C(r^{-1}|v| + r|e(r\xi^{1/2}, r)|),$$

and

$$|v| \leq Cr^{-C}\int_r^\infty s|e(s\xi^{1/2}, s)|ds.$$

Recalling the estimate

$$|e| \lesssim r^{-2}q^{-j_0} = r^{-2-j_0}\xi^{-j_0/2},$$

we get

$$|v| \lesssim r^{-j_0}\xi^{-j_0/2}.$$

For the estimate of higher order derivatives $(r\partial_r)^\alpha(\xi\partial_\xi)^\beta v$, one needs to differentiate (B.2) and repeat above process. The argument is a verbatim repetition of the proof in [34]. □

By the asymptotic expansion of Ψ_0^+, we have $W(\Psi_0^+, \overline{\Psi_0^+}) = -2i$. Concerning the density of spectral measure, we have the following

Proposition B.5. *We have*

$$\Phi^0(r,\xi) = a_0(\xi)\Psi_0^+(r,\xi) + \overline{a_0(\xi)\Psi_0^+(r,\xi)}, \tag{B.3}$$

where

$$|a_0(\xi)| \sim 1.$$

The spectral measure ρ_0 has density estimate

$$\frac{d\rho_0}{d\xi}(\xi) \sim |a_0(\xi)|^{-2} \sim 1.$$

Proof. We follow the argument in [34, Proposition 5.7].

$$a_0(\xi) = \frac{i}{2}W(\Phi^0, \overline{\Psi_0^+}) = \frac{i}{2}(\Phi^0(r,\xi)\partial_r\overline{\Psi_0^+(r,\xi)} - \overline{\Psi_0^+(r,\xi)}\partial_r\Phi^0(r,\xi)).$$

We evaluate the Wronskian in the region where both the $\Psi_0^+(r,\xi)$ and $\Phi^0(r,\xi)$ asymptotics are useful, i.e., where $r^2\xi \sim 1$. Recall that for $r^2\xi \sim 1$, one has

$$\Phi^0(r,\xi) \sim \frac{r^{1/2}(r^2-1)}{r^2+1} + r^{1/2}\Phi_1^0(r^2),$$

and

$$\Phi_1^0(u) \sim \begin{cases} -\frac{1}{4} + \frac{u}{4} + O(u^2) & \text{for } 0 < u \leq 1, \\ \frac{1}{4} - \frac{\log^2 u}{2u} + O\left(\frac{\log u}{u}\right) & \text{for } u \geq 1. \end{cases}$$

For all $\xi > 0$, it follows from Lemma B.4 that

$$|\Psi_0^+(r,\xi)| \lesssim \xi^{-1/4}, \quad |\partial_r \Psi_0^+(r,\xi)| \lesssim \xi^{1/4}.$$

For $r \sim \xi^{-1/2}$, we have

$$|\Phi^0(r,\xi)| \lesssim \xi^{-1/4}, \quad |\partial_r \Phi^0(r,\xi)| \lesssim \xi^{1/4},$$

and thus

$$|a_0(\xi)| \lesssim 1.$$

Therefore, we have

$$\frac{d\rho_0}{d\xi}(\xi) \sim |a_0(\xi)|^{-2} \gtrsim 1.$$

In order to get the lower bound of $|a_0(\xi)|$, by (B.3), we have

$$|a_0(\xi)| \geq \frac{|\Phi^0(r,\xi)|}{2|\Psi_0^+(r,\xi)|}.$$

For $0 < \xi \leq 1$, we take $r = M\xi^{-1/2}$ with a sufficiently large M. Then above estimates imply

$$|\Phi^0(r,\xi)| \gtrsim \xi^{-1/4}.$$

For $\xi \geq 1$, we similarly take $r = m\xi^{-1/2}$ for a sufficiently small m. Then

$$|\Phi^0(r,\xi)| \gtrsim \xi^{-1/4}.$$

Collecting above estimates, we conclude the estimate of density

$$\frac{d\rho_0}{d\xi}(\xi) \sim 1$$

as desired. □

B.2. **Duhamel's representation by DFT.** Now we formulate the Duhamel's formula for the linearization at mode 0

$$\partial_\tau \phi = \partial_{\rho\rho}\phi + \frac{1}{\rho}\partial_\rho \phi + V_0\phi + h,$$

where

$$V_0 = \frac{8}{(1+\rho^2)^2},$$

and h has fast decay in space-time. If we take $\phi = \rho^{-1/2}A$, we get

$$\partial_\tau A = A'' + \frac{A}{4\rho^2} + V_0 A + h\rho^{1/2} = \mathcal{L}_0 A + h\rho^{1/2}.$$

Consider the Cauchy problem

$$\begin{cases} \partial_\tau A = \mathcal{L}_0 A + h\rho^{1/2}, \\ A(\cdot, t_0) = 0. \end{cases} \tag{B.4}$$

For

$$\mathcal{L}_0 \Phi^0 = -\xi \Phi^0(\rho,\xi),$$

the generalized eigenfunction satisfies the following pointwise upper bound

$$|\Phi^0(\rho,\xi)| \lesssim \begin{cases} \rho^{1/2}, & \rho^2\xi \ll 1, \\ \xi^{-1/4}, & \rho^2\xi \gg 1, \end{cases} \tag{B.5}$$

and its associated spectral measure has density estimate

$$\frac{d\rho_0}{d\xi}(\xi) \sim 1. \tag{B.6}$$

Taking distorted Fourier transform on both sides of (B.4), we obtain

$$\begin{cases} \partial_\tau \hat{A}(\xi,\tau) = -\xi \hat{A}(\xi,\tau) + \int_0^\infty h\rho^{1/2}\Phi^0(\rho,\xi)d\rho, \\ \hat{A}(\cdot,\tau_0) = 0. \end{cases}$$

Therefore, one has

$$\phi(\rho,\tau)=\rho^{-1/2}A(\rho,\tau)=\int_{\tau_0}^{\tau}\int_0^\infty\int_0^\infty e^{-\xi(\tau-s)}\rho^{-1/2}\Phi^0(\rho,\xi)\Phi^0(x,\xi)x^{1/2}h(x,s)\frac{d\rho_0}{d\xi}(\xi)d\xi dxds.$$

Now assume the RHS h is smooth enough with the pointwise space-time control

$$|h(\rho,\tau)|\lesssim v(\tau)\langle\rho\rangle^{-\ell}\|h\|_{v,\ell}^{(\tau)},\ \ell>\frac{3}{2},$$

where $v(\tau)$ is a regular function decay in τ. Then

$$|\phi|\lesssim\|h\|_{v,\ell}^{(\tau)}\int_{\tau_0}^{\tau}ds\int_0^\infty v(s)e^{-\xi(\tau-s)}\rho^{-1/2}\Phi^0(\rho,\xi)\frac{d\rho_0}{d\xi}(\xi)d\xi\int_0^\infty\Phi^0(x,\xi)x^{1/2}\langle x\rangle^{-\ell}dx.$$

Before estimating $|\phi|$, we first invoke a lemma that will frequently used below. The following lemma is proved in [55, Lemma A.3].

Lemma B.6 ([55])**.** *Assume constants a, b satisfy either $a>-1$ or $a=-1$ and $b<-1$. For $0\le x_0\le x_1\le\frac{1}{2}$, we have*

$$\int_{x_0}^{x_1}e^{-\lambda x}x^a(-\log x)^bdx\lesssim\begin{cases}\begin{cases}x_1^{a+1}(-\log x_1)^b & \text{if } a>-1\\(-\log x_1)^{b+1}-(-\log x_0)^{b+1} & \text{if } a=-1,b<-1\end{cases} & \text{for } 0\le\lambda\le x_1^{-1}\\ \frac{(\log\lambda)^b}{\lambda^{a+1}}+\begin{cases}0 & \text{if } a>-1\\(\log\lambda)^{b+1}-(-\log x_0)^{b+1} & \text{if } a=-1,b<-1\end{cases} & \text{for } x_1^{-1}\le\lambda\le x_0^{-1}\\ \frac{(\log\lambda)^b}{\lambda^{a+1}}e^{-\frac{x_0\lambda}{2}} & \text{for } \lambda\ge x_0^{-1}\end{cases}. \tag{B.7}$$

The linear theory for the mode 0 is stated as follows, and its proof is in a similar spirit as [55, Proposition 9.8].

Proof of Proposition 7.4. We first estimate the Duhamel's representation without imposing orthogonality on the RHS $h(\rho,\tau)$ and compute

$$\begin{aligned}|F(\xi)|:=\left|\int_0^\infty\Phi^0(x,\xi)x^{1/2}\langle x\rangle^{-\ell}dx\right|&\lesssim\int_0^{\xi^{-1/2}}x\langle x\rangle^{-\ell}dx+\xi^{-1/4}\int_{\xi^{-1/2}}^\infty x^{1/2}\langle x\rangle^{-\ell}dx\\&\lesssim\begin{cases}\xi^{-1/4} & \text{for } \xi\ge1,\\ \begin{cases}\xi^{\frac{\ell}{2}-1}, & \ell<2\\ \log\xi, & \ell=2\\ 1, & \ell>2\end{cases} & \text{for } \xi\le1.\end{cases}\end{aligned}$$

Now we consider the integration in ξ:

$$\left|\int_0^\infty e^{-\xi(\tau-s)}\rho^{-1/2}\Phi^0(\rho,\xi)F(\xi)\frac{d\rho_0}{d\xi}(\xi)d\xi\right|=\left(\int_0^{\rho^{-2}}+\int_{\rho^{-2}}^\infty\right)\cdots:=P_1+P_2.$$

- For P_1, if $\rho>1$, then we have

$$|P_1|\lesssim\int_0^{\rho^{-2}}e^{-\xi(\tau-s)}\begin{cases}\xi^{\frac{\ell}{2}-1} & \text{if } \ell<2\\ \log\xi & \text{if } \ell=2\\ 1 & \text{if } \ell>2\end{cases}d\xi\lesssim\begin{cases}\begin{cases}\rho^{-\ell} & \text{if } \tau-s\le\rho^2\\(\tau-s)^{-\frac{\ell}{2}} & \text{if } \tau-s>\rho^2\end{cases} & \text{if } \ell<2\\ \begin{cases}\rho^{-2}\langle\log\rho\rangle & \text{if } \tau-s\le\rho^2\\(\tau-s)^{-1}\langle\log(\tau-s)\rangle & \text{if } \tau-s>\rho^2\end{cases} & \text{if } \ell=2\\ \begin{cases}\rho^{-2} & \text{if } \tau-s\le\rho^2\\(\tau-s)^{-1} & \text{if } \tau-s>\rho^2\end{cases} & \text{if } \ell>2\end{cases}$$

If $\rho \le 1$, then one has

$$\begin{aligned} P_1 &= \int_0^{\rho^{-2}} e^{-\xi(\tau-s)}\rho^{-1/2}\Phi^0(\rho,\xi)\frac{d\rho_0}{d\xi}(\xi)F(\xi)d\xi \\ &= \left(\int_0^1 + \int_1^{\rho^{-2}}\right) e^{-\xi(\tau-s)}\rho^{-1/2}\Phi^0(\rho,\xi)\frac{d\rho_0}{d\xi}(\xi)F(\xi)d\xi \\ &:= P_{11} + P_{12}, \end{aligned}$$

where we estimate

$$|P_{11}| \lesssim \int_0^1 e^{-\xi(\tau-s)} \begin{cases} \xi^{\frac{\ell}{2}-1} & \text{if } \ \ell < 2 \\ \log\xi & \text{if } \ \ell = 2 \\ 1 & \text{if } \ \ell > 2 \end{cases} d\xi \lesssim \begin{cases} \begin{cases} 1 & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{-\frac{\ell}{2}} & \text{if } \ \tau - s > 1 \end{cases} & \text{for } \ \ell < 2 \\ \begin{cases} 1 & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{-1}\langle\log(\tau-s)\rangle & \text{if } \ \tau - s > 1 \end{cases} & \text{for } \ \ell = 2 \\ \begin{cases} 1 & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{-1} & \text{if } \ \tau - s > 1 \end{cases} & \text{for } \ \ell > 2 \end{cases}$$

and

$$|P_{22}| \lesssim \int_1^{\rho^{-2}} e^{-\xi(\tau-s)}\xi^{-1/4}d\xi \lesssim \begin{cases} \rho^{-\frac{3}{2}} & \text{if } \ \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ \rho^2 < \tau - s \le 1 \\ (\tau-s)^{-\frac{3}{4}}e^{-\frac{\tau-s}{2}} & \text{if } \ \tau - s > 1 \end{cases}.$$

Hence one obtains for $\rho \le 1$

$$|P_1| \lesssim \begin{cases} \begin{cases} \rho^{-\frac{3}{2}} & \text{if } \ \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ \rho^2 < \tau - s \le 1 \\ (\tau-s)^{-\frac{\ell}{2}} & \text{if } \ \tau - s > 1 \end{cases} & \text{for } \ \ell < 2 \\ \begin{cases} \rho^{-\frac{3}{2}} & \text{if } \ \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ \rho^2 < \tau - s \le 1 \\ (\tau-s)^{-1}\langle\log(\tau-s)\rangle & \text{if } \ \tau - s > 1 \end{cases} & \text{for } \ \ell = 2 \\ \begin{cases} \rho^{-\frac{3}{2}} & \text{if } \ \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ \rho^2 < \tau - s \le 1 \\ (\tau-s)^{-1} & \text{if } \ \tau - s > 1 \end{cases} & \text{for } \ \ell > 2 \end{cases}.$$

• For P_2, if $\rho \le 1$, we estimate

$$|P_2| \lesssim \rho^{-1/2} \int_{\rho^{-2}}^{\infty} e^{-\xi(\tau-s)}\xi^{-\frac{1}{2}}d\xi \lesssim \begin{cases} \rho^{-1/2}(\tau-s)^{-\frac{1}{2}} & \text{if } \ \tau - s \le \rho^2 \\ \rho^{-1/2}(\tau-s)^{-\frac{1}{2}}e^{-\frac{\tau-s}{2\rho^2}} & \text{if } \ \tau - s > \rho^2 \end{cases}.$$

If $\rho > 1$, then we have

$$\begin{aligned} P_2 &= \int_{\rho^{-2}}^{\infty} e^{-\xi(\tau-s)}\rho^{-1/2}\Phi^0(\rho,\xi)\frac{d\rho_0}{d\xi}(\xi)F(\xi)d\xi \\ &= \left(\int_{\rho^{-2}}^1 + \int_1^{\infty}\right) e^{-\xi(\tau-s)}\rho^{-1/2}\Phi^0(\rho,\xi)\frac{d\rho_0}{d\xi}(\xi)F(\xi)d\xi \\ &:= P_{21} + P_{22}, \end{aligned}$$

where

$$|P_{21}| \lesssim \rho^{-1/2} \int_{\rho^{-2}}^{1} e^{-\xi(\tau-s)} \xi^{-1/4} \begin{cases} \xi^{\frac{\ell}{2}-1} & \text{if } \ \ell < 2 \\ \log \xi & \text{if } \ \ell = 2 \ \ d\xi \\ 1 & \text{if } \ \ell > 2 \end{cases}$$

$$\lesssim \rho^{-1/2} \begin{cases} \begin{cases} 1 & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } \ 1 < \tau - s \le \rho^2 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} e^{-\frac{\tau-s}{2\rho^2}} & \text{if } \ \tau - s > \rho^2 \end{cases} & \text{for } \ \ell < 2 \\ \begin{cases} 1 & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{-\frac{3}{4}} \langle \log(\tau-s) \rangle & \text{if } \ 1 < \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} \langle \log(\tau-s) \rangle e^{-\frac{\tau-s}{2\rho^2}} & \text{if } \ \tau - s > \rho^2 \end{cases} & \text{for } \ \ell = 2 \\ \begin{cases} 1 & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ 1 < \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} e^{-\frac{\tau-s}{2\rho^2}} & \text{if } \ \tau - s > \rho^2 \end{cases} & \text{for } \ \ell > 2 \end{cases}$$

and

$$|P_{22}| \lesssim \rho^{-1/2} \int_1^{\infty} e^{-\xi(\tau-s)} \xi^{-\frac{1}{2}} d\xi \lesssim \rho^{-1/2} \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{-\frac{1}{2}} e^{-\frac{\tau-s}{2}} & \text{if } \ \tau - s > 1 \end{cases}$$

Therefore, we obtain for $\rho > 1$ that

$$|P_2| \lesssim \rho^{-1/2} \begin{cases} \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } \ 1 < \tau - s \le \rho^2 \\ (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} e^{-\frac{\tau-s}{4\rho^2}} & \text{if } \ \tau - s > \rho^2 \end{cases} & \text{for } \ \ell < 2 \\ \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{-\frac{3}{4}} \langle \log(\tau-s) \rangle & \text{if } \ 1 < \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} \langle \log(\tau-s) \rangle e^{-\frac{\tau-s}{4\rho^2}} & \text{if } \ \tau - s > \rho^2 \end{cases} & \text{for } \ \ell = 2 \, . \\ \begin{cases} (\tau-s)^{-\frac{1}{2}} & \text{if } \ \tau - s \le 1 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ 1 < \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} e^{-\frac{\tau-s}{4\rho^2}} & \text{if } \ \tau - s > \rho^2 \end{cases} & \text{for } \ \ell > 2 \end{cases}$$

Collecting above estimates, we conclude that

- for $\rho \le 1$,

$$|P_1 + P_2| \lesssim \begin{cases} \begin{cases} \rho^{-1/2}(\tau-s)^{-\frac{1}{2}} & \text{if } \ \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ \rho^2 < \tau - s \le 1 \\ (\tau-s)^{-\frac{\ell}{2}} & \text{if } \ \tau - s > 1 \end{cases} & \text{for } \ \ell < 2 \\ \begin{cases} \rho^{-1/2}(\tau-s)^{-\frac{1}{2}} & \text{if } \ \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ \rho^2 < \tau - s \le 1 \\ (\tau-s)^{-1} \langle \log(\tau-s) \rangle & \text{if } \ \tau - s > 1 \end{cases} & \text{for } \ \ell = 2 \, , \\ \begin{cases} \rho^{-1/2}(\tau-s)^{-\frac{1}{2}} & \text{if } \ \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ \rho^2 < \tau - s \le 1 \\ (\tau-s)^{-1} & \text{if } \ \tau - s > 1 \end{cases} & \text{for } \ \ell > 2 \end{cases}$$

• for $\rho > 1$,

$$|P_1 + P_2| \lesssim \begin{cases} \begin{cases} \rho^{-1/2}(\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \le 1 \\ \rho^{-1/2}(\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } 1 < \tau - s \le \rho^2 \\ (\tau-s)^{-\frac{l}{2}} & \text{if } \tau - s > \rho^2 \end{cases} & \text{for } \ell < 2 \\ \begin{cases} \rho^{-1/2}(\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \le 1 \\ \rho^{-1/2}(\tau-s)^{-\frac{3}{4}}\langle \log(\tau-s)\rangle & \text{if } 1 < \tau - s \le \rho^2 \\ (\tau-s)^{-1}\langle \log(\tau-s)\rangle & \text{if } \tau - s > \rho^2 \end{cases} & \text{for } \ell = 2 \\ \begin{cases} \rho^{-1/2}(\tau-s)^{-\frac{1}{2}} & \text{if } \tau - s \le 1 \\ \rho^{-1/2}(\tau-s)^{-\frac{3}{4}} & \text{if } 1 < \tau - s \le \rho^2 \\ (\tau-s)^{-1} & \text{if } \tau - s > \rho^2 \end{cases} & \text{for } \ell > 2 \end{cases}.$$

We now estimate the convolution in time.

• For $\rho \le 1$, one has

$$\begin{aligned} |\phi(\rho,\tau)| &\lesssim \left(\int_{\tau-\rho^2}^{\tau} + \int_{\tau-1}^{\tau-\rho^2} + \int_{\frac{\tau_0}{2}}^{\tau-1}\right) v(s)\Big|P_1(\rho,\tau,s) + P_2(\rho,\tau,s)\Big| ds \\ &\lesssim \rho^{-1/2} v(\tau) \int_{\tau-\rho^2}^{\tau} (\tau-s)^{-\frac{1}{2}} ds + v(\tau)\int_{\tau-1}^{\tau-\rho^2} (\tau-s)^{-\frac{3}{4}} ds \\ &\quad + \begin{cases} \int_{\frac{\tau_0}{2}}^{\tau-1} v(s)(\tau-s)^{-\frac{\ell}{2}} ds & \text{if } \ell < 2 \\ \int_{\frac{\tau_0}{2}}^{\tau-1} v(s)(\tau-s)^{-1}\langle \log(\tau-s)\rangle ds & \text{if } \ell = 2 \\ \int_{\frac{\tau_0}{2}}^{\tau-1} v(s)(\tau-s)^{-1} ds & \text{if } \ell > 2 \end{cases} \\ &\lesssim v(\tau) + \begin{cases} v(\tau)\tau^{1-\frac{\ell}{2}} + \tau^{-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s) ds & \text{if } \ell < 2 \\ v(\tau)(\log\tau)^2 + \tau^{-1}\log\tau \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s) ds & \text{if } \ell = 2 \\ v(\tau)\log\tau + \tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s) ds & \text{if } \ell > 2 \end{cases}. \end{aligned}$$

• For $1 < \rho \le (\frac{\tau}{2})^{1/2}$, we estimate

$$\begin{aligned} |\phi(\rho,\tau)| &\lesssim \left(\int_{\tau-1}^{\tau} + \int_{\tau-\rho^2}^{\tau-1} + \int_{\frac{\tau_0}{2}}^{\tau-\rho^2}\right) v(s)\Big|P_1(\rho,\tau,s) + P_2(\rho,\tau,s)\Big| ds \\ &\lesssim v(\tau)\rho^{-1/2}\int_{\tau-1}^{\tau}(\tau-s)^{-\frac{1}{2}} ds + v(\tau)\rho^{-1/2}\int_{\tau-\rho^2}^{\tau-1} \begin{cases} (\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } \ell < 2 \\ (\tau-s)^{-\frac{3}{4}}\langle \log(\tau-s)\rangle & \text{if } \ell = 2 \\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ell > 2 \end{cases} ds \\ &\quad + \int_{\frac{\tau_0}{2}}^{\tau-\rho^2} v(s) \begin{cases} (\tau-s)^{-\frac{\ell}{2}} & \text{if } \ell < 2 \\ (\tau-s)^{-1}\langle \log(\tau-s)\rangle & \text{if } \ell = 2 \\ (\tau-s)^{-1} & \text{if } \ell > 2 \end{cases} ds \\ &\lesssim \rho^{-1/2}v(\tau) + v(\tau)\begin{cases} \rho^{2-\ell} & \text{if } \ell < 2 \\ \langle \log\rho\rangle & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2 \end{cases} + \begin{cases} v(\tau)\tau^{1-\frac{\ell}{2}} + \tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s) ds & \text{if } \ell < 2 \\ v(\tau)\int_{\rho^2}^{\frac{\tau}{2}} \langle \log z\rangle z^{-1} dz + \tau^{-1}\log\tau \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s) ds & \text{if } \ell = 2 \\ v(\tau)\log(\frac{\tau}{2\rho^2}) + \tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s) ds & \text{if } \ell > 2 \end{cases} \\ &\lesssim \begin{cases} v(\tau)\tau^{1-\frac{\ell}{2}} + \tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s) ds & \text{if } \ell < 2 \\ v(\tau)(\langle \log\rho\rangle + \int_{\rho^2}^{\frac{\tau}{2}} \langle \log z\rangle z^{-1} dz) + \tau^{-1}\log\tau \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s) ds & \text{if } \ell = 2 \\ v(\tau)\langle\log(\frac{\tau}{2\rho^2})\rangle + \tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s) ds & \text{if } \ell > 2 \end{cases}. \end{aligned}$$

• For $(\frac{\tau}{2})^{1/2} \le \rho \le \tau^{1/2}$, we estimate

$$
\begin{aligned}
|\phi(\rho,\tau)| &\lesssim \left(\int_{\tau-1}^{\tau} + \int_{\tau-\rho^2}^{\tau-1} + \int_{\frac{\tau_0}{2}}^{\tau-\rho^2}\right) v(s)\Big|P_1(\rho,\tau,s)+P_2(\rho,\tau,s)\Big|ds \\
&\lesssim \rho^{-1/2}v(\tau)\int_{\tau-1}^{\tau}(\tau-s)^{-\frac{1}{2}}ds + \rho^{-1/2}\int_{\tau-\rho^2}^{\tau-1} v(s)\begin{cases}(\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } \ \ell<2\\ (\tau-s)^{-\frac{3}{4}}\langle\log(\tau-s)\rangle & \text{if } \ \ell=2\\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ \ell>2\end{cases} ds \\
&\quad + \int_{\frac{\tau_0}{2}}^{\tau-\rho^2} v(s)\begin{cases}(\tau-s)^{-\frac{\ell}{2}} & \text{if } \ \ell<2\\ (\tau-s)^{-1}\langle\log(\tau-s)\rangle & \text{if } \ \ell=2\\ (\tau-s)^{-1} & \text{if } \ \ell>2\end{cases} ds \\
&\lesssim \rho^{-1/2}v(\tau) + \rho^{-1/2}v(\tau)\begin{cases}\tau^{\frac{5}{4}-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{\frac{1}{4}}\langle\log\tau\rangle & \text{if } \ \ell=2\\ \tau^{\frac{1}{4}} & \text{if } \ \ell>2\end{cases} + \rho^{-1/2}\int_{\tau-\rho^2}^{\frac{\tau}{2}} v(s)ds\begin{cases}\tau^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{-\frac{3}{4}}\langle\log\tau\rangle & \text{if } \ \ell=2\\ \tau^{-\frac{3}{4}} & \text{if } \ \ell>2\end{cases} \\
&\quad + \int_{\frac{\tau_0}{2}}^{\tau-\rho^2} v(s)ds\begin{cases}\tau^{-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{-1}\langle\log\tau\rangle & \text{if } \ \ell=2\\ \tau^{-1} & \text{if } \ \ell>2\end{cases} \\
&\lesssim \rho^{-1/2}v(\tau)\begin{cases}\tau^{\frac{5}{4}-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{\frac{1}{4}}\langle\log\tau\rangle & \text{if } \ \ell=2\\ \tau^{\frac{1}{4}} & \text{if } \ \ell>2\end{cases} + \rho^{-1/2}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds\begin{cases}\tau^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } \ \ell<2\\ \tau^{-\frac{3}{4}}\langle\log\tau\rangle & \text{if } \ \ell=2\\ \tau^{-\frac{3}{4}} & \text{if } \ \ell>2\end{cases} \\
&\sim \begin{cases}\tau^{1-\frac{\ell}{2}}v(\tau) + \tau^{-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell<2\\ \langle\log\tau\rangle v(\tau) + \tau^{-1}\langle\log\tau\rangle\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell=2\\ v(\tau) + \tau^{-1}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell>2\end{cases}.
\end{aligned}
$$

• For $\rho \ge \tau^{1/2}$, we estimate

$$
\begin{aligned}
|\phi(\rho,\tau)| &\lesssim \left(\int_{\tau-1}^{\tau} + \int_{\frac{\tau_0}{2}}^{\tau-1}\right) v(s)\Big|P_1(\rho,\tau,s)+P_2(\rho,\tau,s)\Big|ds \\
&\lesssim \rho^{-1/2}v(\tau)\int_{\tau-1}^{\tau}(\tau-s)^{-\frac{1}{2}}ds + \rho^{-1/2}\int_{\frac{\tau_0}{2}}^{\tau-1} v(s)\begin{cases}(\tau-s)^{\frac{1}{4}-\frac{\ell}{2}} & \text{if } \ \ell<2\\ (\tau-s)^{-\frac{3}{4}}\langle\log(\tau-s)\rangle & \text{if } \ \ell=2\\ (\tau-s)^{-\frac{3}{4}} & \text{if } \ \ell>2\end{cases} ds \\
&\lesssim \rho^{-1/2}v(\tau) + \rho^{-1/2}\begin{cases} v(\tau)\tau^{\frac{5}{4}-\frac{\ell}{2}} + \tau^{\frac{1}{4}-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell<2\\ v(\tau)\tau^{\frac{1}{4}}\langle\log\tau\rangle + \tau^{-\frac{3}{4}}\langle\log\tau\rangle\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell=2\\ v(\tau)\tau^{\frac{1}{4}} + \tau^{-\frac{3}{4}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell>2\end{cases} \\
&\lesssim \rho^{-1/2}\begin{cases} v(\tau)\tau^{\frac{5}{4}-\frac{\ell}{2}} + \tau^{\frac{1}{4}-\frac{\ell}{2}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell<2\\ v(\tau)\tau^{\frac{1}{4}}\langle\log\tau\rangle + \tau^{-\frac{3}{4}}\langle\log\tau\rangle\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell=2\\ v(\tau)\tau^{\frac{1}{4}} + \tau^{-\frac{3}{4}}\int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} v(s)ds & \text{if } \ \ell>2\end{cases}.
\end{aligned}
$$

Collecting above estimates, we obtain the a priori estimates without orthogonality. Estimates with orthogonality can be derived in a similar manner, where the orthogonality condition

$$\int_0^\infty h(\rho,\tau)\frac{\rho^2-1}{\rho^2+1}\rho d\rho = 0 \quad \forall \tau \in (\tau_0,+\infty)$$

is used when estimating

$$\left|\int_0^\infty \Phi^0(x,\xi)x^{1/2}h(x,s)dx\right| = \left|\int_0^\infty \left(\Phi^0(x,\xi) - \frac{x^{1/2}(x^2-1)}{x^2+1}\right)x^{1/2}h(x,s)dx\right|.$$

We refer the interested readers to [55, Proposition 9.8]. □

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