# SINGULAR RADIAL ENTIRE SOLUTIONS AND WEAK SOLUTIONS WITH PRESCRIBED SINGULAR SET FOR A BIHARMONIC EQUATION

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Abstract. Positive singular radial entire solutions of a biharmonic equation with subcritical exponent are obtained via the entire radial solutions of the equation with supercritical exponent and the Kelvin transformation. The expansions of such singular radial solutions at the singular point 0 are presented. Using these singular radial entire solutions, we construct solutions with a prescribed singular set for the Navier boundary value problem

$$\Delta^2 u = u^p \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega$$

where $\Omega$ is a smooth open set of $\mathbb{R}^n$ with $n \geq 5$.

## 1. Introduction

We study existence, uniqueness, asymptotic behavior and further qualitative properties of radial solutions of the biharmonic equation

$$\begin{cases} \Delta^2 u = u^p \quad \text{in } \mathbb{R}^n \backslash \{0\} \\ u > 0, \quad \text{and } \lim_{|x|\to 0} u(x) = +\infty \end{cases} \tag{1.1}$$

where $n \geq 5$ and $\frac{n}{n-4} < p < \frac{n+4}{n-4}$.

When $p = \frac{n+4}{n-4}$, the equation

$$\Delta^2 u = u^p \quad \text{in } \mathbb{R}^n \tag{1.2}$$

is studied by Lin [14] via the moving-plane method and all the regular solutions are well-established. When $p > \frac{n+4}{n-4}$, (1.2) is studied by Gazzola and Grunau [11], Guo and Wei [10], all the radial entire solutions are classified.

We recall that the corresponding second order equation (when $n \geq 3$ and $\frac{n}{n-2} < p < \frac{n+2}{n-2}$)

$$\begin{cases} -\Delta u = u^p \quad \text{in } \mathbb{R}^n \backslash \{0\} \\ u > 0, \quad \text{and } \lim_{|x|\to 0} u(x) = +\infty \end{cases} \tag{1.3}$$

is studied in [5] and [4]. The following result is established:

1991 *Mathematics Subject Classification.* 35J25, 35J20, 35B33, 35B40.

*Key words and phrases.* Radial singular entire solutions, biharmonic equations, solutions with prescribed isolated singularities.

The research of the first author is supported by NSFC (11171092, 11571093). The research of the second author is partially supported by NSERC of Canada.

**Proposition 1.1.** *([5]) Suppose that* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ *and* $u$ *is a solution of* (1.3). *Then either* $u(x) \equiv c_p|x|^{-\frac{2}{p-1}}$ *or there exists a constant* $\beta > 0$ *such that*

$$\lim_{|x|\to+\infty} |x|^{n-2}u(x) = \beta, \text{ where } c_p = \left[\frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)\right]^{\frac{1}{p-1}}.$$

*Conversely, for any* $\beta > 0$, *there exists a unique solution* $u(x)$ *of* (1.3) *such that* $\lim_{|x|\to+\infty} |x|^{n-2}u(x) = \beta$.

Using this proposition, Chen and Lin [5] constructed positive weak solutions with a prescribed singular set of the Dirichlet problem

$$\Delta u + u^p = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.4}$$

where $\Omega$ is a smooth open set in $\mathbb{R}^n$ with $n \geq 3$. Singular solutions of the equations as (1.4) with various singular sets have been studied by many authors, see, for example, [2, 3, 7, 12, 15, 16, 17, 18, 19, 20, 21, 22].

In this paper, we first obtain the following result.

**Theorem 1.2.** *Suppose that* $\frac{n}{n-4} < p < \frac{n+4}{n-4}$. *Then for any* $\beta > 0$, *there exists a unique radial entire solution* $u(r)$ *of* (1.1) *such that* $\lim_{r\to+\infty} r^{n-4}u(r) = \beta$ *and*

$$\lim_{r\to0^+} r^{\frac{4}{p-1}}u(r) = C_p,$$

*where* $C_p = [K(p,n)]^{1/(p-1)}$ *and*

$$\begin{aligned} K(p,n) \;&=\; \frac{8}{(p-1)^4}\Big[(n-2)(n-4)(p-1)^3 + 2(n^2-10n+20)(p-1)^2 \\ &\quad -16(n-4)(p-1) + 32\Big]. \end{aligned}$$

Moreover, the expansions of the radial singular entire solutions at the singular point are presented. Using such singular radial entire solutions, we will construct positive weak solutions with a prescribed singular set for the Navier boundary value problem:

$$\Delta^2 u = u^p \quad \text{in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega \tag{1.5}$$

where $\Omega$ is a smooth open set in $\mathbb{R}^n$ with $n \geq 5$. $u \in L^p(\Omega)$ is called a weak solution of (1.5) if the equality

$$\int_\Omega \Delta^2\varphi u dx = \int_\Omega u^p \varphi dx,$$

holds for any $\varphi \in C^4(\Omega) \cap C^3(\overline{\Omega})$ and $\varphi = \Delta\varphi = 0$ on $\partial\Omega$.

Let

$$p^c = \frac{n+2+\sqrt{n^2+4-4\sqrt{n^2+H_n}}}{n-6+\sqrt{n^2+4-4\sqrt{n^2+H_n}}}$$

with

$$H_n = \Big(\frac{n(n-4)}{4}\Big)^2.$$

It is known from [6] that $p^c$ is the unique number for $p \in (n/(n-4), (n+4)/(n-4))$ such that $pK(p,n) = H_n$ for $p = p^c$ and $pK(p,n) < H_n$ for $n/(n-4) < p < p^c$; $pK(p,n) > H_n$ for $p^c < p < (n+4)/(n-4)$.

**Theorem 1.3.** *Suppose that $n \geq 5$, $n/(n-4) < p < p^c$, $\Omega$ is a bounded smooth open set in $\mathbb{R}^n$ and $S$ is a closed subset of $\Omega$. Then there exist two distinct sequences of solutions of* (1.5) *having $S$ as their singular set such that one sequence converges to 0 in $L^q(\Omega)$, and the other sequence converges to a smooth solution of* (1.5) *in $L^q(\Omega)$ for $q < p^* := n(p-1)/4$.*

Some special singular solutions of (1.5) have been constructed in [1] by using the special singular solution $u_s(r) = C_p r^{-\frac{4}{p-1}}$ of (1.1).

In order to obtain Theorem 1.2, we need to consider the following equation:

$$\Delta^2 u = |x|^\alpha u^p \quad \text{in } \mathbb{R}^n, \tag{1.6}$$

where $\alpha > -4$, $n \geq 5$ and $p > \frac{n+4+2\alpha}{n-4}$ and use the Kelvin transformation. We need the following theorem.

**Theorem 1.4.** *Suppose that $n \geq 5$, $\alpha > -4$, $p > \frac{n+4+2\alpha}{n-4}$, then for any $a > 0$,* (1.6) *admits a unique positive radial entire solution $u_a(r)$ such that $u_a(0) = a$ and*

$$r^{\frac{4+\alpha}{p-1}} u_a(r) - [K_0(p,n,\alpha)]^{\frac{1}{p-1}} \to 0 \quad \text{as } r \to +\infty \tag{1.7}$$

*where*

$$\begin{aligned} K_0(p,n,\alpha) \;&=\; \frac{(4+\alpha)}{(p-1)^4}\Big[2(n-2)(n-4)(p-1)^3 + (4+\alpha)(n^2-10n+20)(p-1)^2 \\ &\quad -2(4+\alpha)^2(n-4)(p-1) + (4+\alpha)^3\Big]. \end{aligned}$$

*Moreover, if there are $P := P(n,\alpha) > (n+4+2\alpha)/(n-4)$ ($P$ maybe $\infty$) and $p_c := p_c(n,\alpha) \in ((n+4+2\alpha)/(n-4), P)$ such that*

$$\begin{cases} pK_0(p,n,\alpha) - \left(\frac{n(n-4)}{4}\right)^2 > 0 & \text{for } \frac{n+4+2\alpha}{n-4} < p < p_c, \\ pK_0(p,n,\alpha) - \left(\frac{n(n-4)}{4}\right)^2 = 0 & \text{for } p = p_c, \\ pK_0(p,n,\alpha) - \left(\frac{n(n-4)}{4}\right)^2 < 0 & \text{for } p_c < p < P, \end{cases}$$

*then $u_a(r) - [K_0(p,n,\alpha)]^{1/(p-1)} r^{-\frac{4+\alpha}{p-1}}$ changes sign infinitely many times provided $(n+4+2\alpha)/(n-4) < p < p_c$; $u_a(x) < u_s(x) := [K_0(p,n,\alpha)]^{1/(p-1)}|x|^{-(4+\alpha)/(p-1)}$*

*for all* $x \in \mathbb{R}^n$ *and the solutions are strictly ordered with respect to the initial value* $a = u_a(0)$ *provided* $p_c \le p < P$. *Namely, if* $u_1(x)$ *and* $u_2(x)$ *are two radial entire solutions of* (1.1) *with* $u_1(0) < u_2(0)$, *then* $u_1(x) < u_2(x)$ *for* $x \in \mathbb{R}^n$. *Moreover,*

$$u_a(x) \to 0, \;\; \Delta u_a(x) \to 0 \;\text{ for any } x \in \mathbb{R}^n \text{ as } a \to 0$$

*and*

$$u_a(x) \to u_s(x) \;\text{ for } x \in \mathbb{R}^n \backslash \{0\} \text{ as } a \to \infty.$$

In the following, if there is no confusion, we omit $p, n$ on $K(p,n)$ and $p, n, \alpha$ on $K_0(p,n,\alpha)$.

**Theorem 1.5.** *For* $n \ge 13$, *there exists a positive weak solution of*

$$\Delta^2 u = u^{\frac{n+4}{n-4}} \tag{1.8}$$

*in* $L^{\frac{n+4}{n-4}}(\mathbb{R}^n, d\mu)$ *whose singular set is the whole* $\mathbb{R}^n$.

## 2. Classification of radial entire solutions of (1.6): Proof of Theorem 1.4

In this section we classify the radial entire solutions of (1.6) and prove Theorem 1.4.

The proof of existence and uniqueness of entire solutions of (1.6) is similar to that of Theorem 1 of [11]. We only present the main ideas.

In radial coordinates $r = |x|$, the equation (1.6) reads

(2.1)
$$u^{(4)}(r) + \frac{2(n-1)}{r}u'''(r) + \frac{(n-1)(n-3)}{r^2}u''(r) - \frac{(n-1)(n-3)}{r^3}u'(r) = r^\alpha u^p(r).$$

Note that if $p > \frac{n+4+2\alpha}{n-4}$, then

$$(n-4)(p-1) > 2(\alpha+4). \tag{2.2}$$

We set

$$u(r) = r^{-\frac{4+\alpha}{p-1}} v(\ln r), \quad v(t) = e^{\frac{4+\alpha}{p-1}t} u(e^t). \tag{2.3}$$

Then, equation (2.1) can be written as

$$v^{(4)}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = v^p(t) \quad t \in \mathbb{R}, \tag{2.4}$$

where the constants $K_i = K_i(p,n,\alpha)$ $(i = 0, \ldots, 3)$ are given by

$$\begin{aligned} K_0 \;\;=\;\; & \frac{(4+\alpha)}{(p-1)^4}\Big[2(n-2)(n-4)(p-1)^3 + (4+\alpha)(n^2-10n+20)(p-1)^2 \\ & -2(4+\alpha)^2(n-4)(p-1) + (4+\alpha)^3\Big], \end{aligned}$$

$$K_1 = -\frac{2}{(p-1)^3}\Big[(n-2)(n-4)(p-1)^3 + (4+\alpha)(n^2-10n+20)(p-1)^2$$
$$-3(\alpha^2+8\alpha+16)(n-4)(p-1) + 2\alpha(\alpha^2+12\alpha+48)+128\Big],$$

$$K_2 = \frac{1}{(p-1)^2}\Big[(n^2-10n+20)(p-1)^2 - 6(4+\alpha)(n-4)(p-1)$$
$$+6\alpha(\alpha+8)+96\Big],$$

$$K_3 = \frac{2}{p-1}\Big[(n-4)(p-1) - 2(4+\alpha)\Big].$$

By using (2.2), it is not difficult to show that $K_1 = K_3 = 0$ if $p = \frac{n+4+2\alpha}{n-4}$ and that

$$K_0 > 0, \quad K_1 < 0, \quad K_3 > 0, \quad \forall n \geq 5, \alpha > -4, \quad p > \frac{n+4+2\alpha}{n-4}.$$

Note that (2.4) admits two constant solutions $v_0 \equiv 0$ and $v_s \equiv K_0^{1/(p-1)}$, which by (2.3), correspond to the following solutions of (2.1):

$$u_0(r) \equiv 0, \qquad u_s(r) = \frac{K_0^{1/(p-1)}}{r^{\frac{4+\alpha}{p-1}}}.$$

Consider the initial value problem

$$(2.5) \qquad \begin{cases} u_\gamma^{(4)}(r) + \frac{2(n-1)}{r}u_\gamma'''(r) + \frac{(n-1)(n-3)}{r^2}u_\gamma''(r) - \frac{(n-1)(n-3)}{r^3}u_\gamma'(r) = r^\alpha u_\gamma^p(r) \\ u_\gamma(0) = 1, \ \ u_\gamma'(0) = u_\gamma'''(0) = 0, \quad u_\gamma''(0) = \gamma < 0. \end{cases}$$

Arguments similar to those in the proof of Theorem 2 of [11] imply that there exists a unique $\overline{\gamma} < 0$ such that the solution $u_{\overline{\gamma}}$ of (2.5) exists on $[0, \infty)$, is positive everywhere and vanishes at $+\infty$. Moreover, it is seen from the proof of Theorem 1 of [11] that $u_{\overline{\gamma}}$ is the unique entire positive solution of (2.5) which vanishes at $+\infty$. Note that we need a comparison principle as Lemma 2 of [11] here and such comparison principle still holds for our nonlinearity here.

We denote $u_1(r) := u_{\overline{\gamma}}(r)$. Then it is easily seen that $u_1(r)$ satisfies (i) $u_1'(r) < 0$ for all $r > 0$, (ii) $\Delta u_1(r) < 0$ for all $r > 0$, (iii) $(\Delta u_1)'(r) > 0$ for all $r > 0$. Moreover, for any $a > 0$, if we define $u_a(r) = a u_1(a^{(p-1)/(4+\alpha)} r)$, we obtain a unique entire solution of (1.6) satisfying $u_a(0) = a$ and $u_a(r) \to 0$ as $r \to \infty$. Thus we obtain a class of entire solutions $\{u_a\}_{a>0}$ of (1.6). Note that for each $a$, $u_a'(r) < 0$, $\Delta u_a(r) < 0$ and $(\Delta u_a)'(r) > 0$ for $r > 0$. Moreover, arguments similar to those in the proof of Theorem 3 of [11] imply that

$$r^{\frac{4+\alpha}{p-1}} u_a(r) \to K_0^{\frac{1}{(p-1)}} \quad \text{as } r \to +\infty.$$

This completes the proof of the first part of Theorem 1.4.

To prove the second part of Theorem 1.4, we need to analyze the characteristic equations of the linearized equations of (2.4) at $v_0$ and $v_s$. The characteristic polynomials of the linearized equations of (2.4) at $v_0$ and $v_s$ are

$$\lambda \mapsto \lambda^4 + K_3\lambda^3 + K_2\lambda^2 + K_1\lambda + K_0 \tag{2.6}$$

and

$$\nu \mapsto \nu^4 + K_3\nu^3 + K_2\nu^2 + K_1\nu + (1-p)K_0, \tag{2.7}$$

respectively. Then, according to MAPLE, the eigenvalues of (2.6) are given by

$$\lambda_1 = m, \quad \lambda_2 = m+2, \quad \lambda_3 = 4+m-n, \quad \lambda_4 = 2+m-n,$$

here and in the following

$$m = \frac{4+\alpha}{p-1}.$$

The eigenvalues of (2.7) are given by

$$\nu_1 = \frac{N_1 + \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)}, \quad \nu_2 = \frac{N_1 - \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)},$$

$$\nu_3 = \frac{N_1 + \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)}, \quad \nu_4 = \frac{N_1 - \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)},$$

where

$$N_1 := -(n-4)(p-1) + 2(4+\alpha), \quad N_2 := (n^2-4n+8)(p-1)^2,$$

$$\begin{aligned} N_3 : &= (n-2)[n-2+2(n-4)(4+\alpha)](p-1)^4 \\ &+(4+\alpha)[(n^2-10n+20)(4+\alpha)+2(n^2-6n+8)](p-1)^3 \\ &+(4+\alpha)^2[(n^2-10n+20)+2(4-n)(4+\alpha)](p-1)^2 \\ &+(4+\alpha)^3[4+\alpha+2(4-n)](p-1)+(4+\alpha)^4. \end{aligned}$$

Note that $N_1 < 0$ by (2.2).

Let us define

$$\tilde{\nu}_i = \nu_i - \frac{4+\alpha}{p-1}, \quad i = 1,2,3,4.$$

**Proposition 2.1.** *Assume that* $\alpha > -4$ *and* $p > \frac{n+4+2\alpha}{n-4}$, *then*

*(i) For any* $n \geq 5$ *we have* $\tilde{\nu}_2 < 2-n < 0 < \tilde{\nu}_1$.

*(ii) If there are* $P := P(n,\alpha) > (n+4+2\alpha)/(n-4)$ *(*$P$ *maybe* $\infty$*) and* $p_c := p_c(n,\alpha) \in ((n+4+2\alpha)/(n-4), P)$ *such that*

$$\begin{cases} N_4 := 16N_3 - N_2^2 > 0, & \text{for } \frac{n+4+2\alpha}{n-4} < p < p_c, \\ N_4 := 16N_3 - N_2^2 = 0, & \text{for } p = p_c, \\ N_4 := 16N_3 - N_2^2 < 0, & \text{for } p_c < p < P, \end{cases}$$

*then*

-*if* $\frac{n+4+2\alpha}{n-4} < p < p_c$, *then* $\tilde{\nu}_3, \tilde{\nu}_4 \notin \mathbb{R}$ *and* $Re\tilde{\nu}_3 = Re\tilde{\nu}_4 = \frac{4-n}{2} < 0$.

-*if* $p = p_c$, *then* $\tilde{\nu}_3, \tilde{\nu}_4 \in \mathbb{R}$ *and* $\tilde{\nu}_4 = \tilde{\nu}_3 = \frac{4-n}{2} < 0$.

-*if* $p_c < p < P$, *then* $\tilde{\nu}_3, \tilde{\nu}_4 \in R$ *and* $\tilde{\nu}_4 < \tilde{\nu}_3 < 0$ *and*

$$\tilde{\nu}_2 < 4-n < \tilde{\nu}_4 < \frac{4-n}{2} < \tilde{\nu}_3 < 0 < \tilde{\nu}_1, \quad \tilde{\nu}_3 + \tilde{\nu}_4 = 4-n.$$

**Proof.** Note that

$$\frac{N_4}{16(p-1)^4} = pK_0 - \Big(\frac{n(n-4)}{4}\Big)^2.$$

Note also that if $\alpha = 0$, it is known from [11] that for $5 \le n \le 12$, $p_c(n,0) = p^c = +\infty$, i.e., $N_4 > 0$ for all $p > (n+4)/(n-4)$. It is also known from [11], $(n+4)/(n-4) < p_c(n,0) < +\infty$ exists provided $n \ge 13$. Using (2.2), we see that

$$\begin{aligned} N_2 - N_1^2 &= 4(n-2)(p-1)^2 + 4(4+\alpha)(n-4)(p-1) - 4(4+\alpha)^2 \\ &> 4(n-2)(p-1)^2 + 8(4+\alpha)^2 - 4(4+\alpha)^2 \\ &> 0. \end{aligned}$$

Next, we show that

$$N_3 > \frac{(N_2 - N_1^2)^2}{16}. \tag{2.8}$$

Indeed, by using (2.2) again, we have

$$\begin{aligned} N_3 - \frac{(N_2-N_1^2)^2}{16} &\\ &= 2(n-2)(n-4)(4+\alpha)(p-1)^4 + (4+\alpha)^2(n^2-10n+20)(p-1)^3 \\ &\quad -2(4+\alpha)^3(n-4)(p-1)^2 + (4+\alpha)^4(p-1) \\ &> (4+\alpha)^2(n-2)(n-4)(p-1)^3 + 4(4+\alpha)^2(p-1)^3 \\ &\quad -2(4+\alpha)^3(n-4)(p-1)^2 + (4+\alpha)^4(p-1) \\ &> 4(4+\alpha)^2(p-1)^3 + 2(4+\alpha)^3(n-2)(p-1)^2 \\ &\quad -2(4+\alpha)^3(n-4)(p-1)^2 + (4+\alpha)^4(p-1) \\ &= 4(4+\alpha)^2(p-1)^3 + 4(4+\alpha)^3(p-1)^2 + (4+\alpha)^4(p-1) > 0. \end{aligned}$$

Thus, $N_3 > 0$. Note that

$$\tilde{\nu}_1 = \frac{4-n}{2} + \frac{\sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)}, \quad \tilde{\nu}_2 = \frac{4-n}{2} - \frac{\sqrt{N_2+4\sqrt{N_3}}}{2(p-1)}.$$

Then $N_2 > 0$ and $N_3 > 0$ imply that $\tilde{\nu}_2 < 0$. A simple calculation shows that $\tilde{\nu}_1 > 0$. We obtain $\tilde{\nu}_2 < 2-n$ from the fact that $\sqrt{N_3} > (n-2)(p-1)^2$ and it can be obtained from

$$\sqrt{N_3} > \frac{N_2 - N_1^2}{4} > (n-2)(p-1)^2 + (4+\alpha)^2 > (n-2)(p-1)^2.$$

This proves statement (i) in Proposition 2.1. To show (ii), we only need to show $\nu_4 < \nu_3 < 0$ for $p > p_c(n,\alpha)$. This can be easily seen from (2.8). □

We now obtain the expansions of $u_a(r)$ at $r = +\infty$. If there is no confusion, we drop the index $a$. We have the following propositions.

**Proposition 2.2.** *Let $\frac{n+4+2\alpha}{n-4} < p < p_c$ with $\alpha > -4$ and $u$ be the unique entire solution of (1.6) with $u(0) = a$. Then we have for $r$ large:*

$$u(r) = u_s(r) + M_1 r^\tau \cos(\kappa \ln r) + M_2 r^\tau \sin(\kappa \ln r) + O(r^{\max\{\frac{N_1}{(p-1)} - m, \nu_2 - m\}}) \tag{2.9}$$

*where $\tau = \frac{4-n}{2}$, $\kappa = \frac{\sqrt{4\sqrt{N_3} - N_2}}{2(p-1)} > 0$ and $M_1^2 + M_2^2 \neq 0$.*

**Proof.** It is easily seen that $\nu_2 < \frac{N_1}{2(p-1)} < 0$. Using the Emden-Fowler transformation:

$$v(t) = r^m u(r), \quad t = \ln r,$$

and letting $v(t) = (K_0)^{1/(p-1)} + h(t)$, we see that $h(t)$ satisfies

$$h^{(4)}(t) + K_3 h'''(t) + K_2 h''(t) + K_1 h'(t) + (1-p) K_0 h(t) + O(h^2) = 0, \quad t > 1 \tag{2.10}$$

and $\lim_{t\to\infty} h(t) = 0$. By arguments similar to those in the proof of Theorem 3.3 of [10], we have that we can write

$$h(t) = M_1 e^{\frac{N_1}{2(p-1)}t} \cos(\kappa t) + M_2 e^{\frac{N_1}{2(p-1)}t} \sin(\kappa t) + M_3 e^{\nu_2 t} + O(e^{\max\{\frac{N_1}{p-1}, \nu_2\}t}). \tag{2.11}$$

The fact $M_1^2 + M_2^2 \neq 0$ can be obtained by arguments similar to those in the proof of Theorem 3.3 of [10]. Note that if $\phi := \phi(r)$ is a nontrivial solution of the linearized equation

$$\Delta^2 \phi = p r^\alpha u^{p-1} \phi, \quad \phi(r) \to 0 \text{ as } r \to +\infty,$$

then

$$\phi(r) = c\Big(\frac{4+\alpha}{p-1} u(r) + r u'(r)\Big)$$

for some $c \neq 0$. Now, (2.9) can be obtained from (2.11). This completes the proof. □

**Remark 2.3.** Proposition 2.2 implies that $u(r) - u_s(r)$ changes sign infinitely many times in $(0,\infty)$. Moreover, we can also claim that if $u_1(r)$ and $u_2(r)$ are two different regular entire solutions, i.e. $u_1(0) \neq u_2(0)$, then $u_1(r) - u_2(r)$ changes sign infinitely many times in $(0,\infty)$. In fact, if $u_1(r) = r^{-m}v_1(t)$, $u_2(r) = r^{-m}v_2(t)$, then $k(t) := v_1(t) - v_2(t)$ satisfies the equation (for $t \gg 1$)

(2.12)

$$k^{(4)}(r)+K_3k'''(t)+K_2k''(t)+K_1k'(t)+(1-p)K_0k(t)+O(e^{\frac{N_1}{2(p-1)}t})k(t)+O(k^2(t)) = 0.$$

Therefore, $k(t)$ admits a similar expansion to that in (2.11) with $M_1^2 + M_2^2 \neq 0$. Thus, our claim holds.

**Proposition 2.4.** *Assume that $p_c \leq p < P$. Then the set of the solutions $\{u_a(r)\}$ to (1.6) is well ordered. That is, if $a > b$, then $u_a(r) > u_b(r)$ for all $r > 0$. Moreover, the following statements hold.*

*(i) If $k(-\nu_3) < (-\nu_4) < (k+1)(-\nu_3)$ and $\ell(-\nu_4) < (-\nu_2) < (\ell+1)(-\nu_4)$ for some positive integers $k$ and $\ell$, then $u(r)$ has the asymptotic expansion near $\infty$:*

$$\begin{aligned} u(r) &= K_0^{1/(p-1)}r^{-m} + a_1r^{\nu_3-m} + \ldots + a_kr^{k\nu_3-m} \\ &\quad +b_1r^{\nu_4-m} + \ldots + b_\ell r^{\ell\nu_4-m} + c_1r^{\nu_2-m} + O(r^{\nu_2-m-\delta_0}) \end{aligned}$$

*for $a_1 \neq 0$ and some $\delta_0 > 0$ depending only on $p$, $n$ and $\alpha$, which satisfies*

$$\frac{4+\alpha}{p-1} + (-\nu_2) = \frac{n-4}{2} + \delta_0. \qquad \left(\delta_0 = \frac{\sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)}\right)$$

*(ii) If $k(-\nu_3) = -\nu_4$ and $\ell(-\nu_4) < (-\nu_2) < (\ell+1)(-\nu_4)$ for some positive integers $k$ and $\ell$, then $u(r)$ has the asymptotic expansion near $\infty$:*

$$\begin{aligned} u(r) &= K_0^{1/(p-1)}r^{-m} + a_1r^{\nu_3-m} + \ldots + a_{k-1}r^{(k-1)\nu_3-m} + a_kr^{k\nu_3-m}\ln r \\ &\quad +b_1r^{\nu_4-m} + \ldots + b_\ell r^{\ell\nu_4-m} + c_1r^{\nu_2-m} + O(r^{\nu_2-m-\delta_0}) \end{aligned}$$

*for $a_1 \neq 0$ and some $\delta_0 > 0$ depending only on $p$, $n$ and $\alpha$ as the above.*

*(iii) If $k(-\nu_3) < (-\nu_4) < (k+1)(-\nu_3)$ and $\ell(-\nu_4) = (-\nu_2)$ for some positive integers $k$ and $\ell$, then $u(r)$ has the asymptotic expansion near $\infty$:*

$$\begin{aligned} u(r) &= K_0^{1/(p-1)}r^{-m} + a_1r^{\nu_3-m} + \ldots + a_{k-1}r^{(k-1)\nu_3-m} + a_kr^{k\nu_3-m} \\ &\quad +b_1r^{\nu_4-m} + \ldots + b_{\ell-1}r^{(\ell-1)\nu_4-m} + b_\ell r^{\ell\nu_4-m}\ln r + c_1r^{\nu_2-m} + O(r^{\nu_2-m-\delta_0}) \end{aligned}$$

*for $a_1 \neq 0$ and some $\delta_0 > 0$ depending only on $p$, $n$ and $\alpha$ as the above.*

*(iv) If $k(-\nu_3) = (-\nu_4)$ and $\ell(-\nu_4) = (-\nu_2)$ for some positive integers $k$ and $\ell$, then $u(r)$ has the asymptotic expansion near $\infty$:*

$$\begin{aligned} u(r) &= K_0^{1/(p-1)}r^{-m} + a_1r^{\nu_3-m} + \ldots + a_{k-1}r^{(k-1)\nu_3-m} + a_kr^{k\nu_3-m}\ln r \\ &\quad +b_1r^{\nu_4-m} + \ldots + b_{\ell-1}r^{(\ell-1)\nu_4-m} + b_\ell r^{\ell\nu_4-m}\ln r + c_1r^{\nu_2-m} + O(r^{\nu_2-m-\delta_0}) \end{aligned}$$

*for $a_1 \neq 0$ and some $\delta_0 > 0$ depending only on $p$, $n$ and $\alpha$ as the above.*

*The coefficients $a_2,\dots,a_k$ are functions of $a_1$ and $b_2,\dots b_\ell$ are functions of $b_1$. Moreover, if two solutions have the same $a_1$, $b_1$ and $c_1$, then these two solutions must be identical.*

**Proof.** Note that $\nu_2 < \nu_4 < \nu_3 < 0$ for $p > p_c$ and $\nu_2 < \nu_4 = \nu_3$ for $p = p_c$. The first part of this proposition can be obtained by arguments similar to those in the proof of (1) of Corollary 4.3 of [10]. Note that we have $r^m u(r) - K_0^{1/(p-1)} \to 0$ as $r \to \infty$. The expansions of $u(r)$ near $\infty$ can be obtained by (2.10) and some ODE arguments similar to those in [13], [8] and [9]. We refer to Theorem 2.5 of [8]. The fact that $a_1 \neq 0$ can be obtained from Corollary 4.3 of [10]. □

**Proof of Theorem 1.4.** The proof of Theorem 1.4 can be done by arguments as the above. Moreover, we also obtain that $u_a(x) \to 0$, $\Delta u_a(x) \to 0$ for any $x \in \mathbb{R}^n$ as $a \to 0^+$ and $u_a(x) \to u_s(x)$ for any $x \in \mathbb{R}^n \backslash \{0\}$ as $a \to \infty$ provided $p_c \le p < P$. These can be easily seen from the fact $u_a(r) = a u_1(a^{(p-1)/(4+\alpha)} r)$. We also notice that for each $a > 0$, $u_a'(r) < 0$, $(\Delta u_a)(r) < 0$, $(\Delta u_a)'(r) > 0$ for $r > 0$ and $u_a(r) \to 0$, $\Delta u_a(r) \to 0$ as $r \to \infty$. □

## 3. Singular solutions of (1.1): Proof of Theorem 1.2

In this section we will use Kelvin transformation and Theorem 1.4 to obtain a class of singular solutions of (1.1).

We first show the following lemma.

**Lemma 3.1.** *Let $u(x)$ be a solution of* (1.1). *Let*

$$v(y) = |x|^{n-4} u(x) \quad y = \frac{x}{|x|^2}. \tag{3.1}$$

*Then $v(y)$ satisfies the equation*

$$\Delta^2 v(y) = |y|^{(n-4)p-(n+4)} v^p(y) \quad y \in \mathbb{R}^n. \tag{3.2}$$

**Proof.** First of all, let $v(y) = |x|^s u(x)$ for $s > 0$. Then

$$\frac{\partial v}{\partial y_i}(y) = -s|x|^s x_i u + |x|^{s+2} \frac{\partial u}{\partial x_i}(x) - 2|x|^s x \cdot \nabla u x_i, \tag{3.3}$$

$$\Delta_y v(y) = |x|^{s+4} \Delta_x u(x) - (n-s-2)|x|^{s+2}(su + 2x \cdot \nabla u). \tag{3.4}$$

We now take $s = n-4$, then

$$\Delta_y v(y) = |x|^n \Delta u - 2|x|^{n-2}[(n-4)u + 2x \cdot \nabla u],$$

$$\Delta_y(\Delta_y v) = \Delta_y(|x|^n \Delta u) - 2\Delta_y\{|x|^{n-2}[(n-4)u + 2x \cdot \nabla u]\}.$$

To calculate the first term, we take $s = n$ in (3.4). Then

$$\Delta_y(|x|^n \Delta u) = |x|^{n+4}\Delta(\Delta u) + 2|x|^{n+2}[n\Delta u + 2x \cdot \nabla(\Delta u)].$$

Similarly, taking $s = n-2$,

$$-2\Delta_y(|x|^{n-2}(n-4)u) = -2(n-4)|x|^{n+2}\Delta u.$$

From the fact that $\Delta(x \cdot \nabla u) = 2\Delta u + x \cdot \nabla(\Delta u)$,

$$\begin{aligned} -2\Delta_y(|x|^{n-2}2x \cdot \nabla u) &= -4|x|^{n+2}\Delta(x \cdot \nabla u) \\ &= -8|x|^{n+2}\Delta u - 4|x|^{n+2}x \cdot \nabla(\Delta u). \end{aligned}$$

We finally deduce

$$\Delta_y^2 v(y) = |x|^{n+4}\Delta_x^2 u(x).$$

This completes the proof. □

Let $\alpha_* = (n-4)p-(n+4)$. Then $-4 < \alpha_* < 0$ for $n/(n-4) < p < (n+4)/(n-4)$. Moreover,

$$\frac{n+4}{n-4} - p = p - \frac{n+4+2\alpha_*}{n-4}.$$

This implies that if $p \in (n/(n-4), (n+4)/(n-4))$, then $p \in ((n+4+2\alpha_*)/(n-4), (n+8+2\alpha_*)/(n-4))$. Then we obtain the following theorem from Theorem 1.4.

**Theorem 3.2.** *Let $n \geq 5$ and $p \in (\frac{n}{n-4}, \frac{n+4}{n-4})$. Then for any $\beta > 0$, the equation (3.2) admits a unique radial entire solution $v = v_\beta(\rho)$ ($\rho = |y|$) such that $v(0) = \beta$ and*

$$\rho^{\frac{4+\alpha_*}{p-1}} v(\rho) \to [K_0(\alpha_*)]^{\frac{1}{p-1}} \quad \text{as } \rho \to \infty,$$

*where $K_0(\alpha_*)$ is the $K_0$ in Theorem 1.4 with $\alpha = \alpha_*$. Moreover, $v'(\rho) < 0$, $\Delta v(\rho) < 0$ for $\rho > 0$.*

**Proof of Theorem 1.2.** Direct calculations imply

$$K_0(\alpha_*) = K, \tag{3.5}$$

where $K = K(p,n)$ is given in Theorem 1.2. Then, for any $\beta > 0$, the solution $u_\beta(r)$ of (1.1) corresponding to $v_\beta(\rho)$ in Theorem 3.2 satisfies $r^{n-4}u_\beta(r) \to \beta$ as $r \to \infty$ and $r^{4/(p-1)}u_\beta(r) \to C_p$ as $r \to 0^+$. This completes the proof of Theorem 1.2. □

We now obtain the expansions of $u_\beta(r)$ given in Theorem 1.2 near $r = 0$. It is known that

$$r^{4/(p-1)}u_\beta(r) \to C_p \quad \text{as } r \to 0^+.$$

Since

$$\frac{n+4}{n-4} - p = p - \frac{n+4+2\alpha_*}{n-4}$$

and $K_0(\alpha_*) = K$, we see that $p^c > (n+4+2\alpha_*)/(n-4)$ and

$$\begin{cases} pK_0(\alpha_*) < \left(\frac{n(n-4)}{4}\right)^2, & p \in (\frac{n}{n-4}, p^c), \\ pK_0(\alpha_*) = \left(\frac{n(n-4)}{4}\right)^2, & p = p^c, \\ pK_0(\alpha_*) > \left(\frac{n(n-4)}{4}\right)^2, & p \in (p^c, \frac{n+4}{n-4}). \end{cases}$$

Let $N_4(\alpha_*)$ be the $N_4$ in Proposition 2.1 with $\alpha = \alpha_*$. Then

$$\frac{N_4(\alpha_*)}{16(p-1)^4} = p[K_0(\alpha_*)] - \left[\frac{n(n-4)}{4}\right]^2 = pK - \left[\frac{n(n-4)}{4}\right]^2. \tag{3.6}$$

Therefore,

$$\begin{cases} N_4(\alpha_*) > 0, & p \in (p^c, \frac{n+4}{n-4}), \\ N_4(\alpha_*) = 0, & p = p^c, \\ N_4(\alpha_*) < 0, & p \in (\frac{n}{n-4}, p^c). \end{cases}$$

We now obtain the following propositions from Propositions 2.4 and 2.2 (by using the Kelvin transformation).

**Proposition 3.3.** *Assume that $n \geq 5$ and $p \in (\frac{n}{n-4}, p^c]$. Then $0 < u_\beta(r) < u_\gamma(r) < \tilde{u}_s(r) := C_p r^{-\frac{4}{p-1}}$ for $0 < \beta < \gamma < \infty$, and $r > 0$. Moreover, $\lim_{\beta\to\infty} u_\beta(x) = \tilde{u}_s(x)$ and $\lim_{\beta\to 0} u_\beta(x) = 0$ for any $x \in \mathbb{R}^n\backslash\{0\}$. Furthermore, for*

$$\sigma_1 = -\nu_2(\alpha_*), \quad \sigma_2 = -\nu_1(\alpha_*),$$

$$\sigma_3 = -\nu_4(\alpha_*), \quad \sigma_4 = -\nu_3(\alpha_*),$$

*where $\nu_i(\alpha_*)$ $(i = 1,2,3,4)$ are the numbers given in Proposition 2.1 with $\alpha = \alpha_*$, the following statements hold:*

*(i) If $k\sigma_4 < \sigma_3 < (k+1)\sigma_4$ and $\ell\sigma_3 < \sigma_1 < (\ell+1)\sigma_3$ for some positive integers $k$ and $\ell$, then $u(r)$ has the asymptotic expansion near 0,*

$$\begin{aligned} u(r) \;&=\; K^{1/(p-1)} r^{-\frac{4}{p-1}} + a_1 r^{-\frac{4}{p-1}+\sigma_4} + \ldots + a_k r^{-\frac{4}{p-1}+k\sigma_4} \\ &\quad + b_1 r^{-\frac{4}{p-1}+\sigma_3} + \ldots + b_\ell r^{-\frac{4}{p-1}+\ell\sigma_3} + c_1 r^{-\frac{4}{p-1}+\sigma_1} + O(r^{-\frac{4}{p-1}+\sigma_1+\delta_0}) \end{aligned}$$

*for $a_1 \neq 0$ and some $\delta_0 > 0$ depending only on $p$, $n$, which satisfies*

$$-\frac{4}{p-1} + \sigma_1 = -\frac{n-4}{2} + \delta_0.$$

*(ii) If $k\sigma_4 = \sigma_3$ and $\ell\sigma_3 < \sigma_1 < (\ell+1)\sigma_3$ for some positive integers $k$ and $\ell$, then $u(r)$ has the asymptotic expansion near 0,*

$$\begin{aligned} u(r) \;&=\; K^{1/(p-1)} r^{-\frac{4}{p-1}} + a_1 r^{-\frac{4}{p-1}+\sigma_4} + \ldots + a_{k-1} r^{-\frac{4}{p-1}+(k-1)\sigma_4} + a_k r^{-\frac{4}{p-1}+k\sigma_4} \ln r \\ &\quad + b_1 r^{-\frac{4}{p-1}+\sigma_3} + \ldots + b_\ell r^{-\frac{4}{p-1}+\ell\sigma_3} + c_1 r^{-\frac{4}{p-1}+\sigma_1} + O(r^{-\frac{4}{p-1}+\sigma_1+\delta_0}) \end{aligned}$$

*for $a_1 \neq 0$ and some $\delta_0 > 0$ depending only on $p$, $n$ as the above.*

*(iii) If $k\sigma_4 < \sigma_3 < (k+1)\sigma_4$ and $\ell\sigma_3 = \sigma_1$ for some positive integers $k$ and $\ell$, then $u(r)$ has the asymptotic expansion near 0,*

$$\begin{aligned} u(r) &= K^{1/(p-1)} r^{-\frac{4}{p-1}} + a_1 r^{-\frac{4}{p-1}+\sigma_4} + \ldots + a_{k-1} r^{-\frac{4}{p-1}+(k-1)\sigma_4} + a_k r^{-\frac{4}{p-1}+k\sigma_4} \\ &\quad + b_1 r^{-\frac{4}{p-1}+\sigma_3} + \ldots + b_{\ell-1} r^{-\frac{4}{p-1}+(\ell-1)\sigma_3} \\ &\quad + b_\ell r^{-\frac{4}{p-1}+\ell\sigma_3} \ln r + c_1 r^{-\frac{4}{p-1}+\sigma_1} + O(r^{-\frac{4}{p-1}+\sigma_1+\delta_0}) \end{aligned}$$

*for $a_1 \neq 0$ and some $\delta_0 > 0$ depending only on $p$, $n$ as the above.*

*(iv) If $k\sigma_4 = \sigma_3$ and $\ell\sigma_3 = \sigma_1$ for some positive integers $k$ and $\ell$, then $u(r)$ has the asymptotic expansion near 0,*

$$\begin{aligned} u(r) &= K^{1/(p-1)} r^{-\frac{4}{p-1}} + a_1 r^{-\frac{4}{p-1}+\sigma_4} + \ldots + a_{k-1} r^{-\frac{4}{p-1}+(k-1)\sigma_4} + a_k r^{-\frac{4}{p-1}+k\sigma_4} \ln r \\ &\quad + b_1 r^{-\frac{4}{p-1}+\sigma_3} + \ldots + b_{\ell-1} r^{-\frac{4}{p-1}+(\ell-1)\sigma_3} + b_\ell r^{-\frac{4}{p-1}+\ell\sigma_3} \ln r \\ &\quad + c_1 r^{-\frac{4}{p-1}+\sigma_1} + O(r^{-\frac{4}{p-1}+\sigma_1+\delta_0}) \end{aligned}$$

*for $a_1 \neq 0$ and some $\delta_0 > 0$ depending only on $p$, $n$ as the above.*

*The coefficients $a_2, \ldots, a_k$ are functions of $a_1$ and $b_2, \ldots b_\ell$ are functions of $b_1$. Moreover, if two solutions have the same $a_1$, $b_1$ and $c_1$, then these two solutions must be identical.*

**Remark 3.4.** The $\sigma_i$ $(i = 1, 2, 3, 4)$ in Proposition 3.3 are given by

$$\sigma_1 = \frac{\tilde{N}_1 + \sqrt{\tilde{N}_2 + 4\sqrt{\tilde{N}_3}}}{2(p-1)}, \quad \sigma_2 = \frac{\tilde{N}_1 - \sqrt{\tilde{N}_2 + 4\sqrt{\tilde{N}_3}}}{2(p-1)},$$

$$\sigma_3 = \frac{\tilde{N}_1 + \sqrt{\tilde{N}_2 - 4\sqrt{\tilde{N}_3}}}{2(p-1)}, \quad \sigma_4 = \frac{\tilde{N}_1 - \sqrt{\tilde{N}_2 - 4\sqrt{\tilde{N}_3}}}{2(p-1)},$$

the $\delta_0$ is given by $\delta_0 = \frac{\sqrt{\tilde{N}_2 + 4\sqrt{\tilde{N}_3}}}{2(p-1)}$, where

$$\tilde{N}_1 := -(n-4)(p-1) + 8, \quad \tilde{N}_2 := (n^2 - 4n + 8)(p-1)^2,$$

$$\begin{aligned} \tilde{N}_3 : &= (9n-34)(n-2)(p-1)^4 + 8(3n-8)(n-6)(p-1)^3 \\ &\quad + (16n^2 - 288n + 832)(p-1)^2 - 128(n-6)(p-1) + 256. \end{aligned}$$

Note that $\tilde{N}_1 > 0$ for $p \in (n/(n-4), (n+4)/(n-4))$.

**Remark 3.5.** It is clear that $pK < H_n$ for $p \in (n/(n-4), p^c)$. Therefore, the Hardy's inequality implies that for any $a > 0$, we have

$$\begin{aligned}\int_D p u_a^{p-1}\varphi^2 dx &< \int_D p\tilde{u}_s^{p-1}\varphi^2 dx \\ &\le pK\int_D \frac{\varphi^2}{r^4}dx \\ &\le \frac{pK}{H_n}\int_D (\Delta\varphi)^2 dx\end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$.

**Proposition 3.6.** *Assume that $n \ge 5$ and $p \in (p^c, (n+4)/(n-4))$. Then we have for $r$ near 0:*

(3.7)

$$u(r) = \tilde{u}_s(r) + M_1 r^{\frac{4-n}{2}}\cos\left(\tilde{\kappa}\ln\frac{1}{r}\right) + M_2 r^{\frac{4-n}{2}}\sin\left(\tilde{\kappa}\ln\frac{1}{r}\right) + O(r^{\min\{4-n+\frac{4}{p-1},\sigma_1-\frac{4}{p-1}\}})$$

*where*

$$\tilde{\kappa} = \frac{\sqrt{4\sqrt{\tilde{N}_3}-\tilde{N}_2}}{2(p-1)} > 0$$

*and $M_1^2 + M_2^2 \neq 0$. Therefore, $\tilde{u}_s(r) - u(r)$ changes sign infinitely many times for $r \in (0,\infty)$.*

## 4. Constructing weak solutions of (1.5) with prescribed singular set: Proof of Theorem 1.3

In this section we construct weak solutions with prescribed singular set for the Navier boundary value problem:

$$\Delta^2 u = u^p \ \text{ in } \Omega, \quad u = \Delta u = 0 \ \text{ on } \partial\Omega, \tag{4.1}$$

where $n \ge 5$ and $p \in (n/(n-4), p^c)$.

We first construct approximate solutions for (4.1).

A pair of functions $(\overline{u}, \overline{f})$ is called quasi-solution of (4.1) if

$$\Delta^2\overline{u} - \overline{u}^p = \overline{f} \ \text{ in } \Omega \tag{4.2}$$

where $\Omega$ is assumed to be a bounded smooth domain in $\mathbb{R}^n$ throughout this section.

Using the family of radial singular entire solutions $\{u_a\}_{a>0}$ (instead of $\beta$ in Theorem 1.2, we use $a$ here) of (1.1) given in Theorem 1.2 and Proposition 3.3, we have

$$\lim_{a\to 0}\int_{\mathbb{R}^n} u_a^q dx = 0$$

for any $0 < q < p^*$, where $p^*$ is defined by

$$p^* = \frac{n(p-1)}{4}. \tag{4.3}$$

When $\frac{n}{n-4} < p < p^c$, let $\epsilon_0$ be defined by (see Remark 3.5)

$$\epsilon_0 = 1 - \frac{pK}{H_n} > 0.$$

**Lemma 4.1.** *Fix $p_0, q_0$ such that $p < p_0 < p^*$, $\frac{2n}{n+4} < q_0 < \frac{n}{4}$, $\eta > 0$ and $\{\overline{x}_1, \ldots, \overline{x}_k\} \subset \Omega$. Then a quasi-solution $(\overline{u}_k, \overline{f}_k)$ of (4.1) can be constructed to satisfy the followings:*

*(i) $\overline{u}_k$ is smooth except at $\overline{x}_j$, $1 \le j \le k$. At $\overline{x}_j$, $\overline{u}_k(x)$ has the asymptotic behavior*

$$\lim_{x \to \overline{x}_j} |x - \overline{x}_j|^{\frac{4}{p-1}} \overline{u}_k(x) = K^{\frac{1}{p-1}}. \tag{4.4}$$

*(ii)*

$$\Big( \int_\Omega \overline{u}_k^{p_0} dx \Big)^{\frac{1}{p_0}} < \eta, \quad \textit{and} \quad \Big( \int_\Omega \overline{f}_k^{q_0} dx \Big)^{\frac{1}{q_0}} < \eta. \tag{4.5}$$

*(iii) Set*

$$Q_k(\varphi) \equiv \Big(1 + \sum_{j=1}^{k} 3^{-j}\epsilon_0 - \epsilon_0\Big) \int_\Omega (\Delta\varphi)^2 - p \int_\Omega \overline{u}_k^{p-1} \varphi^2$$

*for $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. Then $Q_k$ is positive definite and equivalent to $H^2(\Omega) \cap H_0^1(\Omega)$ norm in $H^2(\Omega) \cap H_0^1(\Omega)$.*

**Proof.** A quasi-solution of (4.1) will be constructed by induction on $k$. Let $\xi(x) = \xi(|x|)$ be a smooth cut-off function such that $\xi(t) = 1$ for $0 \le t \le \frac{1}{2}$, and $\xi(t) = 0$ for $t \ge 1$. For any $r > 0$, we denote $\xi_r(x) \equiv \xi(\frac{x}{r})$. For $k = 0$, $(0, 0)$ is a trivial quasi-solution satisfying (i)-(iii) of Lemma 4.1.

Now suppose that conclusions of Lemma 4.1 hold true for $\{\overline{x}_1, \ldots, \overline{x}_{k-1}\}$ and a quasi-solution $\{\overline{u}_{k-1}, \overline{f}_{k-1}\}$ satisfying (i)-(iii).

Let

$$0 < r_k < \frac{1}{2} \min\{\mathrm{dist}(\overline{x}_k, \partial\Omega),\ |\overline{x}_k - \overline{x}_j|,\ \ 1 \le j \le k-1\},$$

and define

$$\overline{u}_k = \overline{u}_{k-1} + \xi_{r_k}(x - \overline{x}_k) u_a(x - \overline{x}_k) \equiv \overline{u}_{k-1} + \xi_k u_a \tag{4.6}$$

where $a$ will be chosen later.

If $a$ is small enough, we have

$$\|u_a\|_{L^{p_0}} < \frac{1}{2}(\eta - \|\overline{u}_{k-1}\|_{L^{p_0}}). \tag{4.7}$$

Therefore we have

$$\|\overline{u}_k\|_{L^{p_0}} < \frac{1}{2}(\eta + \|\overline{u}_{k-1}\|_{L^{p_0}}) < \eta. \tag{4.8}$$

Set

$$\begin{aligned}
\overline{f}_k &= \Delta^2 \overline{u}_k - \overline{u}_k^p \\
&= \Delta^2 \overline{u}_{k-1} + \Delta^2(\xi_k u_a) - \overline{u}_k^p \\
&= \overline{f}_{k-1} - (\overline{u}_k^p - \overline{u}_{k-1}^p - \xi_k^p u_a^p) \\
&\qquad + [(\xi_k - \xi_k^p) u_a^p + u_a \Delta^2 \xi_k + 2\Delta u_a \Delta \xi_k + 2\nabla \xi_k \nabla(\Delta u_a) \\
&\qquad\quad + 2\Delta(\nabla \xi_k \nabla u_a) + 2\nabla u_a \nabla(\Delta \xi_k)] \\
&= \overline{f}_{k-1} - g_1 + g_2.
\end{aligned}$$

We note that $\overline{u}_{k-1}$ is smooth in $B(\overline{x}_k, r_k)$. Hence

$$\int_\Omega |g_1|^{q_0} = \int_{B(\overline{x}_k, r_k)} |g_1|^{q_0} \le C \int_{B(\overline{x}_k, r_k)} \Big(1 + u_a^{(p-1)q_0}\Big), \tag{4.9}$$

which can be small if both $r_k$ and $a$ are small, and $\frac{2n}{n+4} \le q_0 < \frac{n}{4}$. Also we have

$$\|g_2\|_{L^\infty} \le C \|u_a\|_{C^3(\overline{D}_k)} \tag{4.10}$$

which is small if $a$ is small, where $D_k = \{x \in \mathbb{R}^n : \frac{r_k}{2} < |x| < r_k\}$. Therefore, if $a$ is small, by (4.9) and (4.10), we have

$$\|\overline{f}_k\|_{L^{q_0}} \le \|\overline{f}_{k-1}\|_{L^{q_0}} + \|g_1\|_{L^{q_0}} + \|g_2\|_{L^{q_0}} < \eta.$$

It is obvious that (4.4) holds for $\overline{u}_k$ at $\overline{x}_k$.

We divide the proof of (iii) into two steps.

*Step 1.* For any $\epsilon > 0$, there exists a constant $C > 0$, such that

$$p \int_\Omega \overline{u}_k^{p-1} \varphi^2 \le \Big(1 + \sum_{j=1}^{k-1} 3^{-j} \epsilon_0 - \epsilon_0\Big)(1+\epsilon) \int_\Omega (\Delta \varphi)^2 + C \int_\Omega [\varphi^2 + |\nabla \varphi|^2] \tag{4.11}$$

holds true for any $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, where $C > 0$ depends on $\epsilon$ but is independent of $a$.

Let $\eta_i \in C^\infty(\overline{\Omega})$, $i = 1, 2$, such that $\eta_1^2 + \eta_2^2 \equiv 1$, the support of $\eta_1$ is disjoint from $B(\overline{x}_k, r_k)$, and the support of $\eta_2$ is disjoint from $\{\overline{x}_1, \ldots, \overline{x}_{k-1}\}$. Then it follows from the induction assumption that

$$
\begin{aligned}
p\int_\Omega \overline{u}_k^{p-1}\varphi^2 &= p\int_\Omega \overline{u}_k^{p-1}\eta_1^2\varphi^2 + p\int_\Omega \overline{u}_k^{p-1}\eta_2^2\varphi^2 \\
&\le \delta_0\int_\Omega (\Delta(\eta_1\varphi))^2 + (1-\epsilon_0)\int_\Omega (\Delta(\eta_2\varphi))^2 + C_1\int_\Omega \varphi^2 \\
&\le \delta_0(1+\epsilon)\int_\Omega (\Delta\varphi)^2 + C_2\int_\Omega [\varphi^2 + |\nabla\varphi|^2]
\end{aligned}
$$

for any $\epsilon > 0$, where $\delta_0 = 1 + \sum_{j=1}^{k-1} 3^{-j}\epsilon_0 - \epsilon_0$. Note that $\overline{u}_{k-1}$ is smooth in the support of $\eta_2$ and

$$
\begin{aligned}
\int [\Delta(\eta\phi)]^2 &= \int [\eta(\Delta\phi) + 2\nabla\eta\nabla\phi + \phi(\Delta\eta)]^2 \\
&\le \int [\eta^2(\Delta\phi)^2 + (\Delta\eta)^2\phi^2] + 4\|\eta(\Delta\phi)\|_{L^2}\|\nabla\eta\nabla\phi\|_{L^2} \\
&\quad +2\|\eta(\Delta\phi)\|_{L^2}\|\phi(\Delta\eta)\|_{L^2} + 4\|\nabla\eta\nabla\phi\|_{L^2}\|\phi(\Delta\eta)\|_{L^2} \\
&\quad +4\int [\nabla\eta\nabla\phi]^2 \\
&\le (1+\epsilon)\int [\eta^2(\Delta\phi)^2] + C\int_\Omega [\phi^2 + |\nabla\phi|^2].
\end{aligned}
$$

*Step 2.* Fix a small $\epsilon_1 > 0$. We can find a finite dimensional subspace $\mathcal{N}$ of $H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$
C\int_\Omega [\varphi_2^2 + |\nabla\varphi_2|^2] \le \epsilon_1 Q_{k-1}(\varphi)
$$

for all $\varphi \in H^2(\Omega)\cap H_0^1(\Omega)$, where $\varphi = \varphi_1 + \varphi_2$ $\varphi_1 \in \mathcal{N}$, $\varphi_2 \in \mathcal{N}^\perp$, which is the orthogonal complement of $\mathcal{N}$ with respect to the quadratic form $Q_{k-1}$ and $C$ is the constant stated in (4.11).

For any $\varphi \in H^2(\Omega)\cap H_0^1(\Omega)$, decompose $\varphi = \varphi_1 + \varphi_2$, $\varphi_1 \in \mathcal{N}$ and $\varphi_2 \in \mathcal{N}^\perp$ (with respect to $Q_{k-1}$). Let $\overline{u}_k = \overline{u}_{k-1} + v_k$ and $B_k := B_k(\overline{x}_k, r_k)$. Then

$$
\begin{aligned}
p\int_\Omega \overline{u}_k^{p-1}\varphi^2 &= p\int_\Omega \overline{u}_k^{p-1}(\varphi_1+\varphi_2)^2 \\
&= p\int_\Omega \overline{u}_k^{p-1}\varphi_1^2 + p\int_\Omega \overline{u}_k^{p-1}\varphi_2^2 + 2p\int_\Omega \overline{u}_k^{p-1}\varphi_1\varphi_2 \\
&\le p\int_\Omega \overline{u}_k^{p-1}\varphi_2^2 + p\int_\Omega \overline{u}_{k-1}^{p-1}\varphi_1^2 + 2p\int_\Omega \overline{u}_{k-1}^{p-1}\varphi_1\varphi_2 \\
&\quad +C_3\Big[\int_\Omega v_k^{p-1}\varphi_1^2 + \int_\Omega v_k^{p-1}|\varphi_1\varphi_2| + \int_{B_k}[|\varphi_1|^2 + |\varphi_1||\varphi_2|]\Big] \qquad (4.12)
\end{aligned}
$$

where we notice that $\overline{u}_k \equiv \overline{u}_{k-1}$ outside $\overline{B_k}$, $\overline{u}_{k-1}$ is smooth in $\overline{B_k}$, and the constant $C_3$ depends on $\sup_{B_k}\overline{u}_{k-1}$.

To estimate the sum of the first three terms, we utilize Step 1 and induction step and obtain

$$
\begin{aligned}
&p\int_\Omega \overline{u}_k^{p-1}\varphi_2^2 + p\int_\Omega \overline{u}_{k-1}^{p-1}\varphi_1^2 + 2p\int_\Omega \overline{u}_{k-1}^{p-1}\varphi_1\varphi_2\\
&\quad = \delta_0\Big(\int_\Omega (\Delta\varphi_1)^2 + (\Delta\varphi_2)^2\Big) + 2p\int_\Omega \overline{u}_{k-1}^{p-1}\varphi_1\varphi_2 + C\int_\Omega [\varphi_2^2 + |\nabla\varphi_2|^2]\\
&\qquad +\delta_0\epsilon\int_\Omega (\Delta\varphi_2)^2.
\end{aligned}
$$

Since $\varphi_1$ is orthogonal to $\varphi_2$, we have

$$
\delta_0\int_\Omega (\Delta\varphi_1)(\Delta\varphi_2) = p\int_\Omega \overline{u}_{k-1}^{p-1}\varphi_1\varphi_2.
$$

Therefore we have, by the choice of $\mathcal{N}$,

$$
\begin{aligned}
&p\int_\Omega \overline{u}_k^{p-1}\varphi_2^2 + p\int_\Omega \overline{u}_{k-1}^{p-1}\varphi_1^2 + 2p\int_\Omega \overline{u}_{k-1}^{p-1}\varphi_1\varphi_2\\
&\quad = \delta_0\int_\Omega (\Delta\varphi)^2 + C\int_\Omega [\varphi_2^2 + |\nabla\varphi_2|^2]\\
&\qquad +\delta_0\epsilon\int_\Omega (\Delta\varphi_2)^2\\
&\quad \le (\delta_0 + 3^{-(k+1)}\epsilon_0)\int_\Omega (\Delta\varphi)^2
\end{aligned}
$$

provided that $\epsilon$ and $\epsilon_1$ are sufficiently small. And, for any $\tilde\epsilon > 0$,

$$
\begin{aligned}
\int_\Omega v_k^{p-1}|\varphi_1\varphi_2| &\le \tilde\epsilon\int_\Omega v_k^{p-1}\varphi_2^2 + C_{\tilde\epsilon}\int_\Omega v_k^{p-1}\varphi_1^2\\
&\le \frac{\tilde\epsilon K}{H_n}\int_\Omega (\Delta\varphi_2)^2 + C_{\tilde\epsilon}\int_\Omega v_k^{p-1}\varphi_1^2 \quad \text{(by Remark 3.5)}\\
&\le 2^{-1}3^{-(k+1)}\epsilon_0\int_\Omega (\Delta\varphi_2)^2 + C_{\tilde\epsilon}\int_\Omega v_k^{p-1}\varphi_1^2
\end{aligned}
$$

provided that $\tilde\epsilon$ is sufficiently small.

For the last two terms of (4.12), for any $\tilde\epsilon > 0$,

$$
\begin{aligned}
\int_{B_k} [|\varphi_1|^2 + |\varphi_1||\varphi_2|] &\le C_{\tilde\epsilon}\int_{B_k}\varphi_1^2 + \tilde\epsilon\int_\Omega \varphi^2\\
&\le C_{\tilde\epsilon}\int_{B_k}\varphi_1^2 + C\epsilon\int_\Omega (\Delta\varphi)^2\\
&\le 2^{-1}3^{-(k+1)}\epsilon_0\int_\Omega (\Delta\varphi)^2 + C_{\tilde\epsilon}\int_{B_k}\varphi_1^2.
\end{aligned}
$$

Therefore, (4.12) becomes

$$
p\int_\Omega \overline{u}_k^{p-1}\varphi^2 \le \Big[\delta_0 + \frac{2}{3}3^{-k}\epsilon_0\Big]\int_\Omega (\Delta\varphi)^2 + \tilde C\Big(\int_\Omega v_k^{p-1}\varphi_1^2 + \int_{B_k}\varphi_1^2\Big)
$$

where $\tilde{C}$ is a positive constant independent of $a$ and $r_k$. Since the dimension of $\mathcal{N}$ is finite, $r_k$ can be chosen so small such that

$$\begin{aligned}\tilde{C}\int_{B_k}\varphi_1^2 &\leq 2^{-1}3^{-(k+1)}\epsilon_0 Q_{k-1}(\varphi_1)\\ &\leq 2^{-1}3^{-(k+1)}\epsilon_0 Q_{k-1}(\varphi)\\ &\leq 2^{-1}3^{-(k+1)}\epsilon_0\int_\Omega(\Delta\varphi)^2.\end{aligned}$$

After fixing $r_k$, we may choose $a$ so small such that the left-hand side of (4.10) is small and, by Proposition 3.3 and Remark 3.5, we have

$$\begin{aligned}\tilde{C}\int_\Omega v_k^{p-1}\varphi_1^2 &\leq 2^{-1}3^{-(k+1)}\epsilon_0 Q_{k-1}(\varphi_1)\\ &\leq 2^{-1}3^{-(k+1)}\epsilon_0 Q_{k-1}(\varphi)\\ &\leq 2^{-1}3^{-(k+1)}\epsilon_0\int_\Omega(\Delta\varphi)^2.\end{aligned}$$

This completes the proof of this lemma. □

Let $(\overline{u},\overline{f})$ be a quasi-solution of (4.1) as stated in Lemma 4.1. Suppose that a solution of (4.1) can be written as $u=\overline{u}+v$, then $v$ satisfies

$$\begin{cases}\Delta^2 v+\overline{u}^p-(\overline{u}+v)^p+\overline{f}=0 & \text{in } \Omega,\\ \overline{u}+v>0 & \text{in } \Omega,\\ v=\Delta v=0 & \text{on } \partial\Omega.\end{cases}$$

Note that $\overline{u}=\Delta\overline{u}=0$ on $\partial\Omega$. This can be seen from the construction of $\overline{u}$ in Lemma 4.1.

Define

$$f_+(s,t)=\begin{cases}(s+t)^p-s^p, & \text{for } s,t\geq 0,\\ 0, & \text{for } t<0,\end{cases}$$

$F_+(s,t)=\int_0^t f_+(s,\tau)d\tau$,

$$E(\varphi)=\frac{1}{2}\int_\Omega(\Delta\varphi)^2-\int_\Omega F_+(\overline{u},\varphi)+\int_\Omega\overline{f}\varphi,\quad \varphi\in X$$

where

$$F_+(s,t)=\begin{cases}\frac{1}{p+1}\{|s+t|^p(s+t)-s^{p+1}-(p+1)s^p t\}, & \text{for } s,t\geq 0,\\ 0, & \text{for } t<0,\end{cases}$$

and $X=H^2(\Omega)\cap H_0^1(\Omega)$.

**Lemma 4.2.** *$E\in C^1(X;\mathbb{R})$ and any critical point $v$ of $E$ satisfies*

$$\begin{cases}\Delta^2 v-|\overline{u}+v|^p+\overline{u}^p+\overline{f}=0 & \text{in } \Omega,\\ \overline{u}+v>0 & \text{in } \Omega,\\ v=\Delta v=0 & \text{on } \partial\Omega.\end{cases}\tag{4.13}$$

*Moreover, $E$ satisfies the (P.S.) condition.*

**Proof.** The first part of this lemma is standard. We leave the details of the proof to the reader. To prove the second part of this lemma, suppose that there is a sequence $\{v_j\} \subset X$ such that

$$E(v_j) \to C, \quad \text{and} \tag{4.14}$$

$$E'(v_j) \to 0 \ \text{ in } X \tag{4.15}$$

as $j \to \infty$. We want to show that there exists a strongly convergent subsequence of $v_j$. The derivative of $E$ can be computed as

$$(E'(v_j), \varphi) = \int_\Omega \Delta v_j \Delta \varphi - \int_\Omega (|v_j^+ + \overline{u}|^p - \overline{u}^p)\varphi + \int_\Omega \overline{f}\varphi. \tag{4.16}$$

*Case 1.* $p \le 2$.

A direct calculation shows that

$$\begin{aligned} &E(v_j) - \frac{1}{p+1}(E'(v_j), v_j) \\ &= \frac{(p-1)}{2(p+1)} \int_\Omega (\Delta v_j)^2 - \frac{1}{p+1}\int_\Omega [(v_j^+ + \overline{u})^p \overline{u} - \overline{u}^{p+1} - p\overline{u}^p v_j^+] + \frac{p}{p+1}\int_\Omega \overline{f} v_j. \end{aligned}$$

Since $p \le 2$, the inequality

$$\Big| |\overline{u} + v_j^+|^p \overline{u} - \overline{u}^{p+1} - p\overline{u}^p v_j^+ \Big| \le \frac{p(p-1)}{2} \overline{u}^{p-1}(v_j^+)^2$$

holds. By (4.14) and (4.15), we have

$$\frac{(p-1)}{2(p+1)} \int_\Omega \Big[(\Delta v_j)^2 - p\overline{u}^{p-1}(v_j^+)^2\Big] \le C_4\Big(1 + \|\Delta v_j\|_{L^2}\Big).$$

By Lemma 4.1, we see that $\|\Delta v_j\|_{L^2}$ is bounded. Furthermore, by (4.16), we have

$$(E'(v_i) - E'(v_j), v_i - v_j) = \int_\Omega (\Delta(v_i - v_j))^2 - \int_\Omega (|v_i^+ + \overline{u}|^p - |v_j^+ + \overline{u}|^p)(v_i - v_j). \tag{4.17}$$

Since

$$\begin{aligned} \Big| |1+x|^p - |1+y|^p \Big| &\le p \max\{|1+|x||^{p-1}, |1+|y||^{p-1}\}|x-y| \\ &\le p|x-y| + (|x|^{p-1} + |y|^{p-1})|x-y| \end{aligned}$$

for all $x, y \in \mathbb{R}$, we see from (4.17) that

$$\int_\Omega (\Delta(v_i - v_j))^2 - p\int_\Omega \overline{u}^{p-1}|v_i - v_j|^2 \le \int_\Omega (|v_i|^{p-1} + |v_j|^{p-1})(v_i - v_j)^2 + o(|v_i - v_j|_X).$$

By Hölder inequality, the first term of the right hand side can be estimated by

$$\int_\Omega (|v_i|^{p-1} + |v_j|^{p-1})(v_i - v_j)^2 \le \Big(\int_\Omega \Big[|v_i|^{\frac{2n}{n-4}} + |v_j|^{\frac{2n}{n-4}}\Big]\Big)^{\frac{(n-4)(p-1)}{2n}} \Big(\int_\Omega |v_i - v_j|^{2q}\Big)^{\frac{1}{q}}$$

where

$$\frac{1}{q} = 1 - \frac{(n-4)(p-1)}{2n} > 1 - \frac{4}{n} = \frac{n-4}{n},$$

namely $2q < \frac{2n}{n-4}$. Hence there exists a subsequence of $v_j$ (still denoted by $v_j$) which is convergent in $L^{2q}(\Omega)$. (Note that the boundedness of $\{\|v_j\|_X\}$ and compactness of the embedding $X \hookrightarrow L^{2q}(\Omega)$ imply that the convergent subsequence exists.) Then, by the above inequality, we conclude that $v_j$ is strongly convergent in $X$.

*Case 2.* $p > 2$.

We see that

$$(E'(v_j), v_j) = \int_\Omega (\Delta v_j)^2 - \int_\Omega (|\overline{u} + v_j^+|^p - \overline{u}^p) v_j^+ + \int_\Omega \overline{f} v_j.$$

Since

$$(1+x)^p x - x - \frac{2}{p+1}\Big((1+x)^{p+1} - 1 - (p+1)x\Big) \geq \frac{p-1}{p+1} x^{p+1} \quad \text{for } x \geq 0,$$

we have

$$\frac{(p-1)}{p+1} \int_\Omega |v_j^+|^{p+1} \leq 2E(v_j) - (E'(v_j), v_j) + \int_\Omega \overline{f} v_j. \tag{4.18}$$

For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$(\overline{u} + v_j^+)^p v_j^+ - \overline{u}^p v_j^+ \leq (p+\epsilon)\overline{u}^{p-1}(v_j^+)^2 + C_\epsilon (v_j^+)^{p+1}.$$

By (4.16),

$$\int_\Omega (\Delta v_j)^2 - (p+\epsilon)\int_\Omega \overline{u}^{p-1}(v_j^+)^2 \leq (E'(v_j), v_j) - \int_\Omega \overline{f} v_j + C_\epsilon \int_\Omega (v_j^+)^{p+1}.$$

Together with Lemma 4.1, (4.14), (4.15) and (4.18), we see, for small $\epsilon > 0$,

$$\|\Delta v_j\|_{L^2}^2 \leq C(\|\Delta v_j\|_{L^2} + 1).$$

After establishing boundedness of $\|\Delta v_j\|_{L^2}$, we can use (4.17) and arguments similar to those in Case 1 to obtain a strongly convergent subsequence in $X$. Therefore, the (P.S.)-condition is satisfied and the proof of Lemma 4.2 is complete. □

**Lemma 4.3.** *Let* $n \geq 5$*;* $n/(n-4) < p < p^c$ *and* $\{x_1, \ldots, x_k\}$ *be any set of finite points in* $\Omega$*. Then there exist at least two distinct solutions of* (1.5) *having* $\{x_1, \ldots, x_k\}$ *as their singular set.*

**Proof.** We claim that there exist positive numbers $\eta_0, \rho, \theta > 0$ ($\eta_0$ and $\theta$ depend on $\rho$) such that if $(\overline{u}, \overline{f})$ is a quasi-solution of (4.1) as stated in Lemma 4.1 with $0 < \eta < \eta_0$, then

$$E(u) \geq \theta > 0$$

for $u \in X$ such that $\rho \le \|u\|_X \le 2\rho$. After this claim, the existence's part of Lemma 4.3 follows immediately. Because one solution can be obtained from the minimizing $\min_{\|u\|_X \le \rho} E(u) \le E(0) = 0 < \theta$, and the other solution can be obtained from the Mountain Pass Lemma.

For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|F(s,t)| \le \frac{p}{2}(1+\epsilon)s^{p-1}t^2 + C_\epsilon t^{p+1}.$$

Thus,

$$2E(v) \ge \int_\Omega (\Delta v)^2 - p(1+\epsilon)\int_\Omega \overline{u}^{p-1}v^2 - 2C_\epsilon \int_\Omega v^{p+1} - 2\eta\Big(\int_\Omega v^{\frac{2n}{n-4}}\Big)^{\frac{n-4}{2n}}.$$

By Sobolev's embedding and Lemma 4.1, for small $\epsilon > 0$,

$$2E(v) \ge \tilde{C}_1 \int_\Omega (\Delta v)^2 - \tilde{C}_2\Big[\Big(\int_\Omega (\Delta v)^2\Big)^{\frac{p+1}{2}} + \eta\Big(\int_\Omega (\Delta v)^2\Big)^{1/2}\Big].$$

Then the claim follows easily (note that $\eta$ is small).

Suppose that $u = \overline{u} + v$ is any solution of (4.1) with $v \in X$. Since $p < (n+4)/(n-4)$, then a bootstrap argument can show that $v \in C^\infty(\Omega \backslash \{x_1, x_2, \ldots, x_k\})$. If we assume that $x_j$ is a removable singular point of $u$, by (i) of Lemma 4.1, $-\overline{u} \le v(x) \le C - \overline{u}$ in a neighborhood of $x_j$ which implies that $v \notin L^{p+1}(\Omega)$ (note that $4(p+1)/(p-1) > n$), a contradiction to $v \in X$ (since $p+1 < 2n/(n-4)$). This completes the proof of Lemma 4.3. $\square$

To complete the proof of Theorem 1.3, we need another lemma.

**Lemma 4.4.** *Let $v$ be a solution of (4.13) and $v \in X$. Then $|v|^\alpha \in X$ for some $\alpha > 1$. The constant $\alpha$ depends only on $p, q_0$ and the dimension $n$.*

**Proof.** By Lemma 4.3, we know that $v \in C^\infty$ except at $x_i$, $1 \le i \le k$. We claim that there exists a constant $\delta > 0$ depending on $p, q_0$ and $n$ such that $|x - x_i|^{-\delta} v \in L^{\frac{2n}{n-4}}(\Omega)$, $1 \le i \le k$.

Let $\eta(x) \equiv (|x-x_i|^2 + \sigma^2)^{-\frac{\delta}{2}}$, $1 \le i \le k$ for some sufficiently small $\sigma > 0$. Multiplying (4.13) by $\eta^2 v$, we have

$$\begin{aligned}\int_\Omega \Delta v \Delta(\eta^2 v) &= \int_\Omega [(\overline{u}+v)^p - \overline{u}^p]\eta^2 v - \int_\Omega \overline{f}\eta^2 v \\ &\le (p+\epsilon)\int_\Omega \overline{u}^{p-1}\eta^2 v^2 + C_\epsilon \int_\Omega \eta^2 |v|^{p+1} - \int_\Omega \overline{f}\eta^2 v.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_\Omega \Delta v \Delta(\eta^2 v) &= \int_\Omega (\Delta(\eta v))^2 - \int_\Omega v^2(\Delta\eta)^2 \\ &\quad +2\int_\Omega |\nabla\eta|^2 v\Delta v - 4\int_\Omega (\nabla\eta\cdot\nabla v)^2 - 4\int_\Omega v\Delta\eta(\nabla\eta\cdot\nabla v)\end{aligned}$$

and

$$\begin{aligned}4\int_\Omega[(\nabla\eta\cdot\nabla v)^2 + v\Delta\eta\nabla\eta\cdot\nabla v] &= 4\int_\Omega(\nabla\eta\cdot\nabla v)[\nabla\eta\cdot\nabla v + v\Delta\eta] \\ &= 4\int_\Omega(\nabla\eta\cdot\nabla v)\mathrm{div}(v\nabla\eta) \\ &= -4\Big[\int_\Omega v(\nabla\eta\cdot\nabla v)\Delta\eta + \int_\Omega v|\nabla\eta|^2\Delta v\Big].\end{aligned}$$

The latter identities imply

$$\int_\Omega(\nabla\eta\cdot\nabla v)^2 + 2\int_\Omega v\Delta\eta(\nabla\eta\cdot\nabla v) + \int_\Omega v|\nabla\eta|^2\Delta v = 0.$$

Therefore,

$$\begin{aligned}\int_\Omega \Delta v\Delta(\eta^2 v) &= \int_\Omega(\Delta(\eta v))^2 - \int_\Omega v^2(\Delta\eta)^2 \\ &\quad -6\int_\Omega(\nabla\eta\cdot\nabla v)^2 - 8\int_\Omega v\Delta\eta(\nabla\eta\cdot\nabla v)\end{aligned}$$

Since $|\nabla\eta|^2 \le \delta^2\eta^2|x-x_i|^{-2}$, we have

$$\begin{aligned}\int_\Omega(\nabla\eta\cdot\nabla v)^2 &\le \delta^2\int_\Omega \eta^2|x-x_i|^{-2}|\nabla v|^2 \\ &= \delta^2\int_\Omega[|\nabla(\eta v)|^2 - |\nabla\eta|^2v^2 - 2v\eta(\nabla\eta\cdot\nabla v)]|x-x_i|^{-2} \\ &\le \delta^2\int_\Omega[|\nabla(\eta v)|^2 + |\nabla\eta|^2 v^2]|x-x_i|^{-2} \\ &\quad +\delta^2\int_\Omega v^2\eta^2|x-x_i|^{-4} + \delta^2\int_\Omega(\nabla\eta\cdot\nabla v)^2.\end{aligned}$$

Thus,

$$\begin{aligned}(1-\delta^2)\int_\Omega(\nabla\eta\cdot\nabla v)^2 &\le \delta^2\int_\Omega[\nabla(\eta v)|^2 + |\nabla\eta|^2v^2]|x-x_i|^{-2} \\ &\quad +\delta^2\int_\Omega v^2\eta^2|x-x_i|^{-4}\end{aligned}$$

and we obtain

$$
\begin{aligned}
&\int_\Omega [(\Delta(\eta v))^2 - (\Delta\eta)^2 v^2] \\
&\quad\le \int_\Omega |\overline{f}v|\eta^2 + (p+\epsilon)\int_\Omega \overline{u}^{p-1}(\eta v)^2 + C_\epsilon \int_\Omega \eta^2 |v|^{p+1} \\
&\qquad +6\int_\Omega (\nabla\eta\cdot\nabla v)^2 + 8\int_\Omega |v\Delta\eta||\nabla\eta\cdot\nabla v| \\
&\quad\le \int_\Omega |\overline{f}v|\eta^2 + (p+\epsilon)\int_\Omega \overline{u}^{p-1}(\eta v)^2 + C_\epsilon \int_\Omega \eta^2 |v|^{p+1} \\
&\qquad +10\int_\Omega (\nabla\eta\cdot\nabla v)^2 + 4\int_\Omega (\Delta\eta)^2 v^2 \\
&\quad\le \int_\Omega |\overline{f}v|\eta^2 + (p+\epsilon)\int_\Omega \overline{u}^{p-1}(\eta v)^2 + C_\epsilon \int_\Omega \eta^2 |v|^{p+1} \\
&\qquad +\frac{10\delta^2}{1-\delta^2}\int_\Omega (|\nabla(\eta v)|^2 + |\nabla\eta|^2 v^2)|x-x_i|^{-2} \\
&\qquad +\frac{10\delta^2}{1-\delta^2}\int_\Omega (\eta v)^2 |x-x_i|^{-4} + 4\int_\Omega (\Delta\eta)^2 v^2.
\end{aligned}
$$

Let $w = \eta v$ in $\Omega$, $w \equiv 0$ in $\mathbb{R}^n \backslash \Omega$. Then $w \in H^2(\mathbb{R}^n) \cap H_0^1(\mathbb{R}^n)$. Since

$$
\frac{|\nabla w|^2}{|x|^2} + \frac{w\Delta w}{|x|^2} = \operatorname{div}\Big(\frac{w\Delta w}{|x|^2}\Big) + 2\frac{wx\cdot\nabla w}{|x|^4},
$$

we see that

$$
\int_{\mathbb{R}^n} \frac{|\nabla w|^2}{|x|^2} = \int_{\mathbb{R}^n} \frac{2wx\cdot\nabla w}{|x|^4} - \int_{\mathbb{R}^n} \frac{w\Delta w}{|x|^2}.
$$

Thus, using the Young's inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{|\nabla w|^2}{|x|^2} &\le \frac{1}{2}\int_{\mathbb{R}^n} (\Delta w)^2 + \frac{1}{2}\int_{\mathbb{R}^n} \frac{w^2}{|x|^4} \\
&\qquad + \frac{1}{4}\int_{\mathbb{R}^n} \frac{|\nabla w|^2}{|x|^2} + 4\int_{\mathbb{R}^n} \frac{w^2}{|x|^4}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{3}{4}\int_{\mathbb{R}^n} \frac{|\nabla w|^2}{|x|^2} &\le \frac{1}{2}\int_{\mathbb{R}^n} (\Delta w)^2 + \frac{9}{2}\int_{\mathbb{R}^n} \frac{w^2}{|x|^4} \\
&\le \frac{1}{2}\int_{\mathbb{R}^n} (\Delta w)^2 + \frac{9}{2}\frac{16}{(n(n-4))^2}\int_{\mathbb{R}^n} (\Delta w)^2 \quad \text{(by Hardy inequality)} \\
&= \Big[\frac{1}{2} + \frac{9}{2}\frac{16}{(n(n-4))^2}\Big]\int_{\mathbb{R}^n} (\Delta w)^2 := C(n)\int_{\mathbb{R}^n} (\Delta w)^2.
\end{aligned}
$$

This implies that

$$
\int_\Omega \frac{|\nabla(\eta v)|^2}{|x-x_i|^2} \le C(n)\int_\Omega (\Delta(\eta v))^2.
$$

Therefore,

$$\begin{aligned}
&\int_\Omega [(\Delta(\eta v))^2 - (\Delta\eta)^2 v^2 - (p+\epsilon)\int_\Omega \overline{u}^{p-1}(\eta v)^2]\\
&\quad \le \int_\Omega |\overline{f} v|\eta^2 + C_\epsilon \int_\Omega \eta^2 |v|^{p+1}\\
&\qquad + \frac{10C(n)\delta^2}{1-\delta^2}\int_\Omega (\Delta(\eta v))^2 + \frac{10\delta^2}{1-\delta^2}\int_\Omega |x-x_i|^{-2}|\nabla\eta|^2 v^2\\
&\qquad + \frac{10\delta^2}{1-\delta^2}\int_\Omega |x-x_i|^{-4}(\eta v)^2 + 4\int_\Omega (\Delta\eta)^2 v^2\\
&\quad \le \Big(\int_\Omega (\overline{f}\eta^2)^{\frac{2n}{n+4}}\Big)^{\frac{n+4}{2n}} \Big(|v|^{\frac{2n}{n-4}}\Big)^{\frac{n-4}{2n}} + C_\epsilon \Big(\eta^{2q}\Big)^{\frac{1}{q}} \Big(\int_\Omega |v|^{\frac{2n}{n-4}}\Big)^{\frac{(n-4)(p+1)}{2n}}\\
&\qquad + \frac{10C(n)\delta^2}{1-\delta^2}\int_\Omega (\Delta(\eta v))^2 + \frac{10\delta^2}{1-\delta^2}\int_\Omega |x-x_i|^{-2}|\nabla\eta|^2 v^2\\
&\qquad + \frac{10\delta^2}{1-\delta^2}\int_\Omega |x-x_i|^{-4}(\eta v)^2 + 4\int_\Omega (\Delta\eta)^2 v^2
\end{aligned}$$

where $\frac{1}{q} = 1 - \frac{(p+1)(n-4)}{2n}$. On the other hand, we easily see that

$$(\Delta\eta)^2 \le C\delta^2\eta^2|x-x_i|^{-4}, \quad |\nabla\eta|^2 \le \delta^2\eta^2|x-x_i|^{-2}.$$

Since $\overline{f} \in L^{q_0}(\Omega)$ with $q_0 > \frac{2n}{n+4}$, we can choose $\delta > 0$ and $\epsilon > 0$ sufficiently small, by Lemma 4.1, to obtain

$$\int_\Omega (\Delta(\eta v))^2 \le C.$$

Sending $\sigma$ to 0, our claim is proved.

Since $|v(x)| \le c|x-x_i|^{-\tau}$ in $B_{r_i}(x_i)$ for $1 \le i \le k$ and some $\tau \ge \frac{4}{p-1}$, the claim above implies that $v \in L^{\frac{2n}{n-4}\alpha_0}(\Omega)$ for some $\alpha_0 > 1$ which depends only on $p, q_0$ and $n$. To estimate $\||v|^{\alpha_0}\|_X$, we multiply (4.13) by $|v|^{2\alpha_0-2}v$, then

$$\begin{aligned}
\int_\Omega \Delta v \Delta(|v|^{2\alpha_0-2}v) &= \int_\Omega (|\overline{u}+v|^p - \overline{u}^p) - \overline{f})|v|^{2\alpha_0-2}v\\
&\le (p+\epsilon)\int_\Omega \overline{u}^{p-1}|v|^{2\alpha_0} + \int_\Omega |\overline{f}||v|^{2\alpha_0-1} + C_\epsilon\int_\Omega |v|^{p+2\alpha_0-1}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{2\alpha_0-1}{\alpha_0^2}\int_\Omega (\Delta|v|^{\alpha_0})^2 - (p+\epsilon)\int_\Omega \overline{u}^{p-1}|v|^{2\alpha_0}\\
&\quad \le (\alpha_0-1)^2(2\alpha_0-1)\int_\Omega |v|^{2\alpha_0-4}|\nabla|v||^4 + \int_\Omega |\overline{f}||v|^{2\alpha_0-1} + C_\epsilon\int_\Omega |v|^{p+2\alpha_0-1}\\
&\quad = (\alpha_0-1)^2(2\alpha_0-1)\int_\Omega |v|^{2\alpha_0}|\nabla(\ln|v|)|^4 + \int_\Omega |\overline{f}||v|^{2\alpha_0-1} + C_\epsilon\int_\Omega |v|^{p+2\alpha_0-1}.
\end{aligned}$$

Since $|v(x)| \le c|x-x_i|^{-\tau}$ in $B_{r_i}(x_i)$ for $\tau \ge 4/(p-1)$, we see that

$$\ln|v(x)| \le \ln c - \tau \ln|x-x_i| \ \text{ for } x \in B_{r_i}(x_i).$$

This implies that

$$|\nabla(\ln|v(x)|)|^4 \le \tau^4 |x-x_i|^{-4} \ \text{ for } x \in B_{r_i}(x_i) \tag{4.19}$$

provided $r_i$ sufficiently small. On the other hand, since $\overline{u}+v>0$ in $\Omega$, we can see from the construction of $\overline{u}$ in Lemma 4.1 that $v(x)>0$ for $x \in \Omega \backslash \cup_{i=1}^k B_{r_i}(x_i)$. (Note that we can construct $\overline{u}$ such that $\overline{u}(x) \equiv 0$ for $x \in \Omega \backslash \cup_{i=1}^k B_{r_i}(x_i)$.) Therefore,

$$\begin{aligned}
\frac{2\alpha_0 - 1}{\alpha_0^2} \int_\Omega (\Delta|v|^{\alpha_0})^2 - (p+\epsilon)\int_\Omega \overline{u}^{p-1}|v|^{2\alpha_0} \\
\le C_\epsilon \int_\Omega |v|^{p+2\alpha_0-1} + \int_\Omega |\overline{f}||v|^{2\alpha_0-1} \\
+(\alpha_0-1)^2(2\alpha_0-1)\int_\Omega |v|^{2\alpha_0}|\nabla(\ln|v|)|^4
\end{aligned}$$

This, the Hardy inequality, (4.19) and Lemma 4.1 imply that if both $\alpha_0 - 1$ and $\epsilon$ are small, then

$$\int_\Omega (\Delta|v|^{\alpha_0})^2 \le C$$

where $C$ is a constant depending on $\|\Delta v\|_{L^2(\Omega)}$. The proof of Lemma 4.4 is complete. □

**Proof of Theorem 1.3**

First we give a proof of the existence of weak solutions with prescribed singular set. If the singular set $S$ is a set of finite points, then the existence of two distinct solutions is proved in Lemma 4.3. Let $\{x_1, x_2, \ldots, x_k, \ldots\}$ be a countable dense subset of $S$, $p_k$ is an increasing sequence, $\lim_{k\to\infty} p_k = p^*$. For any $\eta > 0$, by Lemma 4.1, we can construct a sequence of quasi-solutions $(\overline{u}_k, \overline{f}_k)$ with singular set $\{x_1, x_2, \ldots, x_k\}$ such that

$$\int_\Omega |\overline{u}_{k+1} - \overline{u}_k|^{p_k} \le \frac{\eta}{2^k}$$

and

$$\int_\Omega |\overline{f}_{k+1} - \overline{f}_k|^{q_0} \le \frac{\eta}{2^k}$$

where $q_0$ is a fixed constant such that $\frac{2n}{n+4} < q_0 < \frac{n}{4}$. Hence $\overline{u}_k$ converges strongly to $\overline{u}$ in $L^q(\Omega)$ for any $q < p^*$. When $\eta$ is small, by Lemma 4.3, we can find two sequence of solutions $u_k^i$ of (1.5) such that $u_k^i = \overline{u}_k + v_k^i,\ i = 1, 2$ such that

$$\int_\Omega (\Delta v_k^1)^2 \le \rho_0 \le \int_\Omega (\Delta v_k^2)^2 \le \rho_1$$

where $\rho_1 > \rho_0$ are two constants independent of $k$. Let $v_k$ be one of the two solutions obtained in Lemma 4.3. Then by Lemma 4.4, we have

$$\|\Delta |v_k|^\alpha\|_{L^2(\Omega)} \le C_1$$

for some $\alpha > 1$ where $\alpha$ and $C_1$ are independent of $k$. By Sobolev's embedding and the Hölder inequality, we may further assume that $v_k$ converges in $L^{\frac{2n}{n-4}\alpha}(\Omega)$ and weakly converges to $v$ in $X$. We want to prove that $v_k$ strongly converges to $v$ in $X$. By elliptic estimates, it suffices to show that $|\overline{u}_k + v_k|^p - \overline{u}_k^p$ converges strongly in $L^{\frac{2n}{n+4}}(\Omega)$. To prove this statement, we need the following two steps:

*Step 1.* $\overline{u}_k^{p-1} v_k$ strongly converges to $\overline{u}^{p-1} v$ in $L^{\frac{2n}{n+4}}(\Omega)$.

$$\begin{aligned}
&\int_\Omega |\overline{u}_k^{p-1} v_k - \overline{u}^{p-1} v|^{\frac{2n}{n+4}} \\
&\quad \le C\Big[\int_\Omega |\overline{u}_k^{p-1} - \overline{u}^{p-1}|^{\frac{2n}{n+4}} |v_k|^{\frac{2n}{n+4}} + \int_\Omega \overline{u}^{(p-1)(\frac{2n}{n+4})} |v_k - v|^{\frac{2n}{n+4}}\Big] \\
&\quad \le C\Big[\Big(\int_\Omega |\overline{u}_k^{p-1} - \overline{u}^{p-1}|^{\frac{2nq'}{n+4}}\Big)^{\frac{1}{q'}} \Big(\int_\Omega |v_k|^{\frac{2n\alpha}{n-4}}\Big)^{\frac{n-4}{\alpha(n+4)}} \\
&\qquad + \Big(\int_\Omega |\overline{u}^{(p-1)\frac{2nq'}{n+4}}\Big)^{\frac{1}{q'}} \Big(\int_\Omega |v_k - v|^{\frac{2n\alpha}{n-4}}\Big)^{\frac{n-4}{\alpha(n+4)}}\Big],
\end{aligned}$$

where

$$\frac{1}{q'} = 1 - \frac{n-4}{(n+4)\alpha} > \frac{8}{n+4}.$$

Since $\frac{2nq'}{n+4} < \frac{n}{4}$ and $\overline{u}_k$ converges to $\overline{u}$ in $L^q(\Omega)$ for $q < p^* = \frac{n(p-1)}{4}$, we have

$$\lim_{k\to\infty} \Big[\int_\Omega |u_k^{p-1} - \overline{u}^{p-1}|^{\frac{2nq'}{n+4}} + \int_\Omega |v_k - v|^{\frac{2n\alpha}{n-4}}\Big] = 0.$$

Thus, Step 1 is proved.

*Step 2.* Since

$$\Big||\overline{u}_k + v_k|^p - \overline{u}_k^p - p\overline{u}_k^{p-1} v_k\Big|^{\frac{p+1}{p}} \le c(\overline{u}_k^{p-1} v_k^2 + |v_k|^{p+1}),$$

by Step 1 and Lebesgue's dominated theorem, we have

$$\lim_{k\to\infty} \int_\Omega \Big||\overline{u}_k + v_k|^p - \overline{u}_k^p - p\overline{u}_k^{p-1} v_k - |\overline{u} + v|^p + \overline{u}^p + p\overline{u}^{p-1} v\Big|^{\frac{p+1}{p}} = 0.$$

Since $\frac{p+1}{p} = 1 + \frac{1}{p} > 1 + \frac{n-4}{n+4} = \frac{2n}{n+4}$ and $|\overline{u}_k + v_k|^p - \overline{u}_k^p$ can be written as the sum of $|\overline{u}_k + v_k|^p - \overline{u}_k^p - p\overline{u}_k^{p-1} v_k$ and $p\overline{u}_k^{p-1} v_k$, we conclude that $|\overline{u}_k + v_k|^p - \overline{u}_k^p$ strongly converges in $L^{\frac{2n}{n+4}}(\Omega)$. By elliptic estimates, $v_k$ converges to $v$ strongly in $X$. Since $v_k^i,\ i=1,2$ strongly converges to $v^i,\ i=1,2$ in $X$, we have $v^1 \neq v^2$.

Set $u = \overline{u}+v$ where $v$ is one of $v^i$ obtained above. Then $u$ is a solution of (1.5). It is obvious that $u \in C^\infty(\Omega \backslash S)$. For any $x_0 \in S$, any open neighborhood of $x_0$ contains $x_k$ for some $k$. If $u$ is bounded in this neighborhood, $|v| \geq \overline{u} - c \geq c_0 |x-x_k|^{-\frac{4}{p-1}} - c$ by (i) of Lemma 4.1. It is a contradiction to $v \in L^{p+1}(\Omega)$. Therefore, the singular set of $u$ is exactly equal to $S$.

Let $\eta = \frac{1}{k}$ where $k$ is any large positive integer. By the above, we can construct two sequence of solutions $u_k^i,\ i=1,2$ of (1.5) such that $u_k^i = \overline{u}_k^i + v_k^i$ with $v_k^i \in X$ which satisfy

$$\int_\Omega (\overline{u}_k^i)^{p_k} \leq \frac{1}{k} \quad \text{for } i=1,2 \text{ and}$$

$$\int_\Omega (\Delta v_k^1)^2 \leq \rho_k \leq \rho_0 \leq \int_\Omega (\Delta v_k^2)^2 \leq \rho_1$$

where $p_k$ is an increasing sequence converging to $p^*$, $\lim_{k\to\infty} \rho_k = 0$, both $\rho_0$ and $\rho_1$ are two constants independent of $k$. Thus, $u_k^1$ converges to zero in $L^q(\Omega)$ for any $q < p^*$. As in the proofs of step 1 and step 2 above, $v_k^2$, after passing to a subsequence, converges to $v$ in $X$ and $\int_\Omega (\Delta v)^2 \geq \rho_0$. Therefore, $u_k^2$ converges to $v$ in $L^q(\Omega)$ for any $q < p^*$. This implies that $v$ is a weak solution of (1.5). Since $v \in X$, by a bootstrap argument, we can show that $v \in C^\infty(\Omega)$. The proof of Theorem 1.3 is complete. □

**Proof of Theorem 1.5**

Let $m > 4$ be a positive integer such that the following hold:

(4.20) $$pK(p,m) < H_m \quad \Big( := \Big(\frac{m(m-4)}{4}\Big)^2\Big)$$

and

(4.21) $$m > \frac{n+4}{2},$$

where

$$K(p,m) = \frac{8}{(p-1)^4}\Big[(m-2)(m-4)(p-1)^3 + 2(m^2-10m+20)(p-1)^2 - 16(m-4)(p-1) + 32\Big]$$

and $p = (n+4)/(n-4)$ here and in the following. We easily see that (4.20) holds if

$$\frac{m}{m-4} < p := \frac{n+4}{n-4} < \tilde{p}^c(m) := \frac{m+2+\sqrt{m^2+4-4\sqrt{m^2+H_m}}}{m-6+\sqrt{m^2+4-4\sqrt{m^2+H_m}}}.$$

It is not difficult to check that for $n \geq 13$ if $m$ is chosen as $m = \frac{n+5}{2}$ when $n$ is odd, and $m = \frac{n+6}{2}$ when $n$ is even, then $m$ satisfies (4.20) and (4.21).

Let $S^{n-m}$ be an $(n-m)$-dimensional sphere in $\mathbb{R}^n$, and $q_0 \in S^{n-m}$. By using solutions of (1.1) obtained in Theorem 1.2 and the Kelvin transformation, we can construct a family of solutions $u_\beta(x)$, $\beta \in (0, \infty]$, of

$$\Delta^2 u_\beta = u_\beta^p \quad \text{in } \mathbb{R}^n, \tag{4.22}$$

where $p = \frac{n+4}{n-4}$ in the remainder of this section. The family of solutions $u_\beta$ satisfies (4.23)-(4.26) below.

$$\lim_{x \to S^{n-m}} u_\beta(x) d(x)^{\frac{n-4}{2}} = C(p, m) \tag{4.23}$$

uniformly in any compact set of $S^{n-m} \backslash \{q_0\}$, where $d(x)$ denotes the distance between $x$ and $S^{n-m}$ and $C(p, m) = [K(p, m)]^{\frac{1}{(p-1)}}$.

$$u_\beta(x) \text{ is strictly increasing in } \beta, \text{ and } \lim_{\beta \to 0} u_\beta(x) = 0 \tag{4.24}$$

uniformly in any compact set of $\mathbb{R}^n \cup \{\infty\} \backslash S^{n-m}$. Moreover, there exists a constant $c$ independent of $\beta$ such that $u_\beta(x) \leq c|x|^{4-n}$ for $|x|$ large. Therefore, we have

$$\lim_{\beta \to 0} \int_{\mathbb{R}^n} u_\beta^p dx = 0. \tag{4.25}$$

We denote by $D^{2,2}(\mathbb{R}^n)$ the closure of $C_c^\infty(\mathbb{R}^n)$ functions with respect to the norm $\|u\| = (\int_{\mathbb{R}^n} |\Delta u|^2 dx)^{\frac{1}{2}}$. By the Sobolev embedding $D^{2,2}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for $n \geq 5$, it is clear that

$$D^{2,2}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n);\ \Delta u \in L^2(\mathbb{R}^n)\}.$$

Now for $\varphi \in D^{2,2}(\mathbb{R}^n)$, we have by (iii) of Lemma 4.1,

$$p \int_{\mathbb{R}^n} u_\beta^{p-1}(x) \varphi^2(x) dx \leq (1 - \epsilon_0) \int_{\mathbb{R}^n} (\Delta \varphi)^2 dx \tag{4.26}$$

for some positive constant $\epsilon_0$ depending only on $n$ and $m$.

To see this, let $\hat{u}_\beta(x', x'') = w_\beta(x') := w_\beta(|x'|)$ where $w_\beta(r)$ $(r = |x'|)$ is the radial entire solution of

$$\begin{cases} \Delta_m^2 w_\beta = w_\beta^{\frac{n+4}{n-4}} \quad \text{in } \mathbb{R}^m, \\ w_\beta(r) > 0 \quad \text{and} \quad \lim_{r \to \infty} r^{m-4} w_\beta(r) = \beta \end{cases}$$

and $x' \in \mathbb{R}^m$, $x'' \in \mathbb{R}^{n-m}$. Since $m > \frac{n+4}{2}$, we have $\frac{n+4}{n-4} > \frac{m}{m-4}$. Hence, $w_\beta(x')$ is a weak solution of $\Delta^2 w_\beta = w_\beta^{\frac{n+4}{n-4}}$ in $\mathbb{R}^m$ and $\hat{u}_\beta(x', x'')$ is a weak solution of

$\Delta^2\hat{u}_\beta = \hat{u}_\beta^{\frac{n+4}{n-4}}$ in $\mathbb{R}^n$. Let $e = (1, 0, \ldots, 0)$ and $u_\beta(x) = \frac{1}{|x-x_0|^{n-4}}\hat{u}_\beta\Big(\frac{x-x_0}{|x-x_0|^2} + e\Big)$ for $x \in \mathbb{R}^n$ with $x_0 \notin \mathbb{R}^{n-m}$. Then $u_\beta(x)$ is a weak solution of

$$\begin{cases} \Delta^2 u_\beta(x) = u_\beta^{\frac{n+4}{n-4}}(x) \ \text{ in } \mathbb{R}^n, \\ u_\beta(x) = O\Big(\frac{1}{|x|^{n-4}}\Big) \ \text{ at } \infty. \end{cases}$$

Let $\hat{S}^{n-m}$ be the pre-image of $\mathbb{R}^{n-m}$ under the mapping $x \to \frac{x-x_0}{|x-x_0|^2} + e$. It is easy to see that $\hat{S}^{n-m}$ is an $(n-m)$-dimensional sphere in $\mathbb{R}^n$ and $u_\beta$ is a family of weak solutions of $\Delta^2 u = u^{\frac{n+4}{n-4}}$ in $\mathbb{R}^n$ satisfying (4.23)-(4.26). For (4.26), it is easy to check that both sides of (4.26) are invariant under the Kelvin transformation. Since $\hat{S}^{n-m}$ is congruent to any $(n-m)$-sphere, the above claim follows immediately.

*Step 1.* Let $S_1, \ldots, S_k, \ldots$ be a sequence of disjoint $(n-m)$-dimensional spheres. Fix a small positive number $\eta > 0$ which will be chosen later. A sequence of positive approximate solutions $\overline{u}_k$ is constructed to satisfy (4.27)-(4.30).

Fix $q_k \in S_k$, $k = 1, 2, \ldots$. We have

$$\lim_{\substack{x \to S_j \\ x \notin S_j}} \overline{u}_k(x) d_j^{\frac{4}{p-1}}(x) = [K(p,m)]^{\frac{1}{p-1}}, \tag{4.27}$$

uniformly in any compact set of $S_j \backslash \{q_j\}$, $j = 1, 2, \ldots, k$, where $d_j(x)$ denotes the distance between $x$ and $S_j$.

Denote $\overline{f}_k$ by $\overline{f}_k = \Delta^2 \overline{u}_k - \overline{u}_k^p$. We have

$$\Big(\int_{\mathbb{R}^n} \overline{u}_k^p dx\Big)^{\frac{1}{p}} < \eta, \quad \Big(\int_{\mathbb{R}^n} |\overline{f}_k|^{\frac{2n}{n+4}} dx\Big)^{\frac{n+4}{2n}} < \eta. \tag{4.28}$$

$$\overline{u}_k(x) \ \text{ converges to } \overline{u} \text{ in } L^p(\mathbb{R}^n) \text{ and support of } \overline{f}_k \subset \cup_{j=1}^k B(S_j, r_j), \tag{4.29}$$

where $B(S_j, r_j) = \{x \in \mathbb{R}^n : \ d_j(x) \le r_j\}$ and $\lim_{j\to\infty} r_j = 0$. The quadratic form:

$$Q(\varphi) = \Big(1 + \Sigma_{j=1}^k 3^{-j}\epsilon_0 - \epsilon_0\Big) \int_{\mathbb{R}^n} (\Delta\varphi)^2 - p\int_{\mathbb{R}^n} \overline{u}_k^{p-1}\varphi^2 \tag{4.30}$$

is positive definite and its square root is equivalent to the $D^{2,2}-$norm in $D^{2,2}(\mathbb{R}^n)$.

The construction of $\overline{u}_k$ is exactly the same as before except the cut off function $\eta_k(x - \overline{x}_k)$ is replaced by $\eta_k(d_k(x))$. To prove (4.30), it suffices to note that (4.11) becomes

$$p\int_{\mathbb{R}^n} \overline{u}_k^{p-1}\varphi^2 \le [1 + (\Sigma_{j=1}^{k-1} 3^{-j}\epsilon_0 - \epsilon_0)](1+\epsilon)\int_{\mathbb{R}^n} (\Delta\varphi)^2 + C\int_K (\varphi^2 + |\nabla\varphi|^2), \tag{4.31}$$

where $C$ is a positive constant depending on $\epsilon$ but is independent of $\beta$, $K$ is a bounded set independent of $\beta$. Then the rest of the proof of Lemma 4.1 can go through to prove (4.30) without any modification.

Let

$$E(\varphi)=\frac{1}{2}\int_{\mathbb{R}^n}(\Delta\varphi)^2-\int_{\mathbb{R}^n}F(\overline{u}_k,\varphi)+\int_{\mathbb{R}^n}\overline{f}_k\varphi$$

for $\varphi\in D^{2,2}(\mathbb{R}^n)$ where

$$F(s,t)=\frac{1}{p+1}\Big[|s+t|^p(s+t)-s^{p+1}-(p+1)s^pt\Big].$$

It is not difficult to see that $E(\varphi)$ is continuous in the strong topology of $D^{2,2}(\mathbb{R}^n)$. For any $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$|F(s,t)|\le\frac{p}{2}(1+\epsilon)s^{p-1}t^2+C_\epsilon|t|^{p+1}.$$

Thus

$$\begin{aligned}2E(\varphi)\ &\ge\ \int_{\mathbb{R}^n}(\Delta\varphi)^2-p(1+\epsilon)\int_{\mathbb{R}^n}\overline{u}_k^{p-1}\varphi^2-2C_\epsilon\int_{\mathbb{R}^n}|\varphi|^{p+1}\\ &\quad-2\Big(\int_{\mathbb{R}^n}|\overline{f}_k|^{\frac{2n}{n+4}}\Big)^{\frac{n+4}{2n}}\Big(\int_{\mathbb{R}^n}|\varphi|^{p+1}\Big)^{1/(p+1)}.\end{aligned}$$

Fix $\epsilon_1>0$ so that, by (4.30),

$$\int_{\mathbb{R}^n}(\Delta\varphi)^2-p(1+\epsilon_1)\int_{\mathbb{R}^n}\overline{u}_k^{p-1}\varphi^2\ge\epsilon_1\int_{\mathbb{R}^n}(\Delta\varphi)^2. \tag{4.32}$$

By the Sobolev's embedding, we see

$$2E(\varphi)\ge\epsilon_1\int_{\mathbb{R}^n}(\Delta\varphi)^2-c_1\Big[\Big(\int_{\mathbb{R}^n}(\Delta\varphi)^2\Big)^{\frac{p+1}{2}}+2\eta\Big(\int_{\mathbb{R}^n}(\Delta\varphi)^2\Big)^{\frac{1}{2}}\Big]. \tag{4.33}$$

Therefore, there exists small $\rho=\rho(\eta)$ such that

$$\inf_{\|\varphi\|=\rho}E(\varphi)\ge\frac{\epsilon_1\rho^2}{4}>0$$

with $\lim_{\eta\to0}\rho(\eta)=0$.

*Step 2.* We claim that there exists $v_0\in D^{2,2}(\mathbb{R}^n)$ with $\|v_0\|<\rho$ such that $E(v_0)=\inf_{\|v\|\le\rho}E(v)<0$ (note that $\overline{f}_k\not\equiv0$ in $\mathbb{R}^n$). Let $v_j\in D^{2,2}(\mathbb{R}^n)$ with $\|v_j\|<\rho$ and $\lim_{j\to+\infty}E(v_j)=\inf_{\|v\|\le\rho}E(v)$. Since $v_j$ is bounded in $D^{2,2}(\mathbb{R}^n)$, we can assume that $v_j\to v_0$ weakly for some $v_0\in D^{2,2}(\mathbb{R}^n)$. If $v_j$ is strongly convergent to $v_0$, we are done. Hence we may assume that $\varphi_j\equiv v_j-v_0$ is weakly convergent to 0 and $0<\lim_{j\to\infty}\|\varphi_j\|=\tilde{\rho}<\rho$. Without loss of generality, we may assume that $\varphi_j$ is weakly convergent to 0 in $L^{\frac{2n}{n-4}}(\mathbb{R}^n)$ also. To obtain a contradiction, we have

$$\begin{aligned}&E(v_0+\varphi_j)-E(v_0)\\ &\quad=\frac{1}{2}\int_{\mathbb{R}^n}(\Delta\varphi_j)^2-\int_{\mathbb{R}^n}[F(\overline{u},v_0+\varphi_j)-F(\overline{u},v_0)]dx+o(1),\end{aligned}$$

because $\overline{f}_k \in L^{\frac{2n}{n+4}}(\mathbb{R}^n)$ where, for the simplicity of notation, $\overline{u}$ denotes the approximate solution $\overline{u}_k$. We decompose the second term into two terms,

$$\begin{aligned}
&F(\overline{u}, v_0+\varphi_j) - F(\overline{u}, v_0)\\
&\quad = \frac{1}{p+1}\Big[|\overline{u}+v_0+\varphi_j|^p(\overline{u}+v_0+\varphi_j) - |\overline{u}+v_0|^p(\overline{u}+v_0)\\
&\qquad -(p+1)|\overline{u}+v_0|^p\varphi_j\Big] + \Big[|\overline{u}+v_0|^p - \overline{u}^p\Big]\varphi_j\\
&\quad = g_1 + g_2\varphi_j.
\end{aligned}$$

For $g_1$, we have, for any $\epsilon > 0$, there exists $C_\epsilon > 0$,

$$\begin{aligned}
|g_1| &\leq \frac{(p+\epsilon)}{2}|\overline{u}+v_0|^{p-1}\varphi_j^2 + C_\epsilon|\varphi_j|^{p+1}\\
&\leq \Big(\frac{p+2\epsilon}{2}\Big)\overline{u}^{p-1}\varphi_j^2 + C_\epsilon(|v_0|^{p-1}\varphi_j^2 + |\varphi_j|^{p+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n}|g_1|dx &\leq \Big(\frac{p+2\epsilon}{2}\Big)\int_{\mathbb{R}^n}\overline{u}^{p-1}\varphi_j^2 + C_\epsilon\int_{\mathbb{R}^n}[|v_0|^{p-1}\varphi_j^2 + |\varphi_j|^{p+1}]dx\\
&\leq \Big(\frac{p+2\epsilon}{2}\Big)\int_{\mathbb{R}^n}\overline{u}^{p-1}\varphi_j^2 + C_\epsilon\rho^{p-1}\int_{\mathbb{R}^n}(\Delta\varphi)^2.
\end{aligned}$$

For $g_2$, we note that $p := \frac{n+4}{n-4} \leq 2$ for $n \geq 13$ and

$$||\overline{u}+v_0|^p - \overline{u}^p - p\overline{u}^{p-1}v_0| \leq c_2|v_0|^p$$

for some constant $c_2 > 0$. By Sobolev's embedding, we have

$$||\overline{u}+v_0|^p - \overline{u}^p - p\overline{u}^{p-1}v_0| \in L^{\frac{p+1}{p}}(\mathbb{R}^n).$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n} g_2\varphi_j &= \int_{\mathbb{R}^n}[|\overline{u}+v_0|^p - \overline{u}^p - p\overline{u}^{p-1}v_0]\varphi_j + p\int_{\mathbb{R}^n}\overline{u}^{p-1}v_0\varphi_j\\
&= o(1).
\end{aligned}$$

Combining these two estimates, we have

$$\begin{aligned}
&E(v_0+\varphi_j) - E(v_0)\\
&\quad \geq \frac{1}{2}\int_{\mathbb{R}^n}(\Delta\varphi_j)^2 - \frac{p+2\epsilon}{2}\int_{\mathbb{R}^n}\overline{u}^{p-1}\varphi_j^2 - C_\epsilon\rho^{p-1}\int_{\mathbb{R}^n}(\Delta\varphi_j)^2 + o(1).
\end{aligned}$$

Choosing $\epsilon = \frac{p\epsilon_1}{2}$ where $\epsilon_1$ is the constant in (4.32), and $\eta$ is small enough such that $C_\epsilon\rho^{p-1} < \epsilon_1/4$, then we have

$$\lim_{j\to\infty} E(v_0+\varphi_j) > E(v_0)$$

which is a contradiction. The proof of step 2 is complete.

Let $v_k$ be the solution of $E(v_k) = \inf_{\|v\|\le\rho} E(v)$. Then, we can obtain that $u_k = \overline{u}_k + v_k$ is a positive weak solution of

$$\Delta^2 u = u^{\frac{n+4}{n-4}} \quad \text{in } \mathbb{R}^n$$

via the maximum principle. Note that $u_k(x) \to 0$ and $\Delta u_k(x) \to 0$ as $|x| \to \infty$, it follows from the maximum principle that $\Delta(\Delta u_k) \ge 0$ ($\not\equiv 0$) in $\mathbb{R}^n$ and hence $\Delta u_k < 0$, $u_k > 0$ in $\mathbb{R}^n$. Since $v_k$ is bounded in $D^{2,2}(\mathbb{R}^n)$, we can assume that, after passing to a subsequence, $v_k$ converges to $v \in D^{2,2}(\mathbb{R}^n)$ in $L^p(K)$ for any compact set $K \subset \mathbb{R}^n$. Hence $u = \overline{u} + v$ is a nonnegative weak solution of (1.8), where $\overline{u}$ is the limit of $\overline{u}_k$ in $L^p(\mathbb{R}^n)$. Since $v \in D^{2,2}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$, we have $v \in L^{\frac{n+4}{n-4}}(d\mu)$. Hence $u \in L^{\frac{n+4}{n-4}}(d\mu)$. We claim that the singular set of $u$ must include $\cup_{j=1}^{\infty} S_j$. If $q \in S_j \backslash \{q_j\}$, and $q \notin$ singular set of $u$, then there exists a neighborhood $U$ of $q$ such that $u(x) \le c$ in $U$. This implies

$$-\overline{u}(x) \le v(x) \le c - \overline{u}(x), \quad \text{namely,}$$

$|v(x)| \ge \overline{u}(x) - c \ge \overline{u}_j(x) - c$. (Note that if $v(x) > 0$, then $u(x) = \overline{u}(x) + v(x) < c$ implies that $v(x) > 0 > \overline{u}(x) - c$.) However, $v \in L^{\frac{2n}{n-4}}(U)$ implies that $\overline{u}_j \in L^{\frac{2n}{n-4}}(U)$ which is impossible (note that $m - \frac{4(p+1)}{p-1} < 0$). Therefore, $\cup_{j=1}^{\infty} S_j \subset$ the singular set of $u$. Suppose that $\cup_{j=1}^{\infty} S_j$ is dense in $\mathbb{R}^n$, and because the singular set of $u$ is closed, we conclude that the singular set of $u$ is the whole space $\mathbb{R}^n$. The proof of the theorem is complete. □

## References


[1] I. Abid and S. Baraket, Construction of singular solutions for elliptic problem of fourth order derivative with a subcritical nonlinearity, *Diff. Integral Equations* 21 (2008), 653-664.

[2] P. Aviles, On isolated singularities in some nonlinear partial differential equations, *Indiana Univ. Math. J.* 32 (1983), 773-791.

[3] S. Baraket and S. Rebhi, Construction of dipole type singular solutions for a biharmonic equation with critical Sobolev exponent, *Adv. Nonlinear Stud.* 2 (2002), 459-476.

[4] S. Bae, Separation structure of positive radial solutions of a semilinear elliptic equation in $\mathbb{R}^n$, *J. Differential Equations* 194 (2003), 460-499.

[5] C.C. Chen and C.S. Lin, Existence of positive weak solutions with a prescribed singular set of semilinear elliptic equations, *J. Geom. Anal.* 9 (1999), 221-246.

[6] J. Dávila, I. Flores, and I. Guerra, Multiplicity of solutions for a fourth order equation with power-type nonlinearity, *Math. Ann.* 348 (2010), 143-193.

[7] D.L. Finn and R.C. Meowen, Singularities and asymptoties for the equation $\Delta_g u - u^q = Su$, *Indiana Univ. Math. J.* 42 (1993), 1487-1523.

[8] C. Gui, W.M. Ni and X.F. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$, *Comm. Pure Applied Math.* 45 (1992), 1153-1181.

[9] Z.M. Guo, Further study of entire radial solutions of a biharmonic equation with exponential nonlinearity, *Ann. di Matematica* 193 (2014), 187-201.

[10] Z.M. Guo and J.C. Wei, Qualitative properties of entire radial solutions for a biharmonic equation with supercritical nonlinearity, *Proc. Amer. Math. Soc.* 138 (2010), 3957-3964.
[11] F. Gazzola and H.C. Grunau, Radial entire solutions for supercritical biharmonic equations, *Math. Ann.* 334 (2006), 905-936.
[12] N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen, Refined asymptotics for constant scalar curvature metrics with isolated singularities, *Invent. Math.* 135 (1999), 233-272.
[13] Y. Li, Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbb{R}^n$, *J. Differential Equations* 95 (1992), 304-330.
[14] C.S. Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$, *Comment. Math. Helv.* 73 (1998), 206-231.
[15] R. Mazzeo and F. Pacard, A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis, *J. Diff. Geometry* 44 (1996), 331-370.
[16] R. Mazzeo and F. Pacard, Constant scalar curvature metrics with isolated singularities, *Duke Mathematical J.* 99 (1999), 353-418.
[17] R. Mazzeo, D. Pollack and K. Uhlenbeck, Modull spaces of singular Yamabe metrics, *J. Amer. Math. Soc.* 9 (1996), 303-344.
[18] F. Pacard, Existence et convergence de solutions faibles positives de $-\Delta u = u^{\frac{n}{n-2}}$ dans des ouverts bornés $\mathbb{R}^n$, $n \geq 3$, *C.R. Acad. Sci. Paris. Sér. I. Math.*, 314 (1992), 729-734.
[19] F. Pacard, Existence and convergence of positive weak solutions of $-\Delta u = u^{\frac{n}{n-2}}$ in bounded domains of $\mathbb{R}^n$, $n \geq 3$, *Calc. Var. PDEs* 1 (1993), 243-265.
[20] F. Pacard, Existence de solutions faibles positives de $-\Delta u = u^{\alpha}$ dans des ouverts bornés $\mathbb{R}^n$, $n \geq 3$, *C.R. Acad. Sci. Paris. Sér. I. Math.*, 315 (1992), 793-798.
[21] F. Pacard, The Yamabe problem on subdomains of even dimensional spheres, *Topol. Meth. Nonlinear Anal.* 6 (1995), 137-150.
[22] Y. Rébai, Weak solutions of nonlinear elliptic equations with prescribed singular set, *J. Differential Equations* 127 (1996), 439-453.

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