ON NON-RADIAL SINGULAR SOLUTIONS OF SUPERCRITICAL BI-HARMONIC EQUATIONS

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ABSTRACT. We develop gluing method for fourth order ODEs and construct infinitely many non-radial singular solutions for a bi-harmonic equation with supercritical exponent.

2010 Mathematics Subject Classification: Primary 35B40, 35J91; Secondary 58J55

1. INTRODUCTION

In this paper we are concerned with positive singular solution of the following bi-harmonic equation:

$$\Delta^2 u = u^p \quad \text{in } \mathbb{R}^n, \quad n \geq 6, \tag{1.1}$$

where $p > \frac{n+4}{n-4}$.

Equation (1.1) arises both in physics and geometry. In the past decades there have been lots of research activities in classifying solutions to (1.1). When $1 < p \leq \frac{n+4}{n-4}$, all nonnegative solutions to (1.1) have been completely classified (Lin ([18]) and Wei-Xu ([19])): if $p < \frac{n+4}{n-4}$ (1.1) admits no nontrivial nonnegative regular solution, while $p = \frac{n+4}{n-4}$, i.e. the critical case, any positive regular solution of (1.1) can be written as the following form:

$$u_{\lambda,\xi} = [n(n-4)(n-2)(n+2)]^{-\frac{n-4}{8}} \left(\frac{\lambda}{1+\lambda^2|x-\xi|^2}\right)^{\frac{n-4}{2}}, \ \xi \in \mathbb{R}^n.$$

However, the question on complete classification of positive regular solutions of (1.1) in the supercritical case, i.e. $p > \frac{n+4}{n-4}$, remains largely open.

The structure of positive radial solutions of (1.1) with $p > \frac{n+4}{n-4}$ has been studied by Gazzola-Grunau [7] and Guo-Wei [10]. For the following fourth order ODE:

$$\begin{cases} \Delta^2 u(r) = u^p(r), & r \in [0,\infty), \\ u(0) = a, \ u''(0) = b, \ u'(0) = u'''(0) = 0, \end{cases} \tag{1.2}$$

it is known from [7] that for any $a > 0$ there is a unique $b_0 := b_0(a) < 0$ such that the unique solution u_{a,b_0} of (1.2) satisfies that $u_{a,b_0} \in C^4(0,\infty)$, $u'_{a,b_0}(r) < 0$ and

$$\lim_{r\to\infty} r^\alpha u_{a,b_0}(r) = K_0^{\frac{1}{p-1}},$$

where $\alpha = \frac{4}{p-1}$ and

$$K_0 = \frac{8}{(p-1)^4}\Big[(n-2)(n-4)(p-1)^3 + 2(n^2-10n+20)(p-1)^2 - 16(n-4)(p-1) + 32\Big].$$

This implies that $u_{a,b_0}(r) > 0$ for all $r > 0$ and $u_{a,b_0}(r) \to 0$ as $r \to \infty$. Moreover, it is known from [10] that if $5 \leq n \leq 12$ or if $n \geq 13$ and $\frac{n+4}{n-4} < p < p_c(n)$,

$u_{a,b_0} - K_0^{\frac{1}{p-1}} r^{-\alpha}$ changes sign infinitely many times in $(0,\infty)$, if $n \ge 13$ and $p \ge p_c(n)$, then $u(r) < K_0^{\frac{1}{p-1}} r^{-\alpha}$ for all $r > 0$ and the solutions are strictly ordered with respect to the initial value $a = u_{a,b_0}(0)$. Here $p_c(n)$ refers to the unique value of $p > \frac{n+4}{n-4}$ such that

$$p_c(n) = \begin{cases} +\infty, & \text{if } 4 \le n \le 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}, & \text{if } n \ge 13. \end{cases}$$

Very recently, Dávila-Dupaigne-Wang-Wei [5] proved that all stable or finite Morse index solutions of the equation (1.1) are trivial provided $1 < p < p_c(n)$. According to the result by [10] and [16] all radial solutions are stable when $p \ge p_c(n)$. Thus the result in [5] is sharp.

We now turn to the singular solutions of (1.1). It is easily seen that

$$u_s(x) := K_0^{\frac{1}{p-1}} |x|^{-\frac{4}{p-1}} \tag{1.3}$$

is a singular solution of (1.1), in other words, u_s satisfies the equation

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n \setminus \{0\}. \tag{1.4}$$

As far as we know, the radial singular solution in (1.3) is the only singular solution to (1.4) known so far. The question we shall address in this paper is whether or not there are non-radial singular solutions to (1.4). To this end, we first discuss the corresponding second order Lane-Emden equation

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n. \tag{1.5}$$

Equation (1.5) has been widely studied in the past decades. We refer to [1], [2], [3], [6], [9], [11], [12], [14], [15], [17], [22] and the references therein. Farina ([6]) proved that if $\frac{n+2}{n-2} < p < p^c(n)$, the Morse index of any regular solution u of (1.5) is ∞. Here $p^c(n)$ is the Joseph-Lundgren exponent ([15]):

$$p^c(n) = \begin{cases} +\infty, & \text{if } 2 \le n \le 10, \\ \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)}, & \text{if } n \ge 11. \end{cases}$$

In [3], Dancer, Du and Guo showed that if Ω_0 is a bounded domain containing 0, u is a solution of (1.5) in $\Omega_0 \setminus \{0\}$; u has finite Morse index and $\frac{n+2}{n-2} < p < p^c(n)$, then $x = 0$ must be a removable singularity of u. They also show that if Ω_0 is a bounded domain containing 0, u is a solution of (1.5) in $\mathbb{R}^n \setminus \Omega_0$ that has finite Morse index and $\frac{n+2}{n-2} < p < p^c(n)$, then u must be a fast decay solution. It is easily seen that (1.5) has a radial singular solution

$$u^s(x) := u^s(r) = \Big[\frac{2}{p-1}\Big(n-2-\frac{2}{p-1}\Big)\Big]^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}}.$$

In a recent paper [4], Dancer, Guo and Wei obtained infinitely many positive non-radial singular solutions of (1.5) provided $p \in (\frac{n+1}{n-3}, p^c(n-1))$. The proof of [4] is via a gluing of outer and inner solutions.

The main result in this paper is the following theorem.

Theorem 1.1. *Let $n \ge 6$. Assume that*

$$\frac{n+3}{n-5} < p < p_c(n-1).$$

Then (1.1) *admits infinitely many non-radial singular solutions.*

The proof of Theorem 1.1 is via a gluing of inner and outer solutions, as in [4]. In the second order case, one glues $(u(r), u^{'}(r))$ at some intermediate point. However, since equation (1.1) is fourth order, we have to match the inner solution and outer solution up to the third derivative $(u(r), u^{'}(r), u^{''}(r), u^{'''}(r))$. There are some essential obstructions appearing in matching the inner and outer solutions. As far as we know this seems to be the first paper in gluing inner and outer solutions for fourth order ODE problems.

In the following, we sketch the idea of proving Theorem 1.1. After performing a separation of variables for a solution u of (1.1): $u(x) = r^{-\alpha}w(\theta)$, finding a non-radial singular solutions of (1.1) is equivalent to finding a non-constant solution of the following equation:

$$\Delta^2_{S^{n-1}}w + k_1(n)\Delta_{S^{n-1}}w + k_0(n)w = w^p, \tag{1.6}$$

where

$$k_0(n) = (n-4-\alpha)(n-2-\alpha)(2+\alpha)\alpha,$$
$$k_1(n) = -[(n-4-\alpha)(2+\alpha) + (n-2-\alpha)\alpha].$$

It is clear that $w(\theta) = [k_0(n)]^{\frac{1}{p-1}}$ is the constant solution of (1.6), which provides the radial singular solution of (1.1) and it is given in (1.3).

In order to construct positive non-radial singular solutions of (1.1), we need to find positive non-constant solutions of (1.6), which is a fourth order inhomogeneous nonlinear ODE, therefore, we shall construct infinitely many positive non-constant radially symmetric solutions of (1.6), i.e. solutions that only depend on the geodesic distance $\theta \in [0, \pi)$. We only consider the simple case $w(\theta) = w(\pi - \theta)$ for $0 \le \theta \le \frac{\pi}{2}$. In this case, (1.6) can be written as the following form:

$$\begin{cases} T_1 w(\theta) + k_1(n)T_2 w(\theta) + k_0(n)w = w^p, \qquad w(\theta) > 0 \quad 0 < \theta < \frac{\pi}{2}, \\ w'_\theta(0), \quad w'''_\theta(0) \text{ exist}, \quad w'_\theta(\frac{\pi}{2}) = w'''_\theta(\frac{\pi}{2}) = 0, \end{cases} \tag{1.7}$$

where T_1, T_2 are the differential operators defined as:

$$T_1 w(\theta) = \frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d}{d\theta}\Big(\frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{dw(\theta)}{d\theta}\Big)\Big)\Big),$$

and

$$T_2 w(\theta) = \frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{dw(\theta)}{d\theta}\Big).$$

A key observation is that

$$w_*(\theta) = A_p[\sin\theta]^{-\alpha},\ \theta \in (0, \frac{\pi}{2}], \tag{1.8}$$

with

$$A_p^{p-1} = (n-5-\alpha)(n-3-\alpha)(2+\alpha)\alpha\ (:= k_0(n-1))$$

is a singular solution of (1.7) with a singular point at $\theta = 0$. (Note that this is a singular solution in one dimension less.) We will construct the inner and outer solution of (1.7) and glue them at some point close to 0, which gives solutions of (1.7). The main difficulty is the matching of four parameters, which correspond to matching of u and its derivatives up to the third order.

This paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we construct inner solutions of (1.7) by studying an initial value problem of (1.7) with large initial values at $\theta = 0$. In Section 4, we construct outer solutions of (1.7). We first study an initial value problem of (1.7) with the initial values at $\theta = \frac{\pi}{2}$, then we analyze the asymptotic behaviors of the solutions of this initial value problem near $\theta = 0$. Finally, in Section 5, we match the inner and outer solutions constructed in Sections 3 and 4 to obtain solutions of (1.1). This completes the proof of Theorem 1.1. We leave some computation results in the last section.

2. PRELIMINARIES

In this section, we present some known results which will be used in the subsequent sections.

Let $u = u(r)$ be a positive radial solution of (1.1). Using the Emden-Fowler transformation:

$$u(r) = r^{-\alpha} v(t), \quad t = \ln r, \tag{2.1}$$

we see that $v(t)$ satisfies the equation:

$$v^{(4)}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = v^p(t), \quad t \in (-\infty, \infty) \tag{2.2}$$

where the coefficients K_0, K_1, K_2, K_3 are given in [7]:

$$\begin{aligned}
K_0 =& \frac{8}{(p-1)^4}\Big[(n-2)(n-4)(p-1)^3 + 2(n^2-10n+20)(p-1)^2 \\
& - 16(n-4)(p-1) + 32\Big], \\
K_1 =& -\frac{2}{(p-1)^3}\Big[(n-2)(n-4)(p-1)^3 + 4(n^2-10n+20)(p-1)^2 \\
& - 48(n-4)(p-1) + 128\Big], \\
K_2 =& \frac{1}{(p-1)^2}\Big[(n^2-10n+20)(p-1)^2 - 24(n-4)(p-1) + 96\Big], \\
K_3 =& \frac{2}{p-1}\Big[(n-4)(p-1) - 8\Big].
\end{aligned}$$

By direct calculations it is easy to see that $K_0 = k_0$. The characteristic polynomial (linearized at $K_0^{\frac{1}{p-1}}$) of (2.2) is

$$\nu \mapsto \nu^4 + K_3\nu^3 + K_2\nu^2 + K_1\nu + (1-p)K_0$$

and the eigenvalues are given by

$$\nu_1 = \frac{N_1 + \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)}, \quad \nu_2 = \frac{N_1 - \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)},$$
$$\nu_3 = \frac{N_1 + \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)}, \quad \nu_4 = \frac{N_1 - \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)}$$

where

$$\begin{aligned}
N_1 :=& -(n-4)(p-1) + 8, \quad N_2 := (n^2-4n+8)(p-1)^2 \\
N_3 :=& (9n-34)(n-2)(p-1)^4 + 8(3n-8)(n-6)(p-1)^3 \\
& + (16n^2-288n+832)(p-1)^2 - 128(n-6)(p-1) + 256.
\end{aligned}$$

Let $\tilde{\nu}_j = \nu_j - \alpha$ for $j = 1, 2, 3, 4$. We have the following proposition ([10]):

Proposition 2.1. *For any $n \geq 5$ and $p > \frac{n+4}{n-4}$, we have*

$$\tilde{\nu}_2 < 2 - n < 0 < \tilde{\nu}_1. \tag{2.3}$$

(1) *For any $5 \leq n \leq 12$ or $n \geq 13$ and $\frac{n+4}{n-4} < p < p_c(n)$, we have $\tilde{\nu}_3$, $\tilde{\nu}_4 \notin \mathbb{R}$ and $\Re(\tilde{\nu}_3) = \Re(\tilde{\nu}_4) = \frac{4-n}{2} < 0$.*
(2) *For any $n \geq 13$ and $p = p_c(n)$, we have $\tilde{\nu}_3 = \tilde{\nu}_4 = \frac{4-n}{2}$.*
(3) *For any $n \geq 13$ and $p > p_c(n)$, we have*

$$\tilde{\nu}_2 < 4 - n < \tilde{\nu}_4 < \frac{4-n}{2} < \tilde{\nu}_3 < 0 < \tilde{\nu}_1, \quad \tilde{\nu}_3 + \tilde{\nu}_4 = 4 - n. \tag{2.4}$$

Theorem 2.1. *([7]) The following limits hold:*

$$\lim_{t\to\infty} v(t) = K_0^{1/p-1}, \quad \lim_{t\to\infty} v^{(k)}(t) = 0 \tag{2.5}$$

for any $k \geq 1$.

Remark: We see that K_i $(i = 0, 1, 2, 3)$ and ν_j, $\tilde{\nu}_j$ $(j = 1, 2, 3, 4)$ above depend on n and p. In the following, by abuse of notation, we use K_i, ν_j, $\tilde{\nu}_j$ with the dimension n replaced by $n-1$ and denote $k_0 = k_0(n)$, $k_1 = k_1(n)$.

3. Inner solutions

In this section, we construct inner solutions of (1.7).

Let $Q \gg 1$ be a large constant and $\tilde{b}$ be a constant which will be given below. We consider an initial value problem:

$$\begin{cases} T_1 w(\theta) + k_1 T_2 w(\theta) + k_0 w = w^p, \\ w(0) = Q, \ \ w'(0) = 0, \ \ w''(0) = (\tilde{b} + \mu) Q^{1+\frac{2}{\alpha}}, \ \ w'''(0) = 0, \end{cases} \tag{3.1}$$

where $\mu > 0$ is a small constant. Since $Q \gg 1$, we set $Q = \epsilon^{-\frac{4}{p-1}} (:= \epsilon^{-\alpha})$ with $\epsilon > 0$ sufficiently small.

Let $w(\theta) = \epsilon^{-\alpha} v(\frac{\theta}{\epsilon})$, then we have $v(0) = 1$, $v'(0) = 0$, $v''(0) = \tilde{b} + \mu$, $v'''(0) = 0$ and $v(r)$ (for $r = \frac{\theta}{\epsilon}$) satisfies the following equation:

$$\begin{aligned} v^{(4)}(r) + 2(n-2)\epsilon \cot(\epsilon r) v'''(r) + ((n-2)(n-4)\frac{\epsilon^2}{\sin^2(\epsilon r)} - (n-2)^2\epsilon^2 + k_1\epsilon^2) v'' \\ + ((n-2)k_1\epsilon^3 \cot(\epsilon r) - (n-2)(n-4)\epsilon^3 \frac{\cot(\epsilon r)}{\sin^2(\epsilon r)}) v'(r) + k_0\epsilon^4 v(r) = v^p(r), \end{aligned} \tag{3.2}$$

with initial conditions

$$v(0) = 1, \ \ v'(0) = 0, \ \ v''(0) = \tilde{b} + \mu, \ \ v'''(0) = 0.$$

For $\epsilon > 0$ sufficiently small, we have

$$\epsilon\cot(\epsilon r) = \frac{1}{r} - \frac{1}{3}\epsilon^2 r + \sum_{k=1}^{\infty} l_k \epsilon^{2k+2} r^{2k+1},$$

$$\epsilon^2\sin^{-2}(\epsilon r) = \frac{1}{r^2} + \frac{1}{3}\epsilon^2 + \sum_{k=1}^{\infty} m_k \epsilon^{2k+2} r^{2k},$$

$$\epsilon^3\cot(\epsilon r)\sin^{-2}(\epsilon r) = \frac{1}{r^3} + \sum_{k=1}^{\infty} n_k \epsilon^{2k+2} r^{2k-1},$$

then, (3.2) can be written as the following form

$$\begin{aligned} & v^{(4)}(r) + \Big(\frac{2(n-2)}{r} - \frac{2(n-2)}{3}\epsilon^2 r + \sum_{k=1}^{\infty} l'_k \epsilon^{2k+2} r^{2k+1}\Big) v'''(r) \\ & + \Big(\frac{(n-2)(n-4)}{r^2} + \Big[\frac{(n-2)(n-4)}{3} - (n-2)^2 + k_1\Big]\epsilon^2 + \sum_{k=1}^{\infty} m'_k \epsilon^{2k+2} r^{2k+1}\Big) v''(r) \\ & - \Big(\frac{(n-2)(n-4)}{r^3} - (n-2)k_1 r^{-1}\epsilon^2 + \sum_{k=1}^{\infty} n'_k \epsilon^{2k+2} r^{2k-1}\Big) v'(r) + k_0\epsilon^4 v(r) \\ & = v^p(r) \end{aligned} \tag{3.3}$$

with initial conditions

$$v(0) = 1, \;\; v''(0) = \tilde{b} + \mu, \;\; v'(0) = v'''(0) = 0.$$

The first approximation to the solution of (3.3) is the radial solution $v_0(r)$ of the problem

$$\Delta^2 v = v^p \text{ in } \mathbb{R}^{n-1}, \quad v(0) = 1, \;\; v'(0) = 0, \;\; v''(0) = \tilde{b} + \mu, \;\; v'''(0) = 0. \tag{3.4}$$

We write $v_0 = v_{01} + v_{02}$, where v_{01} and v_{02} satisfy the following equations respectively

$$\Delta^2 v = v^p, \quad v(0) = 1, \;\; v'(0) = 0, \;\; v''(0) = \tilde{b}, \;\; v'''(0) = 0 \tag{3.5}$$

and

$$\Delta^2 v = v_0^p - v_{01}^p, \quad v(0) = 0, \;\; v'(0) = 0, \;\; v''(0) = \mu, \;\; v'''(0) = 0. \tag{3.6}$$

We now choose $\tilde{b} < 0$ to be the unique value such that the solution v_{01} is the unique positive radial ground state of (3.5).

Lemma 3.1. *Assume that $v_{01}(r)$ and $v_{02}(r)$ are the solutions to (3.5) and (3.6) respectively. For $\frac{n+3}{n-5} < p < p_c(n-1)$, there exists $R_0 \gg 1$ such that for $r \geq R_0$, the solution $v_{01}(r)$ satisfies*

$$v_{01}(r) = A_p r^{-\alpha} + \frac{a_0\cos(\beta\ln r) + b_0\sin(\beta\ln r)}{r^{\frac{n-5}{2}}} + O(r^{\tilde{\nu}_2}), \tag{3.7}$$

where $\beta = \frac{\sqrt{4\sqrt{N_3} - N_2}}{2(p-1)}$ (with n being replaced by $n-1$ in N_2 and N_3) and $\sqrt{a_0^2 + b_0^2} \neq 0$.

The solution $v_{02}(r)$ satisfies

$$v_{02}(r) = \mu B_p r^{\tilde{\nu}_1} + O\Big(\mu^2 r^{\nu_1 + \tilde{\nu}_1} + \mu r^{\tilde{\nu}_1 + (\alpha - \frac{n-5}{2})}\Big) \tag{3.8}$$

with $B_p \neq 0$ when $\mu = O(\frac{1}{r^{\nu_1 - \sigma}})$ for r in any interval $[e^T, e^{10T}]$ with $T \gg 1$ and $\sigma = \alpha - \frac{n-5}{2}$.

Proof. The proof of this lemma is divided to two steps. We consider $v_{01}(r)$ in the first step. The main arguments in the proof are similar to those in the proof of Theorem 3.1 of [8].

Using the Emden-Fowler transformation:

$$v_{01}(r) = r^{-\alpha} v(t), \quad t = \ln r \ (r > 0), \tag{3.9}$$

and letting $v(t) = A_p - h(t)$, we see that $h(t)$ satisfies

$$h^{(4)}(t) + K_3 h'''(t) + K_2 h''(t) + K_1 h'(t) + (1-p) K_0 h(t) + O(h^2) = 0, \quad t > 1. \tag{3.10}$$

Note that $r^\alpha v_{01}(r) \to A_p$ as $r \to \infty$ and hence $h(t) \to 0$ as $t \to \infty$. It follows from Proposition 2.1 that $\tilde{\nu}_3$, $\tilde{\nu}_4 \notin \mathbb{R}$ and $\Re(\tilde{\nu}_3) = \Re(\tilde{\nu}_4) = \frac{5-n}{2} < 0$ and $\tilde{\nu}_2 < 3 - n < 0 < \tilde{\nu}_1$ provided $\frac{n+3}{n-5} < p < p_c(n-1)$. Let $\nu_3 = \sigma + i\beta$, where $\beta = \frac{\sqrt{4\sqrt{N_3} - N_2}}{2(p-1)}$ and $\sigma = -\frac{n-5}{2} + \alpha < 0$ for $p > \frac{n+3}{n-5}$.

We can write (3.10) as

$$(\partial_t - \nu_4)(\partial_t - \nu_3)(\partial_t - \nu_2)(\partial_t - \nu_1) h(t) = H(h(t)), \tag{3.11}$$

where $H(h(t)) = O(h^2)$. We claim that for any $T \gg 1$, there exist constants A_i and B_i $(i = 1, 2, 3, 4)$ such that

$$\begin{aligned} h(t) =& A_1 e^{\sigma t} \cos \beta t + A_2 e^{\sigma t} \sin \beta t + A_3 e^{\nu_2 t} + A_4 e^{\nu_1 t} \\ &+ B_1 \int_T^t e^{\sigma(t-s)} \sin \beta(t-s) H(h(s)) \mathrm{d}s \\ &+ B_2 \int_T^t e^{\sigma(t-s)} \cos \beta(t-s) H(h(s)) \mathrm{d}s \\ &+ B_3 \int_T^t e^{\nu_2(t-s)} H(h(s)) \mathrm{d}s \\ &+ B_4 \int_T^t e^{\nu_1(t-s)} H(h(s)) \mathrm{d}s. \end{aligned}$$

Moreover, each A_i depends on T and ν_i $(i = 1, 2, 3, 4)$, while each B_i depends only on ν_i $(i = 1, 2, 3, 4)$. In fact, it follows from (3.11) and the ODE theory of second order (see [13]) that

$$\begin{aligned} (\partial_t - \nu_2)(\partial_t - \nu_1) h(t) =& A_1' e^{\sigma t} \cos \beta t + A_2' e^{\sigma t} \sin \beta t \\ &+ \frac{1}{\beta} \int_T^t e^{\sigma(t-s)} \sin \beta(t-s) H(h(s)) \mathrm{d}s, \end{aligned} \tag{3.12}$$

where A_1' and A_2' are constants depending on T, ν_3 and ν_4. Multiplying both sides of (3.12) by $e^{-\nu_2 t}$ and integrating it from T to t, we obtain that

$$\begin{aligned} (\partial_t - \nu_1) h(t) =& A_3' e^{\nu_2 t} + \int_T^t e^{\nu_2(t-s)} (A_1' e^{\sigma s} \cos \beta s + A_2' e^{\sigma s} \sin \beta s) \mathrm{d}s \\ &+ \frac{1}{\beta} \int_T^t e^{\nu_2(t-s)} \int_T^s e^{\sigma(s-\xi)} \sin \beta(s-\xi) H(h(\xi)) \mathrm{d}\xi \mathrm{d}s. \end{aligned}$$

We now switch the order of integrations and find that

$$\begin{aligned}(\partial_t-\nu_1)h(t)=&A_1''e^{\sigma t}\cos\beta t+A_2''e^{\sigma t}\sin\beta t+A_3''e^{\nu_2 t}\\&+B_1'\int_T^t e^{\sigma(t-s)}\sin\beta(t-s)H(h(s))\mathrm{d}s\\&+B_2'\int_T^t e^{\sigma(t-s)}\cos\beta(t-s)H(h(s))\mathrm{d}s\\&+B_3'\int_T^t e^{\nu_2(t-s)}H(h(s))\mathrm{d}s,\end{aligned}$$

where A_1'', A_2'' and A_3'' depend on T, ν_i $(i=2,3,4)$, B_i' $(i=1,2,3)$ depend only on ν_i $(i=2,3,4)$. Repeating the same argument once again, we obtain our claim. We can also write $h(t)$ as

$$\begin{aligned}h(t)=&A_1e^{\sigma t}\cos\beta t+A_2e^{\sigma t}\sin\beta t+A_3e^{\nu_2 t}+M_4e^{\nu_1 t}\\&+B_1\int_T^t e^{\sigma(t-s)}\sin\beta(t-s)H(h(s))\mathrm{d}s+B_2\int_T^t e^{\sigma(t-s)}\cos\beta(t-s)H(h(s))\mathrm{d}s\\&+B_3\int_T^t e^{\nu_2(t-s)}H(h(s))ds-B_4\int_t^\infty e^{\nu_1(t-s)}H(h(s))\mathrm{d}s,\end{aligned}$$

by using the fact that $\int_T^t=\int_T^\infty-\int_t^\infty$. Since $h(t)\to 0$ as $t\to\infty$, we have $M_4=0$ (note $\nu_1>0$). Setting

$$h_1(t)=A_1e^{\sigma t}\cos\beta t+A_2e^{\sigma t}\sin\beta t+A_3e^{\nu_2 t}$$

and

$$\begin{aligned}h_2(t)=&B_1\int_T^t e^{\sigma(t-s)}\sin\beta(t-s)H(h(s))\mathrm{d}s\\&+B_2\int_T^t e^{\sigma(t-s)}\cos\beta(t-s)H(h(s))\mathrm{d}s\\&+B_3\int_T^t e^{\nu_2(t-s)}H(h(s))\mathrm{d}s\\&-B_4\int_t^\infty e^{\nu_1(t-s)}H(h(s))\mathrm{d}s,\end{aligned}$$

we see from the fact $H(h(t))=O(h^2(t))$ that

$$|h_2(t)|\le C[\tilde h_1(t)+\tilde h_2(t)],\tag{3.13}$$

where $C>0$ is independent of T and

$$\begin{aligned}\tilde h_1(t)&=\max\Big\{\int_T^t e^{\sigma(t-s)}|h_1(s)|^2\mathrm{d}s,\ \int_T^t e^{\nu_2(t-s)}|h_1(s)|^2\mathrm{d}s,\ \int_t^\infty e^{\nu_1(t-s)}|h_1(s)|^2\mathrm{d}s\Big\},\\ \tilde h_2(t)&=\max\Big\{\int_T^t e^{\sigma(t-s)}|h_2(s)|^2\mathrm{d}s,\ \int_T^t e^{\nu_2(t-s)}|h_2(s)|^2\mathrm{d}s,\ \int_t^\infty e^{\nu_1(t-s)}|h_2(s)|^2\mathrm{d}s\Big\}.\end{aligned}$$

We now show

$$|h_2(t)|=o(e^{\sigma t}).\tag{3.14}$$

There are three cases to be considered:

(1) $|h_2(t)|\le C\Big[\tilde h_1(t)+\int_T^t e^{\sigma(t-s)}|h_2(s)|^2\mathrm{d}s\Big]$,

(2) $|h_2(t)| \le C\Big[\tilde{h}_1(t) + \int_T^t e^{\nu_2(t-s)}|h_2(s)|^2 \mathrm{d}s\Big]$,

(3) $|h_2(t)| \le C\Big[\tilde{h}_1(t) + \int_t^\infty e^{\nu_1(t-s)}|h_2(s)|^2 \mathrm{d}s\Big]$.

We only consider (1) and (3). Case (2) can be discussed similarly. For case (1), we have

$$|h_2(t)| \le C\Big[\tilde{h}_1(t) + \int_T^t e^{\sigma(t-s)}|h_2(s)|^2 \mathrm{d}s\Big]. \tag{3.15}$$

Thus,

$$|h_2(t)| \le C\Big[\tilde{h}_1(t) + \max_{t\ge T}|h_2(t)| \int_T^t e^{\sigma(t-s)}|h_2(s)| \mathrm{d}s\Big]. \tag{3.16}$$

Let $m(t) = \int_T^t e^{-\sigma s}|h_2(s)|$, then, it is seen from (3.16) that

$$m'(t) \le C\tilde{h}_1(t)e^{-\sigma t} + C\max_{t\ge T}|h_2(t)|m(t). \tag{3.17}$$

For any $\epsilon > 0$ sufficiently small, we can choose T sufficiently large such that $0 < d_T := C\max_{t\ge T}|h_2(t)| < \epsilon$. It follows from (3.17) that

$$m(t) \le Ce^{d_T t}\int_T^t \tilde{h}_1(s)e^{-\sigma s}e^{-d_T s}\mathrm{d}s. \tag{3.18}$$

Substituting $m(t)$ in (3.18) into (3.16), we see that

$$|h_2(t)| \le C\tilde{h}_1(t) + Cd_T e^{(\sigma+d_T)t}\int_T^t \tilde{h}_1(s)e^{-\sigma s}e^{-d_T s}ds. \tag{3.19}$$

Note that $\sigma + d_T < 0$ for T sufficiently large. We also know that $\nu_2 < \sigma$, which implies $\tilde{h}_1(t) = o(e^{\sigma t})$. On the other hand, from (3.19) we can obtain that $|h_2(t)| = o(e^{(\sigma+d_T)t})$. Substituting these into (3.15), we eventually have

$$|h_2(t)| = o(e^{\sigma t}). \tag{3.20}$$

For case (3), we have

$$|h_2(t)| \le C\Big[\tilde{h}_1(t) + \int_t^\infty e^{\nu_1(t-s)}|h_2(s)|^2 \mathrm{d}s\Big]. \tag{3.21}$$

Thus,

$$|h_2(t)| \le C\tilde{h}_1(t) + C\max_{t\ge T}|h_2(t)| \int_t^\infty e^{\nu_1(t-s)}|h_2(s)| \mathrm{d}s. \tag{3.22}$$

Let $l(t) = \int_t^\infty e^{-\nu_1 s}|h_2(s)|\mathrm{d}s$, it is seen from (3.22) that

$$-l'(t) \le C\tilde{h}_1(t)e^{-\nu_1 t} + d_T l(t). \tag{3.23}$$

It follows from (3.23) that

$$l(s) \le Ce^{-d_T t}\int_t^\infty \tilde{h}_1(s)e^{-\nu_1 s}e^{d_T s}\mathrm{d}s. \tag{3.24}$$

Since $\tilde{h}_1(t) = o(e^{\sigma t})$, we obtain from (3.24) that

$$l(s) = o(e^{(\sigma-\nu_1)t}).$$

Substituting this into (3.22), we also have

$$|h_2(t)| = o(e^{\sigma t}).$$

We now write $h(t)$ as

$$\begin{aligned} h(t) =& M_1 e^{\sigma t}\cos\beta t + M_2 e^{\sigma t}\sin\beta t + A_3 e^{\nu_2 t} \\ & - B_1\int_t^\infty e^{\sigma(t-s)}\sin\beta(t-s)H(h(s))\mathrm{d}s \\ & - B_2\int_t^\infty e^{\sigma(t-s)}\cos\beta(t-s)H(h(s))\mathrm{d}s \\ & + B_3\int_T^t e^{\nu_2(t-s)}H(h(s))\mathrm{d}s \\ & - B_4\int_t^\infty e^{\nu_1(t-s)}H(h(s))\mathrm{d}s. \end{aligned}$$

Then, it follows from the facts $H(h(t)) = O(h^2(t))$, $h_1(t) = O(e^{\sigma t})$, $h_2(t) = o(e^{\sigma t})$ and $\nu_2 < 2\sigma$ that

$$h(t) = M_1 e^{\sigma t}\cos(\beta t) + M_2 e^{\sigma t}\sin(\beta t) + A_3 e^{\nu_2 t} + O(e^{2\sigma t}). \tag{3.25}$$

This implies that (3.7) holds for some a_0 and b_0. By a similar argument which was used in the proof of [10, Theorem 3.3], we can show $a_0^2 + b_0^2 \neq 0$. This completes the proof of the first step.

We now proceed the second step. Setting $v_{02} = \mu\tilde{v}_{02}$, we see that $\tilde{v}_{02}(r)$ satisfies the following equation:

$$\Delta^2\tilde{v}_{02} - pv_{01}^{p-1}\tilde{v}_{02} = \mu^{-1}\Big[(v_{01}+\mu\tilde{v}_{02})^p - v_{01}^p - p\mu v_{01}^{p-1}\tilde{v}_{02}\Big] \tag{3.26}$$

with initial conditions

$$\tilde{v}_{02}(0) = 0,\ \ \tilde{v}_{02}'(0) = 0,\ \ \tilde{v}_{02}''(0) = 1,\ \ \tilde{v}_{02}'''(0) = 0.$$

Using the Emden-Fowler transformation:

$$\tilde{v}_{02}(r) = r^{-\alpha}\hat{v}(t),\quad t = \ln r\ (r>0),$$

and the expression obtained for $v_{01}(r)$, we see that $\hat{v}(t)$ satisfies

$$\hat{v}^{(4)} + K_3\hat{v}''' + K_2\hat{v}'' + K_1\hat{v}' + (1-p)K_0\hat{v} = f(r,\mu,\hat{v}), \tag{3.27}$$

where

$$f(r,\mu,\hat{v}) = O\Big([\mu\hat{v} + r^{(\alpha-\frac{n-5}{2})}]\Big)\hat{v}$$

provided that $\mu\hat{v} = o(1)$ for t sufficiently large. It follows from (3.27) that

$$\begin{aligned} \hat{v}(t) =& \hat{A}_1 e^{\sigma t}\cos\beta t + \hat{A}_2 e^{\sigma t}\sin\beta t + \hat{A}_3 e^{\nu_2 t} + \hat{A}_4 e^{\nu_1 t} \\ & + \hat{B}_1\int_T^t e^{\sigma(t-s)}\sin\beta(t-s)f(r,\mu,\hat{v}(s))\mathrm{d}s \\ & + \hat{B}_2\int_T^t e^{\sigma(t-s)}\cos\beta(t-s)f(r,\mu,\hat{v}(s))\mathrm{d}s \\ & + \hat{B}_3\int_T^t e^{\nu_2(t-s)}f(r,\mu,\hat{v}(s))\mathrm{d}s \\ & + \hat{B}_4\int_T^t e^{\nu_1(t-s)}f(r,\mu,\hat{v}(s))\mathrm{d}s, \end{aligned}$$

where $\hat{A}_i = \hat{A}_i(T, \nu_1, \nu_2, \nu_3, \nu_4)$ $(i = 1, 2, 3, 4)$ and $\hat{B}_i = \hat{B}_i(\nu_1, \nu_2, \nu_3, \nu_4)$. We first show that $\tilde{v}_{02}$ is strictly increasing in $(0, \infty)$. Using the initial values, we can find $R \in (0, \infty)$ such that $\tilde{v}_{02}(r) > 0$ for $r \in (0, R)$. Writing (3.26) as

$$\mu \Delta^2 \tilde{v}_{02} = (v_{01} + \mu \tilde{v}_{02})^p - v_{01}^p,$$

we obtain that $(\Delta \tilde{v}_{02})' > 0$, and hence $\Delta \tilde{v}_{02} > \Delta \tilde{v}_{02}(0) = n$ for $r \in (0, R)$, which implies that $(\tilde{v}_{02})'(r) > 0$ for $r \in (0, R)$. Moreover, we can deduce that $R = \infty$ and $\tilde{v}_{02}'(r) > 0$ for $r \in (0, \infty)$. Therefore, $\hat{v}$ is increasing in $(0, \infty)$. Next, we claim that $\hat{A}_4 \neq 0$ for any $T \gg 1$ sufficiently large. Indeed, for $t \in [T, 10T]$,

$$\begin{aligned} e^{-\nu_1 t} \hat{v}(t) = {} & \hat{A}_4 + \tilde{g}(t) \\ & + \hat{B}_1 e^{(\sigma - \nu_1)t} \int_T^t e^{-\sigma s} \sin \beta(t - s) f(r, \mu, \hat{v}(s)) \mathrm{d}s \\ & + \hat{B}_2 e^{(\sigma - \nu_1)t} \int_T^t e^{-\sigma s} \cos \beta(t - s) f(r, \mu, \hat{v}(s)) \mathrm{d}s \\ & + \hat{B}_3 e^{(\nu_2 - \nu_1)t} \int_T^t e^{-\nu_2 s} f(r, \mu, \hat{v}(s)) \mathrm{d}s \\ & + \hat{B}_4 \int_T^t e^{-\nu_1 s} f(r, \mu, \hat{v}(s)) \mathrm{d}s \\ \leq {} & |\hat{A}_4| + |\tilde{g}(t)| + (\Sigma_{j=1}^4 |\hat{B}_j|) \max_{t \in [T, 10T]} \left[\mu \hat{v} + e^{(\alpha - \frac{n-5}{2})t} \right] \int_T^t e^{-\nu_1 s} \hat{v}(s), \end{aligned}$$

where

$$\tilde{g}(t) = \hat{A}_1 e^{(\sigma - \nu_1)t} \cos \beta t + \hat{A}_2 e^{(\sigma - \nu_1)t} \sin \beta t + \hat{A}_3 e^{(\nu_2 - \nu_1)t}.$$

Since

$$(\Sigma_{j=1}^4 |\hat{B}_j|) \max_{t \in [T, 10T]} \left[\mu \hat{v} + e^{(\alpha - \frac{n-5}{2})t} \right] = \tau = o(1),$$

we have

$$e^{-\nu_1 t} \hat{v}(t) \leq |\hat{A}_4| + |\tilde{g}(t)| + \tau \int_T^t e^{-\nu_1 s} \hat{v}(s) \mathrm{d}s. \tag{3.28}$$

Let $\ell(t) = \int_T^t e^{-\nu_1 s} \hat{v}(s) \mathrm{d}s$. We see that

$$(e^{-\tau t} \ell(t))' \leq (|\hat{A}_4| + |\tilde{g}(t)|) e^{-\tau t}. \tag{3.29}$$

Integrating (3.29) in $[T, t]$, we obtain that

$$\ell(t) \leq \frac{|\hat{A}_4| + \max_{t \in [T, 10T]} |\tilde{g}(t)|}{\tau} e^{\tau(t - T)}.$$

If we choose $\tau(t - T) \leq C$ for $t \in [T, 10T]$, i.e., $\tau = O(\frac{1}{T})$, we see that

$$\ell(t) \leq \frac{(|\hat{A}_4| + \max_{t \in [T, 10T]} |\tilde{g}(t)|) C}{\tau}. \tag{3.30}$$

Substituting this into (3.28), we have

$$e^{-\nu_1 t} \hat{v}(t) \leq |\hat{A}_4| (1 + C) + |\tilde{g}(t)| + C \max_{t \in [T, 10T]} |\tilde{g}(t)|. \tag{3.31}$$

Suppose $\hat{A}_4 = 0$. We see from (3.31) and the expression of $|\tilde{g}(t)|$ that

$$\hat{v}(t) = o(1), \quad \forall t \in [T, 10T].$$

This contradicts the fact that $\hat{v}$ is increasing in $(0,\infty)$. Therefore, $\hat{A}_4 \neq 0$ and our claim holds. Moreover, it is known from (3.31) and the expression of $\hat{v}(t)$ that

$$\hat{v}(t) = B_p e^{\nu_1 t} + O\Big(\mu e^{2\nu_1 t} + e^{(\sigma+\nu_1)t}\Big) \tag{3.32}$$

with $B_p \neq 0$ and $\mu = O(e^{(-\nu_1+\sigma)t})$. Therefore,

$$v_{02}(r) = \mu B_p r^{\tilde{\nu}_1} + O\Big(\mu^2 r^{\nu_1+\tilde{\nu}_1} + \mu r^{\tilde{\nu}_1+\sigma}\Big)$$

with $B_p \neq 0$ and $\mu = O(\frac{1}{r^{\nu_1-\sigma}})$. This completes the proof of this lemma. □

Lemma 3.2. *Let p satisfy the conditions of Lemma 3.1 and $v_1(r)$ be the unique solution of the following equation*

$$\begin{cases} v_1^{(4)}(r) + \frac{2(n-2)}{r} v_1'''(r) + \frac{(n-2)(n-4)}{r^2} v_1''(r) - \frac{(n-2)(n-4)}{r^3} v_1'(r) - \frac{2(n-2)}{3} r v_0'''(r) \\ \quad +(\frac{(n-2)(n-4)}{3} - (n-2)^2 + k_1) v_0''(r) + \frac{(n-2)k_1}{r} v_0'(r) = p v_0^{p-1}(r) v_1(r), \\ v_1(0) = 0,\ v_1'(0) = 0,\ v_1''(0) = 0,\ v_1'''(0) = 0. \end{cases} \tag{3.33}$$

Then for $r \in [e^T, e^{10T}]$ with $T \gg 1$ and $\mu = O(\frac{1}{r^{\nu_1-\sigma}})$,

$$\begin{aligned} v_1(r) =& C_p r^{2-\alpha} + r^{2-\frac{n-5}{2}}(a_1\cos(\beta\ln r) + b_1\sin(\beta\ln r)) + \mu D_p r^{2+\tilde{\nu}_1} \\ &+ O(\mu^2 r^{\tilde{\nu}_1+\nu_1+2} + \mu r^{\tilde{\nu}_1+\sigma+2}) + o(r^{2-\frac{n-5}{2}}), \end{aligned} \tag{3.34}$$

where C_p satisfies

$$E_1 C_p - p A_p^{p-1} C_p = F_1 A_p, \tag{3.35}$$

and

$$\begin{aligned} E_1 =& (1+\alpha)(1-\alpha)(2-\alpha)\alpha - 2(n-2)(2-\alpha)(1-\alpha)\alpha - (n-2)(n-4)(2-\alpha) \\ &+ (n-2)(n-4)(2-\alpha)(1-\alpha), \\ F_1 =& ((n-2)^2 - k_1 - \frac{(n-2)(n-4)}{3})\alpha(\alpha+1) - \frac{2(n-2)}{3}\alpha(\alpha+1)(\alpha+2) \\ &+ k_1(n-2)\alpha, \end{aligned}$$

D_p satisfies

$$E_2 D_p = F_2 B_p, \tag{3.36}$$

where

$$E_2 = (2+\tilde{\nu}_1)(\tilde{\nu}_1+n-1)(\tilde{\nu}_1+n-3)\tilde{\nu}_1 - p A_p^{p-1}$$

and

$$\begin{aligned} F_2 =& \frac{2(n-2)(\tilde{\nu}_1-1)(\tilde{\nu}_1-2)\tilde{\nu}_1}{3} + ((n-2)^2 - k_1 - \frac{(n-2)(n-4)}{3})(\tilde{\nu}_1-1)\tilde{\nu}_1 \\ &- k_1(n-2)\tilde{\nu}_1 + p(p-1)A_p^{p-2}C_p, \end{aligned}$$

(a_1, b_1) is the solution of

$$\begin{cases} Aa_1 - Bb_1 = G, \\ Ba_1 + Ab_1 = H, \end{cases}$$

where

$$A = \frac{n^4 - 12n^3 + 14n^2 + 132n - 135}{16} - pA_p^{p-1} + (\frac{n^2-6n-35}{2})\beta^2 + \beta^4,$$

$$B = (2n^2 - 12n - 6)\beta + 8\beta^3,$$

$$
\begin{aligned}
G =& p(p-1)A_p^{p-2}C_p a_0 + \frac{n^4-11n^3+41n^2-61n+30}{12}a_0 + \frac{n^2-6n+5}{4}k_1 a_0 \\
&+ \frac{4n^2+3n-n^3-14}{6}b_0\beta - 2k_1 b_0\beta + \frac{n^2-9n+14}{3}a_0\beta^2 + a_0 k_1\beta^2 \\
&- \frac{2(n-2)}{3}b_0\beta^3,
\end{aligned}
$$

and

$$
\begin{aligned}
H =& p(p-1)A_p^{p-2}C_p b_0 + \frac{n^4-11n^3+41n^2-61n+30}{12}b_0 + \frac{n^2-6n+5}{4}k_1 b_0 \\
&- \frac{4n^2+3n-n^3-14}{6}a_0\beta + 2k_1 a_0\beta + \frac{n^2-9n+14}{3}b_0\beta^2 + b_0 k_1\beta^2 \\
&+ \frac{2(n-2)}{3}a_0\beta^3.
\end{aligned}
$$

Remark: We need to show $E_2 \neq 0$ and the following 2 by 2 matrix is invertible,

$$
K = \begin{bmatrix} A, & -B \\ B, & A \end{bmatrix}.
$$

This will be proved in last section.

Proof. Let

$$
\begin{aligned}
v_1(r) =& C_p r^{2-\alpha} + \tilde{f}(r) r^{2-\frac{n-5}{2}} + \mu D_p r^{2+\tilde{\nu}_1} + o(r^{2-\frac{n-5}{2}}) \\
&+ O(\mu^2 r^{\tilde{\nu}_1+\nu_1+2} + \mu r^{\tilde{\nu}_1+\sigma+2}),
\end{aligned}
$$

where

$$
\tilde{f}(r) = a_1\cos(\beta\ln r) + b_1\sin(\beta\ln r).
$$

Using the expression in (3.7) and (3.8), we can get (3.34) by direct calculations. □

Furthermore, we can obtain the following proposition.

Proposition 3.1. *Let $\frac{n+3}{n-5} < p < p_c(n-1)$ and $v(r)$ be a solution of (3.2). Then for $\epsilon > 0$ sufficiently small,*

$$
v(r) = v_0(r) + \sum_{k=1}^{\infty} \epsilon^{2k} v_k(r).
$$

Moreover, for $r \in [e^T, e^{10T}]$ with $T \gg 1$ and, $\mu = o(\frac{1}{r^{\nu_1-\sigma}})$,

$$
\begin{aligned}
v_k(r) =& \sum_{j=1}^{k} d_j^k r^{2j-\alpha} + \sum_{j=1}^{k} e_j^k r^{2j-\frac{n-5}{2}}\sin(\beta\ln r + E_j^k) + \sum_{j=1}^{k}\mu f_j^k r^{2j+\tilde{\nu}_1} \\
&+ O(\mu^2 r^{\tilde{\nu}_1+\nu_1+2k} + \mu r^{\tilde{\nu}_1+\sigma+2k}) + o(r^{2k-\frac{n-5}{2}}),
\end{aligned}
\tag{3.37}
$$

where $d_j^k, e_j^k, f_j^k, E_j^k$ $(j = 1, 2, \cdots, k)$ are constants. Moreover,

$$
d_1^1 = C_p,\ e_1^1 = \sqrt{a_1^2 + b_1^2},\ f_1^1 = D_p,\ \sin E_1^1 = \frac{a_1}{e_1^1},\ \cos E_1^1 = \frac{b_1}{e_1^1}
$$

where C_p, a_1, b_1, D_p are given in Lemma 3.2.

Proof. Using Taylor's expansion of v^p and the expressions of $v_0(r), v_1(r), \cdots, v_{k-1}(r)$, we can obtain this proposition by the induction argument and direct calculations. Note that

$$O(r^{2-\frac{n-5}{2}}) = o(r^{2-\alpha}).$$

□

Theorem 3.1. *Let $\frac{n+3}{n-5} < p < p_c(n-1)$ and $w^{inn}_{\epsilon,\mu}(\theta)$ be the solution of (1.7) with*

$$w(0) = \epsilon^{-\alpha}, \ \ w_\theta(0) = 0, \ \ w_{\theta\theta}(0) = (\tilde{b}+\mu)\epsilon^{-\alpha-2}, \ \ w_{\theta\theta\theta}(0) = 0.$$

Then for any sufficiently small $\epsilon > 0$, $\frac{\theta}{\epsilon} \in [e^T, e^{10T}]$ with $T \gg 1$ and $\mu = O\Big(\Big(\frac{\epsilon}{\theta}\Big)^{\nu_1-\sigma}\Big)$,

$$\begin{aligned}
w^{inn}_{\epsilon,\mu}(\theta) =& \frac{A_p}{\theta^\alpha} + \frac{C_p}{\theta^{\alpha-2}} + B_p\mu\epsilon^{-\nu_1}\theta^{\tilde{\nu}_1} + \sum_{k=2}^{\infty}\sum_{j=1}^{k} d^k_j\epsilon^{2(k-j)}\theta^{2j-\alpha} \\
&+ \epsilon^{\frac{n-5}{2}-\alpha}\Big[\frac{a_0\cos(\beta\ln\frac{\theta}{\epsilon}) + b_0\sin(\beta\ln\frac{\theta}{\epsilon})}{\theta^{\frac{n-5}{2}}} + \frac{a_1\cos(\beta\ln\frac{\theta}{\epsilon}) + b_1\sin(\beta\ln\frac{\theta}{\epsilon})}{\theta^{\frac{n-5}{2}-2}} \\
&+ \sum_{k=2}^{\infty}\Big(\sum_{j=1}^{k} e^k_j\epsilon^{2(k-j)}\theta^{2j-\frac{n-5}{2}}\sin(\beta\ln\frac{\theta}{\epsilon} + E^k_j) + o(\theta^{2k-\frac{n-5}{2}})\Big) + O(\theta^{2-\frac{n-5}{2}})\Big] \\
&+ \epsilon^{-\alpha}\sum_{k=1}^{\infty}\Big[\sum_{j=1}^{k}(\mu f^k_j\epsilon^{2k-2j-\tilde{\nu}_1}\theta^{2j+\tilde{\nu}_1}) + O\Big(\mu^2\theta^{\tilde{\nu}_1+\nu_1+2k}\epsilon^{-\tilde{\nu}_1-\nu_1} \\
&+ \mu\theta^{\tilde{\nu}_1+\sigma+2k}\epsilon^{-\tilde{\nu}_1-\sigma}\Big)\Big] + O\Big(\mu^2\Big(\frac{\theta}{\epsilon}\Big)^{\tilde{\nu}_1+\nu_1} + \mu\Big(\frac{\theta}{\epsilon}\Big)^{\tilde{\nu}_1+\sigma}\Big).
\end{aligned}$$

Proof. This is a direct consequence of Proposition 3.1 by setting $r = \frac{\theta}{\epsilon}$. □

We now obtain the following lemmas which will be useful in the following proofs.

Lemma 3.3. *Let $\frac{n+3}{n-5} < p < p_c(n-1)$ and*

$$v(Q,\mu,\theta) = Qv_0(Q^{\frac{p-1}{4}}\theta).$$

Then for $Q^{\frac{p-1}{4}}\theta \in [e^T, e^{10T}]$ with $T \gg 1$; $\mu = O(\frac{1}{(Q^{(p-1)/4}\theta)^{\nu_1-\sigma}})$ and $n = 0,1,2$, $v(Q,\mu,\theta)$ satisfies

$$\begin{aligned}
\frac{\partial^n}{\partial Q^n}(v(Q,\mu,\theta)) =& \frac{\partial^n}{\partial Q^n}\Big(\frac{A_p}{\theta^\alpha}\Big) \\
&+ \frac{\partial^n}{\partial Q^n}\Big\{C\theta^{-\frac{n-5}{2}}Q^{-(\frac{(p-1)(n-5)}{8}-1)}\sin(\beta\ln(Q^{\frac{p-1}{4}}\theta) + \kappa)\Big\} \\
&+ Q^{\frac{\tilde{\nu}_2}{\alpha}+1-n}O(\theta^{\tilde{\nu}_2}) + \mu B_pQ^{\frac{\tilde{\nu}_1}{\alpha}+1-n}\theta^{\tilde{\nu}_1} \\
&+ O(\mu^2Q^{\frac{\tilde{\nu}_1+\nu_1}{\alpha}+1-n}\theta^{\tilde{\nu}_1+\nu_1} + \mu Q^{\frac{\tilde{\nu}_1+\sigma}{\alpha}+1-n}\theta^{\sigma+\tilde{\nu}_1}),
\end{aligned}$$

$$\begin{aligned}\frac{\partial^n}{\partial Q^n}(v'_\theta(Q,\mu,\theta)) =&\frac{\partial^n}{\partial Q^n}\Big(-\alpha\frac{A_p}{\theta^{\alpha+1}}\Big)\\&+\frac{\partial^{n+1}}{\partial Q^n\partial\theta}\Big\{C\theta^{-\frac{n-5}{2}}Q^{-(\frac{(p-1)(n-5)}{8}-1)}\sin(\beta\ln(Q^{\frac{p-1}{4}}\theta)+\kappa)\Big\}\\&+Q^{\frac{\tilde\nu_2}{\alpha}+1-n}O(\theta^{\tilde\nu_2-1})+\mu\tilde\nu_1B_pQ^{\frac{\tilde\nu_1}{\alpha}+1-n}\theta^{\tilde\nu_1-1}\\&+O(\mu^2Q^{\frac{\tilde\nu_1+\nu_1}{\alpha}+1-n}\theta^{\tilde\nu_1+\nu_1-1}+\mu Q^{\frac{\tilde\nu_1+\sigma}{\alpha}+1-n}\theta^{\sigma+\tilde\nu_1-1}),\end{aligned}$$

$$\begin{aligned}\frac{\partial^n}{\partial Q^n}(\frac{\partial^2}{\partial\theta^2}v(Q,\mu,\theta)) =&\frac{\partial^n}{\partial Q^n}\Big(\alpha(\alpha+1)\frac{A_p}{\theta^{\alpha+2}}\Big)\\&+\frac{\partial^{n+2}}{\partial Q^n\partial\theta^2}\Big\{C\theta^{-\frac{n-5}{2}}Q^{-(\frac{(p-1)(n-5)}{8}-1)}\sin(\beta\ln(Q^{\frac{p-1}{4}}\theta)+\kappa)\Big\}\\&+Q^{\frac{\tilde\nu_2}{\alpha}+1-n}O(\theta^{\tilde\nu_2-2})+\mu\tilde\nu_1(\tilde\nu_1-1)B_pQ^{\frac{\tilde\nu_1}{\alpha}+1-n}\theta^{\tilde\nu_1-2}\\&+O(\mu^2Q^{\frac{\tilde\nu_1+\nu_1}{\alpha}+1-n}\theta^{\tilde\nu_1+\nu_1-2}+\mu Q^{\frac{\tilde\nu_1+\sigma}{\alpha}+1-n}\theta^{\sigma+\tilde\nu_1-2}),\end{aligned}$$

$$\begin{aligned}\frac{\partial^n}{\partial Q^n}(\frac{\partial^3}{\partial\theta^3}v(Q,\mu,\theta)) =&\frac{\partial^n}{\partial Q^n}\Big(-\alpha(\alpha+1)(\alpha+2)\frac{A_p}{\theta^{\alpha+3}}\Big)\\&+\frac{\partial^{n+3}}{\partial Q^n\partial\theta^3}\Big\{C\theta^{-\frac{n-5}{2}}Q^{-(\frac{(p-1)(n-5)}{8}-1)}\sin(\beta\ln(Q^{\frac{p-1}{4}}\theta)+\kappa)\Big\}\\&+Q^{\frac{\tilde\nu_2}{\alpha}+1-n}O(\theta^{\tilde\nu_2-3})+\mu\tilde\nu_1(\tilde\nu_1-1)(\tilde\nu_1-2)B_pQ^{\frac{\tilde\nu_1}{\alpha}+1-n}\theta^{\tilde\nu_1-3}\\&+O(\mu^2Q^{\frac{\tilde\nu_1+\nu_1}{\alpha}+1-n}\theta^{\tilde\nu_1+\nu_1-3}+\mu Q^{\frac{\tilde\nu_1+\sigma}{\alpha}+1-n}\theta^{\sigma+\tilde\nu_1-3}).\end{aligned}$$

For $n=0,1,$ we have

$$\begin{aligned}\frac{\partial^n}{\partial\mu^n}(v(Q,\mu,\theta)) =&\mu^{1-n}B_pQ^{\frac{\tilde\nu_1}{\alpha}+1}\theta^{\tilde\nu_1}\\&+O(\mu^{2-n}Q^{\frac{\tilde\nu_1+\nu_1}{\alpha}+1}\theta^{\tilde\nu_1+\nu_1}+\mu^{1-n}Q^{\frac{\tilde\nu_1+\sigma}{\alpha}+1}\theta^{\sigma+\tilde\nu_1}),\end{aligned}$$

$$\begin{aligned}\frac{\partial^n}{\partial\mu^n}(\frac{\partial}{\partial\theta}v(Q,\mu,\theta)) =&\mu^{1-n}\tilde\nu_1B_pQ^{\frac{\tilde\nu_1}{\alpha}+1}\theta^{\tilde\nu_1-1}\\&+O(\mu^{2-n}Q^{\frac{\tilde\nu_1+\nu_1}{\alpha}+1}\theta^{\tilde\nu_1+\nu_1-1}+\mu^{1-n}Q^{\frac{\tilde\nu_1+\sigma}{\alpha}+1}\theta^{\sigma+\tilde\nu_1-1}),\end{aligned}$$

$$\begin{aligned}\frac{\partial^n}{\partial\mu^n}\Big(\frac{\partial^2}{\partial\theta^2}v(Q,\mu,\theta)\Big) =&\mu^{1-n}\tilde\nu_1(\tilde\nu_1-1)B_pQ^{\frac{\tilde\nu_1}{\alpha}+1}\theta^{\tilde\nu_1-2}\\&+O(\mu^{2-n}Q^{\frac{\tilde\nu_1+\nu_1}{\alpha}+1}\theta^{\tilde\nu_1+\nu_1-2}+\mu^{1-n}Q^{\frac{\tilde\nu_1+\sigma}{\alpha}+1}\theta^{\sigma+\tilde\nu_1-2}),\end{aligned}$$

$$\begin{aligned}\frac{\partial^n}{\partial\mu^n}\Big(\frac{\partial^3}{\partial\theta^2}v(Q,\mu,\theta)\Big) =&\mu^{1-n}\tilde\nu_1(\tilde\nu_1-1)(\tilde\nu_1-2)B_pQ^{\frac{\tilde\nu_1}{\alpha}+1}\theta^{\tilde\nu_1-3}\\&+O(\mu^{2-n}Q^{\frac{\tilde\nu_1+\nu_1}{\alpha}+1}\theta^{\tilde\nu_1+\nu_1-3}+\mu^{1-n}Q^{\frac{\tilde\nu_1+\sigma}{\alpha}+1}\theta^{\sigma+\tilde\nu_1-3}),\end{aligned}$$

while for $n=2,$ we have

$$\frac{\partial^2}{\partial\mu^2}\Big(\frac{\partial^m}{\partial\theta^m}v(Q,\mu,\theta)\Big)=O(Q^{\frac{\tilde\nu_1+\nu_1}{\alpha}+1}\theta^{\tilde\nu_1+\nu_1-m}),\ m=0,1,2,3,$$

where $\kappa=\tan^{-1}\frac{b_0}{a_0},\ C=\sqrt{a_0^2+b_0^2}.$

Proof. These estimates are obtained by expansions of $v_{01}(r)$ and $v_{02}(r)$ given above and direct calculations. □

Lemma 3.4. *In the region* $\theta = |O(Q^{\frac{\sigma}{(2-\sigma)\alpha}})|$, $\mu = O(\theta^{2-\frac{2\nu_1}{\sigma}})$, $\sigma = -\frac{1}{2}(n-5-2\alpha)$, *the solution* $w(Q,\mu,\theta)$ *of (1.7) with* $w(Q,\mu,0) = Q, w_\theta(Q,\mu,0) = 0, w_{\theta\theta}(Q,\mu,0) = (\tilde{b}+\mu)Q^{1+\frac{2}{\alpha}}, w_{\theta\theta\theta}(Q,\mu,0) = 0$ *satisfies*

(1) $\left|\frac{\partial^{m+n}}{\partial Q^n \partial\theta^m} w(Q,\mu,\theta) - \frac{\partial^{m+n}}{\partial Q^n\partial\theta^m} v(Q,\mu,\theta)\right| = Q^{-\frac{(n-5)(p-1)}{8}-(n-1)}\left|o(\theta^{-\frac{n-5}{2}-m})\right|$,

(2) $\left|\frac{\partial^{m+n}}{\partial\mu^n\partial\theta^m} w(Q,\theta) - \frac{\partial^{n+m}}{\partial\mu^n\partial\theta^m} v(Q,\theta)\right| = \left|O(\mu^{2-n}Q^{\frac{\tilde{\nu}_1+\nu_1}{\alpha}+1}\theta^{\tilde{\nu}_1+\nu_1-m})\right|$.

Proof. This lemma can be obtained from Lemma 3.3 and Theorem 3.1. Note that

$$\epsilon = Q^{-\frac{1}{\alpha}},\ -\frac{\sigma}{\alpha} = \frac{(p-1)(n-5)}{8} - 1.$$

Moreover

$$Q^{\frac{p-1}{4}}\theta \in [e^T, e^{10T}]$$

provided that Q is sufficiently large. □

Now we write the inner solutions obtained in Theorem 3.1 in the form of parameter Q and μ:

Theorem 3.2. *Let* $\frac{n+3}{n-5} < p < p_c(n-1)$ *and* $w^{inn}_{Q,\mu}(\theta)$ *be an inner solution of (1.7) with* $w(0) = Q$, $w_\theta(0) = 0$, $w_{\theta\theta}(0) = (\tilde{b}+\mu)Q^{1+\frac{2}{\alpha}}$, $w_{\theta\theta\theta}(0) = 0$. *Then for any sufficiently large* $Q > 0$ *and* $\theta = |O(Q^{\frac{\sigma}{(2-\sigma)\alpha}})| = |O(\mu^{\frac{\sigma}{2\sigma-2\nu_1}})|$,

$$\begin{aligned}
w^{inn}_{Q,\mu}(\theta) =& \frac{A_p}{\theta^\alpha} + \frac{C_p}{\theta^{\alpha-2}} + B_p\mu Q^{\frac{\nu_1}{\alpha}}\theta^{\tilde{\nu}_1} + \sum_{k=2}^{\infty}\sum_{j=1}^{k} d^k_j Q^{-\frac{(p-1)(k-j)}{2}}\theta^{2j-\alpha} \\
&+ Q^{\frac{\sigma}{\alpha}}\Big[\frac{a_0\cos(\beta\ln(Q^{\frac{p-1}{4}}\theta)) + b_0\sin(\beta\ln(Q^{\frac{p-1}{4}}\theta))}{\theta^{\frac{n-5}{2}}} \\
&+ \frac{a_1\cos(\beta\ln(Q^{\frac{p-1}{4}}\theta)) + b_1\sin(\beta\ln(Q^{\frac{p-1}{4}}\theta))}{\theta^{\frac{n-5}{2}-2}} + O(\theta^{2-\frac{n-5}{2}}) \\
&+ \sum_{k=2}^{\infty}\Big(\sum_{j=1}^{k} e^k_j Q^{-\frac{(p-1)(k-j)}{2}}\theta^{2j-\frac{n-5}{2}}\sin(\beta\ln(Q^{\frac{p-1}{4}}\theta) + E^k_j) + o(\theta^{2k-\frac{n-5}{2}})\Big)\Big] \\
&+ Q\sum_{k=1}^{\infty}\Big[\sum_{j=1}^{k}(\mu f^k_j Q^{-\frac{2k-2j-\tilde{\nu}_1}{\alpha}}\theta^{2j+\tilde{\nu}_1}) + O(\mu^2 Q^{\frac{\tilde{\nu}_1+\nu_1}{\alpha}}\theta^{\tilde{\nu}_1+\nu_1+2k} \\
&+ \mu Q^{\frac{\tilde{\nu}_1+\sigma}{\alpha}}\theta^{\tilde{\nu}_1+\sigma+2k})\Big].
\end{aligned}$$

4. Outer solutions

In this section we construct outer solutions for (1.7). Let $w_*(\theta)$ be the singular solution given in (1.8), and we have the following lemma.

Lemma 4.1. *Equation*

$$T_1\phi(\theta) + k_1T_2\phi(\theta) + k_0\phi = pw_*^{p-1}(\theta)\phi(\theta), \quad 0 < \theta < \frac{\pi}{2}, \tag{4.1}$$

admits a solution which can be written as

$$\phi(\theta)=\theta^{-\frac{n-5}{2}}\Big[c_1\cos(\beta\ln\frac{\theta}{2})+c_2\sin(\beta\ln\frac{\theta}{2})\Big]+O(\theta^{2-\frac{n-5}{2}})\text{ as }\theta\to 0, \tag{4.2}$$

where c_1, c_2 are constants such that $c_1^2+c_2^2\neq 0$, and also admits another solution which can be written as

$$\psi(\theta)=c_0\theta^{\tilde\nu_2}+O(\theta^{\tilde\nu_2+2})\text{ as }\theta\to 0, \tag{4.3}$$

where c_0 is a nonzero constant. Here T_1 and T_2 are differential operators defined in (1.7).

Proof. For the following equations,

$$\begin{cases} T_1\phi_1(\theta)+k_1T_2\phi_1(\theta)+k_0\phi_1(\theta)=pw_*^{p-1}(\theta)\phi_1(\theta), & 0<\theta<\frac{\pi}{2},\\ \phi_1(\frac{\pi}{2})=1,\ \ \phi_1'(\frac{\pi}{2})=0,\ \ \phi_1''(\frac{\pi}{2})=0,\ \ \phi_1'''(\frac{\pi}{2})=0 \end{cases} \tag{4.4}$$

and

$$\begin{cases} T_1\phi_2(\theta)+k_1T_2\phi_2(\theta)+k_0\phi_2(\theta)=pw_*^{p-1}(\theta)\phi_2(\theta), & 0<\theta<\frac{\pi}{2},\\ \phi_2(\frac{\pi}{2})=0,\ \ \phi_2'(\frac{\pi}{2})=0,\ \ \phi_2''(\frac{\pi}{2})=1,\ \ \phi_2'''(\frac{\pi}{2})=0, \end{cases} \tag{4.5}$$

we claim that both $\phi_1\theta)$ and $\phi_2(\theta)$ are strictly decreasing for $\theta\in(0,\frac{\pi}{2})$. We only show the case of $\phi_2(\theta)$, the case of $\phi_1(\theta)$ can be treated similarly.

Let us set $A(\theta)=\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi_2(\theta)}{d\theta}\Big)$. Before proving that $\phi_2(\theta)$ is decreasing, we first present a useful fact that will be used in the following proof. If $A(\theta)>0$ for $\theta\in(\theta_0,\frac{\pi}{2})$, where $\theta_0\in(0,\frac{\pi}{2})$, then $\phi_2'(\theta)<0$ for $\theta\in(\theta_0,\frac{\pi}{2})$ and $\phi_2(\theta)>0$ for $\theta\in(\theta_0,\frac{\pi}{2})$. The proof of this fact is simple, thus we omit it here. Next, we start to show that $\phi_2(\theta)$ is decreasing. By using the boundary condition of ϕ_2 at $\theta=\frac{\pi}{2}$, we have $A(\frac{\pi}{2})=1$ and find $\theta_1\in(0,\frac{\pi}{2})$ such that $A(\theta)>0$ for $\theta\in(\theta_1,\frac{\pi}{2})$, then $\phi_2(\theta)>0$ for $\theta\in(\theta_1,\frac{\pi}{2})$. Using the fact $k_1(n)<0$ and the second conclusion in Lemma 6.1, we have

$$T_1\phi_2(\theta)=[pw_*^{p-1}-k_0]\phi_2(\theta)-k_1\frac{A(\theta)}{\sin^{n-2}\theta}>0,\quad\text{for }\theta\in(\theta_1,\tfrac{\pi}{2}).$$

Now we are going to show that $\theta_1=0$. If not, $\theta_1\in(0,\frac{\pi}{2})$ and $A(\theta_1)=0$. For $\theta\in(\theta_1,\frac{\pi}{2})$, we have

$$\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d}{d\theta}\Big(\frac{A(\theta)}{\sin^{n-2}\theta}\Big)\Big)>0,$$

using this inequality and the following fact

$$\frac{d}{d\theta}\Big(\frac{A(\theta)}{\sin^{n-2}\theta}\Big)\Big|_{\theta=\frac{\pi}{2}}=0,$$

we have

$$\frac{d}{d\theta}\Big(\frac{A(\theta)}{\sin^{n-2}\theta}\Big)<0\quad\text{for }\theta\in(\theta_1,\tfrac{\pi}{2}). \tag{4.6}$$

It follows from (4.6) that

$$\frac{A(\theta)}{\sin^{n-2}\theta}>1\quad\text{for }\theta\in(\theta_1,\tfrac{\pi}{2}), \tag{4.7}$$

which contradicts the fact that $A(\theta_1)=0$. Thus, $A(\theta)>0$ and $\phi_2'(\theta)<0$ for $\theta\in(0,\frac{\pi}{2})$. Hence, we prove the claim.

We now prove that there are $D_1\neq 0$ and $D_2\neq 0$ such that for θ near 0,

$$\phi_1(\theta)=D_1\theta^{\tilde\nu_2}+O(\theta^{2+\tilde\nu_2}) \tag{4.8}$$

and

$$\phi_2(\theta) = D_2\theta^{\tilde{\nu}_2} + O(\theta^{2+\tilde{\nu}_2}). \tag{4.9}$$

We also only show (4.9). The proof of (4.8) is similar. Using Emden-Fowler transformations:

$$\tilde{\phi}(t) = (\sin\theta)^{\alpha}\phi_2(\theta),\ t = \ln(\tan\frac{\theta}{2}),$$

we obtain that for $t \in (-\infty, 0)$, $\tilde{\phi}(t)$ satisfies the following homogeneous equation

$$\tilde{\phi}^{(4)}(t) + a_3(t)\tilde{\phi}'''(t) + a_2(t)\tilde{\phi}''(t) + a_1(t)\tilde{\phi}'(t) + a_0(t)\tilde{\phi}(t) = 0, \tag{4.10}$$

where

$$a_3(t) = K_3 + O(e^{2t}),\ a_2(t) = K_2 + O(e^{2t}),\ a_1(t) = K_1 + O(e^{2t}),\ a_0(t) = (1-p)K_0.$$

Therefore,

$$\begin{aligned}\tilde{\phi}^{(4)}(t) + K_3\tilde{\phi}'''(t) + K_2\tilde{\phi}''(t) + K_1\tilde{\phi}'(t) + (1-p)K_0\tilde{\phi}(t)\\ = O(e^{2t}(\tilde{\phi}'''(t) + \tilde{\phi}''(t) + \tilde{\phi}'(t))).\end{aligned} \tag{4.11}$$

Following the arguments in the proof of Lemma 3.1, we can write the solutions of (4.11) as (for any $T \ll -1$):

$$\begin{aligned}\tilde{\phi}(t) =& A_5e^{\sigma t}\cos\beta t + A_6e^{\sigma t}\sin\beta t + A_7e^{\nu_2 t} + A_8e^{\nu_1 t}\\ &+ B_5\int_{-\infty}^{t} e^{\sigma(t-s)}\sin\beta(t-s)g(s,\tilde{\phi}(s))\mathrm{d}s\\ &+ B_6\int_{-\infty}^{t} e^{\sigma(t-s)}\cos\beta(t-s)g(s,\tilde{\phi}(s))\mathrm{d}s\\ &+ B_7\int_{-\infty}^{t} e^{\nu_2(t-s)}g(s,\tilde{\phi}(s))\mathrm{d}s\\ &+ B_8\int_{T}^{t} e^{\nu_1(t-s)}g(s,\tilde{\phi}(s))\mathrm{d}s,\end{aligned} \tag{4.12}$$

where $g(t,\tilde{\phi}(t))$ is the right hand side of (4.11), A_8 depends on T and each B_{i+4} depends only on ν_i $(i = 1,2,3,4)$. It is known from (4.12) that if $A_7 = 0$, then for $|t|$ sufficiently large,

$$\tilde{\phi}(t) = A_5e^{\sigma t}\cos\beta t + A_6e^{\sigma t}\sin\beta t + O(e^{(2+\sigma)t}) \tag{4.13}$$

with $A_5^2 + A_6^2 \neq 0$ or

$$\tilde{\phi}(t) = A_8e^{\nu_1 t} + O(e^{(2+\nu_1)t}) \tag{4.14}$$

with $A_8 \neq 0$. Otherwise, if $A_5^2 + A_6^2 = 0$ and $A_8 = 0$, we know that $\tilde{\phi}(t) = O(e^{(2+\nu_1)t})$. Substituting this into (4.12), we see that $\tilde{\phi}(t) = O(e^{(4+\nu_1)t})$, repeating this procedure, we eventually obtain that $\tilde{\phi}(t) \equiv 0$. This is impossible. Therefore, for θ near 0,

$$\phi_2(\theta) = A_5\theta^{-\frac{n-5}{2}}\cos(\beta\ln\frac{\theta}{2}) + A_6\theta^{-\frac{n-5}{2}}\sin(\beta\ln\frac{\theta}{2}) + O(\theta^{2-\frac{n-5}{2}})$$

or

$$\phi_2(\theta) = A_8\theta^{\tilde{\nu}_1} + O(\theta^{2+\tilde{\nu}_1}).$$

But these contradict the fact that $\phi_2(\theta)$ is strictly decreasing for $\theta \in (0, \frac{\pi}{2})$. Thus, we prove the claim and get (4.9).

Let $\phi(\theta)=\phi_1(\theta)-\frac{D_1}{D_2}\phi_2(\theta)$. Then $\phi(\theta)$ satisfies the problem

$$\begin{cases} T_1\phi(\theta)+k_1T_2\phi(\theta)+k_0\phi(\theta)=pw_*^{p-1}(\theta)\phi(\theta), \quad 0<\theta<\frac{\pi}{2}, \\ \phi(\frac{\pi}{2})=1,\ \ \phi'(\frac{\pi}{2})=0,\ \ \phi''(\frac{\pi}{2})=-\frac{D_1}{D_2},\ \ \phi'''(\frac{\pi}{2})=0. \end{cases} \tag{4.15}$$

We claim that for θ near 0,

$$\phi(\theta)=\theta^{-\frac{n-5}{2}}\Big[c_1\cos(\beta\ln\frac{\theta}{2})+c_2\sin(\beta\ln\frac{\theta}{2})\Big]+O(\theta^{2-\frac{n-5}{2}}) \tag{4.16}$$

with $c_1^2+c_2^2\neq 0$. Using Emden-Fowler transformations:

$$\hat\phi(t)=(\sin\theta)^{\alpha}\phi(\theta),\ t=\ln(\tan\frac{\theta}{2}) \tag{4.17}$$

and (4.8), (4.9), we obtain that for t near $-\infty$,

$$\hat\phi(t)=e^{\sigma t}[c_1\cos(\beta t)+c_2\sin(\beta t)]+c_3e^{\nu_1 t}+O(e^{(2+\sigma)t}) \tag{4.18}$$

provided $c_1^2+c_2^2\neq 0$ or

$$\hat\phi(t)=c_3e^{\nu_1 t}+O(e^{(2+\nu_1)t}) \tag{4.19}$$

provided $c_1^2+c_2^2=0$ and $c_3\neq 0$. (Note that if both $c_1^2+c_2^2=0$ and $c_3=0$, we can obtain $\hat\phi(t)\equiv 0$. This is impossible.) We now show that (4.19) can not occur. On the contrary, we see that for θ near 0,

$$\phi(\theta)=c_3\theta^{\tilde\nu_1}+O(\theta^{2+\tilde\nu_1}).$$

This implies that $\phi(\theta)\to 0$ as $\theta\to 0$. Since

$$\hat\phi(t)=O(e^{\nu_1 t}),\ \ \hat\phi'(t)=O(e^{\nu_1 t}),\ \ \hat\phi''(t)=O(e^{\nu_1 t}),\ \ \hat\phi'''(t)=O(e^{\nu_1 t}),$$

we obtain from (4.17) that

$$\phi'(\theta)=O(\theta^{\tilde\nu_1-1}),\ \ \sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}=O(\theta^{n-3+\tilde\nu_1}),\ \ \frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)=O(\theta^{n-4+\tilde\nu_1}).$$

The similar arguments imply that

$$\sin^{n-2}\theta\frac{d}{d\theta}\Big(\frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)\Big)=O(\theta^{n-5+\tilde\nu_1}).$$

If we define $e(\theta)=\sin^{n-2}\theta\frac{d}{d\theta}\Big(\frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)\Big)$, we see that $e(0)=0$. Then, we claim that ϕ changes sign in $(0,\frac{\pi}{2})$. Suppose that this is not true, without loss generality, we assume $\phi>0$ in $(0,\frac{\pi}{2})$. Then it follows from the equation of ϕ that for $\theta\in(0,\frac{\pi}{2})$,

$$\frac{d}{d\theta}\Big[e(\theta)-k_1\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big)\Big]=\sin^{n-2}\theta\Big[pw_*^{p-1}-k_0\Big]\phi(\theta)>0. \tag{4.20}$$

But integrating both sides of (4.20) in $(0,\frac{\pi}{2})$ and using the boundary conditions : $\phi'(\frac{\pi}{2})=\phi'''(\frac{\pi}{2})=0$, we obtain that

$$\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\Big[pw_*^{p-1}-k_0\Big]\phi(\theta)d\theta=0.$$

This is clearly impossible. Noticing that $\phi\neq 0$ for θ near 0, we see that there is a minimal zero point $\hat\theta\in(0,\frac{\pi}{2})$ of ϕ. Without loss generality, we assume that $\phi>0$ in $(0,\hat\theta)$. It follows from (4.20) that $E(\theta):=e(\theta)+k_1\Big(\sin^{n-2}\theta\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}\Big)$ is increasing for $\theta \in (0, \hat{\theta})$. Noticing $E(0) = 0$, we then obtain that $E(\theta) > 0$ for $\theta \in (0, \hat{\theta})$. Therefore,

$$\frac{d}{d\theta}\Big[\frac{1}{\sin^{n-2}\theta}\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big) + k_1\phi(\theta)\Big] > 0 \quad \text{for } \theta \in (0, \hat{\theta}). \tag{4.21}$$

Moreover, by a similar argument, we have

$$\frac{d}{d\theta}\Big(\sin^{n-2}\theta\frac{d\phi(\theta)}{d\theta}\Big) > 0 \quad \text{for } \theta \in (0, \hat{\theta}), \tag{4.22}$$

and

$$\frac{d\phi(\theta)}{d\theta} > 0 \quad \text{for } \theta \in (0, \hat{\theta}). \tag{4.23}$$

But (4.23) implies that $\phi(\hat{\theta}) > 0$, which contradicts the fact that $\phi(\hat{\theta}) = 0$. This contradiction implies that (4.19) can not occur and thus (4.18) holds. As a consequence, (4.16) holds and hence (4.2) holds.

Let $\psi(\theta) = \phi_1(\theta)$. We easily know that (4.3) can be obtained from (4.8). This completes the proof of this lemma. □

For any sufficiently small $\delta > \eta > 0$, we set $\psi_1(\theta)$ to be the solution of the problem

$$\begin{cases} T_1\psi_1(\theta) + k_1T_2\psi_1(\theta) + k_0\psi_1(\theta) = \eta^{-2}\Big[(w_* + \Phi + \Psi)^p - w_*^p - pw_*^{p-1}(\Phi + \eta^2\psi)\Big], \\ (\psi_1 + \psi)(\frac{\pi}{2}) = 2, \;\; (\psi_1 + \psi)'(\frac{\pi}{2}) = 0, \;\; (\psi_1 + \psi)''(\frac{\pi}{2}) = \frac{D_1\delta^2}{D_2\eta^2}, \;\; (\psi_1 + \psi)'''(\frac{\pi}{2}) = 0, \end{cases} \tag{4.24}$$

where $\psi(\theta)$ is given in Lemma 4.1, $\Phi = \delta^2\phi(\theta)$ and $\Psi = \eta^2(\psi_1(\theta) + \psi(\theta))$. We can see that Ψ satisfies the problem

$$\begin{cases} T_1\Psi(\theta) + k_1T_2\Psi(\theta) + k_0\Psi(\theta) = (w_* + \Phi + \Psi)^p - w_*^p - pw_*^{p-1}\Phi, \\ \Psi(\frac{\pi}{2}) = 2\eta^2, \;\; \Psi'(\frac{\pi}{2}) = 0, \;\; \Psi''(\frac{\pi}{2}) = \frac{D_1\delta^2}{D_2}, \;\; \Psi'''(\frac{\pi}{2}) = 0. \end{cases} \tag{4.25}$$

This implies that

$$\begin{cases} T_1(\Psi + \Phi) + k_1T_2(\Psi + \Phi) + k_0(\Psi + \Phi) = (w_* + \Phi + \Psi)^p - w_*^p, \\ (\Psi + \Phi)(\frac{\pi}{2}) = 2\eta^2 + \delta^2, \;\; (\Psi + \Phi)'(\frac{\pi}{2}) = 0, \;\; (\Psi + \Phi)''(\frac{\pi}{2}) = 0, \;\; (\Psi + \Phi)'''(\frac{\pi}{2}) = 0. \end{cases} \tag{4.26}$$

Arguments similar to those in the proof of Lemma 4.1 imply that $\Psi(\theta) + \Phi(\theta)$ is strictly decreasing, then

$$\Psi(\theta) + \Phi(\theta) > 0 \quad \text{for } \theta \in (0, \tfrac{\pi}{2}). \tag{4.27}$$

Setting $\psi_2(\theta) = \psi(\theta) + \psi_1(\theta)$, we easily see that ψ_2 satisfies the problem

$$\begin{cases} T_1\psi_2(\theta) + k_1T_2\psi_2(\theta) + k_0\psi_2(\theta) = pw_*^{p-1}\psi_2 \\ \qquad\qquad + \eta^{-2}\Big[(w_* + \Phi + \eta^2\psi_2)^p - w_*^p - pw_*^{p-1}(\Phi + \eta^2\psi_2)\Big], \\ \psi_2(\frac{\pi}{2}) = 2, \;\; \psi_2'(\frac{\pi}{2}) = 0, \;\; \psi_2''(\frac{\pi}{2}) = \frac{D_1\delta^2}{D_2\eta^2}, \;\; \psi_2'''(\frac{\pi}{2}) = 0. \end{cases} \tag{4.28}$$

By the Emden-Fowler transformation:

$$\tilde{\psi}_2(t) = (\sin\theta)^\alpha\psi_2(\theta), \quad t = \ln\tan\frac{\theta}{2},$$

we see that $\tilde{\psi}_2(t)$ satisfies the problem

$$\begin{cases} \tilde{\psi}_2^{(4)}(t)+a_3(t)\tilde{\psi}_2'''(t)+a_2(t)\tilde{\psi}_2''(t)+a_1(t)\tilde{\psi}_2'(t)+a_0(t)\tilde{\psi}_2(t)=G(\tilde{\psi}_2(t)),\ -\infty<t<0,\\ \tilde{\psi}_2'(0)=0,\ \tilde{\psi}_2'''(0)=0, \end{cases} \tag{4.29}$$

where $a_0(t), a_1(t), a_2(t), a_3(t)$ are defined in (4.10), and

$$G(\tilde{\psi}_2(t))=(\sin\theta)^{4+\alpha}\eta^{-2}\Big\{[w_*+\Phi+\eta^2\sin^{-\alpha}\theta\tilde{\psi}_2]^p-w_*^p-pw_*^{p-1}(\Phi+\eta^2\sin^{-\alpha}\theta\tilde{\psi}_2)\Big\}.$$

Moreover, we can rewrite (4.29) as the following form: (see the proof of Lemma 4.1)

$$\tilde{\psi}_2^{(4)}(t)+K_3\tilde{\psi}_2'''(t)+K_2\tilde{\psi}_2''(t)+K_1\tilde{\psi}_2'(t)+(1-p)K_0\tilde{\psi}_2(t)=G(\tilde{\psi}_2(t))+g(t,\tilde{\psi}_2(t)), \tag{4.30}$$

where $g(t,\tilde{\psi}_2(t))=O\Big(e^{2t}(\tilde{\psi}_2'''(t)+\tilde{\psi}_2''(t)+\tilde{\psi}_2'(t))\Big)$ for $t\ll -1$. Therefore, for $t<T$ with any $T\ll -1$,

$$\begin{aligned}\tilde{\psi}_2(t)=&D_5e^{\nu_2 t}+D_6e^{\sigma t}\cos\beta t+D_7e^{\sigma t}\sin\beta t+D_8e^{\nu_1 t}\\ &+B_5\int_{-\infty}^t e^{\sigma(t-s)}\sin\beta(t-s)(G(\tilde{\psi}_2(s))+g(s,\tilde{\psi}_2(s)))\mathrm{d}s\\ &+B_6\int_{-\infty}^t e^{\sigma(t-s)}\cos\beta(t-s)(G(\tilde{\psi}_2(s))+g(s,\tilde{\psi}_2(s)))\mathrm{d}s\\ &+B_7\int_{-\infty}^t e^{\nu_2(t-s)}(G(\tilde{\psi}_2(s))+g(s,\tilde{\psi}_2(s)))\mathrm{d}s\\ &+B_8\int_{T}^t e^{\nu_1(t-s)}(G(\tilde{\psi}_2(s))+g(s,\tilde{\psi}_2(s)))\mathrm{d}s,\end{aligned} \tag{4.31}$$

where B_5, B_6, B_7, B_8 depend only on ν_i $(i=1,2,3,4)$. Using the fact $\Psi(\theta)+\Phi(\theta)$ is strictly decreasing in $(0,\frac{\pi}{2})$ and equation (4.2), we conclude that $D_5\neq 0$. Let $\phi(\theta)=\sin^{-\alpha}\theta\tilde{\phi}(t)$, we see that for $t\in[10T,2T]$ and $\delta^2=O(e^{(2-\sigma)t})$, $\eta^2=O(e^{(2-\nu_2)t})$,

$$G(\tilde{\psi}_2(t))=\eta^{-2}O((\delta^2\tilde{\phi}(t)+\eta^2\tilde{\psi}_2(t))^2)=O(e^{(2+\nu_2)t}). \tag{4.32}$$

Note that $\tilde{\phi}(t)=e^{\sigma t}(c_1\cos(\beta t)+c_2\sin(\beta t))+O(e^{(2+\sigma)t})$ and $\tilde{\psi}_2(t)=D_5e^{\nu_2 t}+O(e^{(2+\nu_2)t})$. Then

$$\delta^2\tilde{\phi}(t)+\eta^2\tilde{\psi}_2(t)=O(e^{2t}).$$

Therefore, it follows from (4.31) and (4.32) that

$$\tilde{\psi}_2(t)=D_5e^{\nu_2 t}+D_6e^{\sigma t}\cos\beta t+D_7e^{\sigma t}\sin\beta t+O(e^{(2+\nu_2)t}) \tag{4.33}$$

provided $\delta^2=O(e^{(2-\sigma)t})$ and $\eta^2=O(e^{(2-\nu_2)t})$. Hence, for θ near 0,

$$\Psi(\theta)=\eta^2\Big[D_5\theta^{\tilde{\nu}_2}+\theta^{-\frac{n-5}{2}}\Big(D_6\cos(\beta\ln\frac{\theta}{2})+D_7\sin(\beta\ln\frac{\theta}{2})\Big)+O(\theta^{2+\tilde{\nu}_2})\Big] \tag{4.34}$$

with $D_5\neq 0$ provided that $\theta=O(\delta^{\frac{2}{2-\sigma}})=O(\eta^{\frac{2}{2-\nu_2}})$. Since $\tilde{\nu}_2<3-n$, we easily know that $\tilde{\nu}_2+2<-(n-5)<-(n-5)/2$. Thus, $\theta^{-\frac{n-5}{2}}=o(\theta^{2+\tilde{\nu}_2})$.

Now we can obtain the following theorem.

Theorem 4.1. *For any $\delta > \eta > 0$ sufficiently small, problem (1.7) admits outer solutions $w^{out}_{\delta,\eta} \in C^4(0, \frac{\pi}{2})$ satisfying*

$$w^{out}_{\delta,\eta}(\theta) = w_*(\theta) + \Phi(\theta) + \Psi(\theta),\ \theta \in (0, \frac{\pi}{2}) \tag{4.35}$$

with $(w^{out}_{\delta,\eta})'(\frac{\pi}{2}) = (w^{out}_{\delta,\eta})'''(\frac{\pi}{2}) = 0$. Moreover,

$$\begin{aligned} w^{out}_{\delta,\eta}(\theta) =& \frac{A_p}{\theta^\alpha} + \frac{2A_p}{3(p-1)}\frac{1}{\theta^{\alpha-2}} + \delta^2\Big[\frac{\vartheta_1\cos(\beta\ln\frac{\theta}{2}) + \vartheta_2\sin(\beta\ln\frac{\theta}{2})}{\theta^{\frac{n-5}{2}}} + O\Big(\frac{1}{\theta^{\frac{n-5}{2}-2}}\Big)\Big] \\ &+ \eta^2\Big[\vartheta_3\theta^{\tilde{\nu}_2} + O(\theta^{\tilde{\nu}_2+2})\Big] \end{aligned} \tag{4.36}$$

provided that $\theta = O(\delta^{\frac{2}{2-\sigma}}) = O(\eta^{\frac{2}{2-\nu_2}})$, where $\vartheta_1, \vartheta_2, \vartheta_3$ are constants independent of δ, η such that $\vartheta_1^2 + \vartheta_2^2 \neq 0, \vartheta_3 \neq 0$.

Proof. The proof can be obtained from the expressions of $w_*(\theta)$, $\Phi(\theta)$ and $\Psi(\theta)$ given in (1.8), (4.16) and (4.34). □

5. Infinitely many solutions of (1.7) and Proof of Theorem 1.1

In this section, we construct infinitely many regular solutions for (1.7) by matching the inner and outer solutions.

We construct solutions of the problem

$$\begin{cases} T_1w + k_1T_2w + k_0w = w^p, \quad w(\theta) > 0,\ 0 < \theta < \frac{\pi}{2}, \\ w(0) = Q(:= \epsilon^{-\alpha}),\ w'(\frac{\pi}{2}) = 0,\ w''(0) = (\tilde{b} + \mu)\epsilon^{-\alpha-2},\ w'''(\frac{\pi}{2}) = 0 \end{cases} \tag{5.1}$$

by matching the inner and outer solutions given in Theorems 3.2 and 4.1. To do so, we will find $\Theta \in (0, \pi/2)$ with

$$\Theta = O(Q^{\frac{\sigma}{(2-\sigma)\alpha}})\ \ (Q \gg 1)$$

such that the following identities hold:

$$[w^{inn}_{Q,\mu}(\theta) - w^{out}_{\delta,\eta}(\theta)]|_{\theta=\Theta} = 0, \tag{5.2}$$

$$[w^{inn}_{Q,\mu}(\theta) - w^{out}_{\delta,\eta}(\theta)]'_\theta|_{\theta=\Theta} = 0, \tag{5.3}$$

$$[w^{inn}_{Q,\mu}(\theta) - w^{out}_{\delta,\eta}(\theta)]''_\theta|_{\theta=\Theta} = 0, \tag{5.4}$$

$$[w^{inn}_{Q,\mu}(\theta) - w^{out}_{\delta,\eta}(\theta)]'''_\theta|_{\theta=\Theta} = 0. \tag{5.5}$$

These will be done by arguments similar to those in the proof of Lemma 6.1 of [1] and Theorem 1.1 of [4]. Then, we obtain a C^4 function $w(\theta)$ defined by $w(\theta) = w^{inn}_{Q,\mu}(\theta)$ for $\theta \leq \Theta$ and $w(\theta) = w^{out}_{\delta,\eta}(\theta)$ for $\theta \geq \Theta$ which is a solution to (5.1).

First, we observe that

$$\frac{2A_p}{3(p-1)} = C_p \tag{5.6}$$

by (3.35), where A_p, C_p are given in Section 3.

Define Q_*, μ_*, δ_*^2 and η_*^2 by

$$\beta\ln Q_*^{\frac{p-1}{4}} + \kappa = \beta\ln 2^{-1} + \omega + 2m\pi, \tag{5.7}$$

$$\delta_*^2 = \sqrt{\frac{a_0^2 + b_0^2}{\vartheta_1^2 + \vartheta_2^2}}Q_*^{\frac{\sigma}{\alpha}}, \tag{5.8}$$

$$\eta_*^2 = O(Q_*^{\frac{(2-\nu_2)\sigma}{(2-\sigma)\alpha}}),\ \mu_* = O(Q_*^{\frac{2\sigma-2\nu_1}{(2-\sigma)\alpha}}), \tag{5.9}$$

$$\mu_* B_p Q_*^{\frac{\nu_1}{\alpha}} = \vartheta_3 \eta_*^2 \Theta_*^{\tilde{\nu}_2-\tilde{\nu}_1}, \tag{5.10}$$

where κ, ω are given by

$$\kappa = \tan^{-1}(\frac{b_0}{a_0}),\ \omega = \tan^{-1}(\frac{\vartheta_2}{\vartheta_1}),$$

and $m \gg 1$ is an integer. The integer m is chosen such that the results in Section 3 and Section 4 hold.

Note that

$$O(\delta_*^{\frac{2}{2-\sigma}}) = O(Q_*^{\frac{\sigma}{\alpha(2-\sigma)}}),$$

$$a_0\cos(\beta\ln(Q^{\frac{p-1}{4}}\theta)) + b_0\sin(\beta\ln(Q^{\frac{p-1}{4}}\theta)) = \sqrt{a_0^2+b_0^2}\sin(\beta\ln\theta + \beta\ln Q^{\frac{p-1}{4}} + \kappa),$$

$$\vartheta_1\cos(\beta\ln\frac{\theta}{2}) + \vartheta_2\sin(\beta\ln\frac{\theta}{2}) = \sqrt{\vartheta_1^2+\vartheta_2^2}\sin(\beta\ln\theta + \beta\ln 2^{-1} + \omega).$$

We will see that Q, μ, δ^2 and η^2 required to satisfy the matching conditions (5.2)-(5.5) can be obtained as small perturbations of Q_*, μ_*, δ_*^2 and η_*^2 given in (5.7)-(5.10), i.e.,

$$Q = Q_*(1 + O(Q_*^{\frac{2\sigma}{(2-\sigma)\alpha}})), \tag{5.11}$$

$$\mu = \mu_*(1 + O(Q_*^{\frac{2\sigma}{(2-\sigma)\alpha}})), \tag{5.12}$$

$$\delta^2 = \delta_*^2(1 + O(Q_*^{\frac{2\sigma}{(2-\sigma)\alpha}})), \tag{5.13}$$

$$\eta^2 = \eta_*^2(1 + O(Q_*^{\frac{2\sigma}{(2-\sigma)\alpha}})). \tag{5.14}$$

To show this we define the function $\mathbf{F}(Q,\mu,\delta,\eta)$ by

$$\mathbf{F}(Q,\mu,\delta^2,\eta^2) = \begin{bmatrix} \Theta^{\frac{n-5}{2}}(w_{Q,\mu}^{inn}(\Theta) - w_{\delta,\eta}^{out}(\Theta)) \\ \Theta[\theta^{\frac{n-5}{2}}(w_{Q,\mu}^{inn}(\theta) - w_{\delta,\eta}^{out}(\theta))]'_\theta \mid_{\theta=\Theta} \\ \Theta^2[\theta^{\frac{n-5}{2}}(w_{Q,\mu}^{inn}(\theta) - w_{\delta,\eta}^{out}(\theta))]''_\theta \mid_{\theta=\Theta} \\ \Theta^3[\theta^{\frac{n-5}{2}}(w_{Q,\mu}^{inn}(\theta) - w_{\delta,\eta}^{out}(\theta))]'''_\theta \mid_{\theta=\Theta} \end{bmatrix}^T.$$

Now, we regard δ^2, η^2 as new variables. Taking Q_*,μ_*,δ_*^2 and η_*^2, we find a bound for $\mathbf{F}(Q_*,\mu_*,\delta_*^2,\eta_*^2)$ by making use of the behaviors of $w_{Q,\mu}^{inn}(\theta)$ and $w_{\delta,\eta}^{out}(\theta)$ given in Theorems 3.2 and Theorem 4.1 respectively. Accordingly we find for some $M > 1$ suitably large,

$$\left|\Theta^{-\frac{n-5}{2}}\mathbf{F}(Q_*,\mu_*,\delta_*^2,\eta_*^2)\right| \le M\Theta^{4-\sigma-\frac{n-5}{2}} + \text{small terms}. \tag{5.15}$$

We now seek values of Q,μ,δ^2,η^2 which are small perturbations of $Q_*,\mu_*,\delta_*^2,\eta_*^2$ and such that $\mathbf{F}(Q,\mu,\delta^2,\eta^2) = 0$. As in [4], we need to evaluate the Jacobian of $\mathbf{F}$ at $(Q_*,\mu_*,\delta_*^2,\eta_*^2)$:

$$\frac{\partial\mathbf{F}(Q,\mu,\delta^2,\eta^2)}{\partial(Q,\mu,\delta^2,\eta^2)} = \begin{bmatrix} I_1+I_3 & I_4 & -D\sin\tau & I_5 \\ \beta I_2 + q_1 I_3 & q_1 I_4 & -\beta D\cos\tau & q_4 I_5 \\ I_6 & q_2 I_4 & I_8 & q_5 I_5 \\ I_7 & q_3 I_4 & I_9 & q_6 I_5 \end{bmatrix} + \text{h.o.t.},$$

where

$$I_1 = C(\frac{\sigma}{\alpha}\sin\tau + \frac{\beta(p-1)}{4}\cos\tau)Q_*^{\frac{\sigma}{\alpha}-1},\ I_2 = C(\frac{\sigma}{\alpha}\cos\tau - \frac{\beta(p-1)}{4}\sin\tau)Q_*^{\frac{\sigma}{\alpha}-1},$$

$$I_3=\frac{\nu_1}{\alpha}B_p\mu_*\Theta^{\tilde\nu_1+\frac{n-5}{2}}Q_*^{\frac{\nu_1}{\alpha}-1},\ I_4=B_pQ_*^{\frac{\nu_1}{\alpha}}\Theta^{\tilde\nu_1+\frac{n-5}{2}},\ I_5=-\vartheta_3\Theta^{\tilde\nu_2+\frac{n-5}{2}},$$
$$I_6=-\beta^2I_1-\beta I_2+q_2I_3,\ I_7=-\beta^3I_2+3\beta^2I_1+2\beta I_2+q_3I_3,$$
$$I_8=\beta^2D\sin\tau+\beta D\cos\tau,\ I_9=\beta^3D\cos\tau-3\beta^2D\sin\tau-2\beta D\cos\tau,$$
$$q_1=\tilde\nu_1+\frac{n-5}{2},\ q_2=(\tilde\nu_1+\frac{n-7}{2})q_1,\ q_3=(\tilde\nu_1+\frac{n-9}{2})q_2,$$
$$q_4=\tilde\nu_2+\frac{n-5}{2},\ q_5=(\tilde\nu_2+\frac{n-7}{2})q_4,\ q_6=(\tilde\nu_1+\frac{n-9}{2})q_5,$$
$$C=\sqrt{a_0^2+b_0^2},\quad D=\sqrt{\vartheta_1^2+\vartheta_2^2},$$
and
$$\tau=\beta\ln\Theta+\beta\ln Q_*^{\frac{p-1}{4}}+\kappa=\beta\ln\Theta+\beta\ln 2^{-1}+\omega+2m\pi.$$

We define the function $\mathbf{G}(x,y)$ by
$$\mathbf{G}(x,y,z,w)=\mathbf{F}(Q_*+xQ_*^{1-\frac{\sigma}{\alpha}},\mu_*+\Theta^{-\tilde\nu_1-\frac{n-5}{2}}Q_*^{-\frac{\nu_1}{\alpha}}y,\delta_*^2+z,\eta_*^2+\Theta^{-\tilde\nu_2-\frac{n-5}{2}}w).$$
Using (5.15), (4.36) and the results in Lemmas 3.3, 3.4, we express $\mathbf{G}(x,y,z,w)$ in the form
$$\mathbf{G}(x,y,z,w)=\mathbf{C}+\left[\begin{array}{cccc} I_1'+I_3' & I_4' & -D\sin\tau & I_5' \\ \beta I_2'+q_1I_3' & q_1I_4' & -\beta D\cos\tau & q_4I_5' \\ I_6' & q_2I_4' & I_8' & q_5I_5' \\ I_7' & q_3I_4' & I_9' & q_6I_5' \end{array}+\text{small terms}\right]\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$$
$$+\mathbf{E}(x,y,z,w,Q_*,\mu_*,\delta_*^2,\eta_*^2),$$
where
$$I_1'=C\Big(\frac{\sigma}{\alpha}\sin\tau+\frac{\beta(p-1)}{4}\cos\tau\Big),\ I_2'=C\Big(\frac{\sigma}{\alpha}\cos\tau-\frac{\beta(p-1)}{4}\sin\tau\Big),$$
$$I_3'=\frac{\nu_1}{\alpha}B_p\mu_*\Theta^{\tilde\nu_1+\frac{n-5}{2}}Q_*^{\frac{\nu_1-\sigma}{\alpha}},\ I_4'=B_p,\ I_5'=-\vartheta_3,$$
$$I_6'=-\beta^2I_1'-\beta I_2'+q_2I_3',I_7'=-\beta^3I_2'+3\beta^2I_1'+2\beta I_2'+q_3I_3',$$
$$I_8'=\beta^2D\sin\tau+\beta D\cos\tau,I_9'=\beta^3D\cos\tau-3\beta^2D\sin\tau-2\beta D\cos\tau,$$
$\mathbf{C}$ is a constant vector independent of (x,y,z,w) which is bounded above by $M\delta_*^4\Theta^\sigma$, and $|\mathbf{E}|$ is bounded independently of $x,y,,z,w,Q,\mu,\delta$ and η. Thus,
$$\mathbf{G}(x,y,z,w)=\mathbf{C}+L\begin{pmatrix}x\\y\\z\\w\end{pmatrix}+\mathbf{T}(x,y,z,w),$$
where L is a linear operator which is invertible, we shall prove this fact in Lemma 6.1). If we define the operator $\mathbf{J}$ mapping $\mathbb{R}^4$ into itself by
$$\mathbf{J}(x,y,z,w)=-(L^{-1}\mathbf{C}+L^{-1}\mathbf{T}(x,y,z,w)),$$
then, provided that Q_* is sufficiently large, a direct calculation shows that $\mathbf{J}$ maps the set I into itself, where I is the ball
$$I=\{(x,y,z,w):\ (x^2+y^2+z^2+w^2)^{\frac12}\le 4M(\det L)^{-1}\Theta^{4-\sigma}\},\tag{5.16}$$

and det$L$ is the determindent of L which depends on $\sqrt{a_0^2+b_0^2}, \beta, D, \alpha, B_p, \vartheta_3$ and $\nu_i, i=1,2,3,4$. An application of the Brouwer Fixed Point Theorem to conclude that $\mathbf{J}$ has a fixed point in I. This point (x,y,z,w) satisfies $\mathbf{G}(x,y,z,w)=0$ and

$$(x^2+y^2+z^2+w^2)^{\frac{1}{2}} \leq M'\Theta^{4-\sigma},$$

where M' is a constant defined in (5.16) and is independent of $Q_*, \mu_*, \delta_*, \eta_*$ and Θ. By substituting for Q, μ δ and η, then taking Θ to have the upper limiting value of $Q_*^{\frac{\sigma}{(2-\sigma)\alpha}}$, we obtain (5.11)- (5.14). Therefore, we can find a solution to (5.1) such that (5.2)-(5.5) holds.

We have shown that (5.2)-(5.5) has a solution for each fixed m large. This yields a solution of (5.1) and also gives the proof of Theorem 1.1. Hence we have

Theorem 5.1. *For $m \gg 1$ large and Q, μ, δ and η being given in (5.11)-(5.14), problem (5.1) admits a classical solution $w_{Q,\mu,\delta,\eta}(\theta)$. Moreover, there is $\Theta = |O(Q^{\frac{\sigma}{(2-\sigma)\alpha}})|$ such that (5.2)-(5.5) hold.*

As a consequence, problem (1.7) admits infinitely many nonconstant positive solutions. Hence, we prove Theorem 1.1.

6. Appendix

In the following lemma, we will show the results which were used in the previous sections.

Lemma 6.1. *For the terms E_2, $k_0(n)$, and matrixes K and L, which were defined in previous sections, we have*

(1) $E_2 \neq 0$,
(2) *If* $p \in (\frac{n+3}{n-5}, p_c(n-1))$, *we have* $pk_0(n-1) \geq k_0(n)$,
(3) det$K \neq 0$,
(4) det$L \neq 0$.

Proof. First, we show that $E_2 \neq 0$, it is known that

$$E_2 = (\tilde{\nu}_1+2)\tilde{\nu}_1(\tilde{\nu}_1+n-3)(\tilde{\nu}_1+n-1) - p(n-5-\alpha)(n-3-\alpha)(2+\alpha)\alpha. \quad (6.1)$$

For convenience, we use n instead of $n-1$ and $\tilde{\nu}_1(n)$ instead of $\tilde{\nu}_1(n-1)$, i.e. we study the following term

$$E_2 = (\tilde{\nu}_1+2)\tilde{\nu}_1(\tilde{\nu}_1+n-2)(\tilde{\nu}_1+n) - p(n-4-\alpha)(n-2-\alpha)(2+\alpha)\alpha. \quad (6.2)$$

Let $f(\alpha) = p(n-4-\alpha)(n-2-\alpha)(2+\alpha)\alpha$, through a simple computation, we get $f(\alpha)$ and its derivative $f'(\alpha)$

$$f(\alpha) = \alpha^4 + (12-2n)\alpha^3 + (n^2-18n+52)\alpha^2 + (6n^2-52n+96)\alpha + 8(n-2)(n-4),$$

and

$$f'(\alpha) = 4\alpha^3 + (36-6n)\alpha^2 + (2n^2-36n+104)\alpha + (6n^2-52n+96).$$

We compute the roots of $f'(\alpha)$ and get all its zero points $\frac{1}{2}(n-6\pm\sqrt{n^2+4}), \frac{1}{2}(n-6)$. It is easy to see that $f(\alpha)$ is strictly increasing in $\alpha \in (0, \frac{1}{2}(n-6))$ and decreasing in $\alpha \in (\frac{1}{2}(n-6), \frac{1}{2}(n-6+\sqrt{n^2+4}))$. We know that $\alpha = \frac{4}{p-1} < \frac{n-4}{2}$ and $\frac{n-4}{2} \in (\frac{1}{2}(n-6), \frac{1}{2}(n-6+\sqrt{n^2+4}))$. As a consequence, we can conclude

$$f(\alpha) \leq f(\frac{n-6}{2}) = \frac{n^4}{16} - \frac{n^2}{2} + 1 \text{ for all } p \in (\frac{n+4}{n-4}, p_c(n)).$$

Let $g(x) = x(x+2)(x+n)(x+n-2) = x^4 + 2nx^3 + (n^2+2n-4)x^2 + (2n^2-4n)x$, we compute its derivative $g'(x) = 4x^3 + 6nx^2 + (2n^2+4n-8)x + (2n^2-4n)$ and find $g'(x) > 0$ for $x > 0$ when $n \geq 5$. On the other hand, using $4\sqrt{N_3} > N_2$ for $p \in (\frac{n+4}{n-4}, p_c(n))$, we find

$$\tilde{\nu}_1 > \frac{\sqrt{2(n^2-4n+8)} - (n-4)}{2},$$

therefore,

$$g(\tilde{\nu}_1) \geq g(\frac{\sqrt{2(n^2-4n+8)} - (n-4)}{2}) = 96 - 40n + 11n^2 - \frac{n^3}{2} + \frac{n^4}{16} + \sqrt{2}(24-4n+n^2)\sqrt{8-4n+n^2}. \tag{6.3}$$

Comparing $\frac{n^4}{16} - \frac{n^2}{2} + 1$ and the right hand side of (6.3), by direct computation, we can get

$$g(\frac{\sqrt{2(n^2-4n+8)} - (n-4)}{2}) > \frac{n^4}{16} - \frac{n^2}{2} + 1 \text{ for } n \in (0,\infty).$$

As a result, $g(\tilde{\nu}_1) > f(\alpha)$. Hence, E_2 is nonzero.

Next, we prove the inequality $pk_0(n-1) \geq k_0(n)$ for $p \in (\frac{n+3}{n-5}, p_c(n-1))$. According to the definition of $k_0(n)$, it is enough for us to show the following

$$p(n-5-\alpha)(n-3-\alpha) \geq (n-4-\alpha)(n-2-\alpha). \tag{6.4}$$

Using the relation $p = \frac{4}{\alpha} + 1$, it is equivalent to show the following (after computation)

$$6\alpha^2 + (39-10n)\alpha + 4n^2 - 32n + 60 \geq 0. \tag{6.5}$$

It is known that (6.5) holds provided

$$\alpha \geq \frac{10n - 39 + \sqrt{4n^2-12n+81}}{12} \text{ or } \alpha \leq \frac{10n - 39 - \sqrt{4n^2-12n+81}}{12}.$$

On the other hand, since $p \in (\frac{n+3}{n-5}, p_c(n-1))$, we have $\alpha < \frac{n-5}{2}$. It is easy to show $\frac{n-5}{2} \leq \frac{10n-39-\sqrt{4n^2-12n+81}}{12}$ when $n \geq 5$. Hence, (6.5) holds. Therefore (6.4) holds.

Then, we show K is invertible, it is enough for us to show $B \neq 0$ or $A \neq 0$. we recall

$$B = (2n^2 - 12n - 6)\beta + 8\beta^3 = (2(n-3)^2 - 24)\beta + 8\beta^3.$$

It is known that $2(n-3)^2 - 24 < 0$ only when $n = 6$. Since $\beta > 0$, therefore $B \neq 0$ when $n \geq 7$. When $n = 6$, we find

$$A = \beta^4 - \frac{35}{2}\beta^2 - \frac{135}{2} - (1-\alpha)(3-\alpha)(2+\alpha)\alpha,\ B = -6\beta + 8\beta^3.$$

If $B \neq 0$ for $n = 6$, we have K is invertible, while if $B = 0$ for $n = 6$, then $A = -21 - (1-\alpha)(3-\alpha)(2+\alpha)\alpha < 0$ for $\alpha \in (0, 1/2)$ and K is also invertible. Therefore, we proved the third conclusion.

Finally, we show the matrix L is invertible. Recall that the matrix L is given by

$$L := \begin{bmatrix} I_1' + I_3' & I_4' & -D\sin\tau & I_5' \\ \beta I_2' + q_1 I_3' & q_1 I_4' & -\beta D\cos\tau & q_4 I_5' \\ I_6' & q_2 I_4' & I_8' & q_5 I_5' \\ I_7' & q_3 I_4' & I_9' & q_6 I_5' \end{bmatrix}, \tag{6.6}$$

where

$$I_1' = C\Big(\frac{\sigma}{\alpha}\sin\tau + \frac{\beta(p-1)}{4}\cos\tau\Big),\ I_2' = C\Big(\frac{\sigma}{\alpha}\cos\tau - \frac{\beta(p-1)}{4}\sin\tau\Big),$$

$$I_3' = \frac{\nu_1}{\alpha}B_p\mu_*\Theta^{\tilde{\nu}_1+\frac{n-5}{2}}Q_*^{\frac{\nu_1-\sigma}{\alpha}},\ I_4' = B_p,\ I_5' = \vartheta_3,$$

$$I_6' = -\beta^2 I_1' - \beta I_2' + q_2 I_3', I_7' = -\beta^3 I_2' + 3\beta^2 I_1' + 2\beta I_2' + q_3 I_3',$$

$$I_8' = \beta^2 D\sin\tau + \beta D\cos\tau, I_9' = \beta^3 D\cos\tau - 3\beta^2 D\sin\tau - 2\beta D\cos\tau.$$

Using simple linear transformations, we see that

$$\begin{bmatrix} I_1' + I_3' & I_4' & -D\sin\tau & I_5' \\ \beta I_2' + q_1 I_3' & q_1 I_4' & -\beta\cos\tau & q_4 I_5' \\ I_6' & q_2 I_4' & I_8' & q_5 I_5' \\ I_7' & q_3 I_4' & I_9' & q_6 I_5' \end{bmatrix} \sim \begin{bmatrix} I_1' & I_4' & -D\sin\tau & I_5' \\ \beta I_2' & q_1 I_4' & -\beta D\cos\tau & q_4 I_5' \\ I_6' - q_2 I_3' & q_2 I_4' & I_8' & q_5 I_5' \\ I_7' - q_3 I_3' & q_3 I_4' & I_9' & q_6 I_5' \end{bmatrix}$$

$$\sim \begin{bmatrix} I_1' & -D\sin\tau & I_4' & I_5' \\ \beta I_2' & -\beta D\cos\tau & q_1 I_4' & q_4 I_5' \\ I_6' - q_2 I_3' & I_8' & q_2 I_4' & q_5 I_5' \\ I_7' - q_3 I_3' & I_9' & q_3 I_4' & q_6 I_5' \end{bmatrix} \sim \begin{bmatrix} I_1' & -D\sin\tau & I_4' & -I_5' \\ \beta I_2' & -\beta D\cos\tau & q_1 I_4' & -q_4 I_5' \\ 0 & 0 & I_{10}' & I_{11}' \\ 0 & 0 & I_{12}' & I_{13}', \end{bmatrix},$$

where

$$I_{10}' = q_2 B_p + q_1 B_p + \beta^2 B_p,\ I_{12}' = q_3 B_p + \beta^2 q_1 B_p - 3\beta^2 B_p - 2q_1 B_p,$$

$$I_{11}' = q_5\vartheta_3 + q_4\vartheta_3 + \beta^2\vartheta_3,\ I_{13}' = q_6\vartheta_3 + \beta^2 q_4\vartheta_3 - 3\beta^2\vartheta_3 - 2q_4\vartheta_3.$$

Since

$$\det\begin{bmatrix} I_1' & -D\sin\tau \\ \beta I_2' & -\beta D\cos\tau \end{bmatrix} \neq 0,$$

showing that L is invertible is equivalent to proving that the following two by two matrix is invertible:

$$\begin{bmatrix} q_2 + q_1 + \beta^2 & q_5 + q_4 + \beta^2 \\ q_3 + \beta^2 q_1 - 3\beta^2 - 2q_1 & q_6 + \beta^2 q_4 - 3\beta^2 - 2q_4 \end{bmatrix}. \tag{6.7}$$

It follows from the definitions of $q_i, (i = 1, 2, 3, 4, 5, 6)$ and β that $q_2 + q_1 + \beta^2 = q_5 + q_4 + \beta^2 \neq 0$. Let

$$\chi_1 = q_3 + \beta^2 q_1 - 3\beta^2 - 2q_1, \quad \chi_2 = q_6 + \beta^2 q_4 - 3\beta^2 - 2q_4.$$

Then

$$\begin{aligned} \chi_1 - \chi_2 =& q_3 - q_6 - (q_1 - q_4)(2 - \beta^2) \\ =& (\tilde{\nu}_1 - \tilde{\nu}_2)[(\tilde{\nu}_1 + \tilde{\nu}_2)^2 - \tilde{\nu}_1\tilde{\nu}_2 + \frac{3n-21}{2}(\tilde{\nu}_1 + \tilde{\nu}_2) + \frac{3n^2 - 42n + 135}{4} + \beta^2] \\ =& (\tilde{\nu}_1 - \tilde{\nu}_2)[\frac{n^2 - 10n + 25}{4} - \tilde{\nu}_1\tilde{\nu}_2 + \beta^2], \end{aligned}$$

where we are using the fact $\tilde{\nu}_1 + \tilde{\nu}_2 = -(n-5)$. It is known that (from Section.2)

$$\tilde{\nu}_1\tilde{\nu}_2 = \frac{n^2 - 10n + 25}{4} - \frac{N_2 + 4\sqrt{N_3}}{4(p-1)^2},$$

and

$$\beta^2 = \frac{4\sqrt{N_3} - N_2}{4(p-1)^2},$$

where N_2 and N_3 (with the dimension n being replaced by $n-1$) are defined in Section 2. Therefore,

$$\chi_1 - \chi_2 = (\tilde{\nu}_1 - \tilde{\nu}_2)\frac{2\sqrt{N_3}}{(p-1)^2} \neq 0.$$

Hence, the above two by two matrix (6.7) is invertible. □

Acknowledgements: The research of the first author is supported by NSFC (11171092) and Innovation Scientists and Technicians Troop Construction Projects of Henan Province (114200510011). The research of J. Wei is partially supported by NSERC of Canada.

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