ON A FOURTH ORDER NONLINEAR ELLIPTIC EQUATION WITH NEGATIVE EXPONENT

ZONGMING GUO AND JUNCHENG WEI

ABSTRACT. We consider the following nonlinear fourth order equation

$$T\Delta u - D\Delta^2 u = \frac{\lambda}{(L+u)^2}, -L < u < 0, \text{ in } \Omega, u = 0, \quad \Delta u = 0 \text{ on } \partial\Omega$$

where $\lambda > 0$ is a parameter. This nonlinear equation models the deflection of charged plates in electrostatic actuators under the pinned boundary condition (Lin and Yang [22]). It has been proved in [22] that there exists a $\lambda_c > 0$ such that for $\lambda > \lambda_c$, there is no solution while for $\lambda < \lambda_c$, there is a branch of maximal solutions. In this paper, we show that in the physical domains (2D or 3D) the maximal solution is unique and regular at $\lambda = \lambda_c$. In a two-dimensional convex smooth domain, we also establish the existence of a second mountain-pass solution for $\lambda \in (0, \lambda_c)$. The asymptotic behavior of the second solution is also studied. The main difficulty is the analysis of the touch-down behavior.

1. INTRODUCTION

We consider the structure of solutions to the following problem

$$(P_\lambda) \qquad \begin{cases} T\Delta u - D\Delta^2 u = \frac{\lambda}{(L+u)^2} & \text{in } \Omega, \\ -L < u \le 0 & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda > 0$ is a parameter, $T > 0$, $D > 0$, $L > 0$ are fixed constants, and $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded smooth domain.

When $D = 0$, problem (P_λ) becomes

$$(Q_\lambda) \qquad \begin{cases} T\Delta u = \frac{\lambda}{(L+u)^2} & \text{in } \Omega, \\ -L < u \le 0 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

which models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid plate located at $-L$. Here $L+u$ represents the distance from membrane to the plate. Recently there have been many studies on (Q_λ). See, for example, [14], [15], [16], [10], [11], [12], [7], [8], [17], [25], [24] and the references therein. These

1991 *Mathematics Subject Classification.* Primary 35B45; Secondary 35J40.
Key words and phrases. Fourth Order, Electrostatic actuation, Touch-down, pull-in threshold.

papers only deal with second order semilinear elliptic equations with singular nonlinearities. Equation (Q_λ) also appears in the study of thin film, see, for example, [2], [3], [5], [19], [20], [21], [18] and the references therein.

In a recent paper [22], Lin and Yang derived the fourth order equation (P_λ) in the study of the deflection of charged plates in electrostatic actuators. Here $\lambda = aV^2$ where V is the electric voltage and a is positive constant. Associated with (P_λ) is the following energy functional

$$E(u) = \int_\Omega \left\{ \frac{T}{2}|\nabla u|^2 + \frac{D}{2}|\Delta u|^2 - \frac{\lambda}{L+u} \right\} \tag{1.1}$$

where $P = \int_\Omega \frac{T}{2}|\nabla u|^2 dx$ is the stretching energy, $Q = \int_\Omega \frac{D}{2}|\Delta u|^2 dx$ corresponds to the bending energy, and $W = -\int_\Omega \frac{\lambda}{L+u(x)} dx$ is the electric potential energy.

Lin and Yang ([22]) considered two kinds of boundary conditions: pinned boundary condition

$$u = \Delta u = 0 \text{ on } \partial\Omega$$

and clamped boundary condition

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

For the pinned boundary condition problem (P_λ), they found that there exists $0 < \lambda_c < \infty$ such that for $\lambda \in (0, \lambda_c)$, (P_λ) has a maximal regular solution u_λ, which can be obtained from an iterative scheme. (By a regular solution u_λ of (P_λ), we mean that $u_\lambda \in C^4(\Omega) \cap C^3(\overline{\Omega})$ satisfies (P_λ).) For $\lambda > \lambda_c$, (P_λ) does not have any regular solution. Moreover, if λ', $\lambda'' \in (0, \lambda_c)$ and $\lambda' < \lambda''$, then the corresponding maximal solutions $u_{\lambda'}$ and $u_{\lambda''}$ satisfy

$$u_{\lambda'} > u_{\lambda''} \quad \text{in } \Omega.$$

Physically, this is a natural relation because a higher supply voltage results in a greater elastic deformation or deflection.

The number λ_c, which determines the pull-in voltage, is called the pull-in threshold. It is known from [22] that, for $\lambda \in (0, \lambda_c)$, $\min_\Omega (L + u_\lambda) > 0$. Let $\Sigma_\lambda = \{x \in \Omega : L + u_\lambda(x) = 0\}$ be the singular set of (P_λ). An interesting question is to study the limit of u_λ as $\lambda \nearrow \lambda_c$. The monotonicity of u_λ with respect to λ implies that there is a well-defined function U so that

$$U(x) = \lim_{\lambda \to \lambda_c^-} u_\lambda(x); \quad -L \leq U(x) < 0, \quad x \in \Omega.$$

However $U(x)$ may touch down to $-L$ and cease to be a regular solution to (P_{λ_c}). (By [22], $U \in W^{2,2}_{loc}(\Omega)$.) For the one-dimensional case, Lin and Yang showed that U is a regular solution, that is, the set $\Sigma_{\lambda_c} = \emptyset$.

In this paper, we will show that *for* $2D$ *and* $3D$*,* U *is a regular solution.* Moreover, we also show that there is a *unique solution for* (P_λ) *at* $\lambda = \lambda_c$. To obtain our results, we first prove that the solutions u_λ for $\lambda \in (0, \lambda_c)$ obtained in [22] are stable in some sense. Furthermore, we also obtain the structure of solutions of (P_λ) in 2D case. Our main results of this paper are:

Theorem 1.1. *For dimension* $N = 2$ *or* 3*, there exists a constant* $0 < C := C(N, L)$ *independent of* λ *such that for any* $0 < \lambda < \lambda_c$*, the maximal solution* u_λ *of* (P_λ) *satisfies* $\min_\Omega (L + u_\lambda) \geq C$.

Consequently, $u_{\lambda_c} = \lim_{\lambda \nearrow \lambda_c} u_\lambda$ *exists in the topology of* $C^4(\Omega)$*. It is the unique regular solution to* (P_{λ_c})*.*

Theorem 1.2. *Let* $N = 2$ *and* Ω *be a bounded, smooth and convex domain in* $\mathbb{R}^2$*. For* $\lambda \in (0, \lambda_c]$*, any solution of the problem* (P_λ) *is regular and*

(i) For $0 < \lambda < \lambda_c$*, problem* (P_λ) *admits two solutions: the maximal solution and a mountain pass solution.*

(ii) For $\lambda = \lambda_c$*, problem* (P_λ) *admits a unique regular solution.*

(iii) For $\lambda > \lambda_c$*, problem* (P_λ) *admits no regular solution.*

Furthermore, the mountain-pass solution V_λ *has the following asymptotic behavior as* $\lambda \to 0$*:*

$$\max_\Omega V_\lambda \to L \quad \text{as } \lambda \to 0, \qquad \lim_{\lambda \to 0+} \frac{[\min_\Omega (L - V_\lambda)]^3}{\lambda} = 0. \tag{1.2}$$

Remark: Theorem 1.2 shows that the bifurcation diagram of (P_λ) changes drastically when $D > 0$. In a nice 2D domain (see [16]), it has been proved in [16] that for λ small, the maximal solution is *unique*, and there exists $0 < \lambda_* < \lambda_c$ such that the solutions to (Q_λ) undergo infinitely many turning points. An interesting question is the asymptotic behavior as $D \to 0$. When $\Omega = B_1 \subset \mathbb{R}^2$, the complete bifurcation picture as well as the asymptotic behavior when $D \to 0$ has been considered in [23].

The organization of the paper is as follows: in Section 2, we present some preliminary results on the first eigenvalue and the corresponding eigenfunction of the problem

$$-T\Delta\phi + D\Delta^2\phi = \sigma\phi \quad \text{in } \Omega, \qquad \phi = \Delta\phi = 0 \quad \text{on } \partial\Omega.$$

In Section 3, we derive a key L^1 bound for $\frac{1}{(L+u)^2}$. In Section 4, we show the stability of the maximal solutions of (P_λ). In Section 5, we show that the solution at the pull-in threshold is regular for $N = 2$ or 3. In Section 6, we show that any weak solution at the pull-in threshold is unique. In Section 7, we present the structure of the solutions of (P_λ) for 2D case. We show that for $0 < \lambda < \lambda_c$, (P_λ) admits at least two solutions: the maximal solution and a mountain pass solution. Finally in Section 8, we give some asymptotic behaviors of the mountain pass solution as $\lambda \to 0^+$.

Acknowledgments: This paper was done while the first author was visiting the Department of Mathematics, Chinese University of Hong Kong. He would like to thank the Department for its hospitality. We thank Professor Michael Ward and Professor Dong Ye for useful discussions. We also thank the referees for their valuable suggestions. The research of the first author is supported by grants of NSFC (10571022) and NSFC (10871060). The research of the second author is partially supported by Earmarked Grants from RGC of Hong Kong and a Direct Grant of CUHK.

2. The First Eigenfunction

In this section, we study the following eigenvalue problem

$$-T\Delta\phi + D\Delta^2\phi = \sigma\phi \quad \text{in } \Omega, \qquad \phi = \Delta\phi = 0 \quad \text{on } \partial\Omega \tag{2.1}$$

where $T,\ D > 0$. We will show that (2.1) has the least eigenvalue σ_1 and the corresponding eigenfunction $\phi_1 > 0$ in Ω. Moreover, ϕ_1 is simple, i.e., all the eigenfunctions corresponding to σ_1 assume the forms of $C\phi_1$ with $C \in \mathbb{R}$.

Proposition 2.1. *Problem (2.1) has the least eigenvalue σ_1 such that all the eigenfunctions corresponding to σ_1 assume the forms of $C\phi_1$, where $\phi_1 \in C^\infty(\Omega)$ and $\phi_1 > 0$ in Ω.*

Proof. This proposition may be known, but we can not find the reference. We give a proof here for completeness.

Consider the following minimization problem

$$\sigma_1 := \inf\Big\{ \int_\Omega [T|\nabla\phi|^2 + D|\Delta\phi|^2]dx : \ \phi \in \mathcal{H}, \ \ \|\phi\|_{L^2(\Omega)} = 1\Big\} \tag{2.2}$$

where $\mathcal{H} = H^2(\Omega) \cap H_0^1(\Omega)$ is the function space obtained by taking the completion under the norm of $H^2(\Omega) \cap H_0^1(\Omega)$ (i.e. $\|\psi\| = \Big(\int_\Omega [T|\nabla\psi|^2 + D|\Delta\psi|^2]dx\Big)^{1/2}$) for the set of smooth functions that satisfy the boundary condition $\phi = \Delta\phi = 0$ on $\partial\Omega$. Since the Sobolev embedding $\mathcal{H} \hookrightarrow L^2(\Omega)$ is compact, by standard direct method of calculus of variations, we have at least one minimizer ϕ_1 for the problem (2.2), where $\phi_1 \in \mathcal{H}$, $\|\phi_1\|_{L^2(\Omega)} = 1$. Furthermore, ϕ_1 is a weak solution to (2.1), namely,

$$\int_\Omega [T\nabla\phi_1\nabla\phi + D\Delta\phi_1\Delta\phi] = \sigma_1 \int_\Omega \phi_1\phi dx, \ \forall \phi \in \mathcal{H}. \tag{2.3}$$

Using the L^p-estimates due to Agmon, Douglis and Nirenberg [1], we conclude that

$$\|\phi_1\|_{W^{4,p}(\Omega)} \le C\|\phi_1\|_{L^p(\Omega)}$$

for any $p > 1$. Thus we have $\phi_1 \in C^4(\Omega) \cap C^3(\bar{\Omega})$ and hence $\Delta\phi_1 = 0$ on $\partial\Omega$ and ϕ_1 satisfies (2.1). (See a similar argument in Lemma B.3 of [27].)

It is clear that

$$\sigma_1 = \frac{\int_\Omega [T|\nabla\phi_1|^2 + D|\Delta\phi_1|^2]dx}{\int_\Omega \phi_1^2 dx} = \inf_{\phi\in\mathcal{H}\setminus\{0\}} \frac{\int_\Omega [T|\nabla\phi|^2 + D|\Delta\phi|^2]dx}{\int_\Omega \phi^2 dx}.$$

In order to show that ϕ_1 is of fixed sign, we consider the following new problem

$$-T\Delta\psi_1 + D\Delta^2\psi_1 = \sigma_1|\phi_1| \quad \text{in } \Omega, \qquad \psi_1 = \Delta\psi_1 = 0 \ \text{ on } \partial\Omega. \tag{2.4}$$

By Maximum Principle, $\psi_1 > 0$, $-D\Delta\psi_1 + T\psi_1 > 0$ in Ω. Furthermore, we have $\psi_1 \ge \phi_1, \psi_1 \ge -\phi_1$ and hence $\psi_1 \ge |\phi_1|$ in Ω.

On the other hand, from (2.4), we obtain

$$\int_\Omega [T|\nabla\psi_1|^2 + D|\Delta\psi_1|^2]dx = \sigma_1 \int_\Omega \psi_1|\phi_1|dx \le \sigma_1 \int_\Omega |\psi_1|^2 dx. \tag{2.5}$$

By the minimality of σ_1, we have

$$\sigma_1 = \frac{\int_\Omega [T|\nabla\psi_1|^2 + D|\Delta\psi_1|^2]dx}{\int_\Omega |\psi_1|^2 dx}. \tag{2.6}$$

Thus ψ_1 also attains σ_1 and hence the inequality of (2.5) is actually an equality. This implies that $\psi_1 = |\phi_1|$ in Ω. Since $\psi_1 > 0$ in Ω, we conclude that ϕ_1 is of fixed sign in Ω.

The above argument actually proves that any nonzero eigenfunction corresponding to σ_1 must be of fixed sign in Ω. So if ϕ_1 and ϕ_2 are two eigenfunctions corresponding to σ_1, we may choose $\phi_1 > 0, \phi_2 > 0$. Let $x_0 \in \Omega$ and $C = \frac{\phi_1(x_0)}{\phi_2(x_0)}$. Then the function $\phi_1 - C\phi_2$ is again an eigenfunction corresponding to σ_1. By the previous argument, we see that $\phi_1 \equiv C\phi_2$ in Ω. This completes the proof. □

3. A Uniform L^1 Bound

In this section, we establish a key uniform L^1 bound for $\frac{1}{(L-v)^2}$, where v satisfies

$$(T_\lambda) \qquad \begin{cases} -T\Delta v + D\Delta^2 v = \frac{\lambda}{(L-v)^2} & \text{in } \Omega, \\ 0 < v < L & \text{in } \Omega, \\ v = 0, \quad \Delta v = 0 & \text{on } \partial\Omega \end{cases}$$

(which is equivalent to (P_λ) by taking $u = -v$). Note that $v \in C^4(\Omega) \cap C^2(\overline{\Omega})$ provided that v satisfies (T_λ).

Theorem 3.1. *Let Ω be a bounded, smooth and convex domain. Then there exists a constant C (independent of λ) such that for any solution v to (T_λ) we have*

$$\int_\Omega \frac{1}{(L-v)^2} \leq \frac{C}{\lambda}. \tag{3.1}$$

As a consequence, we have

$$\int_\Omega (D|\Delta v|^2 + T|\nabla v|^2) \leq C. \tag{3.2}$$

Proof: Let ϕ_1 be given in Proposition 2.1. Multiplying (T_λ) by ϕ_1 and integrating over Ω, we obtain

$$\lambda \int_\Omega \frac{1}{(L-v)^2}\phi_1 = \sigma_1 \int_\Omega v\phi_1 \leq C \tag{3.3}$$

which implies that

$$\int_{\Omega'} \frac{1}{(L-v)^2} \leq \frac{C_{\Omega'}}{\lambda} \tag{3.4}$$

for any $\Omega' \subset\subset \Omega$, where $C_{\Omega'}$ is independent of λ.

We write (T_λ) as

$$\begin{cases} \Delta v + \frac{1}{D}w - \frac{T}{D}v = 0 & \text{in } \Omega \\ \Delta w + \frac{\lambda}{(L-v)^2} = 0 & \text{in } \Omega \\ v = w = 0 & \text{on } \partial\Omega. \end{cases}$$

If we denote $f_1(v,w) = -\frac{T}{D}v + \frac{1}{D}w$, $f_2(v,w) = \frac{\lambda}{(L-v)^2}$, we see that $\frac{\partial f_1}{\partial w} = \frac{1}{D} > 0$ and $\frac{\partial f_2}{\partial v} = \frac{2\lambda}{(L-v)^3} > 0$. Therefore, the convexity of Ω, Lemma 5.1 of [26] and the moving plane method near $\partial\Omega$ as in the Appendix of [13] imply that there exist $t_0 > 0$ and $\alpha > 0$ depending on the domain Ω only, such that $v(x - t\nu)$ *and* $w(x - t\nu)$ *are nondecreasing for* $t \in [0, t_0]$, $\nu \in R^N$ *satisfying* $|\nu| = 1$ *and* $(\nu, n(x)) \geq \alpha$ *and* $x \in \partial\Omega$. Therefore we can find $\gamma, \delta > 0$ such that for any $x \in \Omega_\delta := \{z \in \bar{\Omega} : d(z, \partial\Omega) < \delta\}$ there exists a fixed-sized cone Γ_x (with x as its vertex) with

(i) $meas(\Gamma_x) \geq \gamma$,

(ii) $\Gamma_x \subset \{z \in \Omega : d(z, \partial\Omega) < \delta\}$,

(iii) $v(y) \geq v(x)$ for any $y \in \Gamma_x$.

Then for any $x \in \Omega_\delta$, we have

$$\frac{1}{(L-v(x))^2} \leq \frac{1}{meas(\Gamma_x)} \int_{\Gamma_x} \frac{1}{(L-v)^2} \leq \frac{1}{\gamma} \int_{\Omega_\delta} \frac{1}{(L-v)^2} \leq \frac{C}{\lambda}$$

This implies that $\frac{1}{(L-v)^2} \in L^\infty(\Omega_\delta)$ and there are $C > 0$ independent of λ such that

$$\sup_{x\in\Omega_\delta} v < L - C\sqrt{\lambda}. \tag{3.5}$$

Next, we derive estimates for w near $\partial\Omega$. Multiplying the second equation of the equivalent system of (T_λ) by φ_0-the first eigenfunction of $-\Delta$ and integrating over Ω, we obtain

$$\lambda \int_\Omega \frac{1}{(L-v)^2}\varphi_0 = \lambda_1 \int_\Omega \varphi_0 w \tag{3.6}$$

where λ_1 is the first eigenvalue of $-\Delta$. By (3.3), we have that

$$\lambda_1 \int_\Omega \varphi_0 w = \lambda \int_\Omega \frac{1}{(L-v)^2}\varphi_0 \le \lambda C \int_\Omega \frac{1}{(L-v)^2}\phi_1 \le C \tag{3.7}$$

and hence

$$\int_{\Omega'} w \le C_{\Omega'}$$

for any $\Omega' \subset\subset \Omega$. To see the second inequality of (3.7), we notice that there exist $\ell_i > 0$ $(i = 1, 2, 3, 4)$ such that

$$\ell_1 d(x) \le \varphi_0(x) \le \ell_2 d(x), \quad \ell_3 d(x) \le \phi_1(x) \le \ell_4 d(x)$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$. Hence $\varphi_0(x) \le C\phi_1(x)$. The same reason as above shows that $w \le C(\Omega_\delta)$.

By elliptic regularity applied to the system (T_λ) (noting that $v, w, \frac{1}{(L-v)^2}$ are all bounded in Ω_δ), we have $v \in C^3(\Omega_\delta)$ and hence

$$\lambda \int_\Omega \frac{1}{(L-v)^2} = D \int_{\partial\Omega} \frac{\partial \Delta v}{\partial n} - T \int_\Omega \frac{\partial v}{\partial n} \le C.$$

To prove the inequality (3.2), we multiply (T_λ) by v and integrate over Ω to obtain

$$T \int_\Omega |\nabla v|^2 + D \int_\Omega |\Delta v|^2 = \lambda \int_\Omega \frac{Lv}{(L-v)^2} \le C.$$

□

4. Stability of the maximal solutions of (P_λ)

In this section, we show that the maximal solutions u_λ to (P_λ) obtained in [22] for $\lambda \in (0, \lambda_c)$ are stable in some sense. Let $v_\lambda = -u_\lambda$. Then from [22], for each $\lambda \in (0, \lambda_c)$, (T_λ) has a minimal positive solution v_λ.

We call v_λ stable if the first eigenvalue $\sigma_{1,\lambda}(v_\lambda)$ of the problem

$$-T\Delta h + D\Delta^2 h = \frac{2\lambda}{(L-v_\lambda)^3} h + \sigma h \ \text{ in } \Omega, \quad h = \Delta h = 0 \ \text{ on } \partial\Omega \tag{4.1}$$

is nonnegative. By arguments similar to those in the proof of Proposition 2.1, we see that the first eigenvalue $\sigma_{1,\lambda}(v_\lambda)$ exists and every eigenfunction corresponding to $\sigma_{1,\lambda}(v_\lambda)$ is of fixed sign if $\sigma_{1,\lambda}(v_\lambda) \geq 0$.

Lemma 4.1. *Suppose that v is a regular solution of (T_λ), u is a regular supersolution of (T_λ), that is,*

$$\begin{cases} -T\Delta u + D\Delta^2 u \geq \frac{\lambda}{(L-u)^2} & \text{in } \Omega \\ 0 < u < L & \text{in } \Omega \\ u = 0, \quad \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

If $\sigma_{1,\lambda}(v) > 0$, then $u \geq v$ in Ω, and if $\sigma_{1,\lambda}(v) = 0$ then $u = v$ in Ω.

Proof. For a given λ and $x \in \Omega$, by the fact that $s \to (L-s)^{-2}$ is convex on $(0, L)$, we see that

$$(4.2) \quad -T\Delta(v + \tau(u-v)) + D\Delta^2(v + \tau(u-v)) - \frac{\lambda}{[L - (v + \tau(u-v))]^2} \geq 0 \text{ in } \Omega$$

for $\tau \in [0, 1]$. Note that (4.2) is an identity at $\tau = 0$, which means that the first derivative of the left hand side of (4.2) with respect to τ is nonnegative at $\tau = 0$, i.e.,

$$(4.3) \qquad \begin{cases} -T\Delta(u-v) + D\Delta^2(u-v) - \frac{2\lambda}{(L-v)^3}(u-v) \geq 0 & \text{in } \Omega \\ u - v = 0, \quad \Delta(u-v) = 0 & \text{on } \partial\Omega \end{cases}$$

Thus, the fact $\sigma_{1,\lambda}(v) > 0$ implies that $u \geq v$ in Ω. Indeed, on the contrary, we see that $0 \not\equiv (u-v)^- \in H^2(\Omega) \cap H_0^1(\Omega)$. Multiplying $(u-v)^-$ on both the sides of (4.3) and integrating it on Ω, we see that

$$\begin{aligned} &\sigma_{1,\lambda}(v) \int_\Omega [(u-v)^-]^2 dx \\ &\quad \leq T \int_\Omega |\nabla(u-v)^-|^2 dx + D \int_\Omega |\Delta(u-v)^-|^2 dx - \int_\Omega \frac{2\lambda}{(L-v)^3}[(u-v)^-]^2 dx \\ &\quad \leq 0. \end{aligned}$$

This contradicts $\sigma_{1,\lambda}(v) > 0$.

If $\sigma_{1,\lambda}(v) = 0$, we have

$$-T\Delta(u-v) + D\Delta^2(u-v) - \frac{2\lambda}{(L-v)^3}(u-v) = 0 \text{ in } \Omega.$$

Moreover, the second derivative of the left hand side of (4.2) with respect to τ at $\tau = 0$ is

$$-6\lambda(L-v)^{-4}(u-v)^2 \geq 0$$

which implies that $u \equiv v$ in Ω. This completes the proof. □

Proposition 4.2. *For each $\lambda \in (0, \lambda_c)$, the minimal positive solution v_λ of (T_λ) is stable.*

Proof. Since $\sigma_1 > 0$, we easily see that the first eigenvalue $\sigma_{1,\lambda}(v_\lambda)$ of problem (4.1) is positive provided that λ is sufficiently small. Now we prove that $\sigma_{1,\lambda}(v_\lambda) > 0$ for $\lambda \in (0, \lambda_c)$.

We define

$$\lambda^* = \sup\{\rho : \ v_\lambda \text{ is a stable solution for } \lambda \in (0, \rho)\}.$$

It is clear that $\lambda^* \leq \lambda_c$. To show $\lambda^* = \lambda_c$, it suffices to prove that there is no regular minimal solution for (T_λ) with $\lambda > \lambda^*$. For that, suppose w is regular a minimal solution of $(T_{\lambda^*+\delta})$ with $\delta > 0$, then we would have for $\lambda \leq \lambda^*$,

$$-T\Delta w + D\Delta^2 w = \frac{\lambda^* + \delta}{(L-w)^2} \geq \frac{\lambda}{(L-w)^2} \quad \text{in } \Omega.$$

Since for $0 < \lambda < \lambda^*$ the minimal solution v_λ is stable, it follows from Lemma 4.1 that $L > w \geq v_\lambda$. Consequently, $\overline{v} = \lim_{\lambda \nearrow \lambda^*} v_\lambda$ exists in $C^4(\Omega)$ and it is a regular solution to (T_{λ^*}). Now from the definition of λ^* and the implicit function theorem, we necessarily have $\sigma_{1,\lambda^*}(\overline{v}) = 0$. By Lemma 4.1 again, we obtain that $w \equiv \overline{v}$ in Ω and hence $\delta = 0$. This is a contradiction. Therefore, $\lambda^* = \lambda_c$. This completes the proof. □

5. The regularity of the minimal solution of (T_λ) at $\lambda = \lambda_c$

In this section, we are concerned with the regularity of the minimal solutions of (T_λ) at $\lambda = \lambda_c$. Normally, the minimal solution v_λ at $\lambda = \lambda_c$ may have singular set in Ω, i.e., there exists a set $\Sigma_{\lambda_c} \subset \Omega$ such that $v_{\lambda_c}(x) = L$ for $x \in \Sigma_{\lambda_c}$. But we will see that for lower dimensional case, $v_{\lambda_c} < L$ in Ω.

By a weak solution $v \in \mathcal{H}$ of (T_λ) we mean $0 < v \leq L$ in Ω and $(L-v)^{-2} \in L^1(\Omega)$ such that for any $\varphi \in \mathcal{H}$,

$$\int_\Omega [T\nabla v \cdot \nabla\varphi + D\Delta v \Delta\varphi] dx = \lambda \int_\Omega (L-v)^{-2}\varphi dx.$$

Lemma 5.1. *If $v \in \mathcal{H}$ is a weak solution to (T_λ), then there exists $C := C(\lambda) > 0$ such that*

$$\int_\Omega \frac{dx}{(L-v)^2} \leq C.$$

For $N \geq 2$, any solution v satisfying $(L-v)^{-2} \in L^p(\Omega)$ with $p = N/2$ is a classical solution.

Proof. For the first conclusion, we see that since $v \in \mathcal{H}$ is a solution of (T_λ),

$$\int_\Omega \frac{v}{(L-v)^2} dx = \frac{1}{\lambda}\Big[\int_\Omega (|\nabla v|^2 + |\Delta v|^2) dx\Big] \leq C. \tag{5.1}$$

On the other hand, we see that

$$\frac{v}{(L-v)^2} = \frac{L}{(L-v)^2} - \frac{1}{(L-v)}.$$

Thus, (5.1) implies that

$$\int_\Omega \frac{L}{(L-v)^2} dx = \int_\Omega \frac{1}{(L-v)} dx + \int_\Omega \frac{v}{(L-v)^2} dx. \tag{5.2}$$

By the Young's inequality, we have that

$$\int_\Omega \frac{1}{(L-v)} dx \le \epsilon L \int_\Omega \frac{1}{(L-v)^2} dx + C(\epsilon, L)|\Omega|, \tag{5.3}$$

where $0 < \epsilon < 1/4$ and $C(\epsilon, L) > 0$ is a constant. Our first conclusion can be obtained from (5.1), (5.2) and (5.3).

For $N = 2$, suppose that v is a weak solution such that $\frac{1}{(L-v)^2} \in L^1(\Omega)$. Thus,

$$-D\Delta^2 v = \lambda (L-v)^{-2} - T\Delta v \in L^1(\Omega).$$

This and the Sobolev's embedding imply that $\nabla^3 v \in L^q(\Omega)$ for any $1 < q < 2$. In particular, $\nabla v \in C^{2-\frac{2}{q}}(\Omega)$ for any $1 < q < 2$. This and the fact that $(L-v)^{-2} \in L^1(\Omega)$ clearly imply that $v < L$ in Ω. In fact, on the contrary, suppose that there exists $x_0 \in \Omega$ such that $v(x_0) = \max_\Omega v = L$. Then, $\nabla v(x_0) = 0$ and

$$v(x) - v(x_0) = \nabla v(\xi) \cdot (x - x_0) \ \text{ for } x \in \Omega \text{ near } x_0,$$

where $\xi = t x_0 + (1-t)x$ with $t \in (0,1)$. Moreover, since $\nabla v \in C^{2-\frac{2}{q}}(\Omega)$, we see that

$$|\nabla v(\xi) - \nabla v(x_0)| \le M |\xi - x_0|^{2-\frac{2}{q}} \le M|x - x_0|^{2-\frac{2}{q}}$$

and thus,

$$|v(x) - v(x_0)| \le M|x - x_0|^{3-\frac{2}{q}} \ \text{ for } x \in \Omega \text{ near } x_0.$$

This inequality shows that

$$\infty > \int_\Omega \frac{1}{(L-v)^2} dx \ge M^{-2} \int_\Omega |x - x_0|^{-(6-\frac{4}{q})} dx = \infty,$$

a contradiction, which implies that we must have $\|v\|_{C(\overline{\Omega})} < L$.

For $N \ge 3$, suppose that v is a weak solution such that $\frac{1}{(L-v)^2} \in L^p(\Omega)$ with $p = \frac{N}{2}$. By the regularity of Δ^2, we see that $v \in W^{4,p}(\Omega)$. The Sobolev's embedding theorem then implies that $v \in C^{1,\alpha}(\Omega)$ with $\alpha < 1$ since $4 - \frac{N}{p} = 2$. To show that v is a classical solution, it suffices to show that $v < L$ in Ω. Indeed, on the contrary, there exists $x_0 \in \Omega$ such that $v(x_0) = \max_\Omega v = L$. Then, $\nabla v(x_0) = 0$ and

$$v(x) - v(x_0) = \nabla v(\xi) \cdot (x - x_0) \ \text{ for } x \in \Omega \text{ near } x_0, \tag{5.4}$$

where $\xi = tx_0 + (1-t)x$ with $t \in (0,1)$. Moreover, since $v \in C^{1,1}(\Omega)$, we see that $|\nabla v(\xi) - \nabla v(x_0)| \le M|\xi - x_0| \le M|x - x_0|$. This and (5.4) imply that

$$|v(x) - v(x_0)| \le M|x - x_0|^{1+\alpha} \text{ for } x \in \Omega. \tag{5.5}$$

This inequality shows that

$$\infty > \int_\Omega \Big(\frac{1}{(L-v)^2}\Big)^p dx \ge M^{-2p} \int_\Omega |x - x_0|^{-2(1+\alpha)p} dx = \infty,$$

a contradiction, which implies that we must have $\|v\|_{C(\overline{\Omega})} < L$. This completes the proof. □

Proposition 5.2. *There exists a constant $C := C(L,\lambda) > 0$ such that for each $\lambda \in (0, \lambda_c)$, the minimal solution v_λ satisfies $\|(L - v_\lambda)^{-2}\|_{L^{3/2}(\Omega)} \le C$.*

Proof. Since the minimal solutions v_λ are stable, we have

$$\int_\Omega \frac{2\lambda}{(L-v_\lambda)^3} w^2 dx \le \int_\Omega [T|\nabla w|^2 + D|\Delta w|^2] dx \tag{5.6}$$

for all $0 < \lambda < \lambda_c$ and nonnegative $w \in \mathcal{H}$.

Let $w = v_\lambda$, we then have

$$\int_\Omega \frac{2\lambda}{(L-v_\lambda)^3} v_\lambda^2 dx \le \int_\Omega [T|\nabla v_\lambda|^2 + D|\Delta v_\lambda|^2] dx = \int_\Omega \frac{\lambda v_\lambda}{(L-v_\lambda)^2} \tag{5.7}$$

Since $v_\lambda < L$, this implies that

$$\int_\Omega \frac{v_\lambda^2}{(L-v_\lambda)^3} dx \le C \tag{5.8}$$

and

$$\int_\Omega \frac{L^2}{(L-v_\lambda)^3} dx \le \int_\Omega \frac{v_\lambda^2}{(L-v_\lambda)^3} dx + \int_\Omega \frac{(L-v_\lambda)^2}{(L-v_\lambda)^3} dx \le C + \int_\Omega \frac{1}{L-v_\lambda} dx. \tag{5.9}$$

Hence

$$\int_\Omega \frac{1}{(L-v_\lambda)^3} dx \le C. \tag{5.10}$$

This completes the proof. □

Now we obtain the following theorem, our Theorem 1.1 can be obtained from this theorem.

Theorem 5.3. *For dimension $N = 2$ or 3, there exists a constant $0 < C := C(N,L) < L$ independent of λ such that for any $0 < \lambda < \lambda_c$, the minimal solution v_λ of (T_λ) satisfies $\|v_\lambda\|_{C(\Omega)} \le C$.*

Consequently, $v_{\lambda_c} = \lim_{\lambda \nearrow \lambda_c} v_\lambda$ exists in the topology of $C^4(\Omega)$. It is the unique classical solution to (T_{λ_c}).

Proof. By Proposition 5.2 and (3.2), we see that that there is $C > 0$ independent of λ such that

$$\|v_\lambda\|_{H^2(\Omega)} \le C.$$

Since the mapping $\lambda \mapsto v_\lambda$ is increasing for $\lambda \in (0, \lambda_c)$, we see that there is a function $v_{\lambda_c} \in H^2(\Omega)$ such that

$$\lim_{\lambda \nearrow \lambda_c} v_\lambda = v_{\lambda_c} \text{ weakly in } H^2(\Omega).$$

Consequently, v_{λ_c} is a weak solution of the equation of the equation (T_λ) at the critical parameter λ_c:

$$-T\Delta v_{\lambda_c} + D\Delta^2 v_{\lambda_c} = \frac{\lambda_c}{(L - v_{\lambda_c})^2} \text{ in } \Omega$$

and in the sense of weak solutions, the critical value λ_c is attainable.

Now we show that v_{λ_c} is a classical solution. The implicit function theorem implies that the mapping $\lambda \mapsto v_\lambda$ from $(0, \lambda_c)$ to $C(\overline{\Omega})$ is continuous. Thus, we see that $\sigma_{1,\lambda_c} = 0$. (Otherwise, the implicit function theorem implies that v_λ will exist for $\lambda > \lambda_c$.) By arguments similar to those in the proof of Proposition 5.2, we see that

$$\|(L - v_{\lambda_c})^{-2}\|_{L^{3/2}(\Omega)} \le C(L).$$

Note that (5.6) holds with the inequality replaced by an equality. Then Lemma 5.1 implies that for $N = 2$ and 3, v_{λ_c} is a classical solution. Thus, there exists $C < L$ such that $\|v_{\lambda_c}\|_{C(\overline{\Omega})} \le C$. Note that $\|v_\lambda\|_{C(\overline{\Omega})} \le \|v_{\lambda_c}\|_{C(\overline{\Omega})} \le C < L$ for $\lambda \in (0, \lambda_c)$. The uniqueness of v_{λ_c} of (T_λ) at $\lambda = \lambda_c$ follows from Lemma 4.1. This completes the proof. □

6. Uniqueness of the solution of (T_λ) at $\lambda = \lambda_c$

We first note that the monotonicity with respect to λ and the uniform boundedness of the branch of the minimal solutions imply that the extremal function defined by $v_{\lambda_c} = \lim_{\lambda \nearrow \lambda_c} v_\lambda$ always exists, and can always be considered as a solution for (T_{λ_c}) in a weak sense. On the other hand, if there is a $0 < C < L$ such that $\|v_\lambda\|_{C(\overline{\Omega})} \le C$ for each $\lambda < \lambda_c$-just as in the case $N = 2$ or 3-then we see from Theorem 5.3 that v_{λ_c} is the unique classical solution.

In the following, we only consider the case that v_{λ_c} is a weak solution (i.e., $v_{\lambda_c} \in W^{2,2}_{loc}(\Omega)$, note that we can obtain $v_{\lambda_c} \in \mathcal{H}$ provided that $v_{\lambda_c} \in W^{2,2}_{loc}(\Omega)$ by the moving plane argument) but with the possibility that $\|v_{\lambda_c}\|_{L^\infty(\Omega)} = L$.

Theorem 6.1. *For $\lambda > 0$, assume $v \in \mathcal{H}$ is a weak solution to (T_λ) such that $\|v\|_{L^\infty(\Omega)} = L$. The following assertions are equivalent:*

(i) $\sigma_{1,\lambda}(v) \geq 0$, *that is* v *satisfies*

$$2\lambda \int_\Omega (L-v)^{-3}\phi^2 \leq \int_\Omega [T|\nabla\phi|^2 + D|\Delta\phi|^2]dx, \forall \phi \in \mathcal{H},$$

(ii) $\lambda = \lambda_c$ *and* $v \equiv v_{\lambda_c}$ *in* Ω.

Theorem 6.1 can be easily obtained from the following proposition.

Proposition 6.2. *Let* v_1, v_2 *be two* $\mathcal{H}$*-weak solutions of* (T_λ) *so that* $\sigma_{1,\lambda}(v_i) \geq 0$ *for* $i = 1, 2$. *Then* $v_1 = v_2$ *a.e. in* Ω.

Proof. For any $\theta \in [0,1]$ and $\phi \in \mathcal{H}$, $\phi \geq 0$, we have that

$$\begin{aligned} I_{\theta,\phi} : &= T\int_\Omega \nabla(\theta v_1 + (1-\theta)v_2)\nabla\phi dx + D\int_\Omega \Delta(\theta v_1 + (1-\theta)v_2)\Delta\phi dx \\ &\quad -\lambda \int_\Omega [L - (\theta v_1 + (1-\theta)v_2)]^{-2}\phi dx \\ &= \lambda \int_\Omega \Big[[\theta(L-v_1)^{-2} + (1-\theta)(L-v_2)^{-2}] - [(L-(\theta v_1 + (1-\theta)v_2))^{-2}]\Big] dx \\ &\geq 0 \end{aligned}$$

due to the convexity of $(L-s)^{-2}$ with respect to $s \in (0, L)$. Since $I_{0,\phi} = I_{1,\phi} = 0$, the derivative of $I_{\theta,\phi}$ at $\theta = 0, 1$ provides:

$$\int_\Omega [T\nabla(v_1 - v_2)\nabla\phi + D\Delta(v_1-v_2)\Delta\phi] - 2\lambda\int_\Omega (L-v_2)^{-3}(v_1-v_2)\phi \geq 0$$

$$\int_\Omega [T\nabla(v_1 - v_2)\nabla\phi + D\Delta(v_1-v_2)\Delta\phi] - 2\lambda\int_\Omega (L-v_1)^{-3}(v_1-v_2)\phi \leq 0$$

for any $\phi \in \mathcal{H}$ with $\phi \geq 0$. Testing the first inequality on $\phi = (v_1 - v_2)^-$ and the second one on $(v_1 - v_2)^+$, we obtain that

$$\int_\Omega \Big[T|\nabla(v_1-v_2)^-|^2 + D|\Delta(v_1-v_2)^-|^2\Big] - 2\lambda\int_\Omega (L-v_2)^{-3}((v_1-v_2)^-)^2 \leq 0$$

$$\int_\Omega \Big[T|\nabla(v_1-v_2)^+|^2 + D|\Delta(v_1-v_2)^+|^2\Big] - 2\lambda\int_\Omega (L-v_1)^{-3}((v_1-v_2)^+)^2 \leq 0.$$

Since $\sigma_{1,\lambda}(v_1) \geq 0$, we have:

(1) If $\sigma_{1,\lambda}(v_1) > 0$, then $v_1 \leq v_2$ a.e in Ω.

(2) If $\sigma_{1,\lambda}(v_1) = 0$, which then gives

$$\int_\Omega \Big[T\nabla(v_1-v_2)\nabla\overline{\varphi} + D\Delta(v_1-v_2)\Delta\overline{\varphi}\Big] - 2\lambda\int_\Omega (L-v_1)^{-3}(v_1-v_2)\overline{\varphi} = 0,$$

where $\overline{\varphi} = (v_1 - v_2)^+$. Since $I_{\theta,\overline{\varphi}} \geq 0$ for any $\theta \in [0,1]$ and $I_{1,\overline{\varphi}} = \partial I_{1,\overline{\varphi}} = 0$, we get that $\partial^2_{\theta\theta} I_{1,\overline{\varphi}} = -\int_\Omega \frac{6\lambda}{(L-v_1)^4}((v_1-v_2)^+)^3 \geq 0$. Thus, $(v_1 - v_2)^+ = 0$ a.e. in Ω. Hence, $v_1 \leq v_2$ a.e. in Ω. The same argument applies to prove the reversed inequality, and the proof of the proposition is complete. □

In this section we obtain the structure of positive solutions of (T_λ) in 2-dimensional case. The main theorem of this section is the following theorem. Our Theorem 1.2 can be obtained from this theorem and Theorem 8.2.

Theorem 7.1. *Let Ω be a convex smooth domain in $\mathbb{R}^2$. For $\lambda \in (0, \lambda_c]$, any solution of the problem (T_λ) is regular and*

(i) For $0 < \lambda < \lambda_c$, problem (T_λ) admits two solutions: the minimal solution and a mountain pass solution.

(ii) For $\lambda = \lambda_c$, problem (T_λ) admits a unique regular solution.

(iii) For $\lambda > \lambda_c$, problem (T_λ) admits no regular solution.

To prove this theorem, we first show the following lemma.

Lemma 7.2. *For any fixed $\lambda > 0$, if $v_\lambda \in \mathcal{H}$ is a positive solution of (T_λ) then there exists $0 < \tau_\lambda < L$ such that $v_\lambda \le L - \tau_\lambda$ in Ω. This also implies that v_λ is regular.*

Proof. The embedding theorem implies that $v_\lambda \in C^\alpha(\overline{\Omega})$ for any $0 < \alpha < 1$ and thus the moving plane arguments as in the proof of Theorem 3.1 imply that if $v_\lambda(x_\lambda) = \max_\Omega v_\lambda$, then $x_\lambda \in \Omega_0$, where $\Omega_0 \subset\subset \Omega$. Moreover, by Theorem 3.1, we have

$$\int_\Omega (L - v_\lambda)^{-2} \le \frac{C}{\lambda} \tag{7.1}$$

and

$$\int_\Omega [T|\nabla v_\lambda|^2 + D(\Delta v_\lambda)^2]dx \le C. \tag{7.2}$$

Suppose that there is $\lambda_0 > 0$ and sequences $\{\lambda_i\}$ and $\{v_i\}$ with $\max_\Omega v_i = L - \epsilon_i$ such that $\lambda_i \to \lambda_0$, $\epsilon_i \to 0$ as $i \to \infty$. Making the transformation: $w_i = L - v_i$, we see that w_i with $\min_\Omega w_i = \epsilon_i$ satisfies the problem

$$T\Delta w_i - D\Delta^2 w_i = \lambda_i w_i^{-2} \text{ in } \Omega, \quad w_i = L, \Delta w_i = 0 \text{ on } \partial\Omega.$$

Define $z_i = \Delta w_i$, then

$$-D\Delta z_i + T z_i = \lambda_i w_i^{-2} \text{ in } \Omega, \quad z_i = 0 \text{ on } \partial\Omega. \tag{7.3}$$

It is known from (7.3) that $z_i(x) = \lambda_i \int_\Omega G_{T,D}(x,y) w_i^{-2}(y)dy$, where $G_{T,D}(x,y)$ is the Green's function of the operator $-D\Delta + TId$. Let $w_i(x_i) = \min_\Omega w_i$. Then $x_i \in \Omega_0 \subset\subset \Omega$. Setting $\tilde{w}_i(y) = \frac{w_i}{\epsilon_i}$ and $y = \lambda_i^{1/4}\epsilon_i^{-3/4}(x - x_i)$, we see that $\tilde{w}_i$ with $\tilde{w}_i(0) = \min_{\Omega_i} \tilde{w}_i = 1$ and $\tilde{w}_i$ satisfies the problem

$$\lambda_i^{-1/2}\epsilon_i^{3/2} T\Delta_y \tilde{w}_i - D\Delta_y^2 \tilde{w}_i = \tilde{w}_i^{-2} \text{ in } \Omega_i, \quad \tilde{w}_i = \frac{L_i}{\epsilon_i}, \ \Delta_y \tilde{w}_i = 0 \text{ on } \partial\Omega_i, \tag{7.4}$$

where $\Omega_i = \{y = \lambda_i^{1/4}\epsilon_i^{-3/4}(x - x_i) : \ x \in \Omega\}$. On the other hand,

$$\Delta_y \tilde{w}_i = \lambda_i^{-1/2}\epsilon_i^{1/2}\Delta_x w_i = \lambda_i^{1/2}\epsilon_i^{1/2}\int_\Omega G_{T,D}(x,\xi)w_i^{-2}(\xi)d\xi.$$

Note that $N = 2$ and $w_i \geq \epsilon_i$ in Ω. The Hölder inequality implies that

$$\begin{aligned}
|\Delta_y \tilde{w}_i| &\leq C\epsilon_i^{1/2}\Big(\int_\Omega [G_{T,D}(x,\xi)]^p d\xi\Big)^{1/p}\Big(\int_\Omega w_i^{-2}w_i^{2-2q}(\xi)d\xi\Big)^{1/q} \\
&\leq C\epsilon_i^{1/2}\epsilon_i^{-2/p}\Big(\int_\Omega [G_{T,D}(x,\xi)]^p d\xi\Big)^{1/p}\Big(\int_\Omega w_i^{-2}(\xi)d\xi\Big)^{1/q} \\
&\leq C\epsilon_i^{\frac{1}{2}-\frac{2}{p}}
\end{aligned}$$

where we have applied (7.1). Choosing p sufficiently large, we see that

$$(7.5) \qquad |\Delta_y \tilde{w}_i(y)| \to 0 \ \text{ for } y \in \Omega_i \text{ a.e. as } i \to \infty.$$

On the other hand, it follows from (7.4), (7.5) and the regularity of the operator $T\Delta - D\Delta^2$ that $\tilde{w}_i \to W$ in $C^4_{loc}(\mathbb{R}^2)$ as $i \to \infty$, where W with $W(0) = 1$ and $W \geq 1$ in $\mathbb{R}^2$ satisfies the equation

$$(7.6) \qquad -D\Delta^2 W = W^{-2} \ \text{ in } \mathbb{R}^2, \quad W(0) = 1.$$

Meanwhile, (7.5) implies that $\Delta W = 0$ in $\mathbb{R}^2$. This contradicts (7.6) and completes the proof of this lemma. □

In the remaining of this section, we establish the existence of the second solution. Note that in the energy functional (1.1), the integral $\int_\Omega \frac{1}{L+u(x)}dx$ is not well-defined for $u \in H^2(\Omega)$. Therefore we don't have a good energy functional to work with. Our idea is to modify the nonlinearity so that the mountain-pass lemma works, and then show that the resulting solution has no singularity.

We first modify the nonlinearity. Since the nonlinearity $g(v) = \frac{1}{(L-v)^2}$ is singular at $v = L$, we need to consider a regularized C^1 nonlinearity $g_\epsilon(v)$, $0 < \epsilon < L$, of the following form:

$$g_\epsilon(v) = \begin{cases} \frac{1}{(L-v)^2}, & v \leq L - \epsilon \\ \frac{1}{\epsilon^2} - \frac{(L-\epsilon)}{\epsilon^3} + \frac{1}{\epsilon^3(L-\epsilon)}v^2, & v > L - \epsilon. \end{cases}$$

For $\lambda \in (0, \lambda_c)$, we study the regularized semilinear elliptic problem:

$$(7.7) \qquad -T\Delta v + D\Delta^2 v = \lambda g_\epsilon(v) \ \text{ in } \Omega, \quad v = \Delta v = 0 \ \text{ on } \partial\Omega.$$

From a variational viewpoint, the action functional associated to (7.7) is

$$J_{\epsilon,\lambda}(v) = \frac{1}{2}\int_\Omega [T|\nabla v|^2 + D(\Delta v)^2]dx - \lambda\int_\Omega G_\epsilon(v)dx, \quad v \in \mathcal{H},$$

where $G_\epsilon(v) = \int_{-\infty}^v g_\epsilon(s)ds$.

Fix now $0 < \epsilon < \tau_\lambda/4$, where τ_λ is given in Lemma 7.2. The minimal solution $\underline{v}_\lambda$ of (T_λ) is still a solution of (7.7) so that $\sigma_{1,\lambda}(\underline{v}_\lambda) > 0$. In order to motivate the choice of $g_\epsilon(v)$, we briefly sketch the proof of Theorem 7.1. First, we prove that $\underline{v}_\lambda$ is a local minimum for $J_{\epsilon,\lambda}(v)$. Then, by the well-known Mountain Pass Theorem, we show the existence of a second solution $V_{\epsilon,\lambda}$ for (7.7). (Similar idea has been used in [6].) The subcritical growth:

$$0 \leq g_\epsilon(v) \leq C_\epsilon(1 + |v|^2) \tag{7.8}$$

and the inequality:

$$3G_\epsilon(v) \leq v g_\epsilon(v) \ \text{ for } v \geq L - \theta, \tag{7.9}$$

for some sufficiently small $\theta > 10\epsilon$ independent of ϵ, $C_\epsilon > 0$, will yield that $J_{\epsilon,\lambda}$ satisfies the Palais-Smale condition.

In order to complete the details of the proof of Theorem 7.1, we first need to show the following:

Lemma 7.3. *The minimal solution $\underline{v}_\lambda$ of (T_λ) is a local minimum of $J_{\epsilon,\lambda}$ on $\mathcal{H}$.*

Proof. Since $\mathcal{H} \hookrightarrow C^\alpha(\overline{\Omega})$ for any $0 < \alpha < 1$, we only need to show that $\underline{v}_\lambda$ is a local minimum of $J_{\epsilon,\lambda}$ in $C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$. Indeed, since $\sigma_{1,\lambda}(\underline{v}_\lambda) > 0$, we have the following inequality:

$$\int_\Omega [T|\nabla\varphi|^2 + D(\Delta\varphi)^2]dx - 2\lambda\int_\Omega \frac{1}{(L-\underline{v}_\lambda)^3}\varphi^2 dx \geq \sigma_{1,\lambda}\int_\Omega \varphi^2 dx \tag{7.10}$$

for any $\varphi \in \mathcal{H}$, since $\underline{v}_\lambda \leq L - \tau_\lambda < L - \epsilon$ (see Lemma 7.2). Now, take any $\varphi \in \mathcal{H}\cap C^\alpha(\overline{\Omega})$ such that $\|\varphi\|_{C^\alpha} \leq \delta_\lambda$. Since $\underline{v}_\lambda \leq L-\tau_\lambda$, if $\delta_\lambda \leq \epsilon$, then $\underline{v}_\lambda+\varphi \leq L-\epsilon$ and we have that:

$$\begin{aligned}
&J_{\epsilon,\lambda}(\underline{v}_\lambda + \varphi) - J_{\epsilon,\lambda}(\underline{v}_\lambda)\\
&\quad= \frac{1}{2}\int_\Omega [T|\nabla\varphi|^2 + D(\Delta\varphi)^2]dx + \int_\Omega [T\nabla\underline{v}_\lambda\cdot\nabla\varphi + D\Delta\underline{v}_\lambda\Delta\varphi]dx\\
&\qquad\quad -\lambda\int_\Omega \Big(\frac{1}{L-\underline{v}_\lambda-\varphi} - \frac{1}{L-\underline{v}_\lambda}\Big)\\
&\quad\geq \frac{\sigma_{1,\lambda}}{2}\int_\Omega \varphi^2 - \lambda\int_\Omega\Big(\frac{1}{L-\underline{v}_\lambda-\varphi} - \frac{1}{L-\underline{v}_\lambda} - \frac{\varphi}{(L-\underline{v}_\lambda)^2} - \frac{\varphi^2}{(L-\underline{v}_\lambda)^3}\Big),
\end{aligned}$$

where we have used (7.10). Since now

$$\Big|\frac{1}{L-\underline{v}_\lambda-\varphi} - \frac{1}{L-\underline{v}_\lambda} - \frac{\varphi}{(L-\underline{v}_\lambda)^2} - \frac{\varphi^2}{(L-\underline{v}_\lambda)^3}\Big| \leq C|\varphi|^3$$

for some $C > 0$, we have that

$$J_{\epsilon,\lambda}(\underline{v}_\lambda + \varphi) - J_{\epsilon,\lambda}(\underline{v}_\lambda) \geq \left(\frac{\sigma_{1,\lambda}}{2} - C\lambda\delta_\lambda\right) \int_\Omega \varphi^2 > 0$$

provided δ_λ small enough. This proves that $\underline{v}_\lambda$ is a local minimum of $J_{\epsilon,\lambda}$ in the C^α topology and completes the proof of this lemma. □

Fix some ball $B_{2r} \subset \Omega$ of radius $2r$, $r > 0$. Take a cut-off function χ so that $\chi = 1$ on B_r and $\chi = 0$ outside B_{2r}. Let $w_\epsilon = (L - \epsilon)\chi \in \mathcal{H}$. We have that:

$$J_{\epsilon,\lambda}(w_\epsilon) \leq \frac{(L-\epsilon)^2}{2} \int_\Omega [T|\nabla\chi|^2 + D(\Delta\chi)^2]dx - \frac{\lambda}{\epsilon^2}|B_r| \to -\infty$$

as $\epsilon \to 0$. Moreover, we can find that for $\epsilon > 0$ small, the inequality

$$J_{\epsilon,\lambda}(w_\epsilon) < J_{\epsilon,\lambda}(\underline{v}_\lambda). \tag{7.11}$$

Fix now $\epsilon > 0$ small enough in order that (7.11) holds, and define

$$c_{\epsilon,\lambda} = \inf_{\gamma\in\Gamma} \max_{v\in\gamma} J_{\epsilon,\lambda}(v),$$

where $\Gamma = \{\gamma : [0,1] \to \mathcal{H}; \gamma \text{ continuous and } \gamma(0) = \underline{v}_\lambda,\ \gamma(1) = w_\epsilon\}$. We can then apply the Mountain Pass Theorem to get a solution $V_{\epsilon,\lambda}$ of (7.7), provided the Palais-Smale condition holds at level $c_{\epsilon,\lambda}$. The embedding theorem and the maximum principle imply that $V_{\epsilon,\lambda} > 0$ in Ω.

Lemma 7.4. *Assume that $\{v_n\} \subset \mathcal{H}$ satisfies*

$$J_{\epsilon,\lambda_n}(v_n) \leq C, \quad J'_{\epsilon,\lambda_n}(v_n) \to 0 \ \text{ in } \mathcal{H}' \tag{7.12}$$

for $\lambda_n \to \lambda > 0$. Then the sequence $(v_n)_n$ is uniformly bounded in $\mathcal{H}$ and therefore admits a convergent subsequence in $\mathcal{H}$.

Proof. By (7.12) we have that:

$$\int_\Omega [T|\nabla v_n|^2 + D(\Delta v_n)^2]dx = \lambda_n \int_\Omega g_\epsilon(v_n)v_n dx + o(\|v_n\|_{\mathcal{H}})$$

as $n \to +\infty$ and then,

$$\begin{aligned} C &\geq \frac{1}{2}\int_\Omega [T|\nabla v_n|^2 + D(\Delta v_n)^2]dx - \lambda_n \int_\Omega G_\epsilon(v_n)dx \\ &= \frac{1}{6}\int_\Omega [T|\nabla v_n|^2 + D(\Delta v_n)^2]dx + \lambda_n \int_\Omega \left(\frac{1}{3}v_n g_\epsilon(v_n) - G_\epsilon(v_n)\right)dx + o(\|v_n\|_{\mathcal{H}}) \\ &\geq \frac{1}{6}\int_\Omega [T|\nabla v_n|^2 + D(\Delta v_n)^2]dx + \lambda_n \int_{\{v_n \geq L-\theta\}} \left(\frac{1}{3}v_n g_\epsilon(v_n) - G_\epsilon(v_n)\right)dx \\ &\quad + o(\|v_n\|_{\mathcal{H}}) - C(\theta) \\ &\geq \frac{1}{6}\int_\Omega [T|\nabla v_n|^2 + D(\Delta v_n)^2]dx + o(\|v_n\|_{\mathcal{H}}) - C(\theta) \end{aligned}$$

in view of (7.9). Where $C(\theta) > 0$ depends on θ but is independent of ϵ. Hence, $\sup_{n\in\mathbf{N}} \|v_n\|_{\mathcal{H}} < +\infty$.

The compactness of the embedding $\mathcal{H} \hookrightarrow C^\alpha(\overline{\Omega})$ for any $0 < \alpha < 1$ provides that, up to a subsequence, $v_n \to v$ weakly in $\mathcal{H}$ and strongly in $C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$ and some $v \in \mathcal{H}$. By (7.12) we get that $\int_\Omega [T|\nabla v|^2 + D(\Delta v)^2]dx = \lambda \int_\Omega g_\epsilon(v)v$, and then,

$$\begin{aligned}
&\int_\Omega [T|\nabla(v_n - v)|^2 + D(\Delta(v_n - v))^2]dx \\
&= T\Big[\int_\Omega |\nabla v_n|^2 - \int_\Omega |\nabla v|^2\Big] + D\Big[\int_\Omega (\Delta v_n)^2 - \int_\Omega (\Delta v)^2\Big] + o(1) \\
&= \lambda_n \int_\Omega g_\epsilon(v_n)v_n - \lambda \int_\Omega g_\epsilon(v)v + o(1) \to 0
\end{aligned}$$

as $n \to +\infty$. This completes the proof. □

Proof of Theorem 7.1

We only need to show (i) and it is enough to show that for any fixed $\lambda > 0$, the mountain pass solution $V_{\epsilon,\lambda}$ satisfies $V_{\epsilon,\lambda} \le L - \epsilon$ in Ω.

Since $V_{\epsilon,\lambda} \in \mathcal{H}$, by the same argument as in Theorem 3.1, we easily see that

$$\int_\Omega g_\epsilon(V_{\epsilon,\lambda})dx \le C/\lambda \tag{7.13}$$

where C is independent of ϵ. In fact, we see that

$$J_{\epsilon,\lambda}(V_{\epsilon,\lambda}) \le \max_{v\in\gamma_0} J_{\epsilon,\lambda}(v)$$

where $\gamma_0 : [0,1] \to \mathcal{H}$; $\gamma_0(v) = t\underline{v}_\lambda + (1-t)w_\epsilon$ for $t \in [0,1]$. Thus,

$$J_{\epsilon,\lambda}(V_{\epsilon,\lambda}) \le C$$

where $C > 0$ is independent of ϵ. On the other hand, we see

$$\begin{aligned}
C &\ge \frac{1}{2}\int_\Omega [T|\nabla V_{\epsilon,\lambda}|^2 + D(\Delta V_{\epsilon,\lambda})^2]dx - \lambda_n \int_\Omega G_\epsilon(V_{\epsilon,\lambda})dx \\
&= \frac{1}{6}\|V_{\epsilon,\lambda}\|_{\mathcal{H}}^2 + \lambda \int_\Omega \Big(\frac{1}{3}V_{\epsilon,\lambda} g_\epsilon(V_{\epsilon,\lambda}) - G_\epsilon(V_{\epsilon,\lambda})\Big)dx \\
&\ge \frac{1}{6}\int_\Omega \|V_{\epsilon,\lambda}\|_{\mathcal{H}}^2 + \lambda \int_{\{V_{\epsilon,\lambda} \ge L-\theta\}} \Big(\frac{1}{3}V_{\epsilon,\lambda} g_\epsilon(V_{\epsilon,\lambda}) - G_\epsilon(V_{\epsilon,\lambda})\Big)dx - C(\theta) \\
&\ge \frac{1}{6}\|V_{\epsilon,\lambda}\|_{\mathcal{H}}^2 dx - C(\theta).
\end{aligned}$$

Thus,

$$\|V_{\epsilon,\lambda}\|_{\mathcal{H}} \le C$$

where $C > 0$ is independent of ϵ. The embedding $\mathcal{H} \hookrightarrow C^0(\overline{\Omega})$ implies $V_{\epsilon,\lambda} \leq C$ in Ω. By the moving plane argument as in the proof of Theorem 3.1, we have that $V_{\epsilon,\lambda} \leq L - \theta$ in Ω_δ, where Ω_δ is given in the proof of Theorem 3.1 and θ is given by (7.9). This implies that (7.13) holds.

Let $W_\epsilon = \Delta V_{\epsilon,\lambda}$. Then W_ϵ satisfies the equation

$$-TW_\epsilon + D\Delta W_\epsilon = \lambda g_\epsilon(V_{\epsilon,\lambda}) \in L^1(\Omega).$$

Since $N = 2$, the Brezis-Merle inequality ([4]) implies that

$$\int_\Omega |W_\epsilon|^q dx \leq C, \quad \forall q > 1$$

where C is independent of ϵ. This also yields that

$$\|V_{\epsilon,\lambda}\|_{W^{2,q}(\Omega)} \leq C.$$

By choosing $q > 3$ sufficiently large, we see from the embedding $W_0^{2,q}(\Omega) \hookrightarrow C^{1+\frac{1}{2}}(\Omega)$ that $V_{\epsilon,\lambda} \leq C$ in Ω.

Now we show that $V_{\epsilon,\lambda} < L$ in Ω for ϵ sufficiently small. On the contrary, we suppose that there is a sequence $\{\epsilon_i\}$ with $\epsilon_i \to 0$ as $i \to \infty$ such that $\max_\Omega V_{\epsilon_i,\lambda} \geq L$. Denote $V_{\epsilon_i,\lambda}(x_i) = \max_\Omega V_{\epsilon_i,\lambda}$. By arguments similar to those in the proof of Lemma 5.1, we see that

$$V_{\epsilon_i,\lambda}(x_i) - V_{\epsilon_i,\lambda}(x) \leq C|x - x_i|^{3/2}.$$

Thus,

$$V_{\epsilon_i,\lambda}(x) \geq V_{\epsilon_i,\lambda}(x_i) - C|x - x_i|^{3/2} > L - \epsilon_i$$

provided that $|x - x_i| < (\epsilon_i/C)^{2/3}$. But

$$C \geq \int_\Omega g_{\epsilon_i}(V_{\epsilon_i,\lambda})dx \geq \epsilon_i^{-2} \int_{\{|x-x_i| \leq (\epsilon_i/C)^{2/3}\}} dx = C\epsilon_i^{-2/3} \to \infty$$

as $i \to \infty$. This is a contradiction.

Now we claim that there exists $\delta > 0$ independent of ϵ such that

$$V_{\epsilon,\lambda} \leq L - \delta \ \text{ in } \Omega$$

for ϵ sufficiently small. On the contrary, there are sequences $\{\epsilon_i\}$ and $\{V_i\} \equiv \{V_{\epsilon_i,\lambda}\}$ with $\epsilon_i \to 0$ as $i \to \infty$ such that $\max_\Omega V_i = L - \xi_i$ and $\xi_i \to 0$ as $i \to \infty$. Set $Z_i = L - V_i$. Then $Z_i(x_i) := \min_\Omega Z_i = \xi_i$ and Z_i satisfies

$$T\Delta Z_i - D\Delta^2 Z_i = \lambda h_i(Z_i) \ \text{ in } \Omega, \quad Z_i = T, \ \ \Delta Z_i = 0 \ \text{ on } \partial\Omega,$$

where

$$h_i(Z_i) = \begin{cases} \frac{1}{Z_i^2}, & Z_i \geq \epsilon_i \\ \frac{1}{\epsilon_i^2} + \frac{2(\epsilon_i - Z_i)}{\epsilon_i^3} + \frac{(\epsilon_i - Z_i)^2}{\epsilon_i^3(L-\epsilon_i)}, & Z_i < \epsilon_i. \end{cases}$$

Making the transformations: $\tilde{Z}_i(y) = Z_i/\xi_i$ and $y = \xi_i^{-3/4}(x - x_i)$ we see that $\tilde{Z}_i(0) = \min_\Omega \tilde{Z}_i = 1$ and $\tilde{Z}_i$ satisfies the problem

$$\xi_i^{3/2}\Delta_y \tilde{Z}_i - D\Delta_y^2 \tilde{Z}_i = \lambda \tilde{h}_i(\tilde{Z}_i) \text{ in } \tilde{\Omega}_i, \quad \tilde{Z}_i = T/\xi_i, \quad \Delta_y \tilde{Z}_i = 0 \text{ on } \partial\tilde{\Omega}_i \tag{7.14}$$

where $\tilde{\Omega}_i = \{y = \xi_i^{-3/4}(x - x_i) : x \in \Omega\}$ and

$$\tilde{h}_i(\tilde{Z}_i) = \begin{cases} \frac{1}{\tilde{Z}_i^2}, & \tilde{Z}_i \geq \frac{\epsilon_i}{\xi_i} \\ 3\left(\frac{\xi_i}{\epsilon_i}\right)^2 - 2\left(\frac{\xi_i}{\epsilon_i}\right)^3 \tilde{Z}_i + \frac{\xi_i^2}{\epsilon_i(L-\epsilon_i)} - 2\left(\frac{\xi_i^3}{\epsilon_i^2(L-\epsilon_i)}\right)\tilde{Z}_i + \left(\frac{\xi_i^4}{\epsilon_i^3(L-\epsilon_i)}\right)\tilde{Z}_i^2, & \tilde{Z}_i < \frac{\epsilon_i}{\xi_i}. \end{cases}$$

We consider two cases for $\{\frac{\epsilon_i}{\xi_i}\}$ (we can choose subsequence if necessary):

(i) There is $0 < A < \infty$ such that $\frac{\epsilon_i}{\xi_i} \leq A$ for all i,

(ii) $\frac{\epsilon_i}{\xi_i} \to \infty$ as $i \to \infty$.

For the first case, we have that there is $0 \leq A_1 \leq A$ such that $\lim_{i\to\infty} \frac{\epsilon_i}{\xi_i} = A_1$. If $A_1 < 1$, since $\tilde{Z}_i \geq 1$, we have that $\tilde{h}_i = \tilde{Z}_i^{-2} \leq 1$ in $\tilde{\Omega}_i$ for i sufficiently large. If $1 \leq A_1 \leq A$, we also have that $\tilde{h}_i \leq C$ in $\tilde{\Omega}_i$ for i sufficiently large, where $0 < C < \infty$ is independent of i. Moreover, for $q > 3$,

$$\begin{aligned} \int_{\tilde{\Omega}_i} |\Delta_y \tilde{Z}_i|^q dy &= \xi_i^{q/2} \int_{\tilde{\Omega}_i} |\Delta_x Z_i|^q dy \\ &= \xi^{\frac{q-3}{2}} \int_\Omega |\Delta_x Z_i|^q dx \to 0 \end{aligned}$$

as $i \to \infty$. Thus, the regularity of Δ^2 implies that $\tilde{Z}_i \to \tilde{Z}$ in $C^3_{loc}(\mathbb{R}^2)$ with $\tilde{Z}(0) = \min_{\mathbb{R}^2} \tilde{Z} = 1$ and $\tilde{Z}$ satisfies the equation

$$-D\Delta^2 \tilde{Z} = \lambda \tilde{Z}^{-2} \text{ in } \mathbb{R}^2$$

provided $A_1 \leq 1$ and the equation

$$D\Delta^2 \tilde{Z} = \lambda \tilde{h}(\tilde{Z}) \text{ in } \mathbb{R}^2$$

provided $1 < A_1 \leq A$, where

$$\tilde{h}(\tilde{Z}) = \begin{cases} \tilde{Z}^2 & \tilde{Z} \geq A_1 \\ \frac{3}{A_1^2} - \frac{2}{A_1^3}\tilde{Z}, & \tilde{Z} < A_1. \end{cases}$$

Moreover, for any large ball B_R of $\mathbb{R}^2$, $\int_{B_R} |\Delta \tilde{Z}|^q(y)dy = 0$. This is impossible.

For the second case, we see that $\xi_i = o(\epsilon_i)$ for i sufficiently large. Thus, $Z_i(x_i) = \xi_i = o(\epsilon_i)$. Note that $\int_\Omega |\Delta Z_i|^q dx \leq C$, we see that $Z_i \in W_0^{2,q}(\Omega)$. The embedding $W_0^{2,q}(\Omega) \hookrightarrow C^{1+\frac{1}{2}}(\Omega)$ gives

$$|Z_i(x)| \leq Z_i(x_i) + C|x - x_i|^{3/2} < \epsilon_i$$

provided $|x-x_i| \le \left(\frac{\epsilon_i}{2C}\right)^{2/3}$. Thus

$$C \ge \int_\Omega h_i(Z_i)dx \ge \frac{1}{\epsilon_i^2}\int_{Z_i<\epsilon_i} dx \ge C\epsilon_i^{-2/3} \to \infty$$

as $i \to \infty$. This is a contradiction either. Therefore,

$$V_{\epsilon,\lambda} \le L - \delta \text{ in } \Omega$$

where $\delta > 0$ is independent of ϵ. This also implies that $V_{\epsilon,\lambda}$ is a solution of (T_λ). This completes the proof of (i) of Theorem 7.1 and the proof of Theorem 7.1. □

8. The asymptotic behavior of the mountain pass solution as $\lambda \to 0$

In this section we will study the asymptotic behavior of the mountain pass solution V_λ obtained in Theorem 7.1 as $\lambda \to 0$.

Lemma 8.1.

$$\sigma_{1,\lambda}(V_\lambda) < 0 \ \textit{for}\ 0 < \lambda < \lambda_c.$$

Proof. Let $\underline{v}_\lambda$ be the minimal solution of (T_λ) so that $V_\lambda \ge \underline{v}_\lambda$. If the linearization around V_λ had nonnegative first eigenvalue, then Lemma 4.1 would also yield $V_\lambda \le \underline{v}_\lambda$ so that $\overline{v}_\lambda$ and V_λ necessarily coincide, a contradiction. □

Theorem 8.2.

$$\max_\Omega V_\lambda \to L \ \ \textit{as}\ \lambda \to 0. \tag{8.1}$$

Moreover,

$$\lim_{\lambda\to 0+} \frac{[\min_\Omega(L-V_\lambda)]^3}{\lambda} = 0. \tag{8.2}$$

Proof. Suppose that there are sequences $\{\lambda_i\}$ and $\{V_i\} \equiv \{V_{\lambda_i}\}$ such that $\lambda_i \to 0$ as $i \to \infty$ and $\max_\Omega V_i \le L - \delta$, where $0 < \delta < L$ is independent of i. Then it follows from the equation of V_i that $V_i \to 0$ in $C^0(\overline{\Omega})$ as $i \to \infty$ (we can choose subsequences if necessary). This contradicts the fact that $\sigma_{1,\lambda_i}(V_i) < 0$ for all i. Thus, (8.1) holds.

By Theorem 3.1, we see that

$$\lambda \int_\Omega (L-V_\lambda)^{-2} dx + \int_\Omega |\Delta V_\lambda|^2 \le C$$

for any λ sufficiently small, where C is independent of λ. Since ΔV_λ satisfies

$$-D\Delta(\Delta V_\lambda) = D\Delta V_\lambda + \frac{\lambda}{(L-V_\lambda)^2} \in L^1(\Omega), \Delta V_\lambda = 0 \text{ on } \partial\Omega$$

By the Brezis-Merle inequality ([4]), we have, for any $q > 1$,

$$\int_\Omega |\Delta V_\lambda|^q dx \le C \tag{8.3}$$

for any λ sufficiently small.

Let $V_\lambda(x_\lambda) = \max_\Omega V_\lambda$. Setting $W_\lambda = L - V_\lambda$, we see that $\xi_\lambda := W_\lambda(x_\lambda) = \min_\Omega W_\lambda$ and $\xi_\lambda \to 0$ as $\lambda \to 0$. Now we claim that

$$\lim_{\lambda\to 0} \frac{\xi_\lambda^3}{\lambda} = 0. \tag{8.4}$$

Suppose not, there are sequences $\{\lambda_i\}$ and $\{\xi_i\}$ with $\lambda_i \to 0$ as $i \to \infty$ such that $\frac{\xi_i^3}{\lambda_i} \to C > 0$ or $\frac{\xi_i^3}{\lambda_i} \to \infty$ as $i \to \infty$.

We first consider the case that $\frac{\xi_i^3}{\lambda_i} \to \infty$ as $i \to \infty$. Then defining $\hat{W}_i = W_i/\xi_i$, we see that $\hat{W}_i$ satisfies the problem

$$T\Delta \hat{W}_i - D\Delta^2 \hat{W}_i = \frac{\lambda_i}{\xi_i^3}\hat{W}_i^{-2} \text{ in } \Omega, \quad \hat{W}_i = L/\xi_i, \ \ \Delta W_i = 0 \text{ on } \partial\Omega.$$

Since $\hat{W}_i \ge 1$, we see that $\hat{W}_i \to \hat{W}$ in $C^3_{loc}(\Omega)$ as $i \to \infty$ and $\hat{W}$ with $\hat{W}(0) = \min_\Omega \hat{W} = 1$ satisfies the equation

$$T\Delta \hat{W} - D\Delta^2 \hat{W} = 0 \text{ in } \Omega, \quad \hat{W} = \infty, \ \ \Delta \hat{W} = 0 \text{ on } \partial\Omega. \tag{8.5}$$

Setting $Z = \Delta \hat{W}$, we see from (8.5) that

$$TZ - D\Delta Z = 0 \text{ in } \Omega, \quad Z = 0 \text{ on } \partial\Omega.$$

The strong maximum principle then implies that $Z \equiv 0$ in Ω and hence $\Delta \hat{W} \equiv 0$ in Ω. The maximum principle then implies that $\hat{W} \equiv 1$ in Ω and a contradiction.

Now we consider the case that $\lim_{i\to\infty} \frac{\xi_i^3}{\lambda_i} \to C > 0$. Defining $\hat{W}_i = W_i/\xi_i$ again, we see that $\hat{W}_i$ satisfies the problem

$$T\Delta \hat{W}_i - D\Delta^2 \hat{W}_i = \frac{\lambda_i}{\xi_i^3}\hat{W}_i^{-2} \text{ in } \Omega, \quad \hat{W}_i = L/\xi_i, \ \ \Delta \hat{W}_i = 0 \text{ on } \partial\Omega. \tag{8.6}$$

Setting $\hat{Z}_i = \Delta \hat{W}_i$, we see that $\hat{Z}_i$ satisfies the problem

$$T\hat{Z}_i - D\Delta \hat{Z}_i = \frac{\lambda_i}{\xi_i^3}\hat{W}_i^{-2} \text{ in } \Omega, \quad \hat{Z}_i = 0 \text{ on } \partial\Omega. \tag{8.7}$$

Therefore,

$$\hat{Z}_i = \frac{\lambda_i}{\xi_i^3}\int_\Omega G_{T,D}(x,y)\hat{W}_i^{-2}(y)dy$$

and hence $|\hat{Z}_i| \le C$, where $C > 0$ is independent of i. We now obtain from the regularity of Δ^2 and (8.6) that $\hat{W}_i \to \hat{W}$ in $C^3_{loc}(\Omega)$ and $\hat{W}$ satisfies the equation

$$T\Delta \hat{W} - D\Delta^2 \hat{W} = \frac{1}{C}\hat{W}^{-2} \text{ in } \Omega, \quad \hat{W} = \infty, \ \ \Delta \hat{W} = 0 \text{ on } \partial\Omega.$$

On the other hand, we see from (8.7) that $\hat{Z}_i \to \hat{Z}$ in $C^1(\overline{\Omega})$ as $i \to \infty$ and $\hat{Z} \equiv \Delta \hat{W}$ satisfies the problem

$$T\hat{Z} - D\Delta \hat{Z} = \frac{1}{C}\hat{W}^{-2} \text{ in } \Omega, \quad \hat{Z} = 0 \text{ on } \partial\Omega.$$

Since we easily know that $\Delta \hat{W} \leq C$ on $\overline{\Omega}$ and hence $\Delta(\hat{W} - C\rho) \leq 0$ in Ω, where $-\Delta\rho = 1$ in Ω and $\rho = 0$ on $\partial\Omega$. The maximum principle implies that $\hat{W}$ can not be ∞ on $\partial\Omega$. Thus, (8.2) holds. This completes the proof of Theorem 8.2. □

References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations, satisfying general boundary conditions, I, *Comm. Pure Appl. Math.* 12, 623-727, (1959).

[2] A.L. Bertozzi and M.C. Pugh, Long-wave instabilities and saturation in thin film equations, *Comm. Pure Appl. Math.* 51, 625-661, (1998).

[3] A.L. Bertozzi and M.C. Pugh, Finite-time blow-up of solutions of some long-wave unstable thin film equations, *Indiana Univ. Math. J.* 49, 1323-1366, (2000).

[4] H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equation* 16, 1223-1254, (1991).

[5] J.P. Burelbach, S.G. Bankoff and S.H. Davis, Nonlinear stability of evaporating /condensing liquid films, *J. Fluid Mech.* 195, 463-494, (1988).

[6] M.G. Crandall and P.H. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, *Arch. Rational Mech. Anal.* 58, 207-218, (1975).

[7] P. Esposito, N. Ghoussoub and Y. Guo, Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity, *Comm. Pure Appl. Math.* 60, 1731-1768, (2008).

[8] G. Flores, G.A. Mercado and J.A. Pelesko, Dynamics and touchdown in electrostatic MEMS, *Proceedings of ICMENS* 182-187, (2003).

[9] F. Gazzola and H.-Ch. Grunau, Critical dimensions and higher order Sobolev inequalities with remainder terms, *NoDEA Nonlinear Differential Equations Appl.* 8, 35-44, (2001).

[10] N. Ghoussoub and Y. Guo, On the partial differential equations of electrostatic MEMS devices: stationary case, *SIAM J. Math. Anal* 38, 1423-1449, (2007).

[11] N. Ghoussoub and Y. Guo, On the partial differential equations of electrostatic MEMS devices II: dynamic case, *NoDEA Nonlinear Differential Equations Appl.* 15, 115-145, (2008).

[12] Y. Guo, On the partial differential equations of electrostatic MEMS devices III: refined touchdown behavior, *J. Differential Equations* 244, 2277-2309, (2008).

[13] Z.M. Guo and J.R.L. Webb, Large and small solutions of a class of quasilinear elliptic eigenvalue problems, *J. Differential Equations* 180, 1-50, (2002).

[14] Z.M. Guo and J.C. Wei, Hausdorff domension of ruptures for solutions of a semilinear elliptic equation with singular nonlinearity, *Manuscripta Math.* 120, 193-209, (2006).

[15] Z.M. Guo and J.C. Wei, Symmetry of nonnegative solutions of a semilinear elliptic equation with singular nonlinearity, *Proc. R. Soc. Edinb. A* 137, 963-994, (2007).

[16] Z.M. Guo and J.C. Wei, Infinitely many turning points for an elliptic problem with a singular nonlinearity, *J. London Math. Soc.* 78, 21-35, (2008).

[17] Y. Guo, Z. Pan and M.J. Ward, Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties, *SIAM J. Appl. Math.* 166, 309-338, (2006).

[18] C.C. Hwang, C.K. Lin and W.Y. Uen, A nonlinear three-dimensional rupture theory of thin liquid films, *J. Colloid Interf. Sci* 190, 250-252, (1997).

[19] R.S. Laugesen and M.C. Pugh, Properties of steady states for thin film equations, *Europ. J. Appl. Math.* 11, 293-351, (2000).
[20] R.S. Laugesen and M.C. Pugh, Energy levels of steady-states for thin-film-type equations, *J. Diff. Eqns.* 182, 377-415, (2002).
[21] R.S. Laugesen and M.C. Pugh, Linear stability of steady states for thin film and Cahn-Hilliard type equations, *Arch. Ration. Mech. Anal.* 154, 3-51, (2000).
[22] F.H. Lin and Y.S. Yang, Nonlinear non-local elliptic equation modelling electrostatic actuation, *Proc. R. Soc. Lond., Ser. A Math. Phys. Eng. Sci.* 463, 1323-1337, (2007).
[23] A. Lindsay and M. Ward, in preparation.
[24] J.A. Pelesko, Mathematical modeling of electrostatic MEMS with tailored dielectric properties, *SIAM J. Appl. Math.* 62, 888-908, (2002).
[25] J.A. Pelesko and D.H. Bernstein, Modeling MEMS and NEMS, Chapman Hall and CRC press (2002).
[26] W. C. Troy, Symmetry properties in systems of semilinear elliptic equations, *J. Diff. Eqns.* 42, 400-413, (1981).
[27] R.C.A.M. van der Vorst, Best constant for the embedding of the space $H^2 \cap H_0^1(\Omega)$ into $L^{2N/(N-4)}(\Omega)$, *Diff. Int. Eqns.* 6, 259-276, (1993).

Department of Mathematics, Henan Normal University, Xinxiang, 453007, P.R. China

E-mail address: guozm@public.xxptt.ha.cn

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk