INFINITE TIME BLOW-UP FOR THE 3-DIMENSIONAL ENERGY CRITICAL HEAT EQUATION

MANUEL DEL PINO, MONICA MUSSO, AND JUNCHENG WEI

Abstract. We construct globally defined in time, unbounded positive solutions to the energy-critical heat equation in dimension three

$$u_t = \Delta u + u^5, \quad \text{in} \quad \mathbb{R}^3 \times (0,\infty), \quad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^3.$$

For each $\gamma > 1$ we find initial data (not necessarily radially symmetric) with $\lim\limits_{|x|\to\infty} |x|^\gamma u_0(x) > 0$ such that as $t \to \infty$

$$\|u(\cdot,t)\|_\infty \sim t^{\frac{\gamma-1}{2}}, \quad \text{if} \quad 1<\gamma<2, \quad \|u(\cdot,t)\|_\infty \sim \sqrt{t}, \quad \text{if} \quad \gamma > 2,$$

and

$$\|u(\cdot,t)\|_\infty \sim \sqrt{t}\,(\ln t)^{-1}, \quad \text{if} \quad \gamma = 2.$$

Furthermore we show that this infinite time blow-up is co-dimensional one stable. The existence of such solutions was conjectured by Fila and King [16].

1. Introduction

Let $n \geq 3$. The *energy critical heat equation* in $\mathbb{R}^n$ is the parabolic Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1.1}$$

The *energy*

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 - \frac{n-2}{2n}\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}$$

defines a Lyapunov functional for Problem (1.1). In fact for classical solutions $u(x,t)$ with sufficient decay in space variable we have that

$$\frac{d}{dt} E(u(\cdot,t)) = -\int_{\mathbb{R}^n} |u_t|^2.$$

Classical parabolic theory yields that the Cauchy problem (1.1) is well-posed in its natural finite-energy space for short time intervals.

In this paper we are interested in **positive finite-energy solutions** of (1.1) which are global in time, namely defined and smooth in the entire time interval $(0,\infty)$. The presence of the Lyapunov functional implies that limits of bounded solutions along sequences $t = t_n \to +\infty$ can only be steady states, namely solutions of the Yamabe equation

$$\Delta u + |u|^{\frac{4}{n-2}} u = 0 \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

All **positive** solutions of (1.2) are given by the *Aubin-Talenti bubbles*

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} w\left(\frac{x-\xi}{\mu}\right),$$

where $\mu > 0$, $\xi \in \mathbb{R}^n$ and

$$w(x) \;=\; (n(n-2))^{\frac{n-2}{4}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}.$$

They are precisely the extremals of Sobolev's embedding. The *criticality* of Problem (1.1) refers to the presence of this continuum of steady states which become singular as $\mu \to 0$, in addition to energy invariance. In fact we immediately see that

$$E(U_{\mu,\xi}) = E(U) \quad \text{for all} \quad \xi \in \mathbb{R}^n,\ \mu > 0.$$

A solution $u(x,t)$ of (1.1) which looks around one or more points of space like $u(x,t) \approx U_{\mu(t),\xi(t)}(x)$ with $\mu(t) \to 0$ is called a bubbling blow-up solution. Bubbling phenomena is present in many important time-dependent and stationary setting, usually carrying deep meaning in the global structure of their solutions. Notable examples include the Yamabe and harmonic map flows and the Keller-Segel chemotaxis system. (See [4, 7, 34, 8, 19] and the references therein.) In the last decade or so it has been extensively studied in energy-critical wave equations, Schrodinger maps and other dispersive settings.

Problem (1.1) is a simple looking model which contains much of the complexity of the bubbling blow-up issue. Basic questions have remain unanswered until today. Existence or nonexistence of infinite time bubbling positive solutions in Problem (1.1) is not known. This question has been explicitly stated for instance in [30] and in [32], Remark 22.10. Detecting such solutions rigorously is not easy. Usual behaviors in the flow (1.1) are either asymptotic vanishing or blow-up in finite time. Global solutions with nontrivial asymptotic patterns are typically unstable objects and hence harder to be detected.

In a very interesting paper Fila and King [16] provided insight on the question in the case of a radially symmetric, positive initial condition with an exact power decay rate. Using formal matching asymptotic analysis, they demonstrated that the power decay determines the blow-up rate in a precise manner. Intriguingly enough, their analysis leads them to conjecture that infinite time blow-up **should only happen** in low dimensions 3 and 4, see Conjecture 1.1 in [16].

In this paper we rigorously establish the existence of solutions with infinite time blow-up in dimension 3, confirming the conjecture in [16]. Thus we consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^5 & \text{in } \ \mathbb{R}^3 \ \times(0,\infty), \\[2ex] u(\cdot,0) = u_0 & \text{in } \ \mathbb{R}^3, \end{cases} \tag{1.3}$$

for an initial datum u_0 which we assume first radially symmetric with an exact power decay of the form

$$\lim_{|x|\to\infty} |x|^\gamma u_0(x) \ =: \ A > 0. \tag{1.4}$$

As in [16] we assume that $\gamma > 1$ which means that u_0 decays faster than the bubble

$$w(x) = 3^{\frac14} \left(\frac{1}{1+|x|^2} \right)^{\frac12}. \tag{1.5}$$

Theorem 1.1. *Given $\gamma > 1$, there exists a positive, radially symmetric global solution $u(x,t)$ to problem* (1.3) *whose initial condition $u_0(|x|)$ satisfies* (1.4) *and as $t \to +\infty$*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \sim \begin{cases} t^{\frac{\gamma-1}{2}} & \text{if} \quad 1<\gamma<2, \\ \frac{\sqrt{t}}{\ln t} & \text{if} \quad \gamma = 2, \\ \sqrt{t} & \text{if} \quad \gamma > 2. \end{cases} \tag{1.6}$$

More precisely, the blow-up takes place by bubbling near the origin. The solution of Theorem 1.1 is in the inner self-similar region, $|x| \ll \sqrt{t}$, in leading order of the *bubbling blow-up form*

$$u(x,t) \sim \frac{1}{\mu(t)^{\frac12}} w\left(\frac{x}{\mu(t)} \right),$$

where

$$\mu(t) \sim \begin{cases} t^{1-\gamma} & \text{if} \quad 1<\gamma<2, \\ t^{-1}\ln^2 t & \text{if} \quad \gamma=2, \\ t^{-1} & \text{if} \quad \gamma>2 \end{cases} \tag{1.7}$$

and w is given by (1.5). In the outer self-similar region $|x| \gg \sqrt{t}$, the solution dissipates in the form of a self-similar solution of heat equation $u_t = \Delta u$ in $\mathbb{R}^3 \times (0,\infty)$.

A surprising feature of the construction is the dynamics discovered for the scaling parameter $\mu(t)$. It has a highly non-local character governed by a equation involving a perturbation of the fractional $\frac{1}{2}$-Caputo derivative. In fact, in order to find the precise lower order corrections needed for the scaling parameter $\mu(t)$ we will need to solve linear equations of the type

$$\int_0^t \frac{\beta'(s)}{\sqrt{t-s}} \left(1 - e^{-\frac{M^2}{(t-s)}}\right) ds = h(t),$$

for suitably decaying right hand sides $h(t)$. See (6.8) and (6.13) below.

Problem (1.1) is a special case of the *Fujita equation*

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \mathbb{R}^n \times (0,\infty), \\ \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^n \end{cases} \tag{1.8}$$

with $p>1$. Blow-up phenomena in Problem (1.8) is extremely sensitive to the values of the exponent p. A vast literature has been devoted to this problem after Fujita's seminal work [18]. We refer the reader for instance to the book [32] for background and a comprehensive account of results until 2007 and to the more recent works [21, 22, 23] and references therein. The case $p = \frac{n+2}{n-2}$ is special in many ways. Positive steady states do not exist when $p < \frac{n+2}{n-2}$. Positive radial global solutions must be bounded and go to zero, see [26, 28, 32]. They exist when $p > \frac{n+2}{n-2}$ but they have infinite energy, see [20]. Infinite time blow-up exists in that case but it has an entirely different nature, see [29, 30].

The study of energy critical problems has attracted much attention in the last decade. For energy-critical wave equations, blow-up solutions have been characterized and constructed in [10, 11, 12, 13, 15]. In [36] Type-II sign changing, finite time blow-up for (1.1) is constructed, first formally predicted in [17]. Threshold dynamics around the steady states of (1.1) has been characterized in large dimensions $n \geq 7$ in [5]. Also in large dimensions $n \geq 5$ in [6] infinite time bubbling solutions of (1.1) in a bounded domain under Dirichlet boundary conditions are constructed for $n \geq 5$. The cases $n = 3, 4$ are indeed considerably more delicate and not treated there. The solutions in Theorem 1.1 are specially meaningful for the full dynamics since they are *threshold solutions* in the sense that the solution of (1.3) with initial condition λu_0 goes to zero as $t \to \infty$ if $\lambda < 1$ while it blows-up in finite time if $\lambda > 1$. Radial threshold solutions for various ranges of exponents in (1.3) are analyzed in [32].

We recall that from [16], it is not expected to have this blow-up in entire space in dimensions $n \geq 5$. Our approach is entirely different from that in [36] for $n = 4$ in which a finite-time type II blow-up solution of (1.1) is constructed on the basis of the modulation equation methods developed for critical dispersive equations in [9, 25, 24, 33, 34].

Our approach has a parabolic-elliptic flavor, in line with the recent works [6, 8]. Since our proofs only rely on elliptic and parabolic estimates, we can easily modify the proof to deal with nonradial and general initial data, in particular establishing *codimension 1 stability* of the solution built. This is concordant with a result on [14] on the corresponding wave analogue. In Section 10 we prove the following

Theorem 1.2. *Let $\bar{v}_0 = \bar{v}_0(x)$ be a positive continuous function, uniformly bounded for $x \in \mathbb{R}^3$. Let $\gamma > 1$ and $\kappa > \max\{\frac{\gamma+3}{2}, \gamma\}$. Then, there exists a positive global solution $u(x,t)$ to problem* (1.3) *with initial condition*

$$u(x,0) = u_0(|x|) + \frac{\bar{v}_0(x)}{|x|^\kappa}\left[1 - \eta\left(\frac{|x|}{t_0}\right)\right]$$

where u_0 is positive, radially symmetric, satisfies (1.4)*, $t_0 > 0$ is a fixed large number and η is a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ for $s > 2$. As $t \to +\infty$, $u(x,t)$ satisfies* (1.6)*.*

Furthermore, there exists a codimension 1 manifold of functions in $C^1(\mathbb{R}^3)$ converging to 0 at infinity with a sufficiently fast decay, that contains $u_0(|x|)+\frac{\bar v_0(x)}{|x|^\kappa}(1-\eta(\frac{|x|}{t_0}))$ such that if $\bar u_0$ lies in that manifold and it is sufficiently close to $u_0(|x|)+\frac{\bar v_0(x)}{|x|^\kappa}(1-\eta(\frac{|x|}{t_0}))$ in the sense that $\bar u_0 = u_0(|x|)+\frac{\bar v_0(x)}{|x|^\kappa}(1-\eta(\frac{|x|}{t_0}))+\mathcal{O}(|x|e^{-b|x|})$ for some $b > 0$, then the solution $\bar u(x,t)$ to (1.3) *with $\bar u(x,0) = \bar u_0(x)$ is global in time and satisfies* (1.6)*.*

In the non-radial setting, the profile of the solution in the inner self-similar regime is

$$u(x,t) \sim \frac{1}{\mu(t)^{\frac12}} w\left(\frac{x-p(t)}{\mu(t)}\right), \quad \frac{|p(t)|}{\mu(t)} \to 0, \quad \text{as} \quad t\to\infty$$

where w is given by (1.5) and μ satisfies the asymptotics (1.7). Precise description of the dynamics of the center $p = p(t)$ is provided.

A surprising feature of the construction is the dynamics discovered for the scaling parameter $\mu(t)$. It has a highly non-local character governed by a equation involving a perturbation of the fractional $\frac12$-Caputo derivative. In fact, in order to find the precise lower order corrections needed for the scaling parameter $\mu(t)$ we will need to solve linear equations of the type

$$\int_0^t \frac{\beta'(s)}{\sqrt{t-s}} \left(1 - e^{-\frac{M^2}{(t-s)}}\right) ds = h(t),$$

for suitably decaying right hand sides $h(t)$. See (6.8) and (6.13) below.

We believe that an approach similar to that in this paper could be used to prove the existence of global unbounded solution when $N = 4, p = 3$ as conjectured in [16]. We will undertake that issue in a future work.

The proof of Theorem 1.1 starts with the construction of an approximate solution to Problem (1.3) with the asymptotic behavior described in (1.6). This is done in full details in Section 2. We then show the existence of an actual solution to Problem (1.3) deforming the approximation, by means of a *inner-outer gluing* procedure. This scheme is described in Section 3, and its proof is addressed in Sections 4 to 9. In Section 10 we prove Theorem 1.2. Sections 11 to 13 gather some technical results needed to prove the Theorems.

In the rest of the paper, we shall denote by C a generic positive constant, whose value may change from line to line, and within the same line. We shall use the notation $\mathbf{c}$ to indicate a positive constant, with $\mathbf{c} < 1$, whose explicit value may change from line to line. Furthermore, t_0 will denote a large fixed positive number and

$$\eta : \mathbb{R} \to \mathbb{R}, \tag{1.9}$$

a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$.

Acknowledgements: We are indebted to Marek Fila for introducing this problem to us and for many useful discussions. M. del Pino has been supported by a UK Royal Society Research Professorship and Grant PAI AFB-170001, Chile. M. Musso has been partly supported by Fondecyt grant 1160135, Chile. The research of J. Wei is partially supported by NSERC of Canada.

2. Construction of an approximate solution and estimate of the associated error

After shifting the initial time to $t_0 > 0$, Problem (1.3) takes the form

$$u_t = \Delta u + u^5, \quad \text{in} \quad \mathbb{R}^3 \times (t_0, \infty), \tag{2.1}$$

with initial condition $u_0(r) = u(r, t_0)$ satisfying

$$\lim_{r\to\infty} r^\gamma u_0(r) = A > 0, \quad \text{for some} \quad \gamma > 1. \tag{2.2}$$

This section is devoted to the construction of a first approximation for a solution to (2.1)-(2.2), and to the description of the associated error.

The first approximation is build by matching an inner profile, made upon solving the elliptic problem

$$\Delta u + u^5 = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{2.3}$$

and an outer profile, made upon solving the heat equation in the whole space

$$u_t = \Delta u \quad \text{in} \quad \mathbb{R}^3, \tag{2.4}$$

in the set of functions satisfying the decaying conditions (2.2). It is constructed in Subsections 2.1 (for the inner profile), 2.2 (for the outer profile), and in Subsection 2.3 we derive a precise description of the *error* of approximation. In [16], this approximate solution was already derived. We realize though that, for our rigorous construction to work, we need a further improvement of the approximation. This is done in Subsection 2.4, where we introduce a next correction term, and describe the associated error. It turns out that this next correction term gives the right dynamics for the blow-up rate which turns out to be governed by a nonlocal differential equation with a fractional time-derivative closely related to the so-called 1/2-Caputo derivative. See (6.13).

2.1. **Construction of the first inner profile.** We recall that all positive radially symmetric solutions to (2.3) constitute a one-parameter family of functions, which are given explicitly by

$$w(r) = 3^{\frac14}\left(\frac{1}{1+r^2}\right)^{\frac12}, \quad w_\mu(r) = \mu^{-\frac12} w(\frac{r}{\mu}), \tag{2.5}$$

for any positive number $\mu > 0$. (See [1, 2].) We denote by Z_0 the only bounded and radial function belonging to the kernel of the linear operator

$$L_0(\phi) = \Delta\phi + 5w^4\phi. \tag{2.6}$$

See [35]. The function Z_0 is explicitly defined by

$$Z_0(r) = -\left[\frac{w}{2} + w'(r)r\right] = \frac{3^{\frac14}}{2}\,\frac{r^2-1}{(1+r^2)^{\frac32}}. \tag{2.7}$$

Given Z_0, we denote by $\Phi_1(r)$ the solution to

$$\Delta\Phi_1 + 5w^4\Phi_1 = Z_0, \tag{2.8}$$

defined as

$$\Phi_1(r) = \Phi_0(r) + \pi_0 + \bar\Phi_1(r), \quad \text{where} \quad \Phi_0(r) = \frac{3^{\frac14}}{4}\, r, \tag{2.9}$$

$$\left(5\int_0^\infty w^4 Z_0 r^2\, dr\right)\pi_0 = \int_0^\infty (Z_0 - \frac{3^{\frac14}}{2r})Z_0 r^2\, dr - 5\int_0^\infty w^4\Phi_0 Z_0 r^2\, dr$$

and $\bar\Phi_1$ being the unique solution to

$$\Delta\phi + 5w^{p-1}\phi = \underbrace{(Z_0 - \frac{3^{\frac14}}{2r}) - 5w^4(\Phi_0+\pi_0)}_{:=\Pi_0(r)},$$

explicitly given by

$$\bar\Phi_1(r) = \tilde Z(r)\int_0^r \Pi_0(s)Z_0(s)s^2\, ds - Z_0(r)\int_0^r \Pi_0(s)\tilde Z(s)s^2\, ds.$$

In the above expression, $\tilde Z$ denoted another solution to $\Delta\phi + 5w^4\phi = 0$, linearly independent to Z_0. $\tilde Z$ satisfies the asymptotic behavior $\tilde Z(s) \sim s^{-1}$, as $s \to 0$, and $\tilde Z(s) \sim 1$, as $s \to \infty$.

A closer look at the expression of $\bar\Phi_1$ gives that,

$$\|r^{2-\sigma}\bar\Phi_1(r)\|_\infty < C,$$

for some fixed positive constant C, and any $\sigma > 0$ small.

Remark 2.1. The solution to (2.8) is not unique. (In fact one can add any multiple of Z_0.) The choice we made in (2.9) is used to match the outer solution in the next section.

We have now the elements to define the first inner profile. We introduce a smooth positive function $\mu(t)$ of the form

$$\mu(t) = \mu_0(t)\,(1+\Lambda(t))^2\,, \quad \text{where} \quad \mu_0(t)>0, \quad \lim_{t\to\infty}\mu_0(t)=0. \tag{2.10}$$

The function μ_0 will be defined below, (see (2.23), (2.32), (2.36)), as an explicit function of t depending on the decay rate γ. On the other hand, the function $\Lambda=\Lambda(t)$ will be left as a parameter in the construction, and it will be determined in the final argument to get an actual solution to the problem. In the meanwhile, we shall assume that $\Lambda=\Lambda(t)$ is a smooth function in (t_0,∞), defined by

$$\begin{aligned} &\Lambda(t) := \int_t^\infty \lambda(s)ds, \quad \text{where} \quad \lambda \quad \text{satisfies} \\ &\|\lambda\|_\sharp := \sup_{t>t_0}\ \mu_0(t)^{-1}t\ \left[\|\lambda\|_{\infty,[t,t+1]} + [\lambda]_{0,\sigma,[t,t+1]}\right]\ \le \ell, \end{aligned} \tag{2.11}$$

for $\sigma=\frac12+\sigma'$, with $\sigma'>0$ small, and for some fixed constant ℓ. Here we intend

$$\|f\|_{\infty,[t,t+1]} = \sup_{s\in[t,t+1]}|f(s)|, \quad [f]_{0,\sigma,[t,t+1]} = \sup_{s_1\neq s_2\in[t,t+1]}\frac{|f(s_1)-f(s_2)|}{|s_1-s_2|^\sigma}.$$

For later purpose we introduce the space

$$X_\sharp = \{\lambda\in C(t_0,\infty)\ :\ \|\lambda\|_\sharp \quad \text{is bounded}\}. \tag{2.12}$$

With this in mind, we define the inner approximation to be

$$u_{\text{in}}(r,t) = w_\mu(r) + \mu_0'\psi_1(r,t), \quad \psi_1(r,t) = \mu^{\frac12}\Phi_1(\frac{r}{\mu}). \tag{2.13}$$

A direct computation gives that

$$\Delta\psi_1 + 5w_\mu^4\psi_1 = -\mu^{-\frac32}Z_0(\frac{r}{\mu}) = \frac{\partial w_\mu}{\partial\mu}(r).$$

In the region $\{r\ :\ r>R\mu_0\}$, where R is any large but fixed positive number, the inner approximation looks like

$$u_{\text{in}}(r,t) = 3^{\frac14}\frac{\mu^{\frac12}}{r} - \frac{3^{\frac14}}{4}\mu_0'\mu^{-\frac12}\,r + \mu_0^{\frac12}\mu_0'\,\Theta[\mu](r,t) + \frac{\mu_0^{\frac12}}{r}\left(\frac{\mu_0}{r}\right)^2\Theta[\mu](r,t) \tag{2.14}$$

where $\Theta[\mu](r,t)$ denotes a generic function, which depends smoothly on μ, and on (r,t), and which is uniformly bounded, for parameters μ satisfying (2.10), for r in the considered region, and any t large.

2.2. **Construction of the first outer profile and choice of $\mu_0(t)$.**

The outer profile is chosen to satisfy the heat equation $u_t=\Delta u$, in the whole space $\mathbb{R}^3$, and to fit the requested decaying property for the initial condition (2.2). Its properties and exact definitions change depending on the value of the decay rate γ of the initial condition u_0, see (2.2). We consider three different situations: $1<\gamma<2$, $\gamma=2$ and $\gamma>2$.

Case $1<\gamma<2$. In this case we define u_{out} as

$$u_{\text{out}}(r,t) = t^{-\frac{\gamma}{2}}g(\frac{r}{\sqrt t}) \tag{2.15}$$

with g the positive solution to

$$g''(s) + \left(\frac2s+\frac s2\right)g'(s) + \frac{\gamma}{2}g(s) = 0 \quad s\in(0,\infty) \tag{2.16}$$

that satisfies the properties

(1) $\lim_{s\to\infty}s^\gamma g(s)=A$,
(2) $\lim_{s\to0^+}sg(s)=d$, for a certain positive constant d for which $\lim_{s\to0^+}\left[g(s)-\frac ds\right]=0$.

Such a function g indeed exists. Let

$$L_\nu(g) = g'' + (\frac{2}{s} + \frac{s}{2})g' + \nu g, \quad s \in (0, \infty).$$

In Section 11, we prove the following

Lemma 2.2. *If $\frac{1}{2} < \nu < 1$, there exist two positive linearly independent solutions $y_1 = y_1(s)$ and $y_2 = y_2(s)$ to*

$$L_\nu(g) = 0, \quad s \in (0, \infty) \tag{2.17}$$

that satisfy respectively

$$y_1(s) = \frac{1}{s} + (\nu - 1)\left(\int_0^\infty s y_1(s)\, ds\right) + \frac{1-2\nu}{4} s + O(s^2), \quad \text{if} \quad s \to 0^+, \tag{2.18}$$

$$y_2(s) = c_2 + o(s) \quad \text{if} \quad s \to 0^+, \tag{2.19}$$

$$y_1(s) = c_1 e^{-\frac{s^2}{4}} s^{4\nu - 3}, \quad y_2(s) = \frac{1}{s^{2\nu}}(1 + o(\frac{1}{s})) \quad \text{if} \quad s \to \infty, \tag{2.20}$$

for some positive constants c_1, c_2.

Thanks to the Lemma, which we apply to solve (2.16) when $\nu = \frac{\gamma}{2}$, we get that the function g we are looking for in (2.15) is thus given by

$$g(s) = d y_1(s) + A y_2(s), \quad \text{with} \quad d = \frac{2A y_2(0)}{(2-\gamma)\left(\int_0^\infty s y_1(s)\, ds\right)} > 0. \tag{2.21}$$

We observe that, in a region like $r < R^{-1}\sqrt{t}$, for some large but fixed R, we get

$$u_{\text{out}}(r,t) = d\frac{t^{-\frac{\gamma-1}{2}}}{r} + t^{-\frac{\gamma+1}{2}} A \frac{(1-\gamma)y_2(0)}{2(2-\gamma)\int_0^\infty z y_1(z)\, dz} r + t^{-\frac{\gamma}{2}} O(\frac{r^2}{t}). \tag{2.22}$$

We next choose the function $\mu_0(t)$ in the definition of $\mu(t)$, (2.10), in such a way that the functions u_{in} and u_{out} automatically match in the whole region $R\mu_0 < r < R^{-1}\sqrt{t}$, for some R large, but fixed independent of t. This is possible if

$$\mu_0(t) = \frac{d^2}{\sqrt{3}} t^{1-\gamma}. \tag{2.23}$$

Indeed, with this choice for $\mu_0(t)$, and given the bound (2.11), there exists a constant C so that

$$\left|u_{\text{in}}(r,t) - u_{\text{out}}(r,t)\right| \le C\frac{\mu_0^{\frac{1}{2}}}{r}, \quad \left|\nabla u_{\text{in}}(r,t) - \nabla u_{\text{out}}(r,t)\right| \le C\frac{\mu_0^{\frac{1}{2}}}{r^2} \tag{2.24}$$

for any $R\mu_0 < r < R^{-1}\sqrt{t}$, and t large enough.

Case $\gamma = 2$. In this case, we define u_{out} as

$$u_{\text{out}}(r,t) = t^{-1}(\log t) k A g_0(\frac{r}{\sqrt{t}}) + t^{-1} h(\frac{r}{\sqrt{t}}) \tag{2.25}$$

where $g_0(s) = s^{-1} e^{-\frac{s^2}{4}}$ is a solution to

$$g''(s) + \left(\frac{2}{s} + \frac{s}{2}\right) g'(s) + g(s) = 0 \tag{2.26}$$

and h solves

$$h''(s) + \left(\frac{2}{s} + \frac{s}{2}\right) h'(s) + h(s) = k A g_0(s) \tag{2.27}$$

with $\lim_{s\to\infty} s^\gamma h(s) = A$, and $\lim_{s\to 0^+} s h(s) = d$, so that $\lim_{s\to 0^+}\left[h(s) - \frac{d}{s}\right] = 0$. The function h can be described explicitly. Let $g_1(s) = s^{-1} e^{-\frac{s^2}{4}} \int_0^s e^{\frac{z^2}{4}} dz$. This function solves (2.26). Since g_1 and g_0 are linearly independent, the variation of parameters formula gives that, for any constants d and b

$$h(s) = g_0(s)\left[d - kA\int_0^s z g_1(z)\, dz\right] + g_1(s)\left[b + kA\int_0^s z g_0(z)\, dz\right] \tag{2.28}$$

solves (2.27). In order to have $\lim_{s\to\infty} s^\gamma h(s) = A$, we need $2\left[b + kA\int_0^\infty zg_0(z)\,dz\right] = A$. Furthermore, to have $\lim_{s\to 0^+}\left[h(s) - \frac{d}{s}\right] = 0$, we need $b = 0$. Thus we select

$$b = 0, \quad k = \frac{1}{2\int_0^\infty zg_0(z)\,dz}. \tag{2.29}$$

Observe that, up to this moment, the constant d is arbitrary. Nevertheless, we remind that u_{out} wants to be a solution to $u_t = \Delta u = u_{rr} + \frac{2}{r}u_r$. Multiplying this equation by r, and integrating in $(0,R)$, for some fixed, large R, we get

$$\frac{d}{dt}\left(\int_0^R ru(r,t)\,dr\right) = Ru_r(R,t) + u(R,t),$$

where we use the fact that $\lim_{r\to 0}[ru_r(r,t) + u(r,t)] = 0$. Next, we integrate the above equation in t, from 0 to ∞, and using the fact that $\lim_{t\to\infty}\int_0^R ru(r,t)\,dt = 0$, we get

$$-\int_0^R ru(r,0)\,dr = \int_0^\infty [Ru_r(R,t) + u(R,t)]\,dt. \tag{2.30}$$

Take now $u = u_{\text{out}}$ and compute the right hand side of (2.30)

$$\begin{aligned}\int_0^\infty [Ru_r(R,t) + u(R,t)]\,dt &= Ak\int_0^\infty t^{-1}(\log t)[\frac{R}{\sqrt{t}}g_0'(\frac{R}{\sqrt{t}}) + g_0(\frac{R}{\sqrt{t}})]\,dt \\ &+ \int_0^\infty t^{-1}[\frac{R}{\sqrt{t}}h'(\frac{R}{\sqrt{t}}) + h(\frac{R}{\sqrt{t}})]\,dt \quad s := \frac{R}{\sqrt{t}} \\ &= \left(4Ak\int_0^\infty s^{-1}[sg_0'(s) + g_0(s)]\,ds\right)\log R \\ &+ \bar{d} + \left(2\int_0^\infty s^{-1}[sh'(s) + h(s)]\,ds\right)\end{aligned}$$

where $\bar{d}$ is the constant defined by

$$\bar{d} = -\left(4Ak\int_0^\infty s^{-1}(\log s)[sg_0'(s) + g_0(s)]\,ds\right).$$

We can simplify the expression of the constant in front of $\log R$. Indeed, multiplying (2.26) against s, we get that $(sg'(s) + g + \frac{s^2}{2}g)' = 0$. For $g = g_0$, and using the fact that g_0 decays very fast as $s\to\infty$, we get that $sg_0'(s) + g_0(s) = -\frac{s^2}{2}g_0(s)$ for any s, thus

$$4Ak\int_0^\infty s^{-1}[sg_0'(s) + g_0(s)]\,ds = Ak\left(-2\int_0^\infty sg_0(s)ds\right) = -A$$

since (2.29). On the other hand, the decaying condition $\lim_{r\to\infty} r^2u(r,0) = A$ gives

$$-\int_0^R ru(r,0)\,dr = -A\log R + B(R),$$

with $\lim_{R\to\infty} B(R) = B$, being B a real constant. Plugging this information in (2.30), we get that

$$\bar{d} + \left(2\int_0^\infty s^{-1}[sh'(s) + h(s)]\,ds\right) = B.$$

This last relation defines in a unique way the constant $d > 0$ in the definition of h, (2.28). Indeed, a direct computation gives that

$$\int_0^\infty s^{-1}[sh'(s) + h(s)]\,ds = -\frac{d}{2}\left(\int_0^\infty sg_0(s)\,ds\right) + \omega,$$

with

$$\omega = \frac{kA}{2}\int_0^\infty sg_0(s)(\int_0^s zg_1(z)\,dz)\,ds + \int_0^s s^{-1}[sg_1' + g_1](kA\int_0^s zg_0(z)\,dz)\,ds,$$

from which we deduce that

$$d = \frac{\bar d - 2\omega - B}{\int_0^\infty s g_0(s)\, ds}.$$

With this choice for the function h in (2.25), we get

$$h(s) = \frac{d}{s} - \frac{s}{4}[d + 10kA] + O(s^3), \quad \text{as} \quad s \to 0^+$$

and

$$\begin{aligned} u_{\text{out}}(r,t) &= \frac{t^{-\frac12}}{r}\ [kA(\log t) + d] \\ &\quad + t^{-1}\left[-\frac{kA(\log t)}{4} - \frac{d + 10kA}{4} \right] \frac{r}{\sqrt{t}} + O\left((\log t)\, \frac{r^3}{t^3\sqrt{t}}) \right) \end{aligned} \tag{2.31}$$

in the region $r < R^{-1}\sqrt{t}$, for some large but fixed R, as $t \to \infty$.

In this case, namely when $\gamma = 2$, we choose μ_0 in (2.10) as

$$\mu_0(t) = \frac{[d + kA(\log t)]^2}{\sqrt{3}}\, t^{-1}, \tag{2.32}$$

and thanks to this choice, and to the bound (2.11) on λ, we find a constant C so that

$$\left| u_{\text{in}}(r,t) - u_{\text{out}}(r,t) \right| \le C \frac{\mu_0^{\frac12}}{r}, \quad \left| \nabla u_{\text{in}}(r,t) - \nabla u_{\text{out}}(r,t) \right| \le C \frac{\mu_0^{\frac12}}{r^2} \tag{2.33}$$

for any $R\mu_0 < r < R^{-1}\sqrt{t}$, for some fixed and large R, and for all t large enough.

Case $\gamma > 2$. In this case, we define u^1_{out} as

$$u^1_{\text{out}}(r,t) = t^{-1}\, d g_0(\frac{r}{\sqrt{t}}), \quad d = \left(\frac{\int_0^\infty r u_0(r) dr}{\int_0^\infty s g_0(s)\, ds} \right)$$

where $g_0(s) = s^{-1} e^{-\frac{s^2}{4}}$ solves (2.26), and $u_0(r)$ is the initial condition for (2.1)-(2.2). Observe that, in a region like $r < R^{-1}\sqrt{t}$, for some large but fixed R, we get

$$u^1_{\text{out}}(r,t) = d\, \frac{t^{-\frac12}}{r} - t^{-1}\, \frac{d}{4}\, \frac{r}{\sqrt{t}} + t^{-1} O(\frac{r^2}{t^{\frac32}}). \tag{2.34}$$

For a given time t, the function u^1_{out} is decaying very fast as $r \to \infty$. For this reason, we modify u^1_{out} with a function that has the right decay to match the initial condition $u_0(r)$, for r large. Define

$$u_{\text{out}}(r,t) = \eta(\frac{r}{t}) u^1_{\text{out}}(r,t) + (1 - \eta(\frac{r}{t})) u^2_{\text{out}}(r), \quad \text{with} \quad u^2_{\text{out}}(r) = \frac{A}{r^\gamma}, \tag{2.35}$$

where η is the cut off function defined in (1.9).

In this case, $\gamma > 2$, we choose μ_0 in (2.10) as

$$\mu_0(t) = \frac{d^2}{\sqrt{3}}\, t^{-1}. \tag{2.36}$$

With this choice for $\mu_0(t)$, and thanks to (2.11), given any large but fixed number $R > 0$, there exists a constant C so that

$$\left| u_{\text{in}}(r,t) - u_{\text{out}}(r,t) \right| \le C \frac{\mu_0^{\frac12}}{r}, \quad \left| \nabla u_{\text{in}}(r,t) - \nabla u_{\text{out}}(r,t) \right| \le C \frac{\mu_0^{\frac12}}{r^2} \tag{2.37}$$

for any $R\mu_0 < r < R^{-1}\sqrt{t}$, and for all t large.

2.3. **Construction of the first global approximation and estimate of the error.** Let $r_0 > 0$ be a small and fixed number, define

$$U_1(r,t) = \eta(\frac{r}{r_0\sqrt{t}})u_{\rm in}(r,t) + \left(1-\eta(\frac{r}{r_0\sqrt{t}})\right)u_{\rm out}(r,t) \tag{2.38}$$

where η is given by (1.9). For any smooth function $u = u(r,t)$, we define the Error Function as

$$\mathcal{E}[u](r,t) = \Delta u + u^5 - u_t. \tag{2.39}$$

Our next purpose is to describe

$$\mathcal{E}_1(r,t) = \mathcal{E}[U_1](r,t) \tag{2.40}$$

with U_1 given by (2.38). To this end, we introduce the function $\alpha = \alpha(t)$, $t > t_0$,

$$\alpha(t) = 3^{\frac14}\,\mu_0^{-\frac12}\,(\mu_0\Lambda)'. \tag{2.41}$$

Since Λ satisfies (2.11), definition (2.41) defines a linear homeomorphism $\mathcal{A} : X_\sharp \to X_\flat$, $\mathcal{A}(\lambda) = \alpha$, where

$$X_\flat = \{\alpha \in C(t_0,\infty) \,:\, \|\alpha\|_\flat \quad \text{is bounded}\}, \tag{2.42}$$

and

$$\|\alpha\|_\flat := \sup_{t>t_0} \mu_0^{-\frac32}(t)\,t\,\left[\|\alpha\|_{\infty,[t,t+1]} + |\alpha|_{0,\sigma,[t,t+1]}\right]. \tag{2.43}$$

Here σ is the number introduced in (2.11). Let us denote by $h_0 : (0,\infty) \to (0,\infty)$ a smooth function with the properties that

$$h_0(s) = \begin{cases} \frac{1}{s} & \text{for} \quad s \to 0 \\ \frac{1}{s^3} & \text{for} \quad s \to \infty, \end{cases} \tag{2.44}$$

and define the following norm for any function $f : \mathbb{R}^3 \times (t_0,\infty) \to \mathbb{R}$

$$\|f\|_* := \sup_{x\in\mathbb{R}^3, t>t_0} \mu_0^{-\frac12}\,t^{\frac32}\,h_0^{-1}(\frac{r}{\sqrt{t}}) \quad \Big[\|f\|_{\infty,B(x,1)\times[t,t+1]} + [f]_{0,\sigma,B(x,1)\times[t,t+1]}\Big], \quad r = |x|. \tag{2.45}$$

Here σ is defined in (2.11),

$$\|f\|_{\infty,B(x,1)\times[t,t+1]} = \sup_{y\in B(x,1),\quad s\in[t,t+1]} |f(y,s)| \tag{2.46}$$

and

$$[f]_{0,\sigma,B(x,1)\times[t,t+1]} = \sup_{y_1\neq y_2\in B(x,1),\quad s_1\neq s_2\in[t,t+1]} \frac{|f(y_1,s_1)-f(y_2,s_2)|}{|y_1-y_2|^{2\sigma}+|s_1-s_2|^{\sigma}}. \tag{2.47}$$

We have the validity of the following estimates, whose proof is quite technical and delayed to Section 12.

Lemma 2.3. *Assume $\lambda = \lambda(t)$ satisfies* (2.11)*. The error function defined in* (2.40) *can be described as follows*

$$\mathcal{E}_1(r,t) = \frac{\alpha(t)}{\mu+r}\eta(\frac{r}{r_0\sqrt{t}}) + \mathcal{E}_{1,*}[\lambda](r,t), \tag{2.48}$$

where η is the smooth cut off function defined in (1.9)*, α is the function defined in* (2.41)*, and r_0 is a given fixed small number. The function $\mathcal{E}_{1,*}[\lambda](r,t)$ depends smoothly on λ. Furthermore, there exists $C > 0$ such that*

$$\|\mathcal{E}_{1,*}\|_* \le C. \tag{2.49}$$

If the initial time t_0 in Problem (2.1) *is large enough, there exist $\mathbf{c} \in (0,1)$ so that, for any λ_1, λ_2 satisfying* (2.11)*, we have*

$$\|\mathcal{E}_{1,*}[\lambda_1] - \mathcal{E}_{1,*}[\lambda_2]\|_{\infty,B(x,1)\times[t,t+1]} \le \mathbf{c}\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt{t}})\,\|\lambda_1-\lambda_2\|_\sharp \tag{2.50}$$

and

$$[\mathcal{E}_{1,*}[\lambda_1]-\mathcal{E}_{1,*}[\lambda_2]]_{0,\sigma,B(x,1)\times[t,t+1]} \leq \mathbf{c}\mu_0^{\frac{1}{2}} t^{-\frac{3}{2}}\, h_0(\frac{r}{\sqrt{t}})\, \|\lambda_1-\lambda_2\|_\sharp, \tag{2.51}$$

for any $r=|x|$ and any t. The definition of the function h_0 and of the norm $\|\cdot\|_*$ are given respectively in (2.44) *and in* (2.45)*. Furthermore the constant $\mathbf{c}$ in* (2.50) *and* (2.51) *can be made as small as one needs, provided that the initial time t_0 is chosen large enough.*

2.4. **Construction of the second global approximation and estimate of the new error.** Taking into account the expression of the error function given in (2.48), we introduce a correction function ϕ_0 to partially get rid of the term $\frac{\alpha(t)}{\mu+r}$. More precisely, let

$$\bar\alpha(t)=\begin{cases}\alpha(t_0) & \text{for} \quad t<t_0\\ \alpha(t) & \text{for} \quad t\geq t_0\end{cases}, \tag{2.52}$$

and introduce the function ϕ_0 solution to

$$\partial_t\phi_0=\Delta\phi_0+\frac{\bar\alpha(t)}{\mu+r}\,\mathbf{1}_{\{r<M\}}, \quad \text{in} \quad \mathbb{R}^3\times(0,\infty), \quad \phi_0(x,t_0-1)=0, \quad \text{in} \quad \mathbb{R}^3, \quad M^2=t_0. \tag{2.53}$$

Here, for a set K, we mean

$$1_K(x)=1, \quad \text{if} \quad x\in K, \quad =0, \quad \text{if} \quad x\notin K.$$

Duhamel's formula provides an explicit expression for ϕ_0

$$\phi_0(x,t)=\int_{t_0-1}^t \frac{1}{(4\pi(t-s))^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4(t-s)}}\,\frac{\bar\alpha(s)}{\mu+|y|}\,\mathbf{1}_{\{|y|<M\}}\,dy\,ds. \tag{2.54}$$

Since λ satisfies (2.11), classical parabolic estimates give that ϕ_0 is locally $C^{2+2\sigma,1+\sigma}$, where σ is the Hölder exponent in (2.11). In the interval (t_0,∞), the function ϕ_0 solves

$$\partial_t\phi_0=\Delta\phi_0+\frac{\alpha(t)}{\mu+r}\,\mathbf{1}_{\{r<M\}}, \quad \text{in} \quad \mathbb{R}^3\times(t_0,\infty), \tag{2.55}$$

and at time $t=t_0$, the function $\phi_0(x,t_0)$ is radial in x and decays fast as $|x|\to\infty$, that is

$$|\phi_0(x,t_0)|\leq ce^{-a|x|^2}, \quad \text{as} \quad |x|\to\infty \tag{2.56}$$

for some positive, fixed constants a and c. Indeed, let $x=\ell\, e$, with $\|e\|=1$, and assume that $\ell>\max\{1,2M\}$. Thus $|x-y|^2>\frac{\ell^2}{4}$, for any $|y|<M$, and

$$|\phi_0(x,t_0)|\leq C|\alpha(t_0)|\left(\int_{t_0-1}^{t_0}\frac{e^{-\frac{\ell^2}{16(t_0-s)}}}{(t_0-s)^{\frac{3}{2}}}\,ds\right)\left(\int_{|y|<M}\frac{dy}{|y|}\right)\leq C|\alpha(t_0)|M^2e^{-\frac{\ell^2}{16}}.$$

Taking $\ell\to\infty$, estimate (2.56) thus follows from (2.41).

The second approximation is given by

$$U_2[\lambda](r,t)=U_1(r,t)+\phi_0(r,t) \tag{2.57}$$

where U_1 is in (2.38). Observe that U_2 satisfies the decaying conditions (2.2) at the initial time t_0 as consequence of (2.56). The new Error Function

$$\mathcal{E}_2[\lambda](r,t)=\mathcal{E}[U_2](r,t)$$

is thus

$$\mathcal{E}_2[\lambda](r,t)=\underbrace{\mathcal{E}_{1,*}+\frac{\alpha(t)}{r}\left(\eta(\frac{r}{r_0\sqrt{t}})-\mathbf{1}_{\{r<2M\}}\right)}_{:=\mathcal{E}_{21}}+\underbrace{(U_1+\phi_0)^5-U_1^5}_{\mathcal{E}_{22}}. \tag{2.58}$$

The function $\mathcal{E}_{1,*}$ is defined in (2.48). For later purpose, it is useful to estimate, in the $\|\cdot\|_*$-norm introduced in (2.45), the function

$$\bar{\mathcal{E}}_2 := \mathcal{E}_{21} + (1-\eta_R(x,t))\mathcal{E}_{22} \quad \text{where} \quad \eta_R(x,t) = \eta\left(\frac{x}{R\mu_0}\right). \tag{2.59}$$

Here $\eta(s)$ is given by (1.9), while the number R is a large number, whose definition will depend on t_0, but it will not dependent on t.

We have the validity of the following lemma, whose proof is given in Section 13.

Lemma 2.4. *Assume $\lambda = \lambda(t)$ satisfies* (2.11). *The error function defined in* (2.58) *depends smoothly on λ and it satisfies the following estimates: there exists $C > 0$*

$$\|\bar{\mathcal{E}}_2\|_* \le C. \tag{2.60}$$

If the initial time t_0 is large enough, there exist small positive number $\mathbf{c} \in (0,1)$ such that, for any λ_1, λ_2 satisfying (2.11)*, we have*

$$\|\bar{\mathcal{E}}_2[\lambda_1] - \bar{\mathcal{E}}_2[\lambda_2]\|_{\infty, B(x,1)\times[t,t+1]} \le \mathbf{c}\mu_0^{\frac12} t^{-\frac32} h_0(\frac{r}{\sqrt{t}})\, \|\lambda_1-\lambda_2\|_\sharp, \quad r=|x|, \tag{2.61}$$

and

$$\left[\bar{\mathcal{E}}_2[\lambda_1](r,t) - \bar{\mathcal{E}}_2[\lambda_2](r,t)\right]_{0,\sigma,[t,t+1]} \le \mathbf{c}\mu_0^{\frac12} t^{-\frac32}\, h_0(\frac{r}{\sqrt{t}})\, \|\lambda_1-\lambda_2\|_\sharp, \tag{2.62}$$

for any x and $t > t_0$, provided the initial time t_0 in Problem (2.1) *is chosen large enough. The definition of the function h_0 is given in* (2.44)*, and the definition of the $\|\cdot\|_*$-norm is given in* (2.45)*.*

Remark 2.5. From the proof of the result, we also get that the constant $\mathbf{c}$ in (2.61) and (2.62) can be made as small as one needs, provided that the initial time t_0 is chosen large enough.

3. The inner-outer gluing

We recall the reader that our ultimate purpose is to construct a global unbounded solution u to (2.1)-(2.2) of the form

$$u = U_2[\lambda](r,t) + \tilde\phi, \quad t > t_0 \tag{3.1}$$

where U_2 is defined in (2.57), while $\tilde\phi(x,t)$ is a smaller perturbation. The rest of the paper is thus devoted to find $\tilde\phi(x,t)$. The construction of $\tilde\phi(x,t)$ is done by means of a *inner-outer gluing* procedure. This procedure consists in writing

$$\tilde\phi(x,t) = \psi(x,t) + \phi^{in}(x,t) \quad \text{where} \quad \phi^{in}(x,t) := \eta_R(x,t)\hat\phi(x,t) \tag{3.2}$$

with

$$\hat\phi(x,t) := \mu_0^{-\frac12}\phi\left(\frac{x}{\mu_0}, t\right), \quad \eta_R(x,t) = \eta\left(\frac{x}{R\mu_0}\right), \tag{3.3}$$

where $\eta(s)$ is given in (1.9).

In terms of $\tilde\phi$, Problem (2.1)-(2.2) reads as

$$\partial_t\tilde\phi = \Delta\tilde\phi + 5U_2^4\tilde\phi + N(\tilde\phi) + \mathcal{E}_2 \quad \text{in } \mathbb{R}^3\times[t_0,\infty), \tag{3.4}$$

where $\mathcal{E}_2$ is defined in (2.58) and

$$N(\tilde\phi) = (U_2+\tilde\phi)^5 - U_2^5 - 5\,U_2^4\,\tilde\phi.$$

Recalling that $w_\mu = \mu^{-\frac12} w(\frac{r}{\mu})$, we let

$$V[\lambda](r,t) = 5\left(U_2^4 - w_\mu^4\right)\eta_R + 5U_2^4\,(1-\eta_R) \tag{3.5}$$

and write $5U_2^4 = 5w_\mu^4\eta_R + V[\lambda](r,t)$. A main observation we make is that $\tilde\phi$ solves Problem (3.4) if the tuple (ψ,ϕ) solves the following coupled system of nonlinear equations

$$\begin{aligned}\partial_t\psi = \Delta\psi + V[\lambda]\psi + [2\nabla\eta_R\nabla_x\hat\phi + \hat\phi(\Delta_x - \partial_t)\eta_R]\\ + N[\lambda](\tilde\phi) + \mathcal{E}_{21} + \mathcal{E}_{22}(1-\eta_R) \quad \text{in } \mathbb{R}^3\times[t_0,\infty),\end{aligned} \tag{3.6}$$

and

$$\partial_t\hat\phi = \Delta\hat\phi + 5w_\mu^4\hat\phi + 5w_\mu^4\psi + \mathcal{E}_{22} \quad \text{in } B_{2R\mu_0}(0)\times[t_0,\infty). \tag{3.7}$$

We refer to (2.58) for the definition of $\mathcal{E}_{21}$ and $\mathcal{E}_{22}$. In terms of ϕ, see (3.3), equation (3.7) becomes

$$\begin{aligned}\mu_0^2\partial_t\phi =&\Delta_y\phi + 5w^4\phi + \mu_0^{\frac52}\mathcal{E}_{22}(\mu_0 y,t) + 5\,\frac{\mu_0^{\frac12}}{(1+\Lambda)^4}\,w^4\Big(\frac{y}{(1+\Lambda)^2}\Big)\psi(\mu_0 y,t)\\ &+ B[\phi] + B^0[\phi] \quad \text{in } B_{2R}(0)\times[t_0,\infty)\end{aligned} \tag{3.8}$$

where

$$B[\phi] := \mu_0\,(\partial_t\mu_0)\,\left(\frac{\phi}{2} + y\cdot\nabla_y\phi\right) \tag{3.9}$$

and

$$B^0[\phi] := 5\left[w^4\left(\frac{y}{(1+\Lambda)^2}\right) - w^4(y)\right]\,\phi + 5\,\left(\frac{1-(1+\Lambda)^4}{(1+\Lambda)^4}\right)\,w^4\left(\frac{y}{(1+\Lambda)^2}\right)\phi. \tag{3.10}$$

We call (3.6) the *outer problem* and (3.8) the *inner problem(s)* .

We next describe precisely our strategy to solve (3.6)-(3.8). For given parameter λ satisfying (2.11), and function ϕ fixed in a suitable range, we first solve for ψ the outer Problem (3.6), in the form of a (nonlocal) nonlinear operator $\psi = \Psi(\lambda,\phi)$. This is done in full details in Section 4.

We then replace this ψ in equation (3.8). At this point we consider the change of variable,

$$t = t(\tau), \quad \frac{dt}{d\tau} = \mu_0^2(t),$$

that reduces (3.8) to

$$\partial_\tau\phi = \Delta_y\phi + 5w^4\phi + H[\psi,\lambda,\phi](y,t(\tau)), \quad y\in B_{2R}(0), \quad \tau\ge\tau_0 \tag{3.11}$$

where τ_0 is such that $t(\tau_0) = t_0$, and

$$\begin{aligned}H[\psi,\lambda,\phi](y,t(\tau)) = \mu_0^{\frac52}\mathcal{E}_{22}(\mu_0 y,t) + 5\,\frac{\mu_0^{\frac12}}{(1+\Lambda)^4}\,w^4\Big(\frac{y}{(1+\Lambda)^2}\Big)\psi(\mu_0 y,t)\\ + B[\phi] + B^0[\phi]\end{aligned} \tag{3.12}$$

Next step is to construct a solution ϕ to Problem (3.11). We can do this for functions ϕ which furthermore satisfy

$$\phi(y,\tau_0) = e_0 Z(y), \quad y\in B_{2R}(0), \tag{3.13}$$

for some constant e_0. Here Z is the positive radially symmetric bounded eigenfunction associated to the only negative eigenvalue λ_0 to the problem

$$L_0(\phi) + \lambda\phi = 0, \quad \phi\in L^\infty(\mathbb{R}^3). \tag{3.14}$$

Here L_0 is the linear operator around the standard bubble w in $\mathbb{R}^3$. We refer to (2.6) for the definition of L_0. Furthermore, it is known that λ_0 is simple and Z decays like

$$Z(y) \sim |y|^{-1}e^{-\sqrt{|\lambda_0|}\,|y|} \quad \text{as} \quad |y|\to\infty.$$

To be more precise, we prove that Problem (3.11)-(3.13) is solvable in ϕ, provided that in addition the parameter λ is chosen so that $H[\psi,\lambda,\phi](y,t(\tau))$ satisfies the orthogonality condition

$$\int_{B_{2R}} H[\psi,\lambda,\phi](y,t(\tau))Z_0(y)dy = 0, \quad \text{for all} \quad t > t_0. \tag{3.15}$$

We recall that $Z_0(y)$, defined in (2.7), is the only bounded radial element in the kernel of the linear elliptic operator L_0.

Equation (3.15) becomes a non-linear, non-local problem in λ, for any fixed ϕ. We attack this problem in Sections 5, 6, 7. In Section 5, we get the precise form of Equation (3.15) as a non local non linear operator in λ. The principal part of the operator in λ defined by Equation (3.15) is a linear non-local operator which turns out to be a perturbation of the $\frac{1}{2}$-Caputo derivative. We refer to [3] for the original definition of Caputo derivatives. In Section 6 we develop an invertibility theory for such linear operator. In Section 7 we fully solve Equation (3.15) in λ, by means of a Banach fixed point argument. The solution $\lambda = \lambda[\phi]$ is a non linear operator in ϕ, and we also describe the Lipschitz dependence of λ with respect to ϕ, which is a key property for our final argument.

At this point, one realizes that a central point of our complete proof is to design a linear theory that allows us to solve in ϕ Problem (3.11)-(3.13). To this purpose, we shall construct a solution to an initial value problem of the form

$$\phi_\tau = \Delta\phi + 5w^4\phi + h(y,\tau) \quad \text{in } B_{2R}\times(\tau_0,\infty), \quad \phi(y,\tau_0) = e_0 Z(y) \quad \text{in } B_{2R}. \tag{3.16}$$

And then we solve Problem (3.11)-(3.13) by means of a contraction mapping argument.

Let a be a fixed number with $a\in(0,2)$, and let $\nu>0$ so that, for t large,

$$\tau^{-\nu} \sim \mu_0^{\frac{3}{2}} t^{-1}, \quad \text{if} \quad \gamma\neq 2, \quad \text{and} \quad \tau^{-\nu}\sim \mu_0^{\frac{3}{2}} t^{-1+\nu'}, \quad \text{if} \quad \gamma = 2,$$

for some $\nu'>0$ that can be fixed arbitrarily small. We solve (3.16) for functions h with $\|h\|_{\nu,2+a}$-norm bounded, where

$$\|h\|_{\nu,2+a} := \sup_{\tau>\tau_0,\, y\in\mathbb{R}^3} \tau^\nu (1+|y|^{2+a}) \left[\|h\|_{\infty,B(y,1)\times[\tau,\tau+1]} + [h]_{0,\sigma,B(y,1)\times[\tau,\tau+1]} \right], \tag{3.17}$$

and we construct solutions ϕ in the class of functions with $\|\phi\|_{\nu,a}$-norm bounded, where

$$\begin{aligned} \|\phi\|_{\nu,a} \quad &:= \sup\nolimits_{\tau>\tau_0,y\in\mathbb{R}^3} \tau^\tau (1+|y|^a) \left[\|\phi\|_{\infty,B(y,1)\times[\tau,\tau+1]} + [\phi]_{0,\sigma,B(y,1)\times[\tau,\tau+1]} \right] \\ &+ \sup\nolimits_{\tau>\tau_0,y\in\mathbb{R}^3} \tau^\nu (1+|y|^{1+a}) \left[\|\nabla\phi\|_{\infty,B(y,1)\times[\tau,\tau+1]} + [\nabla\phi]_{0,\sigma,B(y,1)\times[\tau,\tau+1]} \right] \end{aligned} \tag{3.18}$$

We have the validity of the following result

Proposition 3.1. *Let ν, a be given positive numbers with $0<a<2$. Then, for all sufficiently large $R>0$ and function $h=h(y,\tau)$, with $h(y,\tau)=h(|y|,\tau)$ and $\|h\|_{\nu,2+a}<+\infty$ that satisfies*

$$\int_{B_{2R}} h(y,\tau)\, Z_0(y)\, dy \;=\; 0 \quad \textit{for all} \quad \tau\in(\tau_0,\infty) \tag{3.19}$$

there exist $\phi\in C^{2+2\sigma,1+\sigma}$-loc., which is radial in y, and e_0 which solve Problem (3.16). Moreover, $\phi=\phi[h]$, and $e_0=e_0[h]$ define linear operators of h that satisfy the estimates

$$|\phi(y,\tau)| \;\le C\, \tau^{-\nu} \frac{R^{4-a}}{1+|y|^3} \|h\|_{\nu,2+a}, \quad |\nabla_y\phi(y,\tau)| \;\le C\, \tau^{-\nu} \frac{R^{4-a}}{1+|y|^4} \|h\|_{\nu,2+a}, \tag{3.20}$$

and

$$|e_0[h]| \;\le C\, \|h\|_{\nu,2+a},$$

for some fixed constant C.

We postpone the proof of this Proposition to Section 9. Section 8 is devoted to solve Problem (3.11)-(3.13) and this concludes the proof of Theorem 1.1.

4. Solving the outer problem

The aim of this section is to solve the *outer problem* (3.6) for given parameter λ satisfying (2.11), and for given small functions ϕ, in the form of a nonlinear nonlocal operator

$$\psi(x,t) = \Psi[\lambda,\phi](x,t).$$

We recall that $\phi^{in}(x,t) = \eta_R(x,t)\hat{\phi}(x,t)$ with

$$\hat{\phi}(x,t) := \mu_0^{-\frac{1}{2}}\phi\left(\frac{x}{\mu_0},t\right), \quad \text{and} \quad \eta_R(x,t) = \eta\left(\frac{x}{R\mu_0}\right).$$

Here $\eta(s)$ is defined in (1.9), and number R is a sufficiently large number, independent of t. We assume that

$$\|\phi\|_{\nu,a} \quad \text{is bounded.} \tag{4.1}$$

Let $\varphi_0 : (0,\infty) \to (0,\infty)$ be a smooth and bounded given function with the property that

$$\varphi_0(s) = \begin{cases} s & \text{for} \quad s \to 0^+ \\ \frac{1}{s^3} & \text{for} \quad s \to \infty \end{cases}. \tag{4.2}$$

We introduce the following L^∞-weighted norms for functions $f = f(r,t)$

$$\|f\|_{**} := \|f\|_1 + \|Df\|_2 \tag{4.3}$$

$$\|f\|_1 := \sup_{x\in\mathbb{R}^3, t>t_0} \mu_0^{-\frac{1}{2}}\, t^{\frac{1}{2}}\, \varphi_0^{-1}(\frac{r}{\sqrt{t}}) \quad \Big[\|f\|_{\infty,B(x,1)\times[t,t+1]} + [f]_{0,\sigma,B(x,1)\times[t,t+1]}\Big], \quad r = |x|. \tag{4.4}$$

$$\|f\|_2 := \sup_{x\in\mathbb{R}^3, t>t_0} \mu_0^{-\frac{1}{2}}\, t\, (\varphi_0')^{-1}(\frac{r}{\sqrt{t}}) \quad \Big[\|f\|_{\infty,B(x,1)\times[t,t+1]} + [f]_{0,\sigma,B(x,1)\times[t,t+1]}\Big], \quad r = |x|. \tag{4.5}$$

Refer to (2.46) and (2.47) for the definitions of $\|f\|_{\infty,B(x,1)\times[t,t+1]}$ and $[f]_{0,\sigma,B(x,1)\times[t,t+1]}$.

Proposition 4.1. *Assume that λ satisfies* (2.11)*, and that the function ϕ satisfies the bound* (4.1)*. Let $\psi_0 \in C^2(\mathbb{R}^3)$, radially symmetric so that*

$$|y|\,|\psi_0(y)| + |y|\,|\nabla\psi_0(y)| \le t_0^{-a} e^{-b|y|}, \tag{4.6}$$

for some positive constants a and b. There exists t_0 large so that Problem (3.6) *has a unique solution $\psi = \Psi[\lambda,\phi]$ so that*

$$\psi(r,t_0) = \psi_0(r), \quad \|\psi\|_1 + \|D\psi\|_2 \le C. \tag{4.7}$$

Proof. Let f be a given function with $\|f\|_*$-norm bounded. Classical parabolic estimates give that any solution to $\partial_t\psi = \Delta\psi + f$ is locally $C^{2+2\sigma,1+\sigma}$. Furthermore, consequence of Lemma 11.1 is that the function $\bar{\varphi}_0(r,t) = \mu_0^{\frac{1}{2}} t^{-\frac{1}{2}}\varphi_0(\frac{r}{\sqrt{t}})$ is a positive supersolution for $\partial_t\psi \ge \Delta\psi + f(r,t)$. Observe also that $\bar{\varphi}_0(r,t_0) \ge \psi_0(r)$. Combining these facts with the maximum principle, we see that, for a function f with $\|f\|_*$-norm bounded, the unique solution to $\partial_t\psi = \Delta\psi + f$, with $\psi(r,t_0) = \psi_0$, has $\|\psi\|_{**}$-norm bounded. We claim that a possibly large multiple of $\bar{\varphi}_0$ works as a supersolution also for Problem

$$\partial_t\psi \ge \Delta\psi + V(r,t)\psi + f(r,t). \tag{4.8}$$

Indeed, recalling the definition of V in (3.5), we write

$$V = V_1 + V_2, \quad V_1 = 5\left(U_2^4 - w_\mu^4\right)\eta_R, \quad V_2 = 5U_2^4\left(1 - \eta_R\right).$$

In the region where $\eta_R \neq 0$, namely when $r < 2R\mu_0$, we expand in Taylor the function V_1 and we find $s^* \in (0,1)$ so that

$$V_1(r,t) = 20\left(w_\mu + s^*(\mu_0'\Psi_1(r,t) + \phi_0(r,t))\right)^3 [\mu_0'\Psi_1(r,t) + \phi_0(r,t))]\eta_R.$$

From here, we see that, in this region, $|V_1(r,t)| \lesssim Rt^{-1}\,\eta_R$, so that

$$|V_1(r,t)\psi_0(r,t)| \lesssim \mu_0^{\frac12} t^{-\frac32}\, h_0(\frac{r}{\sqrt t}). \tag{4.9}$$

Let us now consider V_2. This function is not zero only when $r > R\mu_0$, and in this region we have that $|V_2(r,t)| \lesssim \frac{\mu_0^2}{r^4}\,(1-\eta_R)$, so that

$$|V_2(r,t)\psi_0(r,t)| \lesssim \frac{\mu^2}{r^4}\mu_0^{\frac12} t^{-\frac12}\varphi_0\left(\frac{r}{\sqrt t}\right)(1-\eta_R) \lesssim R^{-2}\mu_0^{\frac12} t^{-\frac32} h_0(\frac{r}{\sqrt t}). \tag{4.10}$$

Choosing R large, but independent of t, we thus find that a multiple of $\bar\varphi_0$ is a supersolution for (4.8).

We call $T_o : (f,\psi_0) \to \psi$ the linear operator that to any f with $\|f\|_*$-norm bounded and any initial condition ψ_0 satisfying (4.6) associates the unique solution to

$$\partial_t\psi = \Delta\psi + V[\lambda](r,t)\psi + f(r,t), \quad \psi(r,t_0) = \psi_0(r), \tag{4.11}$$

which has bounded $\|\psi\|_{**}$-norm. Define $\bar\psi = T_o(0,\psi_0)$. We observe that $\psi + \bar\psi$ is a solution to (3.6) if ψ is a fixed point for the operator

$$\mathcal{A}_o(\psi) = T_o\left([2\nabla\eta_R\nabla_x\hat\phi + \hat\phi(\Delta_x - \partial_t)\eta_R] + N[\lambda](\tilde\phi + \bar\psi) + \mathcal{E}_{21} + \mathcal{E}_{22}(1-\eta_R)\right) \tag{4.12}$$

We shall show the existence and uniqueness of such fixed point as consequence of the Contraction Mapping Theorem. We perform a fixed point argument in the set of functions ψ in

$$B_o = \{\psi \in L^\infty \,:\, \|\psi\|_{**} < r\} \tag{4.13}$$

for some $r > 0$.

From Lemma 2.3 we have that there exists a constant c_1 so that

$$\|\mathcal{E}_{21} + \mathcal{E}_{22}(1-\eta_R)\|_* \leq c_1. \tag{4.14}$$

We now claim that there exists constant c_2 such that, if the parameter λ satisfies (2.11), and if the function ϕ satisfies the bound (4.1), then

$$\left\|2\nabla\eta_R\nabla_x\hat\phi + \hat\phi(\Delta_x - \partial_t)\eta_R\right\|_* + \left\|N(\tilde\phi + \bar\psi)\right\|_* \leq c_2 \tag{4.15}$$

Furthermore, we claim that there exists a constant $\mathbf{c} \in (0,1)$ so that, for any ψ_1, $\psi_2 \in B_0$,

$$\|\mathcal{A}_o(\psi_1) - \mathcal{A}_o(\psi_2)\|_{**} \leq \mathbf{c}\,\|\psi_1 - \psi_2\|_{**}. \tag{4.16}$$

If we assume, for the moment, the validity of (4.14), (4.15) and (4.16), we get the existence of a fixed point for problem (4.12) in the set (4.13), provided r is chosen large enough.

Proof of (4.15). We start with the estimate of the first term in (4.15). Since we assume the validity of the bound (4.1) on ϕ, we write

$$\left|\hat\phi\Delta_x\eta_R\right| \lesssim \frac{|\eta''(\frac{|x|}{R\mu_0})|}{R^2\mu_0^2}\,|\hat\phi| \lesssim \frac{|\eta''(\frac{|x|}{R\mu_0})|}{R^2\mu_0^2}\,\frac{\mu_0^{\frac32}t^{-1}}{(1+|\frac{x}{\mu_0}|^a)}\,\|\phi\|_{\nu,a}$$

see (3.18) for the notation $\|\phi\|_{\nu,a}$. Thus, we get

$$\left|\hat\phi\Delta_x\eta_R\right| \lesssim \frac{|\eta''(\frac{|x|}{R\mu_0})|}{R^{2+a}}\mu_0^{-\frac12}t^{-1}\frac{r}{\sqrt t}h_0(\frac{r}{\sqrt t})\,\|\phi\|_{\nu,a} \lesssim \frac{|\eta''(\frac{|x|}{R\mu_0})|}{R^{1+a}}\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\,\|\phi\|_{\nu,a}$$
$$\lesssim \mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\frac{\|\phi\|_{\nu,a}}{R^{1+a}}.$$

Arguing similarly, we get

$$\left|\hat\phi\partial_x\eta_R\right| \lesssim \mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\frac{\|\phi\|_{\nu,a}}{R^{1+a}}, \quad \text{and} \quad \left|\nabla\hat\phi\nabla\eta_R\right| \lesssim \mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\frac{\|\phi\|_{\nu,a}}{R^{1+a}},$$

which proves the L^∞ bound in the first estimate in (4.15). To check the Hölder bound for this term, we focus the analysis on the term $g(x,t) := \hat\phi\Delta_x\eta_R$. The others terms can be treated in a similar way. We write

$$\begin{aligned}\frac{|g(x_1,t_1)-g(x_2,t_2)|}{|x_1-x_2|^{2\sigma}+|t_1-t_2|^{\sigma}} &= |\Delta_x\eta_R(x_1,t_1)|\frac{|\hat\phi(x_1,t_1)-\hat\phi(x_2,t_2)|}{|x_1-x_2|^{2\sigma}+|t_1-t_2|^{\sigma}}\\ &+|\hat\phi(x_2,t_2)|\frac{|\Delta_x\eta_R(x_1,t_1)-\Delta_x\eta_R(x_2,t_2)|}{|x_1-x_2|^{2\sigma}+|t_1-t_2|^{\sigma}}\end{aligned}$$

In order to control the first term, we use the definition in (3.18) of $\|\phi\|_{\nu,a}$ and we argue as before. The second term can be easily treated using the L^∞-bound on $\hat\phi$ and the smoothness of the function $\Delta_x\eta_R$. This complete the analysis of the first estimate in (4.15).

We continue with the proof of the second estimate in (4.15). We recall that $N(\tilde\phi) = (U_2+\tilde\phi)^5 - U_2^5 - 5\,U_2^4\,\tilde\phi$. It is convenient to estimate this function in three different regions: where $r < \bar M^{-1}\mu_0$, where $\bar M^{-1}\mu_0 < r < \bar M\sqrt t$ and where $r > \bar M\sqrt t$, with $\bar M$ a large positive number.

From the definition of U_2 in (2.57), we see that, if $r < \bar M^{-1}\mu_0$, then

$$|N(\tilde\phi)| \lesssim \mu_0^{-\frac32}|\tilde\phi|^2 \lesssim \mu_0^{-\frac32}\left[|\psi|^2 + |\eta_R\hat\phi|^2\right].$$

We recall that

$$|\psi| \lesssim \|\psi\|_{**}\,\mu_0^{\frac12}t^{-\frac12}\varphi_0(\frac{r}{\sqrt t}), \quad \left|\eta_R\hat\phi\right| \lesssim \mu_0^{\frac32}t^{-1}|\eta_R|\,\|\phi\|_{\mu,a} \tag{4.17}$$

so that we get, for $r < \bar M^{-1}\mu_0$,

$$|N(\tilde\phi+\bar\psi)| \lesssim \mu_0^2t^{-1}\left[\|\psi+\bar\psi\|_{**}^2 + \|\phi\|_{\mu,a}^2\right]\left(\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\right) \tag{4.18}$$

Let us now consider the region $\bar M^{-1}\mu_0 < r < \bar M\sqrt t$. Here, after a Taylor expansion, we get that

$$\left|N(\tilde\phi+\bar\psi)\right| \lesssim w_\mu^3\left[|\psi+\bar\psi|^2 + |\eta_R\hat\phi|^2\right] \lesssim \frac{\mu_0^{\frac32}}{r^3}\left[|\psi|^2+|\eta_R\hat\phi|^2\right].$$

Using again (4.17), we obtain, for $\bar M^{-1}\mu_0 < r < \bar M\sqrt t$,

$$|N(\tilde\phi+\bar\psi)| \lesssim \mu_0^2t^{-1}\left[\|\psi+\bar\psi\|_{**}^2 + \|\phi\|_{\mu,a}^2\right]\left(\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\right). \tag{4.19}$$

Let us now consider $r > \bar M\sqrt t$. Observe that in this region $\eta_R = 0$, $|(\psi+\bar\psi)(r,t)| \lesssim \mu_0^{\frac12}t^{-\frac12}\varphi_0(\frac{r}{\sqrt t})$ and, from (13.3), also $|U_2(r,t)| \lesssim \frac{\mu_0}{r}$. Thus we have

$$\left|N(\tilde\phi+\bar\psi)\right| \lesssim \left(\frac{\mu_0}{r}\right)^5 \lesssim \mu_0^{\frac92}t^{-\frac12}\left(\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\right). \tag{4.20}$$

From (4.18), (4.19), (4.20), we get the L^∞ bound for the second estimate in (4.15).

Proof of (4.16). For any ψ_1, $\psi_2 \in B_o$, we have that

$$\mathcal A_o(\psi_1) - \mathcal A_o(\psi_2) = T_0\left(N(\psi_1+\bar\psi+\phi^{in}) - N(\psi_2+\bar\psi+\phi^{in})\right)$$

thus

$$\|\mathcal A_o(\psi_1) - \mathcal A_o(\psi_2)\|_{**} \le C\|N(\psi_1+\bar\psi+\phi^{in}) - N(\psi_2+\bar\psi+\phi^{in})\|_{*}.$$

We write

$$
\begin{aligned}
N(\psi_1+\phi^{in})-N(\psi_2+\phi^{in}) &= (U_2+\psi_1+g)^5-(U_2+\psi_2+g)^5-5U_2^4(\psi_1-\psi_2)\\
&= \underbrace{(U_2+\psi_1+g)^5-(U_2+\psi_2+g)^5-5(U_2+g)^4(\psi_1-\psi_2)}_{:=N_1}\\
&+\underbrace{5[(U_2+g)^4-U_2^4](\psi_1-\psi_2)}_{:=N_2}, \quad g:=\phi^{in}+\bar\psi
\end{aligned}
$$

In the region where $r<\bar M\sqrt{t}$, we have that

$$|N_1(x,t)|\lesssim w_\mu^3|\psi_1-\psi_2|^2$$

which yields to

$$|N_1(x,t)|\lesssim \mu_0^2t^{-1}\left[\|\psi_1-\psi_2\|_{**}^2\right]\left(\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\right).$$

while N_2 can be estimated as

$$|N_2(x,t)|\lesssim \left[\mu_0^2t^{-1}\|\bar\psi\|_{**}+\mu_0^2t^{-1}\ \|\phi^{in}\|_{\nu,a}\right]\|\psi_1-\psi_2\|_{**}\left(\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\right).$$

On the other hand, if $r>\bar M\sqrt t$, we have that $\phi^{in}\equiv 0$, so that

$$|N_2(x,t)|\lesssim \mu_0^2\|\bar\psi\|_{**}\|\psi_1-\psi_2\|_{**}\left(\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\right).$$

On the other hand N_1 can be estimates as follows

$$|N_1(x,t)|\lesssim|\psi_1-\psi_2|^5,\quad \text{from which}\quad |N_1(x,t)|\lesssim\mu_0^2\ \mu_0^{\frac12}t^{-\frac12}\,h_0(\frac{r}{\sqrt t})\|\psi_1-\psi_2\|_{**}.$$

In summary, we get that

$$\|N(\psi_1+\phi^{in}+\bar\psi)-N(\psi_2+\phi^{in}+\bar\psi)\|_{*,\beta}\le C\mu_0^2\,\|\psi_1-\psi_2\|_{**}$$

where $C=\max\{\|\psi_1-\psi_2\|_{**},\|\phi^{in}\|_{\nu,a}\}$. Thus we get the validity of (4.16) provided that t_0 is large enough.

□

Remark 4.2. Proposition 4.1 defines the solution to Problem (3.6) as a function of the initial condition ψ_0, in the form of an operator $\psi=\bar\Psi[\psi_0]$, from a small neighborhood of 0 in the Banach space $L^\infty(\Omega)$ equipped with the norm

$$\sup_{y\in\mathbb{R}^3}\left[|y|\,|e^{b|y|}\psi_0(y)|+|y|\,|e^{b|y|}\nabla\psi_0(y)|\right] \tag{4.21}$$

into the Banach space of functions $\psi\in L^\infty(\Omega)$ equipped with the norm $\|\psi\|_{**}$, defined in (4.3). A closer look to the proof of Proposition 4.1, and the Implicit Function Theorem give that $\psi_0\to\bar\Psi[\psi_0]$ is a diffeomorphism, and that

$$\|\bar\Psi[\psi_0^1]-\bar\Psi[\psi_0^2]\|_{**}\le c\left[\sup_{y\in\mathbb{R}^3}\left||y|\,e^{b|y|}[\psi_0^1-\psi_0^2]\right|+\sup_{y\in\mathbb{R}^3}\left||y|\,e^{b|y|}[\nabla\psi_0^1-\nabla\psi_0^2]\right|\right],$$

for some positive constant c.

Proposition 4.3. *Assume the validity of the assumptions of Proposition 4.1. Then the function $\psi=\Psi(\lambda,\phi)$ depends smoothly on λ and ϕ, and we have the validity of the following estimates: for any initial time t_0 in Problem* (2.1) *sufficiently large, and any sufficiently large radius R in the cut off function η_R introduced in* (3.2) *and there exist* $\mathbf{c}$ *such that, given λ_1, λ_2 satisfying* (2.11) *one has*

$$\|\Psi[\lambda_1,\phi]-\Psi[\lambda_2,\phi]\|_{**}\le\mathbf{c}\|\lambda_1-\lambda_2\|_\sharp \tag{4.22}$$

and for any ϕ satisfying (4.1)*. Moreover, given ϕ_1, ϕ_2 satisfying* (4.1)*, one has*

$$\|\Psi[\lambda,\phi_1]-\Psi[\lambda,\phi_2]\|_{**}\leq \mathbf{c}\|\phi_1-\phi_2\|_{\nu,a} \tag{4.23}$$

for any λ satisfying (2.11)*.*

Proof. Fix ϕ and define $\bar\psi=\psi[\lambda_1,\phi]-\psi[\lambda_2,\phi]$, for λ_1 and λ_2 satisfying (2.11). Then $\bar\psi$ solves

$$\partial_t\bar\psi=\Delta\bar\psi+\left(V[\lambda_1]+N'[\lambda]\right)(\bar\psi)+F,\quad \mathbb{R}^3\times(t_0,\infty),\quad \bar\psi(r,t_0)=0$$

for $\lambda=s\lambda_1+(1-s)\lambda_2$, $s\in(0,1)$, where

$$\begin{aligned}F&=\mathcal{E}_{21}[\lambda_1]-\mathcal{E}_{21}[\lambda_2]+(1-\eta_R)\left[\mathcal{E}_{22}[\lambda_1]-\mathcal{E}_{22}[\lambda_2]\right]\\&\quad+\left[V[\lambda_1]-V[\lambda_2]\right]\psi_2+\left[N[\lambda_1]-N[\lambda_1]\right](\psi_2+\phi^{in})\end{aligned}$$

where $\psi_j=\psi[\lambda_j,\phi]$, $j=1,2$. From Lemma 2.3, estimates (2.61)-(2.62), we get that

$$\|\mathcal{E}_{21}[\lambda_1]-\mathcal{E}_{21}[\lambda_2]\|_*\leq\mathbf{c}\|\lambda_1-\lambda_2\|_\sharp$$

and

$$\|(1-\eta_R)\left[\mathcal{E}_{22}[\lambda_1]-\mathcal{E}_{22}[\lambda_2]\right]\|_*\leq\mathbf{c}\|\lambda_1-\lambda_2\|_\sharp,$$

provided t_0 is large enough. One also checks that, for some $\mathbf{c}\in(0,1)$

$$\|\left[V[\lambda_1]-V[\lambda_2]\right]\psi_2\|_*\leq\mathbf{c}\|\lambda_1-\lambda_2\|_\sharp,\quad \|\left[N[\lambda_1]-N[\lambda_2]\right](\psi_2+\phi^{in})\|_*\leq\mathbf{c}\|\lambda_1-\lambda_2\|_\sharp.$$

The constant c_1 can be made arbitrarily small provided t_0 is large. Arguing as in (4.9) and (4.10), one can show that a certain multiple of the function $\|\lambda_1-\lambda_2\|_\sharp\bar\varphi_0(r,t)$, where $\bar\varphi_0=\mu_0^{\frac12}t^{-\frac12}\varphi_0(\frac{r}{\sqrt t})$, serves as supersolution for $\bar\psi$. This proves (4.22).

Let us now fix λ, and take ϕ_1, ϕ_2 satisfying (4.1). Denote by $\phi_j^{in}=\eta_R\hat\phi_j$, and $\hat\phi_j(x,t)=\mu_0^{-\frac12}\phi_j(\frac{x}{\mu_0},t)$, for $j=1,2$, as natural. Let $\bar\psi=\psi(\lambda,\phi_1)-\psi(\lambda,\phi_2)$. We have $\bar\psi(r,t_0)=0$ and

$$\begin{aligned}\partial_t\bar\psi&=\Delta\bar\psi+V[\lambda]\bar\psi+(\psi_1+\phi_1^{in})^5-(\psi_2+\phi_1^{in})^5\\&\quad+[2\nabla\eta_R\nabla_x(\hat\phi_1-\hat\phi_2)+(\hat\phi_1-\hat\phi_2)(\Delta_x-\partial_t)\eta_R]\\&\quad+(\psi_2+\phi_1^{in})^5-(\psi_2+\phi_2^{in})^5-5U_2^4(\phi_1^{in}-\phi_2^{in}).\end{aligned}$$

Arguing as in (4.6)-(4.21), we get

$$\begin{aligned}\left|[2\nabla\eta_R\nabla_x(\hat\phi_1-\hat\phi_2)+(\hat\phi_1-\hat\phi_2)(\Delta_x-\partial_t)\eta_R]\right|&\leq\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\frac{\|\phi_1-\phi_2\|_{\nu,a}}{R^{1+a}}\\&\leq\mathbf{c}\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\|\phi_1-\phi_2\|_{\nu,a}\end{aligned}$$

and also

$$\begin{aligned}\left|(\psi_2+\phi_1^{in})^5-(\psi_2+\phi_2^{in})^5-5U_2^4(\phi_1^{in}-\phi_2^{in})\right|&\leq\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\frac{\|\phi_1-\phi_2\|_{\nu,a}}{R^{1+a}}\\&\leq\mathbf{c}\mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t})\|\phi_1-\phi_2\|_{\nu,a}.\end{aligned}$$

The constant c_1 in the last two formulas can be made arbitrarily small provided R is chosen large enough. This concludes the proof. $\square$

5. Choice of λ: Part I

Let $\psi = \Psi[\lambda, \phi]$ be the solution to Problem (3.6) predicted by Proposition 4.1, and satisfying the properties described in Proposition 4.3. We substitute ψ in equations (3.11) and (3.12), and we want to solve, in ϕ, Problem (3.11), satisfying the initial condition (3.13). As we stated in Proposition 3.1, Problem (3.11)-(3.13) can be solved for functions ϕ satisfying (4.1), provided that

$$\int_{B_{2R}} H[\psi, \lambda, \phi](y, t(\tau))\, Z_0(y)\, dy = 0, \quad \text{for all} \quad t > t_0, \tag{5.1}$$

where $H[\psi, \lambda, \phi]$ is defined in (3.12).

Next Lemma states that (5.1) is a non linear, non local equation in λ, at any fixed ϕ.

Lemma 5.1. *Assume that λ satisfies* (2.11)*, and that the function ϕ satisfies the bound* (4.1)*. Let $\psi = \Psi[\lambda, \phi]$ be the solution to Problem* (3.6) *predicted by Proposition 4.1. Then Equation* (5.1) *is equivalent to*

$$[1 + \mu_0 \mu_0' b(t) + q_1(\lambda)]\, \phi_0(0, t) = g(t) + G[\lambda, \phi](t). \tag{5.2}$$

Here ϕ_0 is the function defined in (2.53) *and also in* (2.54)*, thus*

$$\phi_0(0, t) = \int_{t_0 - 1}^{t} \frac{1}{(4\pi(t-s))^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{|y|^2}{4(t-s)}} \frac{\bar\alpha(s)}{\mu + |y|} \mathbf{1}_{\{r<M\}}\, dy\, ds. \tag{5.3}$$

The function $b = b(t)$ is a smooth function in (t_0, ∞). With $q_1(s)$ we denote a smooth function so that $q_1(0) = 0$, and $q_1'(0) \neq 0$. Moreover,

$$\|b\|_\infty < C, \quad \|g\|_\flat \leq C, \quad \|G[\lambda, \phi]\|_\flat \leq C. \tag{5.4}$$

Furthermore, if the initial time t_0 in Problem (2.1) *is chosen large enough, there exists R in the definition of the cut off function in* (3.2) *sufficiently large and there exist constant $\mathbf{c} \in (0,1)$ so that, for any ϕ,*

$$\|G[\lambda_1, \phi] - G[\lambda_2, \phi]\|_\flat \leq \mathbf{c} \|\lambda_1 - \lambda_2\|_\sharp \tag{5.5}$$

and, for any λ,

$$\|G[\lambda, \phi_1] - G[\lambda, \phi_2]\|_\flat \leq \mathbf{c} \|\phi_1 - \phi_2\|_{\nu, a}. \tag{5.6}$$

The constants $\mathbf{c}$ in (5.5) *and* (5.6) *can be made as small as one needs, provided that the initial time t_0 is chosen large enough. We refer to* (2.43) *and* (3.18) *for the definitions of $\|\cdot\|_\flat$ and $\|\cdot\|_{\nu,a}$ respectively.*

Proof. Throughout the proof, we denote by $q_i = q_i(s)$, for any interegr i, a smooth real function, with the property that $\frac{d}{(ds)^j} q_i(0) = 0$, for $j < i$, and $\frac{d}{(ds)^i} q_i(0) \neq 0$.

We decompose

$$\begin{aligned}
\int_{B_{2R}} H[\psi, \lambda, \phi](y, t(\tau))\, Z_0(y)\, dy &= \mu_0^{\frac{5}{2}} \int_{B_{2R}} \mathcal{E}_{22}(\mu_0 y, t)\, Z_0(y)\, dy \\
&+ 5 \int_{B_{2R}} \frac{\mu_0^{\frac{1}{2}}}{(1+\lambda)^2}\, w^4\Big(\frac{y}{1+\lambda}\Big) \psi(\mu_0 y, t)\, Z_0(y)\, dy \\
&+ \int_{B_{2R}} B[\phi] Z_0(y)\, dy \int_{B_{2R}} B^0[\phi] Z_0(y)\, dy \\
&= i_1 + i_2 + i_3 + i_4.
\end{aligned}$$

For any $j = 1, \ldots, 4$, i_j is a function of t, and depends also on λ and ϕ. To emphasize this dependence, we write $i_j = i_j[\lambda, \phi](t)$.

We claim that

$$\mu_0^{-\frac12}\, i_1[\lambda,\phi](t) = \mu_0^2\mu^{-2}\Bigg[\left(5\int_{B_{2R}} w^4(y)Z_0(y)\,dy\right)\,\phi_0(0,t) \\ + \left(q_1(\lambda)+\mu_0\mu_0' q_0(\lambda)\right)\,\phi_0(0,t) + \mu_0^\sigma\alpha(t)b(t)\Bigg], \tag{5.7}$$

where $b(t)$ is a smooth function in (t_0,∞), which is uniformly bounded as $t\to\infty$.

Observe that i_1 does not depend on ϕ. From the equation (2.53) satisfied by ϕ_0, and Lemma 11.1, we get the existence of a positive constant c so that $|\phi_0(\mu_0 y,t)|\le c\alpha(t)\mu_0(t)$ for any $y\in B_{2R}$. Thus, we Taylor expand $\mathcal{E}_{22}$ in the region $y\in B_{2R}$ as follows

$$\mathcal{E}_{22}(\mu_0 y,t) = 5U_1^4\phi_0 + 4(U_1+s\phi_0)^3\phi_0^2 = a+b$$

for some $s\in(0,1)$. Let us first analyze a. We write

$$a = 5\mu^{-2}w^4(y)\phi_0(0,t) + \underbrace{5[U_1^4(\mu_0 y)-\mu^{-2}w^4(y)]\phi_0(0,t)}_{:=a_1} + \underbrace{5U_1^4[\phi_0(\mu_0 y,t)-\phi_0(0,t)]}_{:=a_2}$$

Observe that, by definition of U_1 in (2.38), and (2.13), we have

$$\begin{aligned} U_1^4(\mu_0 y)-\mu^{-2}w^4(y) &= \left[w_\mu(\mu_0 y)+\mu_0'\mu^{\frac12}\Phi_1(\frac{\mu_0 r}{\mu})\right]^4 - \mu^{-2}w^4(y)\\ &= \mu^{-2}\left[w(y)+\left(w(\frac{y}{(1+\Lambda)^2})-w(y)\right)+\mu_0'\mu\Phi_1(\frac{\mu_0 r}{\mu})\right]^4 - \mu^{-2}w^4(y)\\ &= 4\mu^{-2}\,w^3(y)s\left[\left(w(\frac{y}{(1+\Lambda)^2})-w(y)\right)+\mu_0'\mu\Phi_1(\frac{\mu_0 r}{\mu})\right] \end{aligned}$$

for some $s\in(0,1)$. Observe that

$$w(\frac{y}{(1+\Lambda)^2})-w(y) = \nabla w(y)\cdot y + \nabla w(y)\cdot yz[-2\Lambda-\Lambda^2] \tag{5.8}$$

for some $z\in(0,1)$. Taking into account also the description of Φ_1 in (2.9), we get that

$$\int_{B_{2R}} a_1 Z_0\,dy = \mu^{-2}\left[q_1(\Lambda)+\mu_0\mu_0' q_0(\Lambda)\right]\phi_0(0,t). \tag{5.9}$$

We next claim that, for $y\in B_{2R}$, we have

$$\phi_0(\mu_0 y,t)-\phi_0(0,t) = \alpha(t)|\mu_0 y|^\sigma\,\Pi(t)\,\Theta(|y|), \tag{5.10}$$

for some $\sigma\in(0,1)$. We postpone the proof of (5.10) to the Appendix. We thus get

$$\int_{B_{2R}} a_2 Z_0\,dy = \mu^{-2}\mu_0^\sigma\alpha(t)b(t). \tag{5.11}$$

Collecting estimates (5.9)-(5.11) we get (5.7).

We claim that

$$\mu_0^{-\frac12} i_2[\lambda,\phi](t) = g(t)+G[\lambda,\phi](t) \tag{5.12}$$

with

$$\|g\|_\flat\le c,\quad \|G[\lambda,\phi]\|_\flat\le c$$

for some constant c. We refer to (2.43) for the definition of $\|\cdot\|_\flat$. Furthermore, we claim that G satisfies estimates (5.5) and (5.6), for some constant $c_1\in(0,1)$. To prove the above assertion, we write

$$
\begin{aligned}
\mu_0^{-\frac12} i_2[\lambda,\phi](t) &= 5\int_{B_{2R}} w^4(y)\psi[0,0](\mu_0 y,t)\, Z_0(y)\,dy\\
&+5\int_{B_{2R}} w^4(y)[\psi[\lambda,0]\ -\psi\,[0,0]](\mu_0 y,t)Z_0(y)\,dy\\
&+5\int_{B_{2R}} w^4(y)[\psi[\lambda,\phi]-\psi[\lambda,0]](\mu_0 y,t)Z_0(y)\,dy\\
&+5\int_{B_{2R}} [w^4(\frac{y}{(1+\Lambda)^2})-w^4(y)]\psi[\lambda,\phi](\mu_0 y,t)Z_0(y)\,dy\\
&+5[\frac{1}{(1+\Lambda)^4}-1]\int_{B_{2R}} w^4(\frac{y}{(1+\Lambda)^2})\psi[\lambda,\phi](\mu_0 y,t)\, Z_0(y)\,dy\\
&=\sum_{j=1}^5 g_j.
\end{aligned}
$$

The first term,

$$
g_1(t)=5\int_{B_{2R}} w^4(y)\psi(\mu_0 y,t)\,[0,0]Z_0(y)\,dy,
$$

is an explicit smooth function, globally defined in (t_0,∞), which satisfies the bound

$$
\|g_1\|_\flat\le c\left(5\int_{B_{2R}} w^4(y)|y|Z_0(y)\,dy\right) \tag{5.13}
$$

for some constant $c>0$, as direct consequence of (4.7). Let us analyze the term g_5. We see that $g_5=g_5[\lambda,\phi](t)$. Let us first assume that λ and ϕ are fixed. From (4.7), we get

$$
|g_5(t)|\le cq_1(\lambda)\int_{B_{2R}}\left|w^4(y)\psi[\lambda,\phi](\mu_0 y,t)Z_0(y)\right|\,dy\le c\mu_0^{\frac32}t^{-1}q_1(\lambda)\int\frac{|y|}{(1+|y|^5)}\,dy.
$$

Using again (4.7) and the assumptions on λ and on ϕ, we get $[g_5]_{0,\sigma,[t,t+1]}\le c\mu_0^{\frac32}t^{-1}$, from which we conclude that $\|g_5\|_\flat\le c$, for some constant $c>0$. Let us now fix ϕ and take λ_1, λ_2 satisfying (2.11). We write

$$
\begin{aligned}
g_5[\lambda_1,\phi]-g_5[\lambda_2,\phi] &= 5[\frac{1}{(1+\Lambda_1)^4}-\frac{1}{(1+\Lambda_2)^4}]\int_{B_{2R}} w^4(\frac{y}{(1+\Lambda_1)^2})\psi[\lambda_1,\phi](\mu_0 y,t)\, Z_0(y)\,dy\\
&+5[\frac{1}{(1+\Lambda_2)^4}-1]\int_{B_{2R}}[w^4(\frac{y}{(1+\Lambda_1)^2})-w^4(\frac{y}{(1+\Lambda_2)^2})]\psi[\lambda_1,\phi](\mu_0 y,t)\, Z_0(y)\,dy\\
&+5[\frac{1}{(1+\Lambda_2)^4}-1]\int_{B_{2R}} w^4(\frac{y}{(1+\Lambda_2)^2})[\psi[\lambda_1,\phi]-\psi[\lambda_2,\phi]](\mu_0 y,t)\, Z_0(y)\,dy\\
&=e_1+e_2+e_3.
\end{aligned}
$$

Thanks to (2.11), and arguing as before, we see that

$$
\begin{aligned}
|e_1(t)| &\le c|\Lambda_1(t)-\Lambda_2(t)|\int_{B_{2R}}\left|w^4(y)\psi[\lambda_1,\phi](\mu_0 y,t)Z_0(y)\right|\,dy\\
&\le c\mu_0(t)^{\frac32}t^{-1}\left(\int_t^\infty s^{-1}\mu_0(s)\,ds\right)\|\lambda_1-\lambda_2\|_\sharp\\
&\le[\mu_0(t_0)]\mu_0(t)^{\frac32}t^{-1}\|\lambda_1-\lambda_2\|_\sharp\le c_1\mu_0(t)^{\frac32}t^{-1}\|\lambda_1-\lambda_2\|_\sharp
\end{aligned}
$$

where c_1 is a positive number, which can be chosen arbitrarily small, in particular $c_1<1$, provided t_0 is chosen large enough. Similarly one can show that, thanks to (2.11),

$$
[e_1]_{0,\sigma,[t,t+1]}\le c_1\mu_0(t)^{\frac32}t^{-1}\|\lambda_1-\lambda_2\|_\sharp.
$$

We thus can conclude that there exists a positive small number $c_1 < 1$ so that

$$\|e_1\|_\flat \le c_1 \|\lambda_1 - \lambda_2\|_\sharp.$$

A similar argument allow us to say that also $\|e_2\|_\flat \le c_1\|\lambda_1 - \lambda_2\|_\sharp$. We next analyze e_3. From (4.22) we get that

$$\begin{aligned}|e_3(t)| &\le \mu_0^{\frac{3}{2}} t^{-1} \left(\int w^4(y) \frac{|y|}{1+|y|}\, dy \right) \|\psi[\lambda_1, \phi] - \psi[\lambda_2, \phi]\|_{**} \\ &\le c_1 \mu_0^{\frac{3}{2}} t^{-1} \|\lambda_1 - \lambda_2\|_\sharp,\end{aligned}$$

and also

$$[e_3]_{0,\sigma,[t,t+1]} \le c_1 \mu_0^{\frac{3}{2}} t^{-1} \|\lambda_1 - \lambda_2\|_\sharp,$$

for some constant $c_1 \in (0,1)$. We can conclude that

$$\|g_5[\lambda_1, \phi] - g_5[\lambda_2, \phi]\|_\flat \le c_1 \|\lambda_1 - \lambda_2\|_\sharp.$$

The same estimate can be obtained for g_4, arguing in a similar way.

Let us now consider g_2. This term does not depend on ϕ, namely $g_2[\lambda, \phi](t) = g_2[\lambda](t)$. From Proposition 4.3, we get

$$|g_2(t)| \le \mu_0^{\frac{3}{2}} t^{-2} \left(\int w^4 \frac{|y|}{1+|y|}\, dy \right) \|\lambda\|_\sharp \le c \mu_0^{\frac{3}{2}} t^{-2} \|\lambda\|_\sharp,$$

and similarly

$$[g_2(t)]_{0,\sigma,[t,t+1]} \le c \mu_0^{\frac{3}{2}} t^{-2} \|\lambda\|_\sharp.$$

Furthermore, if t_0 is large enough, there exists $c_1 \in (0,1)$ so that

$$\begin{aligned}|g_2[\lambda_1](t) - g_2[\lambda_2](t)| &\le 5 \int_{\mathbb{R}^3} w^4(y) \left| [\psi[\lambda_1, 0] - \psi[\lambda_2, 0]] (\mu_0 y, t) \right| Z_0 dy \\ &\le C t_0^{-1} \mu_0^{\frac{3}{2}} t^{-2} \|\lambda_1 - \lambda_2\|_\sharp \le c_1 \mu_0^{\frac{3}{2}} t^{-2} \|\lambda_1 - \lambda_2\|_\sharp\end{aligned}$$

and also

$$[g_1[\lambda_2] - g_2[\lambda_2]]_{0,\sigma,[t,t+1]} \le c_1 \mu_0^{\frac{3}{2}} t^{-2} \|\lambda_1 - \lambda_2\|_\sharp$$

thanks to the results of Proposition 4.3. Arguing in the same way, one gets similar estimates for g_3.

Collecting all the above arguments, we conclude that $\mu_0^{-\frac{1}{2}} i_2[\lambda, \phi](t)$ can be written as in (5.12), with g and G satisfying (5.4), (5.5) and (5.6).

Next we claim that

$$\mu_0^{-\frac{1}{2}} i_j[\lambda, \phi](t) = G[\lambda, \phi](t), \quad j = 3, 4, \tag{5.14}$$

and G satisfies (5.4), (5.5) and (5.6). We start with $j = 3$. First, we see that i_3 does not depend on λ, and it is linear in ϕ. Since we are assuming that ϕ satisfies (4.1), we have

$$\left| \mu_0^{-\frac{1}{2}} i_3(t) \right| \le \left(\mu_0 \mu_0' R^{2-a} \right) \mu_0^{\frac{3}{2}}(t) t^{-1} \|\phi\|_{\nu,a} \le c \mu_0^{\frac{3}{2}}(t) t^{-1} \|\phi\|_{\nu,a}$$

and

$$\left[\mu_0^{-\frac{1}{2}} i_3(t) \right]_{0,\sigma,[t,t+1]} \le c \mu_0^{\frac{3}{2}}(t) t^{-1} \|\phi\|_{\nu,a}$$

for some constant $c > 0$. Let us know take ϕ_1, and ϕ_2, and we get that, if $\mu_0(t_0)\mu_0'(t_0)R^{2-a}$ is small enough,

$$\left| \mu_0^{-\frac{1}{2}} \left(i_3[\phi_1] - i_3[\phi_2] \right)(t) \right| \le c_1 \mu_0^{\frac{3}{2}}(t) t^{-1} \|\phi_1 - \phi_2\|_{\nu,a}$$

and

$$\left[\mu_0^{-\frac{1}{2}} \left(i_3[\phi_1] - i_3[\phi_2] \right)(t) \right]_{0,\sigma,[t,t+1]} \le c_1 \mu_0^{\frac{3}{2}}(t) t^{-1} \|\phi_1 - \phi_2\|_{\nu,a}$$

for some $c_1 \in (0,1)$. Estimate (5.14) for $j = 4$ can be proved in a very similar way. We leave the details to the interested reader. Combining (5.7), (5.12) and (5.14), we complete the proof of (5.2). This concludes the proof of the Lemma.

□

6. Solving a non local linear problem

Let ϕ_0 be the function introduced in (2.53). Later in our argument we will need to solve in λ, a non local equation of the form

$$\phi_0(0,t) = h(t), \quad t \in (t_0, \infty) \tag{6.1}$$

for a certain right hand side h. We see from (5.3) that $\phi_0(0,t)$, defined as

$$\phi_0(0,t) = \int_{t_0-1}^{t} \int_{\mathbb{R}^3} \frac{\bar\alpha(s)}{(4\pi(t-s))^{\frac32}} \frac{e^{-\frac{|y|^2}{4(t-s)}}}{\mu + |y|} \mathbf{1}_{\{|y|<M\}}\, dy\, ds,$$

defines a non-local non-linear operator in λ. For convenience we recall that

$$\alpha(t) = 3^{\frac14}\, \mu_0^{-\frac12}\, (\mu_0\Lambda)', \quad \bar\alpha(t) = \begin{cases} \alpha(t_0) & \text{for} \quad t < t_0 \\ \alpha(t) & \text{for} \quad t \ge t_0 \end{cases}, \quad \Lambda(t) = \int_t^\infty \lambda(s)\, ds.$$

We write

$$\phi_0(0,t) = T[\lambda](t) + \hat T[\lambda](t), \tag{6.2}$$

where T is

$$T[\lambda](t) = \int_{t_0-1}^{t} \int_{\mathbb{R}^3} \frac{\bar\alpha(s)}{(4\pi(t-s))^{\frac32}} \frac{e^{-\frac{|y|^2}{4(t-s)}}}{|y|} \mathbf{1}_{\{|y|<M\}}\, dz\, ds. \tag{6.3}$$

We shall see that $\hat T$ is a small perturbation of T, in some sense we will precise later. In this section, we start with the analysis of Problem

$$T[\lambda](t) = h(t), \quad t > t_0. \tag{6.4}$$

Straightforward computations give that

$$T[\lambda](t) = -\frac{\bar\omega_3}{4} \int_{t_0-1}^{t} \frac{\bar\alpha(s)}{\sqrt{t-s}} \left(1 - e^{-\frac{M^2}{(t-s)}}\right) ds. \tag{6.5}$$

Indeed, letting $z = \frac{y}{2\sqrt{t-s}}$, one gets

$$\begin{aligned} T[\lambda](t) &= \int_{t_0-1}^{t} \int_{\mathbb{R}^3} \frac{\bar\alpha(s)}{2\sqrt{t-s}} \frac{e^{-|z|^2}}{|z|} \mathbf{1}_{\{|z|<\frac{M}{\sqrt{t-s}}\}}\, dz\, ds \\ &= \frac{\bar\omega_3}{2} \int_{t_0-1}^{t} \int_0^\infty \frac{\bar\alpha(s)}{\sqrt{t-s}} e^{-\rho^2} \rho \mathbf{1}_{\{\rho<\frac{M}{\sqrt{t-s}}\}}\, d\rho\, ds = \frac{\bar\omega_3}{4} \int_{t_0-1}^{t} \frac{\bar\alpha(s)}{\sqrt{t-s}} \int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2} 2\rho\, d\rho \\ &= -\frac{\bar\omega_3}{4} \int_{t_0-1}^{t} \frac{\bar\alpha(s)}{\sqrt{t-s}} \left(1 - e^{-\frac{M^2}{(t-s)}}\right) ds. \end{aligned} \tag{6.6}$$

Introduce the function $\beta = \beta(t)$ as

$$\beta(t) = \frac{\bar\omega_3}{4} \int_t^\infty \bar\alpha(s)\, ds. \tag{6.7}$$

If $\beta = \beta(t)$ solves

$$\int_{t_0-1}^{t} \frac{\beta'(s)}{\sqrt{t-s}} \left(1 - e^{-\frac{M^2}{(t-s)}}\right) ds = h(t), \tag{6.8}$$

then the function $\Lambda(t) = \int_t^\infty \lambda(s)\, ds$, defined as

$$\bar\omega \Lambda(t) = \mu_0^{-\frac12}(t)\beta(t) + \frac{\mu_0^{-1}(t)}{2} \int_t^\infty \beta(s) \mu_0^{-\frac12}\, \mu_0'(s)\, ds, \quad \bar\omega = \frac{\bar\omega_3}{4} 3^{\frac14}, \tag{6.9}$$

solves (6.4).

Next Lemma constructs a solution to (6.8). If we formally let $M \to \infty$ in (6.8), we get that the left hand side of (6.8) is nothing but the $\frac12$-Caputo derivative of β. This fact inspires the proof of the following

Lemma 6.1. *Let $\sigma = \frac{1}{2} + \sigma'$, with $\sigma' > 0$ small, be the number fixed in* (2.11)*, and $h : (t_0, \infty) \to \mathbb{R}$ a smooth function satisfying*

$$\sup_{t>t_0} \mu_0^{-\frac{3}{2}} t \left[\|h\|_{0,[t,t+1]} + [h]_{0,\sigma,[t,t+1]} \right] \leq C, \tag{6.10}$$

for some constant C. Then there exist a constant C_1 and a unique smooth function $\beta : (t_0 - 1, \infty) \to \mathbb{R}$ which solves (6.8)*, $\beta \in C^1$ and satisfies the bounds*

$$\sup_{t>t_0} \mu_0^{-\frac{3}{2}} t \left[\|\beta'\|_{0,[t,t+1]} + [\beta']_{0,\sigma,[t,t+1]} \right] \leq C_1 M^{-1}. \tag{6.11}$$

We recall that $M^2 = t_0$, was first introduced in (2.53)*.*

Observe that a direct consequence of this Lemma, together with (6.9) and (2.41) is the invertibility theory for Problem (6.4) that will be used in next Section to solve (5.1). This is contained in the following

Proposition 6.2. *The function $T : X_\sharp \to X_\flat$, defined in* (6.3) *is a linear, non-local, homeomorphism so that*

$$\|T^{-1}(h)\|_\sharp \leq C M^{-1} \|h\|_\flat, \quad \textit{for any} \quad h \in X_\flat \tag{6.12}$$

for some fixed positive constant C. We refer to (2.11) *and to* (2.12) *for the definition of the $\|\cdot\|_\sharp$-norm and of the set $X_\sharp$, and to* (2.43) *and* (2.42) *for the definition of the norm $\|\cdot\|_\flat$ and of the space $X_\flat$.*

We devote the rest to the Section to the

Proof of Lemma 6.1. We start performing a change of variables, to transform Problem (6.8) into an equivalent one with simpler form: let

$$s = t_0 - 1 + M^2 a, \quad t = t_0 - 1 + M^2 b, \quad \tilde\beta(a) = \beta(s), \quad \tilde h(b) = h(t).$$

After this change of variables, Problem (6.8) takes the form

$$\int_0^b \frac{\tilde\beta'(a)}{\sqrt{b-a}} \left(1 - e^{-\frac{1}{b-a}}\right) da = M \, \tilde h(b). \tag{6.13}$$

Let $K(\eta) = \frac{1 - e^{-\frac{1}{\sqrt{\eta}}}}{\sqrt{\eta}}$ and take the Laplace transform of both sides in (6.13), thus getting

$$\mathcal{L}\left(\tilde\beta'\right)(\xi) \mathcal{L}(K)(\xi) = M \mathcal{L}\left(\tilde h\right)(\xi).$$

Since $\mathcal{L}\left(\tilde\beta'\right) = \xi \mathcal{L}\left(\tilde\beta\right)(\xi) - \tilde\beta(0)$, we get

$$\mathcal{L}\left(\tilde\beta\right)(\xi) = \frac{\tilde\beta(0)}{\xi} + M \frac{\mathcal{L}\left(\tilde h\right)(\xi)}{\xi \mathcal{L}(K)(\xi)} \tag{6.14}$$

Observe now that

$$\mathcal{L}(K)(\xi) = \int_0^\infty e^{-\xi\eta} \left(\frac{1 - e^{-\frac{1}{\eta}}}{\sqrt{\eta}} \right) d\eta = \frac{2}{\sqrt{\xi}} \int_0^\infty e^{-p^2} \left(1 - e^{-\frac{\xi}{p^2}}\right) dp.$$

We readily get that

$$\mathcal{L}(K)(\xi) = \frac{1}{\sqrt{\xi}} \left(2 \int_0^\infty e^{-p^2} \, dp \right) (1 + o(1)), \quad \text{as} \quad \xi \to \infty. \tag{6.15}$$

To describe the behavior of $\mathcal{L}(K)(\xi)$, for $\xi \to 0$, we first notice that

$$\int_0^{\frac{1}{\xi}} e^{-\xi\eta} \left(\frac{1 - e^{-\frac{1}{\eta}}}{\sqrt{\eta}} \right) d\eta = \int_0^\infty \frac{1 - e^{-\frac{1}{\eta}}}{\sqrt{\eta}} \, d\eta + O(\sqrt{\xi}).$$

On the other hand,

$$\int_{\frac{1}{\xi}}^{\infty} e^{-\xi\eta}\left(\frac{1-e^{-\frac{1}{\eta}}}{\sqrt{\eta}}\right) d\eta = \int_{\frac{1}{\xi}}^{\infty} e^{-\xi\eta}\left(\frac{1-\frac{1}{\eta}-e^{-\frac{1}{\eta}}}{\sqrt{\eta}}\right) d\eta + \int_{\frac{1}{\xi}}^{\infty} \frac{e^{-\xi\eta}}{\eta\sqrt{\eta}}\, d\eta = O(\sqrt{\xi}).$$

Thus we conclude that

$$\mathcal{L}\left(K\right)(\xi) = \int_0^\infty \frac{1-e^{-\frac{1}{\eta}}}{\sqrt{\eta}}\, d\eta + O(\sqrt{\xi}), \quad \text{as} \quad \xi \to 0. \tag{6.16}$$

From (6.15) and (6.16), we conclude that

$$\frac{1}{\xi\mathcal{L}\left(K\right)(\xi)} = \begin{cases} \frac{c_1}{\xi} + \frac{c_2}{\sqrt{\xi}} + O(1) & \text{if} \quad \xi \to 0 \\ \frac{c_3}{\sqrt{\xi}}\left(1+o(1)\right) & \text{if} \quad \xi \to \infty. \end{cases}$$

Let now $G = G(t)$ be so that $\mathcal{L}\left(G\right)(\xi) = \frac{1}{\xi\mathcal{L}(K)(\xi)}$. Standard arguments on Laplace transformation imply that

$$G(t) = \begin{cases} \tilde c_1 + \frac{\tilde c_2}{\sqrt{t}} + O(\frac{1}{t}) & \text{if} \quad t \to \infty \\ \frac{\tilde c_3}{\sqrt{t}}\left(1+o(1)\right) & \text{if} \quad t \to 0. \end{cases},$$

for certain constants $\tilde c_1$, $\tilde c_2$ and $\tilde c_3$. From (6.14), taking the anti-Laplace transform of both sides, we get

$$\begin{aligned} \tilde\beta(b) &= \tilde\beta(0) + M\int_0^b \tilde h(a) G(b-a)\, da \\ &= \tilde\beta(0) + M\tilde c_1 \int_0^\infty \tilde h(a)\, da + M\tilde c_1 \int_b^\infty \tilde h(a)\, da + M\int_0^b \tilde h(a)\left[G(b-a) - \tilde c_1\right] da. \end{aligned}$$

We select the solution to Problem (6.13) so that

$$\tilde\beta(0) + M\tilde c_1 \int_0^\infty \tilde h(a)\, da = 0.$$

In the original variables, we thus obtain an explicit solution to (6.8)

$$\beta(t) = \underbrace{\frac{\tilde c_1}{M}\int_t^\infty h(s)\, ds}_{:=\beta_1(t)} + \underbrace{\frac{1}{M}\int_{t_0-1}^t h(s)\left[G\left(\frac{t-s}{M^2}\right) - \tilde c_1\right] ds}_{:=\beta_2(t)}. \tag{6.17}$$

Let us now check (6.11). Since (6.10) holds, we easily get that

$$\sup_{t>t_0} \mu_0^{-\frac{3}{2}}\left|\beta_1(t)\right| \lesssim M^{-1}.$$

To control the second term in (6.17), we change variable $t = M^2\bar t$, $s = M^2\bar s$, so that

$$\beta_2(t) = M\int_{\frac{t_0-1}{M^2}}^{\bar t} h(M^2\bar s)\left[G\left(\bar t - \bar s\right) - \tilde c_1\right] d\bar s.$$

Since $t_0 = M^2$ and since (6.10) holds, we get

$$|\beta_2(t)| \lesssim \frac{1}{M}\int_{1-\frac{1}{t_0}}^{\bar t} \frac{\mu_0^{\frac{3}{2}}(\bar s)}{\bar s}\left[G\left(\bar t - \bar s\right) - \tilde c_1\right] d\bar s \lesssim \frac{1}{M}\mu_0^{\frac{3}{2}}(\bar t) \lesssim M^{-1}\mu_0^{\frac{3}{2}}(t),$$

from which we get the validity of (6.11).

The assumption that $\mu_0^{-\frac{3}{2}} t[h]_{0,\sigma,[t,t+1]}$ is bounded guarantees that the function β defined in (6.17) is differentiable. Indeed, trivially one has $\beta_1'(t) = -\frac{\tilde c_1}{M} h(t)$. Let us write β_2 in the following way

$$\beta_2(t) = \frac{1}{M}\int_{t_0}^t (h(s)-h(t)) \left[G\left(\frac{t-s}{M^2}\right) - \tilde c_1\right] ds + \frac{h(t)}{M}\int_{t_0}^t \left[G\left(\frac{t-s}{M^2}\right) - \tilde c_1\right] ds.$$

Thus we have

$$\begin{aligned}
\beta_2'(t) &= \underbrace{\frac{1}{M}\lim_{s\to t}\left[(h(s)-h(t))[G\left(\frac{t-s}{M^2}\right) - \tilde c_1]\right]}_{=0} + \frac{1}{M^3}\int_{t_0}^t (h(s)-h(t)) G'\left(\frac{t-s}{M^2}\right) ds \\
&\quad + \underbrace{\frac{h'(t)}{M}\int_{t_0}^t \left[G\left(\frac{t-s}{M^2}\right) - \tilde c_1\right] ds - \frac{h'(t)}{M}\int_{t_0}^t \left[G\left(\frac{t-s}{M^2}\right) - \tilde c_1\right] ds}_{=0} \\
&\quad + \frac{h(t)}{M}\frac{d}{dt}\left(\int_{t_0}^t \left[G\left(\frac{t-s}{M^2}\right) - \tilde c_1\right] ds\right) \\
&= \frac{1}{M^3}\int_{t_0}^t (h(s)-h(t)) G'\left(\frac{t-s}{M^2}\right) ds + \frac{h(t)}{M}\frac{d}{dt}\left(\int_{t_0}^t \left[G\left(\frac{t-s}{M^2}\right) - \tilde c_1\right] ds\right).
\end{aligned}$$

Both the last two integrals are well defined, as consequence of the behavior of $G(\eta)$, as $\eta\to 0$, and the assumption that $\mu_0^{-\frac{3}{2}} t[h]_{0,\sigma,[t,t+1]}$ is bounded. Since $G(\eta)\sim \eta^{-\frac{1}{2}}$, as $\eta \to 0$, direct computations give the bounds in (6.11) for $\beta'(t)$. This concludes the proof of the Lemma. □

7. Choice of λ: Part II

This Section is devoted to solve in λ Equation (5.1), for fixed ϕ satisfying (4.1). We have the validity of the following

Proposition 7.1. *For any ϕ satisfying* (4.1)*, there exists $L>0$ and a unique solution $\lambda=\lambda[\phi]$ to Equation* (5.1)*, with*

$$\|\lambda\|_\sharp \le LM^{-1} \tag{7.1}$$

where $M=\sqrt{t_0}$, provided the initial time t_0 in Problem (2.1) *is chosen large enough. Furthermore, there exists a constant $\mathbf{c}\in(0,1)$ such that, for any ϕ_1, ϕ_2 satisfying* (4.1)*, we have*

$$\|\lambda[\phi_1]-\lambda[\phi_2]\|_\sharp \le \mathbf{c}\|\phi_1-\phi_2\|_{\nu,a}. \tag{7.2}$$

Proof of Proposition 7.1. Lemma 5.1 states that solving Equation (5.1) is equivalent to solve (5.2). We write (5.2) as follows

$$T[\lambda](t) + \hat T[\lambda](t) = (1+\mu_0\mu_0' b(t) + q_1(\lambda))^{-1}\,[g(t)+G[\lambda,\phi](t)], \tag{7.3}$$

where T and $\hat T$ are defined in (6.2) and (6.3), while b, g and G satisfy the bounds in (5.4),(5.5) and (5.6). Here $q_1=q_1(s)$ denotes a smooth function such that $q_1(0)=0$ and $q_1'(0)\neq 0$. We observe first that

$$(1+\mu_0\mu_0' b(t) + q_1(\lambda))^{-1}\,[g(t)+G[\lambda,\phi](t)] = g_1(t) + G_1[\lambda,\phi](t),$$

for some new functions g_1 and G_1 that also satisfy (5.4), (5.5), and (5.6).

Thanks to the result of Proposition 6.2, solving in λ Equation (7.3) reduces to find the fixed point problem

$$\lambda(t) = \mathcal{F}(\lambda)(t), \quad \mathcal{F}(\lambda) := T^{-1}\left(g_1 + G_1[\lambda,\phi] - \hat T[\lambda]\right) \tag{7.4}$$

where T^{-1} is the operator introduced in Proposition 6.2.

Step 1. First we show that, for any fixed ϕ satisfying (4.1), there exists a unique fixed point $\lambda=\lambda[\phi]$ of contraction type for $\mathcal{F}$ in the set

$$B = \{\lambda\in X_\sharp \,:\, \|\lambda\|_\sharp \le LM^{-1}\}$$

for some $L > 0$ large.

In order to prove this fact, we claim that, if the initial time t_0 in Problem (2.1) is large enough, there are positive constants $\bar c_1$, $\bar c_2 \in (0,1)$ so that, for any $\lambda \in B$,

$$\|\hat T[\lambda]\|_\flat \le \bar c_1 M \|\lambda\|_\sharp, \quad \text{with} \quad \bar c_1 C < 1 \tag{7.5}$$

and

$$\|\hat T[\lambda_1] - \hat T[\lambda_2]\|_\flat \le \bar c_2 \|\lambda_1 - \lambda_2\|_\sharp \quad \text{with} \quad CM^{-1}(\mathbf{c} + \bar c_2) < 1, \tag{7.6}$$

for any λ_1, $\lambda_2 \in B$. The constant C is the constant appearing in (6.12), Proposition 6.2, while $\mathbf{c}$ is the constant is the one appearing in (5.5).

Assume for the moment the validity of (7.5) and (7.6). For any $\lambda \in B$, we have

$$\begin{aligned}\|\mathcal F(\lambda)\|_\sharp &\le CM^{-1}\|g_1 + G_1[\lambda,\phi] - \hat T[\lambda]\|_\flat \le CM^{-1}\left(\|g_1\|_\flat + \|G_1[\lambda,\phi]\|_\flat + \|\hat T[\lambda]\|_\flat\right) \\ &\le CM^{-1}\left(2c + \bar c_1 L\right) \le LM^{-1}\end{aligned}$$

provided $L > \frac{2cC}{1-\bar c_1 C}$, where C is the constant in (6.12), c are the constants in (5.4), and $\bar c_1$ is the constant in (7.5), which satisfies $\bar c_1 C < 1$.

Let us take now λ_1, $\lambda_2 \in B$. We have

$$\begin{aligned}\|\mathcal F(\lambda_1) - \mathcal F(\lambda_2)\|_\sharp &= \|T^{-1}(G_1[\lambda_1,\phi] - G_1[\lambda_2,\phi]) - T^{-1}(\hat T[\lambda_1] - \hat T[\lambda_2])\|_\sharp \\ &\le CM^{-1}\left(\|G_1[\lambda_1,\phi] - G_1[\lambda_2,\phi]\|_\flat + \|\hat T[\lambda_1] - \hat T[\lambda_2]\|_\flat\right] \\ &\le CM^{-1}(c_1 + \bar c_2)\|\lambda_1 - \lambda_2\|_\sharp < \varepsilon \|\lambda_1 - \lambda_2\|_\sharp,\end{aligned}$$

for some $\varepsilon < 1$, thanks to the choice of $\bar c_2$ in (7.6).

A direct application of Banach fixed point gives the existence and uniqueness of a solution λ to Equation (5.1), satisfying (7.1). We complete the first part of the proof of the Proposition with the proofs of (7.5) and (7.6).

Proof of (7.5). Let $\lambda \in B$. From (6.2) and (6.3), we get

$$\begin{aligned}\hat T[\lambda](t) &= -\int_{t_0-1}^t \int_{\mathbb R^3} \frac{\bar\alpha(s)}{(4\pi(t-s))^{\frac32}} \, \frac{e^{-\frac{|y|^2}{4(t-s)}}}{|y|} \, \frac{\mu(s)}{\mu(s) + |y|} \mathbf{1}_{\{|y|<M\}} \, dz\, ds \\ &= \bar c \int_{t_0-1}^t \frac{\bar\alpha(s)\mu(s)}{\sqrt{t-s}} \int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2} \frac{\rho}{\mu+\rho}\, d\rho ds,\end{aligned}$$

for some explicit constant $\bar c$. Since $\left|\int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2} \frac{\rho}{\mu+\rho}\, d\rho\right| \le c\frac{M}{\sqrt t}$, for any t large, we observe that

$$\left|\hat T[\lambda](t)\right| \le A\frac{M}{\sqrt t}\left|\int_{t_0-1}^t \frac{\bar\alpha(s)\mu(s)}{\sqrt{t-s}}\, ds\right|, \tag{7.7}$$

for some fixed constant A. We claim that

$$\int_{t_0-1}^t \frac{\bar\alpha(s)\mu(s)}{\sqrt{t-s}}\, ds = \bar\alpha(t)\mu(t)\sqrt{t-t_0+1}\,\Pi(t), \quad t > t_0, \tag{7.8}$$

for some smooth and uniformly bounded function $\Pi(t)$. Indeed, we write, for $\beta_*(s) = \bar\alpha(s)\mu(s)$,

$$\int_{t_0-1}^t \frac{\beta_*(s)}{\sqrt{t-s}}\, ds = \int_{t_0-1}^t \frac{\beta_*(s) - \beta_*(t)}{\sqrt{t-s}}\, ds + 2\beta_*(t)\sqrt{t-t_0+1} = i + 2\beta_*(t)\sqrt{t-t_0+1}. \tag{7.9}$$

Use the change of variables $x = \sqrt{t-s}$

$$i = -2\int_0^{\sqrt{t-t_0+1}} \left[\beta_*(t) - \beta_*(t-x^2)\right]\, dx = -2\beta_*(t)\int_0^{\sqrt{t-t_0+1}} \frac{\left[\beta_*(t) - \beta_*(t-x^2)\right]}{\beta_*(t)}\, dx.$$

We now observe that the function $x \to \frac{\left[\beta_*(t)-\beta_*(t-x^2)\right]}{\beta_*(t)}$ is uniformly bounded in $x \in [0, \sqrt{t-t_0+1}]$, since

$$\frac{\left[\beta_*(t)-\beta_*(t-x^2)\right]}{\beta_*(t)} = \begin{cases} 1-(1-\frac{x^2}{t})^{-1}(1-\frac{x^2}{t})^{-\frac{3}{2}\bar\gamma-1} & \text{for} \quad \gamma \neq 2 \\ 1-(1-\frac{x^2}{t})^{-\frac{5}{2}}[1+\log(1-\frac{x^2}{t})]^3 & \text{for} \quad \gamma = 2 \end{cases}$$

where $\bar\gamma = 1$ if $\gamma > 2$, and $\bar\gamma = \gamma - 1$ if $1 < \gamma < 2$. With this in mind, we conclude that

$$i = \beta_*(t)\sqrt{t-t_0+1}\;\Pi(t) \tag{7.10}$$

for some smooth and bounded function Π. Inserting (7.10) into (7.9), we get (7.8).

Using (7.8) in (7.7), we conclude that

$$\left|\hat T[\lambda](t)\right| \le A\mu_0(t) M \|\lambda\|_\sharp \left[\mu_0^{\frac{3}{2}}(t)t^{-1}\right],$$

for some fixed constant A, independent of t and of M. Thus, for t large, if we choose t_0 sufficiently large, there exists a constant $c_1 \in (0,1)$ such that

$$\left|\hat T[\lambda](t)\right| \le c_1\, M\, \|\lambda\|_\sharp \left[\mu_0^{\frac{3}{2}}(t)t^{-1}\right].$$

Let now consider t_1 and $t_2 \in [t, t+1]$. We write

$$\begin{aligned} \hat T[\lambda](t_1) - \hat T[\lambda](t_2) &= \bar c \int_{t_0-1}^{t_1} \left[\frac{\bar\alpha(s)}{\sqrt{t_1-s}} - \frac{\bar\alpha(s)}{\sqrt{t_2-s}}\right] \int_0^{\frac{M}{\sqrt{t_1-s}}} e^{-\rho^2} \frac{\rho\mu}{\mu+\rho}\, d\rho ds \\ &- \bar c \int_{t_0-1}^{t_1} \frac{\bar\alpha(s)}{\sqrt{t_2-s}} \int_{\frac{M}{\sqrt{t_2-s}}}^{\frac{M}{\sqrt{t_1-s}}} e^{-\rho^2} \frac{\rho\mu}{\mu+\rho}\, d\rho ds \\ &- \bar c \int_{t_1}^{t_2} \frac{\bar\alpha(s)}{\sqrt{t_2-s}} \int_0^{\frac{M}{\sqrt{t_2-s}}} e^{-\rho^2} \frac{\rho\mu}{\mu+\rho}\, d\rho ds = \sum_{j=1}^{3} i_j \end{aligned}$$

Observe that, for t_1, $t_2 \in [t, t+1]$, for t large, we have

$$\begin{aligned} \sup_{t_1,t_2\in[t,t+1]} \frac{|\mu(t_1)-\mu(t_2)|}{|t_1-t_2|^\sigma} &\le C\mu_0(t) \sup_{t_1,t_2\in[t,t+1]} \tfrac{|\Lambda(t_1)-\Lambda(t_2)|}{|t_1-t_2|^\sigma} \\ &\le CM^{-1}\mu_0(t)\left(\mu_0^{\frac{3}{2}}(t)t^{-1}\right) \end{aligned} \tag{7.11}$$

for some constant C. With this, we can estimate i_1 and i_5, as follows

$$[i_j]_{0,\sigma,[t,t+1]} \le CM^{-1}\mu_0(t)\left(\mu_0^{\frac{3}{2}}(t)t^{-1}\right), \quad \text{for} \quad j = 1, 5.$$

Straightforward computation gives

$$[i_j]_{0,\sigma,[t,t+1]} \le CM^{-1}\mu_0(t)t^{-\sigma}\,\|\lambda\|_\sharp\left(\mu_0^{\frac{3}{2}}(t)t^{-1}\right), \quad \text{for} \quad j = 1, 2, 3.$$

These estimates, together with the ones we obtained before, constitute the proof of (7.5).

Proof of (7.6). Let λ_1, $\lambda_2 \in B$. From (6.2) and (6.3),

$$\hat T[\lambda_1](t) - \hat T[\lambda_2](t) = \bar c \int_{t_0-1}^{t} \frac{\bar\alpha(s)}{\sqrt{t-s}} \int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2}\left[\frac{\rho\mu[\lambda_1]}{\mu[\lambda_1]+\rho} - \frac{\rho\mu[\lambda_2]}{\mu[\lambda_2]+\rho}\right] d\rho ds.$$

Observe that

$$\begin{aligned} |(\mu[\lambda_1]-\mu[\lambda_2])(s)| &\le A\mu_0(s)\,|\Lambda_1(s)-\Lambda_2(s)| \\ &\le A\mu_0(s)\int_s^\infty |\lambda_1-\lambda_2|(x)\,dx \le A\mu_0^2(s)\|\lambda_1-\lambda_2\|_\sharp \end{aligned}$$

for some constant A, whose value may change from one line to the other, and which is independent of t and t_0. A Taylor expansion gives

$$|\hat T[\lambda_1](t) - \hat T[\lambda_2](t)| \le \int_{t_0-1}^t \frac{|\bar\alpha(s)|}{\sqrt{t-s}} \int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2} \frac{\rho}{(\tilde\mu+\rho)^2} \, |\mu[\lambda_1](s) - \mu[\lambda_2](s)| \, d\rho ds$$

for some $\tilde\mu$ between $\mu[\lambda_1]$ and $\mu[\lambda_2]$. Thus we get

$$|\hat T[\lambda_1](t) - \hat T[\lambda_2](t)| \le A\mu_0^2(t) M \, [\mu_0^{\frac32}(t)t^{-1}] \, \|\lambda_1 - \lambda_2\|_\sharp,$$

where A is a constant independent of t_0 and t. Using again (7.11), we can show that

$$\left| \hat T[\lambda_1] - \hat T[\lambda_2] \right|_{0,\sigma,[t,t+1]} \le A\mu_0^2(t) M \, [\mu_0^{\frac32}(t)t^{-1}] \, \|\lambda_1 - \lambda_2\|_\sharp,$$

where A is a constant independent of t_0 and t. Choosing t_0 large enough, we can find $\bar c_2$ small enough so that (7.6) holds true.

Step 2. In the second part of the proof, we show the validity of (7.2). For this purpose, we fix ϕ_1 and ϕ_2 satisfying (4.1), and we let $\lambda_j = \lambda[\phi_j]$, $j = 1, 2$. If $\bar\lambda = \lambda_1 - \lambda_2$, then we see that $\bar\lambda$ solves

$$\begin{aligned} \bar\lambda &= T^{-1} \left(G_1[\lambda_1, \phi_1] - G_1[\lambda_2, \phi_2] \right) \\ &= T^{-1} \left(G_1[\bar\lambda_1, \phi_1] - G[\bar\lambda_1, \phi_2] \right) + T^{-1} \left(G_1[\lambda_1, \phi_2] - G[\lambda_2, \phi_2] \right). \end{aligned}$$

Thus

$$\begin{aligned} \|\bar\lambda\|_\sharp &\le CM^{-1} \left(\|G_1[\bar\lambda_1, \phi_1] - G[\bar\lambda_1, \phi_2]\|_\flat + \|G_1[\lambda_1, \phi_2] - G[\lambda_2, \phi_2]\|_\flat \right) \\ &\le CM^{-1} \left(\mathbf{c} \|\phi_1 - \phi_2\|_{\nu,a} + \mathbf{c} \|\lambda_1 - \lambda_2\|_\sharp \right), \end{aligned}$$

where C is the constant in (6.12), $M^2 = t_0$, $\mathbf{c}$ are the constants defined respectively in (5.5) and (5.6). We now observe that the proof of Lemma 5.1 also gives that the constants $\mathbf{c}$ in (5.5) and (5.6) can be such that $CM^{-1}\mathbf{c} < 1$. Thus the proof of (7.2) readily follows.

This concludes the proof of the Proposition. □

Remark 7.2. Recall that the function $\psi = \bar\Psi[\psi_0]$ solution to Problem (3.6) depends smoothly on the initial condition ψ_0, provided ψ_0 belongs to a small neighborhood of 0 in the Banach space $L^\infty(\Omega)$ equipped with the norm defined in (4.21), as observed in Remark 4.2. This fact implies that also $\lambda = \lambda[\psi_0]$ solution to (5.1) depends on ψ_0. A closer look at the definitions of $\lambda = \lambda[\psi_0]$ gives that

$$\|\lambda[\psi_0^{(1)}] - \lambda[\psi_0^{(2)}]\|_\sharp \lesssim \|e^{b|y|}[\psi_0^{(1)} - \psi_0^{(1)}]\|_{L^\infty(\mathbb{R}^3)} + \|e^{b|y|}[\nabla\psi_0^{(1)} - \nabla\psi_0^{(1)}]\|_{L^\infty(\mathbb{R}^3)}.$$

This fact will be useful in the final argument of finding ϕ solution to (3.13).

8. Final argument: solving (3.8)

We are constructing a global unbounded solution to Problem (2.1)-(2.2) of the form (3.1)

$$u = U_2[\lambda](r,t) + \tilde\phi.$$

The function U_2 is defined in (2.57), while $\tilde\phi$ is given in (3.2). The function ψ which enters in the definition of $\tilde\phi$ solves the *outer problem* (3.6), and its properties are contained in Proposition 4.1 and 4.3. The parameter $\lambda = \lambda(t)$ belongs to the space $X_\sharp$, (2.12), and has been chosen to solve Equation (5.1). The properties of this $\lambda = \lambda(t)$ are collected in Proposition 7.1. What is left is to solve in ϕ the *inner problem* (3.8). Thanks to the choice of $\lambda = \lambda(t)$, the orthogonality condition (3.19) is satisfied, so that we can use the result of Proposition 3.1 to solve in ϕ Problem (3.8).

In other words, we want to find ϕ, with its $\|\phi\|_{\nu,a}$-bounded, solution to Problem (3.8). The function $\psi = \Psi[\lambda[\phi], \phi]$ solves (3.6), while $\lambda = \lambda[\phi]$ solves Equation (5.1).

At this point, we fix a in the definition od the $\| \star \|_{\nu,a}$ to be equal to 1. Proposition 3.1 defines a linear operator $\phi = \mathcal{T}(h)$, where ϕ is the solution to (3.16) so that

$$\|\phi\|_{\nu,1} \leq C_0 \, R^4 \, \|h\|_{\nu,3}$$

for some fixed constant C_0. We refer to (3.17) for $\|h\|_{\nu,2+a}$ and to (3.18) for $\|\phi\|_{\nu,a}$, for $a = 1$. Thus we can say that ϕ solves (3.11)-(3.13) if and only if ϕ is a fixed point for the Problem

$$\phi = \mathcal{T}\left(\mathbf{H}[\phi]\right), \quad \text{where} \quad \mathbf{H}[\phi] = H(\psi[\phi], \lambda[\phi], \phi), \tag{8.1}$$

and H is defined in (3.12). Choose the number R in the cut off function η_R, defined in (2.59) and appearing in the ansatz (3.2), to be sufficiently large in terms of t_0, say $R^6 \mu_0^{\frac{1}{2}}(t_0) = 1$. We claim that there exists a unique ϕ solution to (8.1) in the set

$$B_1 = \{\phi \, : \, \|\phi\|_{\nu,1} \leq L_1\}$$

for some $L_1 > 0$, fixed.

From (2.59) and (4.7), we see that

$$\left|\mu_0^{\frac{5}{2}} \mathcal{E}_{22}(\mu_0 y, t)\right| \lesssim \mu_0^{\frac{1}{2}} \frac{\mu_0^{\frac{3}{2}} t^{-1}}{(1+|y|^2)^2}, \quad \left| 5 \, \frac{\mu_0^{\frac{1}{2}}}{(1+\Lambda)^4} \, w^4 (\frac{y}{(1+\Lambda)^2}) \psi(\mu_0 y, t) \right| \lesssim \frac{\mu_0^2(t) t^{-1}}{(1+|y|^3)}.$$

Furthermore,

$$|B[\phi](t)| \leq C R^2 \mu_0 \mu_0' \frac{\mu_0^{\frac{3}{2}} t^{-1}}{(1+|y|^{2+a})}, \quad \left|B^0[\phi](t)\right| \leq C \Lambda(t) \frac{\mu_0^{\frac{3}{2}} t^{-1}}{(1+|y|^{4+a})}.$$

In fact, one can prove that

$$\|\mathbf{H}[\phi]\|_{\nu,2+a} \leq C_1 \, R^{-4}$$

for some fixed number C_1, independent from t and of t_0. This implies that, if $\phi \in B_1$, then $\mathcal{T}(\phi) \in B_1$ provided L_1 is chosen large. Furthermore, combining (2.61), the result of Proposition 4.3, and the result of Proposition 7.1, we get the existence of a number $\mathbf{c} \in (0,1)$, so that

$$\|\mathcal{T}[\phi_1] - \mathcal{T}[\phi_2]\|_{\nu,a} \leq \mathbf{c} \|\phi_1 - \phi_2\|_{\nu,a}$$

for any ϕ_1 and $\phi_2 \in B_1$. We apply Banach fixed point theorem to get the existence of a unique solution to (8.1) with $\| \cdot \|_{\nu,a}$-bounded.

This concludes the proof of the existence of the solution to Problem (2.1)-(2.2), or equivalently Problem (1.3)-(1.4), as predicted by Theorem 1.1. □

9. Basic linear theory for the inner problem

Let $R > 0$ be a fixed large number. This section is devoted to construct a solution to the initial value problem

$$\phi_\tau = \Delta \phi + 5 w^4 \phi + h(y, \tau) \quad \text{in } B_{2R} \times (\tau_0, \infty), \quad \phi(y, \tau_0) = e_0 Z(y) \quad \text{in } B_{2R}, \tag{9.1}$$

for any given function h with $\|h\|_{\nu,2+a} < +\infty$, not necessarily radial in the y variable. We refer to (3.17) for the explicit definition of the $\| \cdot \|_{\nu,2+a}$-norm. The corresponding problem in dimension $n \geq 5$ has already been treated in [6], Section 7. We follow the same strategy in the procedure to construct the solution to (9.1), but in dimension 3 we get a decay estimate for the solution different from the one valid for dimensions $n \geq 5$.

We recall that the operator $L_0(\phi) = \Delta \phi + 5 w^4 \phi$ has an 4 dimensional kernel generated by the bounded functions Z_0 defined in (2.7) and also by

$$Z_i(y) = \frac{\partial w}{\partial y_i}, \quad i = 1, 2, 3. \tag{9.2}$$

In the class of radially symmetric functions, the only element in the kernel of L_0 is Z_0. To describe our construction, we consider an orthonormal basis ϑ_m, $m = 0, 1, \dots,$ in $L^2(S^2)$ of spherical harmonics, namely eigenfunctions of the problem

$$\Delta_{S^2}\vartheta_m + \lambda_m\vartheta_m = 0 \quad \text{in } S^2$$

so that $0 = \lambda_0 < \lambda_1 = \ldots = \lambda_3 = 2 < \lambda_4 \leq \ldots$. Let $h(\cdot, \tau) \in L^2(B_{2R})$, for any $\tau \in [\tau_0, \infty)$. We decompose it into the form

$$h(y,\tau) = \sum_{j=0}^{\infty} h_j(r,\tau)\vartheta_j(y/r), \quad r = |y|, \quad h_j(r,\tau) = \int_{S^2} h(r\theta,\tau)\vartheta_j(\theta)\, d\theta.$$

In addition, we write $h = h^0 + h^1 + h^\perp$ where

$$h^0 = h_0(r,\tau), \quad h^1 = \sum_{j=1}^{3} h_j(r,\tau)\vartheta_j, \quad h^\perp = \sum_{j=4}^{\infty} h_j(r,\tau)\vartheta_j.$$

Observe that $h^1 = h^\perp = 0$ if h is radially symmetric in the y variable. Consider also the analogous decomposition for ϕ into $\phi = \phi^0 + \phi^1 + \phi^\perp$. We build the solution ϕ of Problem (9.1) by doing so separately for the pairs (ϕ^0, h^0), (ϕ^1, h^1) and $(\phi^\perp, h^\perp)$.

Our main result in this section is the following proposition.

Proposition 9.1. *Let ν, a be given positive numbers with $0 < a < 2$. Then, for all sufficiently large $R > 0$ and any $h = h(y,\tau)$ with $\|h\|_{\nu,2+a} < +\infty$ that satisfies for all $j = 0, 1, \dots, 3$*

$$\int_{B_{2R}} h(y,\tau)\, Z_j(y)\, dy \;=\; 0 \quad \textit{for all} \quad \tau \in (\tau_0, \infty) \tag{9.3}$$

there exist $\phi = \phi[h]$ and $e_0 = e_0[h]$ which solve Problem (9.1). They define linear operators of h that satisfy the estimates

$$|\phi(y,\tau)| \;\lesssim\; \tau^{-\nu}\Big[\frac{R^{4-a}}{1+|y|^3}\,\|h^0\|_{\nu,2+a} + \frac{R^{4-a}}{1+|y|^4}\,\|h^1\|_{\nu,2+a} + \frac{\|h\|_{\nu,2+a}}{1+|y|^a}\Big], \tag{9.4}$$

$$|\nabla_y\phi(y,\tau)| \;\lesssim\; \tau^{-\nu}\Big[\frac{R^{4-a}}{1+|y|^4}\,\|h^0\|_{\nu,2+a} + \frac{R^{4-a}}{1+|y|^5}\,\|h^1\|_{\nu,2+a} + \frac{\|h\|_{\nu,2+a}}{1+|y|^{a+1}}\Big], \tag{9.5}$$

and

$$|e_0[h]| \;\lesssim\; \|h\|_{\nu,2+a}. \tag{9.6}$$

Proposition 3.1 is a direct consequence of Proposition 9.1. Indeed, if h is radially symmetric in the y variable, (9.3) is authomatically satified for $j = 1, \dots, 3$, and $h \equiv h^0$.

The result contained in Proposition 9.1 follows from next Proposition, which refers to the following problem

$$\phi_\tau = \Delta\phi + 5w(y)^4\phi + h(y,\tau) - c(\tau)Z \quad \text{in } B_{2R}\times(\tau_0,\infty), \quad \phi(y,\tau_0) = 0 \quad \text{in } B_{2R}. \tag{9.7}$$

Proposition 9.2. *Let ν, a be given positive numbers with $0 < a < 2$. Then, for all sufficiently large $R > 0$ and any h with $\|h\|_{\nu,2+a} < +\infty$ and satisfying the orthogonality conditions* (3.19)*, there exist $\phi = \phi[h]$ and $c = c[h]$ which solve Problem* (9.7)*, and define linear operators of h. The function $\phi[h]$ satisfies estimate* (9.4)*,* (9.5) *and for some $\Gamma > 0$*

$$\left|c(\tau) - \int_{B_{2R}} hZ\right| \lesssim \tau^{-\nu}\left[R^{2-a}\left\|h - Z\int_{B_{2R}} hZ\right\|_{\nu,2+a} + e^{-\Gamma R}\|h\|_{\nu,2+\alpha}\right]. \tag{9.8}$$

Assuming the validity of Proposition 9.2, we proceed with

Proof of Proposition 9.1. Let ϕ_1 be the solution of Problem (9.7) predicted by Proposition 9.2. Let us write

$$\phi(y,\tau) = \phi_1(y,\tau) + e(\tau)Z(y). \tag{9.9}$$

for some $e \in C^1([\tau_0,\infty))$. We find

$$\partial_\tau \phi = \Delta\phi + 5w^4\phi + h(y,\tau) + [\, e'(\tau) - \lambda_0 e(\tau) - c(\tau)\,]\, Z(y).$$

We choose $e(\tau)$ to be the unique bounded solution of the equation

$$e'(\tau) \,-\, \lambda_0 e(\tau) \,=\, c(\tau), \quad \tau \in (\tau_0,\infty)$$

which is explicitly given by

$$e(\tau) = \int_\tau^\infty \exp(\sqrt{\lambda_0}(\tau - s))\, c(s)\, ds\,.$$

The function e depends linearly on h. Besides, we clearly have from (9.8), $|e(\tau)| \lesssim \tau^{-\nu}\|h\|_{\nu,2+a}$. and thus, from the fact that ϕ_1 satisfies estimates (9.4), (9.5), so does ϕ given by (9.9). Thus ϕ satisfies Problem (9.1) with initial condition $\phi(y,\tau_0) = e(\tau_0)Z(y)$. The proof is concluded. □

The rest of the Section is devoted to the

Proof of Proposition 9.2. The proof is divided in two steps. In the first step, we construct a solution to (9.7) which has value zero on the boundary ∂B_{2R}, at any time τ, for a right hand side h not necessarily satisfying the orthogonality conditions (9.3). In the second step, we make use of this construction to solve (9.7), for a right hand side satisfying (9.3), and to obtain estimates (9.4), (9.5) and (9.6).

Step 1. We claim that for all sufficiently large $R > 0$ and any H with $\|H\|_{\nu,a} < +\infty$ there exists $\phi = \phi(y,\tau)$ and $c = c(\tau)$ which solve Problem

$$\phi_\tau = \Delta\phi + 5w^4\phi + H(y,\tau) - c(\tau)Z(y) \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{9.10}$$
$$\phi = 0 \quad \text{on } \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R}.$$

The functions ϕ and c are linear operators of h and satisfy the estimates

$$(1+|y|)\,|\nabla\phi(y,\tau)| + |\phi(y,\tau)| \;\lesssim$$
$$\tau^{-\nu}\Big[\frac{R^{4-a}\|H^0\|_{\nu,a}}{1+|y|} + \frac{R^{4-a}\|H^1\|_{\nu,a}}{1+|y|^2} + R^2\,\frac{\|H\|_{\nu,a}}{1+|y|^a}\Big] \tag{9.11}$$

and for some $\Gamma > 0$

$$\left| c(\tau) - \int_{B_{2R}} HZ \right| \lesssim \tau^{-\nu}\,\Big[\, R^2 \left\| H - Z\int_{B_{2R}} HZ \right\|_{\nu,a} + e^{-\Gamma R}\|H\|_{\nu,a}\Big]. \tag{9.12}$$

We construct the solution ϕ mode by mode, considering first mode 0, then modes $1,2,3$ and finally modes greater or equal to 4. For each mode, we get the corresponding estimates.

Construction at mode 0. Consider Problem (9.10) for a right hand side $H = H_0(r,\tau)$ radially symmetric. Let $\eta(s)$ be the smooth cut-off function in (1.9), and consider $\eta_\ell(y) = \eta(|y|-\ell)$, for a large but fixed number ℓ independently of R. By standard parabolic theory, there exists a unique solution $\phi_*[\bar h_0]$ to

$$\phi_\tau = \Delta\phi + 5w(r)^4(1-\eta_\ell)\phi + \bar H_0(y,\tau) \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{9.13}$$
$$\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R},$$

where

$$\bar H_0 = H_0 - c_0(\tau)Z, \quad c_0(\tau) = \int_{B_{2R}} H_0(y,\tau)Z(y)\,dy.$$

The function $\phi_*[\bar h_0]$ is radial and satisfies the bound

$$\big|\phi_*[\bar H_0]\big| \lesssim \tau^{-\nu}\, R^{2-a}\, \|H\|_{\nu,a}.$$

This can be proven with the use of a special super solution, arguing as in Lemma 7.3 in [6]. Setting $\phi = \phi_*[\bar H_0] + \tilde\phi$ and $c(\tau) = c_0(\tau) + \tilde c(\tau)$, Problem (9.10) gets reduced to

$$\tilde\phi_\tau = \Delta\tilde\phi + 5w(r)^4\tilde\phi + \tilde H_0(r,\tau) - \tilde c(\tau)Z \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{9.14}$$
$$\tilde\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \tilde\phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R}.$$

where $\tilde H_0 = 5w^4\eta_\ell\phi_*[\bar H_0]$. Observe that $\tilde H_0$ is radial, it is compactly supported and with size controlled by that of $\bar H_0$. In particular we have that for any $m > 0$,

$$|\tilde H_0(r,\tau)| \lesssim \frac{\tau^{-\nu}}{1+r^m}\left[\sup_{\tau>\tau_0}\tau^\nu\|\phi_*[\bar H_0](\cdot,\tau)\|_{L^\infty}\right] \lesssim \frac{\tau^{-\nu}}{1+r^m}\,R^{2-a}\,\|H\|_{\nu,a}. \tag{9.15}$$

We shall next solve Problem (9.14) under the additional orthogonality constraint

$$\int_{B_{2R}}\tilde\phi(\cdot,\tau)\,Z \;=\; 0 \quad \text{for all} \quad \tau\in(\tau_0,\infty). \tag{9.16}$$

Problem (9.14)-(9.16) is equivalent to solving just (9.14) for $\tilde c$ given by the explicit linear functional $\tilde c := \tilde c[\tilde\phi, \tilde H_0]$ determined by the relation

$$\tilde c(\tau)\int_{B_{2R}} Z^2 = \int_{B_{2R}}\tilde H_0(\cdot,\tau)Z + \int_{\partial B_{2R}}\partial_r\tilde\phi(\cdot,\tau)Z. \tag{9.17}$$

If the function $\tilde c = \tilde c(\tau)$ defined by (9.17) were independent of ϕ, standard linear parabolic theory would give the existence of a unique solution. On the other hand, a close look to (9.17) shows that the dependence of $\tilde c = \tilde c(\tau)$ on ϕ is small in an L^∞-$C^{1+\alpha,\frac{1+\alpha}{2}}$ setting, since $Z(R) = O(e^{-\Gamma R})$ for some $\Gamma > 0$. A contraction argument applies to yield existence of a unique solution to (9.14)-(9.16) defined at all times. To get the estimates, we assume smoothness of the data so that integrations by parts and differentiations can be carried over, and then arguing by approximations. Testing (9.14)-(9.16) against $\tilde\phi$ and integrating in space, we obtain the relation

$$\partial_\tau\int_{B_{2R}}\tilde\phi^2 + Q(\tilde\phi,\tilde\phi) = \int_{B_{2R}} g\tilde\phi, \quad g = \tilde H_0 - \tilde c(\tau)Z_0,$$

where Q is the quadratic form defined by

$$Q(\phi,\phi) := \int\left[|\nabla\phi|^2 - 5w^4|\phi|^2\right]. \tag{9.18}$$

Since dimension is 3, there exists $\beta > 0$ such that, for any ϕ with $\int\phi Z = 0$, the following inequality holds

$$Q(\phi,\phi) \ge \frac{\beta}{R^2}\int\phi^2.$$

The proof of this inequality is a slight modification of the proof for the corresponding inequality in dimensions $n \ge 5$ that can be found in Lemma 7.2 [6], considering that $\int_{B_R} Z_0^2 = O(R)$, as $R\to\infty$, when dimension is 3. Thus we have, for some $\beta' > 0$,

$$\partial_\tau\int_{B_{2R}}\tilde\phi^2 + \frac{\beta'}{R^2}\int_{B_{2R}}\tilde\phi^2 \lesssim R^2\int_{B_{2R}} g^2. \tag{9.19}$$

We observe that from (9.17) and (9.15) for $m = 0$ we get that

$$|\tilde c(\tau)| \le \tau^{-\nu}K, \quad K := \left[\sup_{\tau>\tau_0}\tau^\nu\|\phi_*[\bar H_0](\cdot,\tau)\|_{L^\infty}\right] + e^{-\Gamma R}\left[\sup_{\tau>\tau_0}\tau^\nu\|\nabla\phi_*[\bar H_0](\cdot,\tau)\|_{L^\infty}\right].$$

Besides, using again estimate (9.15) for a sufficiently large m, we get

$$\int_{B_{2R}} g^2 \;\lesssim\; \tau^{-2\nu}K^2.$$

Using that $\tilde\phi(\cdot,\tau_0) = 0$ and Gronwall's inequality, we readily get from (9.19) the L^2-estimate

$$\|\tilde\phi(\cdot,\tau)\|_{L^2(B_{2R})} \;\lesssim\; \tau^{-\nu}R^2K, \tag{9.20}$$

for all $\tau > \tau_0$. Now, using standard parabolic estimates in the equation satisfied by $\tilde\phi$ we obtain then that on any large fixed radius $\ell > 0$,

$$\|\tilde\phi(\cdot,\tau)\|_{L^\infty(B_M)} \lesssim \tau^{-\nu} R^2 K \quad \text{for all} \quad \tau > \tau_0.$$

Since the right hand side has a fast decay at infinity and taking into account that we are in dimension 3, outside B_ℓ we can dominate the solution by a barrier of the order $\tau^{-\nu}|y|^{-1}$. As a conclusion, also using local parabolic estimates for the gradient, we find that

$$(1+|y|)\,|\nabla_y\tilde\phi(y,\tau)| + |\tilde\phi(y,\tau)| \lesssim \tau^{-\nu}\frac{R^2}{1+|y|}\left[\sup_{\tau>\tau_0}\tau^\nu\|\phi_*[\bar H_0](\cdot,\tau)\|_{L^\infty}\right]. \tag{9.21}$$

It clearly follows from this estimate and inequality (9.15) that the function

$$\phi_0[h_0] := \tilde\phi + \phi_*[\bar H_0] \tag{9.22}$$

solves Problem (9.10) for $H = H_0$ and satisfies

$$(1+|y|)\,|\nabla_y\phi_0(y,\tau)| + |\phi_0(y,\tau)| \lesssim \tau^{-\nu}\frac{R^{4-a}}{1+|y|}\|H\|_{\nu,a}$$

Finally, from (9.17) we see that we have that

$$c(\tau) = \int_{B_{2R}} HZ + \int_{B_{2R}} 5w^4\eta_\ell\phi_*[\bar H_0]\,Z + O(e^{-\Gamma R})\|H\|_{\nu,a}.$$

From here we find the validity of estimate

$$\left|c(\tau) - \int_{B_{2R}} H_0 Z\right| \lesssim \tau^{-\nu}\left[R^2\left\|H_0 - Z\int_{B_{2R}} H_0 Z\right\|_{\nu,a} + e^{-\Gamma R}\|H_0\|_{\nu,a}\right].$$

Hence estimates (9.11) and (9.12) hold. The construction of the solution at mode 0 is concluded.

Construction at modes 1 *to* 3. Here we consider the case $H = H^1$ where $H^1(y,\tau) = \sum_{j=1}^3 H_j(r,\tau)\vartheta_j$. The function

$$\phi^1[H^1] := \sum_{j=1}^n \phi_j(r,\tau)\vartheta_j. \tag{9.23}$$

solves the initial-boundary value problem

$$\phi_\tau = \Delta\phi + 5w^4\phi + H^1(y,\tau) \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{9.24}$$
$$\phi = 0 \quad \text{on } \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R},$$

if the functions $\phi_j(r,\tau)$ solves

$$\partial_\tau\phi_j = \mathcal{L}_1[\phi_j] + H_j(r,\tau) \quad \text{in } (0,2R)\times(\tau_0,\infty) \tag{9.25}$$
$$\partial_r\phi_j(0,\tau) = 0 = \phi_j(R,\tau) \quad \text{for all} \quad \tau\in(\tau_0,\infty), \quad \phi_j(r,\tau_0) = 0 \quad \text{for all} \quad r\in(0,R),$$

where

$$\mathcal{L}_1[\phi_j] := \partial_{rr}\phi_j + 2\frac{\partial_r\phi_j}{r} - 2\frac{\phi_j}{r^2} + 5w^4\phi_j. \tag{9.26}$$

Let us consider the solution of the stationary problem $\mathcal{L}_1[\phi] + (1+r)^{-a} = 0$ given by the variation of parameters formula

$$\bar\phi(r) = Z(r)\int_r^{2R}\frac{1}{\rho^2 Z(\rho)^2}\int_0^\rho (1+s)^{-a}Z(s)s^2\,ds$$

where $Z(r) = w_r(r)$. Since $w_r(r)\sim r^{-2}$ for large r, we find the estimate $|\bar\phi(r)| \lesssim \frac{R^{4-a}}{1+r^2}$. Then, provided that τ_0 was chosen sufficiently large, the function $2\|H_j\|_{\nu,a}\tau^{-\nu}\bar\phi(r)$ is a positive super-solution of Problem (9.25) and thus we find $|\phi_j(r,\tau)| \lesssim \tau^{-\nu}\frac{R^{4-a}}{1+r^2}\|H_j\|_{\nu,a}$. Hence $\phi^1[H^1]$ given by (9.23) satisfies

$$|\phi^1[H^1](y,\tau)| \lesssim \frac{R^{4-a}}{1+|y|^2}\|H^1\|_{\nu,a}.$$

A corresponding estimate for the gradient follows.

Construction at higher modes. We consider now the case of higher modes,

$$\phi_\tau = \Delta\phi + 5w^4\phi + H^\perp \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{9.27}$$
$$\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,\tau_0)=0 \quad \text{in } B_{2R},$$

where $H = H^\perp = \sum_{j=4}^\infty H_j(r)\Theta_j$ whose solution has the form $\phi^\perp = \sum_{j=4}^\infty \phi_j(r,\tau)\Theta_j$. Given the quadratic form in (9.18), for $\phi^\perp \in H_0^1(B_{2R})$

$$\int_{B_{2R}} \frac{|\phi^\perp|^2}{r^2} \lesssim Q(\phi^\perp,\phi^\perp). \tag{9.28}$$

The proof of this fact is elementary. The interested reader can find it in [6]. Let $\phi_*[H^\perp]$ be the solution to

$$\phi_\tau = \Delta\phi + 5w(r)^4(1-\eta_\ell)\phi + \bar H^\perp(y,\tau) \quad \text{in } B_{2R}\times(\tau_0,\infty)$$
$$\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,0)=0 \quad \text{in } B_{2R},$$

where $\bar H^\perp = H^\perp - c^\perp Z$, and $c^\perp = \int_{B_{2R}} H^\perp Z$. By writing $\phi = \phi_*[H^\perp] + \tilde\phi$, Problem (9.27) reduces to solving

$$\tilde\phi_\tau = \Delta\tilde\phi + 5w(y)^4\tilde\phi + \tilde H \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{9.29}$$
$$\tilde\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \tilde\phi(\cdot,\tau_0)=0 \quad \text{in } B_{2R},$$

where $\tilde H = 5w(y)^4\eta_\ell\phi_*[H^\perp]$, for a sufficiently large ℓ. Arguing as in (9.19) we now get

$$\partial_\tau \int_{B_{2R}} \tilde\phi^2 + c\int_{B_{2R}} \frac{|\tilde\phi|^2}{|y|^2} \lesssim \int_{B_{2R}} |y|^2|\tilde H|^2. \tag{9.30}$$

Similarly to (9.20) we get

$$\|\,|y|^{-1}\tilde\phi(\cdot,\tau)\|_{L^2(B_{2R})} \lesssim \tau^{-\nu}R^{2-a}\|H\|_{\nu,a} \tag{9.31}$$

From elliptic estimates we then get that

$$\|\tilde\phi(\cdot,\tau)\|_{L^\infty(B_{2R})} \lesssim \tau^{-\nu}R^{2-a}\|H^\perp\|_{\nu,a}. \quad \text{for all} \quad \tau>\tau_0,$$

so that with the aid of a barrier we obtain

$$|\tilde\phi(y,\tau)| \lesssim \tau^{-\nu}R^{2-a}\|H^\perp\|_{\nu,a}\,(1+|y|)^{-1}.$$

It follows that the function

$$\phi^\perp[H^\perp] := \tilde\phi + \phi_*[H^\perp] \tag{9.32}$$

satisfies

$$|\phi^\perp[H^\perp](y,\tau)| \lesssim \tau^{-\nu}\,R^2\,\left[(1+|y|)^{-1} + (1+|y|)^{-a}\right]\,\|H^\perp\|_{\nu,a} \quad \text{in } B_{2R}.$$

Similar estimates for the gradient follow. Conclusion: let

$$\phi[h] := \phi^0[h^0] + \phi^1[h^1] + \phi^\perp[h^\perp]$$

for the functions defined in (9.22), (9.23), (9.32). By construction, $\phi[h]$ solves Equation (9.10). It defines a linear operator of h and satisfies (9.11). The proof of *Step 1* is concluded.

Step 2. To complete the proof of Proposition 9.2, we decompose the right hand side h in (9.7) in modes, $h = h^0+h^1+h^\perp$ as before, and define separately associated solutions of (9.7) in a decomposition $\phi = \phi^0+\phi^1+\phi^\perp$.

Construction at mode 0. For a bounded radial $h = h(|y|)$ defined in B_{2R} with $\int_{B_{2R}} hZ_0 = 0$, let $\tilde h$ designate the extension of h as zero outside B_{2R}. The equation

$$\Delta H + 5w^4(y)H + \tilde h(|y|) = 0 \quad \text{in } \mathbb{R}^3, \quad H(y)\to 0 \quad \text{as} \quad |y|\to\infty$$

has a solution $H =: L_0^{-1}[h]$ represented by the variation of parameters formula

$$H(r) = \tilde Z(r)\int_r^\infty \tilde h(s)\,Z_0(s)\,s^2\,ds + Z_0(r)\int_r^\infty \tilde h(s)\,\tilde Z(s)\,s^2\,ds \tag{9.33}$$

where $\tilde{Z}(r)$ is a suitable second radial solution of $L_0[\tilde{Z}] = 0$, linearly independent with Z_0. Mode 0 function $h_0 = h_0(|y|, \tau)$ is defined in B_{2R}, and satisfies $\|h_0\|_{\nu,2+a} < +\infty$ and $\int_{B_{2R}} h_0 Z_0 = 0$ for all τ. Then $H_0 := L_0^{-1}[h_0(\cdot, \tau)]$ satisfies the estimate

$$|H_0(r, \tau)| \lesssim \frac{\tau^{-\nu}}{(1+r)^a} \|h_0\|_{\nu,2+a}.$$

Let $\Phi_0[h_0]$ be the radial solution in B_{3R} to

$$\Phi_\tau = \Delta\Phi + 5w^4(y)\Phi + H_0(|y|, \tau) - c_0(\tau)Z \quad \text{in } B_{3R} \times (\tau_0, \infty) \tag{9.34}$$
$$\Phi = 0 \quad \text{on} \quad \partial B_{3R} \times (\tau_0, \infty), \quad \Phi(\cdot, \tau_0) = 0 \quad \text{in } B_{3R},$$

that we discussed in Step 1. $\Phi_0[h_0]$ defines a linear operator of h_0 and satisfies the estimates

$$|\Phi_0(y, \tau)| \ \lesssim \ \frac{\tau^{-\nu} R^{4-a}}{(1+|y|)} \|H_0\|_{\nu,a}, \tag{9.35}$$

where for some $\Gamma > 0$

$$\left| c_0(\tau) - \int_{B_{2R}} H_0 Z \right| \lesssim \tau^{-\nu} \Big[R^2 \left\| H_0 - Z \int_{B_{2R}} H_0 Z \right\|_{\nu,a} + e^{-\Gamma R} \|H_0\|_{\nu,a} \Big]. \tag{9.36}$$

Since $L_0[Z] = \lambda_0 Z$ then

$$\lambda_0 \int_{B_{2R}} H_0 Z = \int_{B_{2R}} H_0 L_0[Z] = \int_{B_{2R}} L_0[H_0]\, Z + \int_{\partial B_{2R}} (Z \partial_\nu H_0 - H_0 \partial_\nu Z),$$

and hence

$$\int_{B_{2R}} H_0 Z = \lambda_0^{-1} \int_{B_{2R}} h_0\, Z + O(e^{-\Gamma R}) \tau^{-\nu} \|h_0\|_{\nu,2+a}.$$

Also, from the definition of the operator L_0^{-1} we see that $Z = \lambda_0 L_0^{-1}[Z]$. Thus

$$\left\| H_0 - Z \int_{B_{2R}} H_0 Z \right\|_{\nu,a} = \left\| L_0^{-1} \Big[h_0 - \lambda_0 Z \int_{B_{2R}} H_0 Z \Big] \right\|_{\nu,a} \lesssim \left\| h_0 - Z \int_{B_{2R}} h_0 Z \right\|_{\nu,2+a} + e^{-\Gamma R} \|h_0\|_{\nu,2+a}.$$

Next, we discuss estimates on the first and second derivatives of Φ_0. Let us fix now a vector e with $|e| = 1$, a large number $\rho > 0$ with $\rho \le 2R$ and a number $\tau_1 \ge \tau_0$. Consider the change of variables

$$\Phi_\rho(z, t) := \Phi_0(\rho e + \rho z, \tau_1 + \rho^2 t), \quad H_\rho(z, t) := \rho^2 [H_0(\rho e + \rho z, \tau_1 + \rho^2 t) - c_0(\tau_1 + \rho^2 t) Z(\rho e + \rho z)\,].$$

Then $\Phi_\rho(z, t)$ satisfies an equation of the form

$$\partial_t \Phi_\rho = \Delta_z \Phi_\rho + B_\rho(z, t) \Phi_\rho + H_\rho(z, t) \quad \text{in } B_1(0) \times (0, 2).$$

where $B_\rho = O(\rho^{-2})$ uniformly in $B_2(0) \times (0, \infty)$. Standard parabolic estimates yield that for any $0 < \alpha < 1$

$$\|\nabla_z \Phi_\rho\|_{L^\infty(B_{\frac{1}{2}}(0) \times (1,2))} \lesssim \|\Phi_\rho\|_{L^\infty(B_1(0) \times (0,2))} + \|H_\rho\|_{L^\infty(B_1(0) \times (0,2))}.$$

Moreover

$$\|H_\rho\|_{L^\infty(B_1(0) \times (0,2))} \ \lesssim \ \rho^{2-a} \tau_1^{-\nu} \|H_0\|_{\nu,a}, \quad \|\Phi_\rho\|_{L^\infty(B_1(0) \times (0,2))} \lesssim \tau_1^{-1} K(\rho)$$

where

$$K(\rho) = \frac{R^{2-a}}{1+\rho} R^2 \|h^0\|_{\nu,2+a} \tag{9.37}$$

This yields in particular that

$$\rho |\nabla_y \Phi(\rho e, \tau_1 + \rho^2)| \ = \ |\nabla \tilde{\phi}(0, 1)| \lesssim \tau_1^{-\nu} K(\rho).$$

Hence if we choose $\tau_0 \ge R^2$, we get that for any $\tau > 2\tau_0$ and $|y| \le 3R$

$$(1 + |y|)\, |\nabla_y \Phi(y, \tau)| \ \lesssim \ \tau^{-\nu} K(|y|) \tag{9.38}$$

We obtain that these bounds are as well valid for $\tau < 2\tau_0$ by the use of similar parabolic estimates up to the initial time (with condition 0).

Now, we observe that the function H_0 is of class C^1 in the variable y and $\|\nabla_y H_0\|_{\nu,1+a} \le \|h^0\|_{\nu,2+a}$. It follows that we have the estimate

$$(1+|y|^2)\,|D_y^2\Phi(y,\tau)| \;\lesssim\; \tau^{-\nu}K(|y|)$$

for all $\tau > \tau_0$, $|y| \le 2R$. where K is the function in (9.37). The proof follows simply by differentiating the equation satisfied by Φ, rescaling in the same way we did to get the gradient estimate, and apply the bound already proven for $\nabla_y\Phi$. Thus we have in B_{2R}

$$(1+|y|^2)|D^2\Phi(y,\tau)| + (1+|y|)|\nabla\Phi(y,\tau)| \;+\; |\Phi(y,\tau)| \lesssim \tau^{-\nu}\|h^0\|_{\nu,2+a}\frac{R^{4-a}}{1+|y|}.$$

This yields in particular

$$|L_0[\Phi](\cdot,\tau)| \;\lesssim\; \tau^{-\nu}\|h^0\|_{\nu,2+a}\frac{R^{4-a}}{1+|y|^3} \quad \text{in } B_{2R}$$

We define

$$\phi^0[h_0] := L_0[\Phi]\Big|_{B_{2R}}.$$

Then $\phi^0[h_0]$ solves Problem (9.7) with

$$c(\tau) := \lambda_0 c_0(\tau). \tag{9.39}$$

$\phi^0[h_0]$ satisfies the estimate

$$|\phi^0[h_0](y,\tau)| \;\lesssim\; \tau^{-\nu}\|h_0\|_{\nu,2+a}\frac{R^{4-a}}{1+|y|^3} \quad \text{in } B_{2R}. \tag{9.40}$$

and from (9.36), estimate (9.8) holds too.

Construction for modes 1 *to* 3. We consider now $h^1(y,\tau) = \sum_{j=1}^3 h_j(r,\tau)\vartheta_j$ with $\|h^1\|_{\nu,2+a} < +\infty$ that satisfies for all $i = 1,\dots,3$ $\int_{B_{2R}} h^1 Z_i = 0$ for all $\tau \in (\tau_0,\infty)$. We will show that there is a solution

$$\phi^1[h^1] = \sum_{j=1}^3 \phi_j(r,\tau)\vartheta_j(\frac{y}{r})$$

to Problem (9.7) for $h = h^1$, which define a linear operator of h^1 and satisfies the estimate

$$|\phi^1(y,\tau)| \;\lesssim\; \frac{R^4}{1+|y|^4}R^{-a}\|h\|_{\nu,2+a}. \tag{9.41}$$

Let us fix $1 \le j \le 3$. For a function $h = h_j(r)\vartheta_j(\frac{y}{r})$ defined in B_{2R}, we let $H = L_0^{-1}[h] := H_j(r)\vartheta_j(\frac{y}{r})$ be the solution of the equation

$$\Delta H + pU^{p-1}H + \tilde h_j\vartheta_j = 0 \quad \text{in } \mathbb{R}^n, \quad H(y)\to 0 \quad \text{as} \quad |y|\to\infty$$

where $\tilde h_j$ designates the extension of h_j as zero outside B_{2R}, represented by the variation of parameters formula

$$H_j(r) = w_r(r)\int_r^{2R}\frac{1}{\rho^{n-1}w_r(\rho)^2}\int_\rho^\infty \tilde h_j(s)\,w_r(s)s^{n-1}\,ds$$

If we consider a function $h^j = h_j(r,\tau)\vartheta_j$ defined in B_{2R} with $\|h^j\|_{2+a,\nu} < +\infty$ and $\int_{B_{2R}} h^j Z_j = 0$ for all τ, then $H_j = L_0^{-1}[h^j(\cdot,\tau)]$ satisfies the estimate $\|H_j\|_{\nu,a} \lesssim \|h_j\|_{\nu,2+a}$. Let us consider the boundary value problem in B_{3R}

$$\Phi_\tau = \Delta\Phi + pU(y)^{p-1}\Phi + H_j(r)\vartheta_j(y) \quad \text{in } B_{3R}\times(\tau_0,\infty) \tag{9.42}$$

$$\Phi = 0 \quad \text{on } \partial B_{3R}\times(\tau_0,\infty), \quad \Phi(\cdot,\tau_0) = 0 \quad \text{in } B_{3R}.$$

As consequence of Step 1, we find a solution $\Phi_j[h]$ to this problem, which defines a linear operator of h_j and satisfies the estimates

$$|\Phi_j(y,\tau)| \lesssim \frac{\tau^{-\nu}R^{3-a}}{1+|y|^2}R^1\|h_j\|_{\nu,2+a}, \tag{9.43}$$

Arguing by scaling and parabolic estimates, we find as in the construction for mode 0,

$$|L[\Phi_j](\cdot,\tau)| \lesssim \tau^{-\nu}\|h\|_{\nu,2+a}\frac{R^{4-a}}{1+|y|^4} \quad \text{in } B_{2R}.$$

We define $\phi_j[h_j] := L[\Phi_j]\Big|_{B_{2R}}$. and $\phi^1[h^1] := \sum_{j=1}^3 \phi_j[h_j]\vartheta_j$. This function solves (9.7) for $h = h^1$ and satisfies

$$|\phi^1[h^1](y,\tau)| \lesssim \tau^{-\nu}\|h_j\|_{2+a,\nu}\frac{R^{4-a}}{1+|y|^4} \quad \text{in } B_{2R}. \tag{9.44}$$

Construction at higher modes. In order to deal with the higher modes, for $h = h^\perp = \sum_{j=4}^\infty h_j(r)\Theta_j$ we let $\phi^\perp[h^\perp]$ be just the unique solution of the problem

$$\phi_\tau = \Delta\phi + pU(y)^{p-1}\phi + h^\perp \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{9.45}$$
$$\phi = 0 \quad \text{on} \quad \partial B_{2R}\times(\tau_0,\infty), \quad \phi(\cdot,\tau_0) = 0 \quad \text{in } B_{2R},$$

which is estimated as

$$|\phi^\perp[h^\perp](y,\tau)| \lesssim \tau^{-\nu}\frac{\|h^\perp\|_{\nu,2+a}}{1+|y|^a} \quad \text{in } B_{2R}. \tag{9.46}$$

We just let

$$\phi[h] := \phi^0[h^0] + \phi^1[h^1] + \phi^\perp[h^\perp]$$

be the functions constructed above. According to estimates (9.40) and (9.46) we find that this function solves Problem (9.7) for $c(\tau)$ given by (9.17), with bounds (9.4), (9.5), (9.8) as required. The proof is concluded.

□

10. Non radially symmetric case

In this section, we discuss the existence of solutions for Problem (2.1) when the initial condition is not radially symmetric, and we discuss the co-dimension 1 stability. Let $\bar v_0$ be a positive, uniformly bounded smooth function, not radially symmetric and define

$$v_0(x) = \frac{\bar v_0(x)}{|x|^\kappa}, \quad \text{with} \quad \kappa > \max\{\frac{\gamma+3}{2},\gamma\}. \tag{10.1}$$

We construct a solution to the initial value Problem

$$\begin{cases} u_t = \Delta u + u^5, & \text{in} \quad \mathbb{R}^3\times(t_0,\infty), \\ u(x,t_0) = u_0(|x|) + v_0(x) \end{cases} \tag{10.2}$$

where u_0 is radial and satisfies the decay condition (2.2), while v_0 is a non radial function of the form (10.1).

Since the strategy of the proof is similar to the one already performed in details for $\bar v_0(x)\equiv 0$, we shall indicate the changes in the argument that are required when the initial condition is not radially symmetric.

We start with a slightly different first approximation. Let $p = p(t) : [t_0,\infty) \to \mathbb{R}^3$ be a smooth function so that

$$p(t_0) = \mathbf{0}, \quad p(t) = \int_{t_0}^t P(s)\,ds, \quad \text{where} \quad P \quad \text{satisfies}$$

$$\|P\|_\diamond := \sup_{t>t_0} \mu_0(t)^{-\frac12}t^{\kappa-1}\left[\|P(s)\|_{\infty,[t,t+1]} + [P]_{0,\sigma,[t,t+1]}\right] \le \ell, \tag{10.3}$$

with σ the number fixed in (2.11), and ℓ a positive fixed number. Observe that, under these assumptions, and the bound on κ in (10.1), we have $\frac{|p(t)|}{\mu_0(t)} \to 0$ as $t \to \infty$. Define

$$U[\lambda, P](x,t) = \hat{U}_2(x,t) + U_3(x,t), \quad \hat{U}_2(x,t) := U_2(|x-p(t)|, t), \tag{10.4}$$

where U_2 is given by (2.57) and

$$U_3(x,t) = \left(1 - \eta(\frac{|x|}{t})\right) v_0(x). \tag{10.5}$$

If we call $\mathcal{E}[\lambda, P](x,t) := \Delta U + U^5 - U_t$, we can write

$$\begin{aligned} \mathcal{E}[\lambda, P](x,t) &= \mathcal{E}_2[\lambda](|x-p|, t) - \nabla U_2(|x-p|, t) \cdot \dot{p}(t) \\ &+ \underbrace{\Delta U_3 - \frac{\partial U_3}{\partial t} + (\hat{U}_2 + U_3)^5 - (\hat{U}_2)^5}_{:=\mathcal{E}_3}. \end{aligned}$$

Define

$$\begin{aligned} \bar{\mathcal{E}}(x,t) &= \mathcal{E}_{21}(|x-p|, t) \\ &+ \left(1 - \eta_R(\frac{|x|}{R\mu_0})\right) [\mathcal{E}_{22}(|x-p|, t) - \nabla U_2(|x-p|, t) \cdot \dot{p}(t)] \end{aligned} \tag{10.6}$$

where η_R is defined in (2.59). We have that

$$\left|\bar{\mathcal{E}}(x,t)\right| \le C\mu_0^{\frac12} t^{-\frac32} h_0(\frac{|x|}{\sqrt{t}}), \quad |\mathcal{E}_3(x,t)| \le C t^{-\kappa + \frac12} h_0(\frac{|x|}{\sqrt{t}}). \tag{10.7}$$

A solution to (10.2) does exist and has the form

$$u = U[\lambda, P](r,t) + \tilde{\phi}, \quad t > t_0 \tag{10.8}$$

where U is defined in (10.4), while $\tilde{\phi}(x,t)$ is given as in (3.2)

$$\tilde{\phi}(x,t) = \psi(x,t) + \phi^{in}(x,t) \quad \text{where} \quad \phi^{in}(x,t) := \eta_R(x,t)\hat{\phi}(x,t)$$

and $\hat{\phi}(x,t) := \mu_0^{-\frac12} \phi\left(\frac{x}{\mu_0}, t\right)$. For any $\psi_0 \in C^2(\mathbb{R}^3)$ so that

$$|y|\, |\psi_0(y)| + |y|\, |\nabla \psi_0(y)| \le t_0^{-a} e^{-b|y|}, \tag{10.9}$$

for some positive constants a and b, the function ψ is the solution to

$$\begin{aligned} \partial_t \psi &= \Delta \psi + V\psi + [2\nabla \eta_R \nabla_x \hat{\phi} + \hat{\phi}(\Delta_x - \partial_t)\eta_R] \\ &+ N[\lambda](\tilde{\phi}) + \bar{\mathcal{E}}_+ \mathcal{E}_3 \quad \text{in } \mathbb{R}^3 \times [t_0, \infty), \\ \psi(x, t_0) &= \psi_0, \end{aligned} \tag{10.10}$$

where V is defined as in (3.5) with U instead of U_2, and $N(\tilde{\phi}) = (U + \tilde{\phi})^5 - U^5 - 5\,U^4\,\tilde{\phi}$. This solution ψ can be described as follows

$$\psi(x,t) = \psi_r(x,t) + \psi_{nr}(x,t), \tag{10.11}$$

where ψ_r is a radial function in $|x - p(t)|$, for any t, and

$$|\psi_r(x,t)| \le C\mu_0^{\frac12} t^{-\frac12}\, \varphi_0(\frac{|x|}{\sqrt{t}}), \quad |\psi_{nr}(x,t)| \le C t^{-\eta + \frac32}\, \varphi_0(\frac{|x|}{\sqrt{t}}). \tag{10.12}$$

We refer to (4.2) for the definition of φ_0.

On the other hand, the function $\hat{\phi}$ satisfies

$$\partial_t \hat{\phi} = \Delta \hat{\phi} + 5w_\mu^4 \hat{\phi} + 5w_\mu^4 \psi + \mathcal{E}_{22}(|x-p(t)|, t) - \nabla U_2(|x-p(t)|, t) \cdot \dot{p}(t) \quad \text{in } B_{2R\mu_0}(0) \times [t_0, \infty),$$

with $\hat\phi(x,t_0)=\mu_0^{-\frac12}(t_0)e_0Z(\frac{x}{\mu_0(t_0)})$. In terms of ϕ, this equation becomes

$$\begin{aligned}\mu_0^2\partial_t\phi =&\Delta_y\phi+5w^4\phi+f(y,t)\quad \text{in } B_{2R}(0)\times[t_0,\infty)\\ \phi(y,t_0)&=e_0Z(y)\end{aligned}\tag{10.13}$$

where

$$\begin{aligned}f(y,t)&=\mu_0^{\frac52}\mathcal{E}_{22}(|\mu_0y-p(t)|,t)-\nabla U_2(|\mu_0y-p(t)|,t)\,\cdot\,\dot p(t)\\ &+5\,\frac{\mu_0^{\frac12}}{(1+\Lambda)^4}\,w^4(\frac{y}{(1+\Lambda)^2})\psi(\mu_0y,t)+B[\phi]+B^0[\phi].\end{aligned}$$

In the above expression, ψ is the solution to (10.10), while B and B^0 are defined respectively in (3.9) and (3.10). The solution ϕ exists in the class of functions with $\|\cdot\|_{\nu,a}$-norm bounded (see (4.1)), as consequence of Proposition 9.1, and a contraction type argument, provided the parameter functions λ and P can be chosen so that

$$\int_{B_R}f(y,t)Z_j(y)\,dy=0,\quad \text{for all}\quad t>t_0,\quad j=0,1,\dots,n.\tag{10.14}$$

The system of $(n+1)$ non linear, non local equations in λ and P is solvable for λ and P satisfying (2.11) and (10.3). Indeed, equation (10.14), for $j=0$, can be treated as we did for equation (5.1) in Sections 5, 6, 7. On the other hand, when $j=1,\dots,n$, equations (10.14) are perturbations of

$$\dot p(t)=\mu_0^{\frac12}t^{-\kappa+1}\bar u$$

for some fixed vector $\bar u\in\mathbb{R}^3$. Thus it can be solved for parameters $p(t)=\int_{t_0}^t P(s)\,ds$ satisfying (10.3). This concludes the proof of existence of a positive global solution to (10.2).

Next we discuss the co-dimension 1 stability. Let us observe that the construction of ϕ, and e_0 solution to (10.13) is possible for any initial condition ψ_0 to the outer Problem (10.10). We have the validity of Lipschitz dependence of $\phi=\phi[\psi_0]$, and $e_0=e_0[\psi_0]$ in the C^1-topology described in (10.9). As a consequence of the Implicit Function Theorem the maps $\phi[\psi_0]$, and $e_0[\psi_0]$ depends in C^1-sense on ψ_0 in our C^1-topology (10.9), thanks to the corresponding dependence for ψ, λ and p.

Let us consider the following map defined in a small neighborhood of 0 in $X=C^1(\bar\Omega)$.

$$F(\psi_0)=\psi_0-(e_0[\psi_0]-e_0[t_0])Z_0$$

so that $F[0]=0$, F is differentiable and

$$D_{\psi_0}F(0)[h]=h-\langle D_{\psi_0}e_0[0],h\rangle Z_0,\quad h\in X.$$

We have a solution which blows-up as $t\to+\infty$ provided that

$$u(\cdot,t_0)=u^*(\cdot,t_0)-e_0[0]Z_0+g\tag{10.15}$$

where u^* is the solution corresponding to $\psi_0=0$, and $g=F[\psi_0]$ for any small ψ_0.

The vector space of the functionals in X given by $D_{\psi_0}e_0[0]$ has dimension 1. We write $W:=\operatorname{Ker}(D_{\psi_0}e_0[0])$ is a space with codimension 1. Indeed, we can find a non zero function u such that

$$X=W\oplus<u>.$$

We consider the operator in a neighborhood of 0 in X given by

$$G\big(w+\alpha u\big)\;=\;\alpha u+F(w),\quad \alpha_j\in\mathbb{R},\quad w\in W.$$

Then G is of class C^1 near the origin, $G(0)=0$ and $D_{\psi_0}G(0)[h]=h$. By the local inverse theorem, G defines a local C^1 diffeormorphism onto a neighborhood of the origin. For all small g we can find smooth functions $\alpha(g)$, $w(g)$ with

$$\alpha(g)u+F(w(g))=g.$$

Thus the set $\mathcal{M}$ of functions $F[w]$, $w\in W$ can be described in a neighborhood of 0 exactly as those $g\in X$ such that

$$\alpha(g)=0.$$

This says precisely that $\mathcal{M}$ is locally a codimension 1 C^1-manifold, such that if g in (10.15) is selected there, then the desired phenomenon takes place. The proof is concluded. □

11. Appendix A

Proof of Lemma 2.2. We denote by $y_2(s)$ the solution to (2.17) with $\lim_{s\to\infty} s^{2\nu} y_2(s) = 1$, and by $y_1(s)$ another solution, linearly independent from y_2, defined explicitly by

$$y_1(s) = c\, y_2(s) \int_s^\infty \frac{e^{-\frac{z^2}{4}}}{y_2(z)^2\, z^2}\, dz, \tag{11.1}$$

for some positive constant c we fix later. The function $y_1(s)$ decays fast at infinity, since $y_1(s) = c_1 e^{-\frac{s^2}{4}}\, s^{4\nu-3}\left(1+o(s^{-1})\right)$, as $s\to\infty$, for some positive constant c_1, as a direct consequence from (11.1). The function $y_2(s)$ is definite for any $s\in(0,\infty)$, and it is positive. Indeed, we first observe that the operator L_ν satisfies the maximum principe. This is consequence of the fact that the positive function $g_0(s) = \frac{e^{-\frac{s^2}{4}}}{s}$, which solves $L_1(g_0)=0$, satisfies $L_\nu(g_0)<0$ in $(0,\infty)$. With this is mind, we define $\bar g_0(s) = \int_s^\infty \frac{e^{-\frac{z^2}{4}}}{z^2}\, dz$. This is a positive function, which satisfies $L_\nu(\bar g_0) = \nu \bar g_0 > 0$ in $(0,\infty)$. Thus $\bar g_0$ is a sub solution. Moreover, it is easy to see that $\bar g_0(R) < y_2(R)$ for any R large enough. A standard application of the maximum principle thus gives that y_2 is positive in $(0,\infty)$.

We now claim that $\lim_{s\to 0^+} s\, y_1(s)$ exists and it is positive. Write $y_1(s) = \phi(\frac{s^2}{4})$, $x = \frac{s^2}{4}$, from which we get that

$$x\phi'' + (\frac{3}{2}+x)\phi' + \nu\phi = 0, \quad x\in(0,\infty).$$

Performing the further change of variables $\phi(x) = e^{-x}\varphi(x)$, we get that φ satisfies

$$x\varphi'' + (\frac{3}{2}-x)\varphi' - (\frac{3}{2}-\nu)\varphi = 0, \quad x\in(0,\infty). \tag{11.2}$$

In [17], Appendix A, it is proven that (11.2) admits polynomial solutions if and only if $\frac{3}{2}-\nu = -k$, $k=0,1,2,\ldots$. Since $\frac{1}{2}<\nu<1$, this never happens, thus φ can not be bounded, as $x\to 0^+$. On the other hand, the behavior of the solutions to (11.2), as $x\to 0^+$, are determined by $x\varphi'' + \frac{3}{2}\varphi' = 0$, which implies that the solutions to (11.2) are bounded around $x=0$, or they behave like $x^{-\frac{1}{2}}$ as $x\to 0^+$. Combining all the above information, we showed that, for a proper choice of the constant c in (11.1), we get that

$$y_1(s) = \frac{1}{s}(1+o(1)), \quad \text{as} \quad s\to 0.$$

To understand further the behavior of y_1 around $s=0$, we write $sy_1(s) = f(s)$, so that

$$f'' + \frac{s}{2}f' + (\nu - \frac{1}{2})f = 0, \quad s\in(0,\infty). \tag{11.3}$$

Integrating (11.3) between 0 and ∞, and using the fast decay of y_1 to 0 as $s\to\infty$, we compute

$$f'(0) = (\nu-1)\int_0^\infty f(s)\, ds < 0, \quad f''(0) = \frac{1}{2}-\nu. \tag{11.4}$$

With this information, we get the estimates (2.18) and (2.20) for $y_1(s)$.

Since the Wronskian associated to Problem (2.17) is given by a multiple of $\frac{e^{-\frac{s^2}{4}}}{s^2}$, we conclude that, since y_1 is unbounded as $s\to 0^+$, we have that $y_2(s)$ is bounded, as $s\to 0^+$. This concludes the proof of the Lemma.

□

Lemma 11.1. *Let $h=h(s)$ be a smooth function defined for $s\geq 0$ so that*

$$h(s) = \begin{cases} \frac{1}{s} & \text{for} \quad s\to 0 \\ \frac{1}{s^3} & \text{for} \quad s\to\infty \end{cases}$$

Then there exists a solution to

$$\partial_t \psi = \Delta \psi + t^{-\beta} h(\frac{r}{\sqrt{t}}), \tag{11.5}$$

of the form

$$\psi(r,t) = t^{-\beta+1} \varphi(\frac{r}{\sqrt{t}}), \quad \text{with} \quad \varphi(s) = \begin{cases} s & \text{for} \quad s \to 0 \\ \frac{1}{s^3} & \text{for} \quad s \to \infty \end{cases}. \tag{11.6}$$

Proof. We look for a solution to (11.5) of the form $\psi(r,t) = t^{-(\beta-1)} \varphi(\frac{r}{\sqrt{t}})$. Thus φ satisfies

$$\varphi'' + \left(\frac{2}{s} + \frac{s}{2} \right) \varphi' + (\beta - 1)\varphi + h(s) = 0.$$

We look for a solution of the above equation of the form

$$\varphi(s) = z(s)\, y_1(s)$$

where y_1 solves $y_1'' + \left(\frac{2}{2} + \frac{s}{2}\right) y_1' + (\beta - 1) y_1 = 0$, and $y_1(s) \sim \begin{cases} \frac{1}{s} & \text{as} \quad s \to 0 \\ e^{-\frac{s^2}{4}} s^{4(\beta-1)-3} & \text{as} \quad s \to \infty \end{cases}$. The existence of y_1 is consequence of Lemma 2.2. A direct computation gives

$$z(s) = -\int_0^s \frac{e^{-\frac{\eta^2}{4}}}{y_1(\eta)^2 \eta^2} \left(\int_0^\eta h(x) y_1(x)\, x^2\, e^{\frac{x^2}{4}}\, dx \right) d\eta.$$

One can easily see that

$$z(s) \sim \begin{cases} s^2 & \text{as} \quad s \to 0 \\ e^{\frac{s^2}{4}} s^{-4(\beta-1)} & \text{as} \quad s \to \infty \end{cases}.$$

This fact gives (11.6), and concludes the proof of the Lemma. $\square$

Proof of (5.10). For $x \in B_{2R}$, we shall prove

$$\phi_0(\mu_0 x, t) - \phi_0(0,t) = \alpha(t) |\mu_0 x|^\sigma\, \Pi(t)\, \Theta(|x|), \tag{11.7}$$

for some $\sigma \in (0,1)$. Here $\Pi = \Pi(t)$ denotes a smooth and bounded function of t, and Θ a smooth and bounded function of x.

We have

$$\begin{aligned} \phi_0(\mu_0 x, t) - \phi_0(0,t) &= \int_{t_0}^t \frac{1}{(4\pi(t-s))^{\frac{3}{2}}} \int_{\mathbb{R}^3} \left[e^{-\frac{|x-y|^2}{4(t-s)}} - e^{\frac{-|y|^2}{4(t-s)}} \right] \frac{\alpha(s)}{|y|} \mathbf{1}_{\{r<M\}}\, dy\, ds \\ &= \frac{1}{2} \int_{t_0}^t \int \frac{\beta'(s)}{(t-s)^{\frac{1}{2}}} \left[e^{-|z-\frac{\mu_0 x}{2\sqrt{t-s}}|^2} - e^{-|z|^2} \right] \frac{1}{|z|} \mathbf{1}_{\{|z|<\frac{M}{2\sqrt{t-s}}\}}\, dy\, ds \\ &= I + II \end{aligned}$$

where

$$I = \int_{t_0}^{t-(\frac{\mu_0 x}{2m})} \int \frac{\beta'(s)}{(t-s)^{\frac{1}{2}}} \left[e^{-|z-\frac{\mu_0 x}{2\sqrt{t-s}}|^2} - e^{-|z|^2} \right] \frac{1}{|z|} \mathbf{1}_{\{|z|<\frac{M}{2\sqrt{t-s}}\}}\, dy\, ds.$$

We start estimating II. We observe that, if $t - (\frac{\mu_0 x}{2m}) < s < t$, then $\frac{\mu_0 |x|}{2\sqrt{t-s}} > m$. We write

$$II = II_1 + II_2 + II_3$$

where

$$II_j = \int_{t-(\frac{\mu_0 x}{2m})}^t \int_{D_j} \frac{\beta'(s)}{(t-s)^{\frac{1}{2}}} \left[e^{-|z-\frac{\mu_0 x}{2\sqrt{t-s}}|^2} - e^{-|z|^2} \right] \frac{1}{|z|} \mathbf{1}_{\{|z|<\frac{M}{2\sqrt{t-s}}\}}\, dy\, ds.$$

with

$$D_1 = \{z \,:\, |z - \frac{\mu_0 x}{2\sqrt{t-s}}| < \frac{1}{4} \frac{\mu_0 |y|}{2\sqrt{t-s}}\}, \quad D_2 = \{z \,:\, |z| < \frac{1}{4} \frac{\mu_0 |y|}{2\sqrt{t-s}}\}$$

and D_3 the complement of the two above regions.

We start estimating II_1. We see that

$$\int_{D_1} e^{-|z-\frac{\mu_0 x}{2\sqrt{t-s}}|^2}\,\frac{1}{|z|}\,\mathbf{1}_{\{|z|<\frac{M}{2\sqrt{t-s}}\}}\,dy=\int e^{-|\bar z|}\frac{1}{|\bar z+\frac{\mu_0 x}{2\sqrt{t-s}}|}\,d\bar z=c\frac{2\sqrt{t-s}}{\mu_0|x|},$$

for some constant c, as a direct application of Dominated Convergence Theorem. Thus

$$\int_{t-(\frac{\mu_0 x}{2m})}^{t}\int_{D_1} e^{-|z-\frac{\mu_0 x}{2\sqrt{t-s}}|^2}\,\frac{1}{|z|}\,\mathbf{1}_{\{|z|<\frac{M}{2\sqrt{t-s}}\}}\,dy\,ds=\frac{2c}{\mu_0|x|}\int_{t-(\frac{\mu_0 x}{2m})}^{t}\sqrt{t-s}ds=c'(\mu_0|x|)^{\frac12}.$$

On the other hand, for any z in D_1, one has $|z|>\frac14\frac{\mu_0|x|}{2\sqrt{t-s}}$, and hence we can bound

$$\left|\int_{D_1} e^{-|z|^2}\frac{1}{|z|}\,dz\right|\le c\left[\frac{\sqrt{t-s}}{\mu_0|x|}\right]^{\sigma},$$

for any $\sigma>0$. We take $\sigma>1$, so that

$$|\int_{t_0}^{t-(\frac{\mu_0 x}{2m})}\int_{D_1} e^{-|z|^2}\,\frac{1}{|z|}\,\mathbf{1}_{\{|z|<\frac{M}{2\sqrt{t-s}\}}}\,dy\,ds|\le\frac{1}{(\mu_0|x|)^{\sigma}}\left|\int_{t_0}^{t-(\frac{\mu_0 x}{2m})}(t-s)^{\frac{\sigma}{2}-\frac12}\,ds\right|\le c'\mu_0|x|$$

Thus we conclude that

$$|II_1|\lesssim\beta'(t)(\mu_0|x|)^{\frac12}.$$

Arguing in a similar way, one finds the same type of estimate for II_2. In the third region D_3, we have that

$$|z|>\frac14\frac{\mu_0|x|}{2\sqrt{t-2}},\quad |z-\frac{\mu_0 x}{2\sqrt{t-s}}|>\frac14\frac{\mu_0|x|}{2\sqrt{t-s}},$$

so that again one gets the estimate

$$|II_3|\lesssim\beta'(t)\mu_0|x|.$$

Let us now consider the interval of time $t_0<s<t-\left(\frac{\mu_0|x|}{2m\sqrt{t-s}}\right)^2$, region where one has $\frac{\mu_0|x|}{2\sqrt{t-s}}<m$. We decompose

$$I=III+IV$$

where

$$III=\int_{t_0}^{t-1}\int\frac{\beta'(s)}{(t-s)^{\frac12}}\,\left[e^{-|z-\frac{\mu_0 x}{2\sqrt{t-s}}|^2}-e^{-|z|^2}\right]\,\frac{1}{|z|}\,\mathbf{1}_{\{|z|<\frac{M}{2\sqrt{t-s}\}}}\,dy\,ds$$

We start with IV, where we expand in Taylor

$$\begin{aligned}IV&=\int_{t-1}^{t-\left(\frac{\mu_0|x|}{2m\sqrt{t-s}}\right)^2}\int\frac{\beta'(s)}{(t-s)^{\frac12}}\,\left[e^{-|z-\frac{\mu_0 x}{2\sqrt{t-s}}|^2}-e^{-|z|^2}\right]\,\frac{1}{|z|}\,\mathbf{1}_{\{|z|<\frac{M}{2\sqrt{t-s}\}}}\,dy\,ds\\&=\beta'(t)\int_{t-1}^{t-\left(\frac{\mu_0|x|}{2m\sqrt{t-s}}\right)^2}\frac{\mu_0|x|}{t-s}\left(\int\frac{e^{-|z|^2}}{|z|}\,dz\right)\,ds=\beta'(t)\log\left(\frac{t-\left(\frac{\mu_0|x|}{2m\sqrt{t-s}}\right)^2}{t}\right)\mu_0|x|\\&=\beta'(t)\mu_0|x|[\log(\mu_0|x|)]=\beta'(t)(\mu_0|x|)^{\sigma},\end{aligned}$$

for some positive $\sigma<1$. Finally, we consider III. Again, after a Taylor expansion, we have

$$III=\mu_0|x|\int_{t_0}^{t-1}\frac{\beta'(s)}{(t-s)}\,ds=\mu_0|x|\int_{t_0}^{t-1}\frac{\beta'(s)}{t-s}\,ds.$$

Collecting the previous estimates, we conclude with the validity of (11.7). $\square$

12. Appendix B

Proof of Lemma 2.3. Throughout the proof of the Lemma, we denote by $q_i = q_i(s)$, for any interegr i, a smooth real function, with the property that $\frac{d}{(ds)^j} q_i(0) = 0$, for $j < i$, and $\frac{d}{(ds)^i} q_i(0) \neq 0$. With $\Theta = \Theta(r)$ we intend a smooth function of the space variable, which is uniformly bounded. Also, $\Pi = \Pi(t)$ stands for a smooth function of the time variable, which is uniformly bounded in $t \in (0, \infty)$. The explicit expressions of these functions change from line to line, and also within the same line.

Let $R_0 = r_0\sqrt{t}$. A simple computation gives the explicit expression of the error $\mathcal{E}_1$ in (2.40)

$$
\begin{aligned}
\mathcal{E}_1(r,t) &= \mathcal{E}^1_{\text{in}} \eta(\frac{r}{R_0}) + \mathcal{E}^1_{\text{out}} \left(1 - \eta(\frac{r}{R_0})\right) \\
&+ \underbrace{R_0^{-2} \left(u_{\text{in}} - u_{\text{out}}\right) \Delta\eta(\frac{r}{R_0}) + 2R_0^{-1} \nabla \left(u_{\text{in}} - u_{\text{out}}\right) \cdot \nabla\eta(\frac{r}{R_0})}_{:=\bar{\mathcal{E}}_1} \\
&+ \underbrace{\left(u_{\text{in}} - u_{\text{out}}\right) \frac{R_0'}{R_0^2} \eta'(\frac{r}{R_0})}_{:=\hat{\mathcal{E}}_1}
\end{aligned}
\tag{12.1}
$$

where

$$
\mathcal{E}^1_{\text{in}} = \Delta u_{\text{in}} + u^5_{\text{in}} - \partial_t u_{\text{in}}, \quad \text{and} \quad \mathcal{E}^1_{\text{out}} = \Delta u_{\text{out}} + u^5_{\text{out}} - \partial_t u_{\text{out}}. \tag{12.2}
$$

We start analyzing $\mathcal{E}^1_{\text{in}}$, getting

$$
\begin{aligned}
\mathcal{E}^1_{\text{in}}(r,t) &= \mu_0' \left[\Delta\psi_1 + 5w_\mu^4 \psi_1\right] - \mu' \frac{\partial w_\mu}{\partial \mu} \\
&+ (w_\mu + \mu_0'\psi_1)^5 - w_\mu^4 - 5w_\mu^4 \mu_0' \psi_1 - \mu_0''\psi_1 - \mu_0'\mu' \frac{\partial \psi_1}{\partial \mu} \\
&= (\mu' - \mu_0')\, \mu^{-\frac{3}{2}} Z_0(\frac{r}{\mu}) + \left[(w_\mu + \mu_0'\psi_1)^5 - w_\mu^5 - 5w_\mu^4 \mu_0'\psi_1\right] \\
&- \mu_0''\psi_1 - \mu_0'\mu' \frac{\partial \psi_1}{\partial \mu}.
\end{aligned}
\tag{12.3}
$$

Now we write

$$
\begin{aligned}
(\mu' - \mu_0')\, \mu^{-\frac{3}{2}} Z_0(\frac{r}{\mu}) &= \left[2(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})' + (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})\mu_0^{-1}\mu_0'\right] \mu^{-1} Z_0(\frac{r}{\mu}) \\
&- \frac{(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})^2}{\mu^{\frac{1}{2}}} \mu_0^{-1}\mu_0'\, \mu^{-1} Z_0(\frac{r}{\mu}).
\end{aligned}
$$

Taking into account that $Z_0(s) = \frac{3^{\frac{1}{4}}}{2} \frac{1}{s} + O(\frac{1}{s^3})$, as $s \to \infty$, it is convenient to write

$$
\begin{aligned}
\left[2(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})' + (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})\mu_0^{-1}\mu_0'\right] \quad \mu^{-1} Z_0(\frac{r}{\mu}) &= \frac{\alpha(t)}{\mu + r} \\
&+ \left[2(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})' + (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})\mu_0^{-1}\mu_0'\right] \mu^{-1} \left[Z_0(\frac{r}{\mu}) - \frac{3^{\frac{1}{4}}}{2} \frac{\mu}{\mu + r}\right],
\end{aligned}
$$

where α is defined in (2.41). We decompose (12.3) as

$$
\mathcal{E}^1_{\text{in}}(r,t) = \frac{\alpha(t)}{\mu + r} + \bar{\mathcal{E}}^1_{\text{in}}(r,t), \tag{12.4}
$$

where $\bar{\mathcal{E}}^1_{\rm in}$ is explicitly given by

$$\begin{aligned}\bar{\mathcal{E}}^1_{\rm in}(r,t) &= -\frac{(\mu^{\frac12}-\mu_0^{\frac12})^2}{\mu^{\frac12}}\mu_0^{-1}\mu_0'\,\mu^{-1}Z_0(\frac{r}{\mu}) - \mu_0''\psi_1 + \left[(w_\mu+\mu_0'\psi_1)^5 - w_\mu^5 - 5w_\mu^4\mu_0'\psi_1\right]\\ &+ \left[2(\mu^{\frac12}-\mu_0^{\frac12})' + (\mu^{\frac12}-\mu_0^{\frac12})\mu_0^{-1}\mu_0'\right]\mu^{-1}\left[Z_0(\frac{r}{\mu}) - \frac{3^{\frac14}}{2}\frac{\mu}{\mu+r}\right] - \mu_0'\mu'\frac{\partial\psi_1}{\partial\mu}\\ &= \sum_{j=1}^5 e_j. \end{aligned}\tag{12.5}$$

We observe now that $(e_1+e_2+e_3)\eta(\frac{r}{R_0})$ can be described as sum of functions of the form

$$\frac{\mu_0^{\frac12}t^{-2}R_0^2}{\mu_0+r}\,q_0(\Lambda)\,\Pi(t)\,\Theta(r),\qquad \frac{\mu_0^{-\frac12}t^{-1}}{\mu_0+r}\,q_2(\Lambda)\,\Pi(t)\,\Theta(r),\tag{12.6}$$

where q_0 is a smooth function with $q(0)\neq 0$, while q_2 is a smooth function with $q_2(0)=q_2'(0)=0$, and $q_2''(0)\neq 0$. On the other hand, we see that

$$e_4 = \frac{\alpha(t)\mu_0^2}{\mu_0^3+r^3}\,\Pi(t)\,\Theta(r),\tag{12.7}$$

and e_5

$$\frac{\mu_0^{\frac12}t^{-1}}{\mu_0+r}\left[R_0^2\Lambda' + R_0^2t^{-1}q_1(\Lambda)\right]\,\Pi(t)\,\Theta(r),\tag{12.8}$$

where q_1 is a smooth function with $q_1(0)=0$, $q_1'(0)\neq 0$. Under assumption (2.11) and combining (12.4)-(12.6)-(12.7)-(12.8), we find that

$$\left|\bar{\mathcal{E}}^1_{\rm in}\eta\right|_{\infty,B(x,1)\times[t,t+1]} \lesssim \mu_0^{\frac12}t^{-\frac32}h_0(\frac{r}{\sqrt t}),\quad r=|x|.$$

Since (2.41), we observe that

$$\left|\frac{\alpha(t)}{\mu+r}\left(1-\eta(\frac{r}{R_0})\right)\right| \lesssim \mu_0^{\frac32}t^{-\frac32}h_0(\frac{r}{\sqrt t}),\quad r=|x|.$$

Let us fix λ_1 and λ_2 satisfying (2.11). We write, for some $\bar\lambda = s\lambda_1+(1-s)\lambda_2$, $s\in(0,1)$,

$$\left(\bar{\mathcal{E}}^1_{\rm in}[\lambda_1]-\bar{\mathcal{E}}^1_{\rm in}[\lambda_2]\right)\eta(\frac{r}{R_0}) = \left(D_\lambda\bar{\mathcal{E}}^1_{\rm in}[\bar\lambda][\lambda_1-\lambda_2]\right)\eta(\frac{r}{R_0}),\quad \text{with}\quad D_\lambda\bar{\mathcal{E}}^1_{\rm in}[\bar\lambda] = \sum_{j=1}^5 (D_\lambda e_j)[\bar\lambda],$$

where the e_j are defined in (12.5). Let us consider e_1. We have that

$$(D_\lambda e_1)[\bar\lambda] = 2\mu_0(1+\Lambda)D_\mu(e_1)[\bar\lambda].$$

Direct computation give that

$$\left|D_\mu(e_1)[\bar\lambda](r,t)\right| \lesssim \frac{\mu_0^{-\frac12}t^{-1}}{\mu_0+r}q_0(\bar\lambda)\,\Pi(t)\,\Theta(r).$$

We combine the above estimates to get

$$\begin{aligned}|e_1[\lambda_1]-e_1[\lambda_2]|\,\eta(\frac{r}{R_0}) &\le \mu_0\frac{\mu_0^{-\frac12}t^{-1}}{\mu_0+r}\,|\lambda_1-\lambda_2|\eta(\frac{r}{R_0})\\ &\le C\left(\mu_0(t)t^{-1}\right)\,\mu_0^{\frac32}(t)t^{-\frac32}h_0(\frac{r}{\sqrt t})\|\lambda_1-\lambda_2\|_\sharp\\ &\le C\left(\mu_0(t_0)t_0^{-1}\right)\,\mu_0^{\frac32}(t)t^{-\frac32}h_0(\frac{r}{\sqrt t})\|\lambda_1-\lambda_2\|_\sharp.\end{aligned}$$

Choosing t_0 large if necessary, we get $C\left(\mu_0(t_0)t_0^{-1}\right) < 1$. Similar estimates can be obtained for the other terms $e_2, \ldots, e_5$. Thus we get

$$\left|\left(\bar{\mathcal{E}}^1_{\text{in}}[\lambda_1] - \bar{\mathcal{E}}^1_{\text{in}}[\lambda_2]\right)\chi\right|_{\infty, B(x,1)\times[t,t+1]} \leq c_1^o \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0(\frac{r}{\sqrt{t}})\,\|\lambda_1 - \lambda_2\|_\sharp,$$

for some constant c_1^o which can be made arbitrarily small, if t_0 is chosen large. Also, we have

$$[\bar{\mathcal{E}}^1_{\text{in}}[\lambda_1] - \bar{\mathcal{E}}^1_{\text{in}}[\lambda_2]]_{0,\sigma,[t,t+1]} \leq c_1^o \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0(\frac{r}{\sqrt{t}})\,[\lambda_1 - \lambda_2]_{0,\sigma,[t,t+1]}.$$

Let us now describe $\mathcal{E}^1_{\text{out}}$. A first observation is that, for any value of γ, we immediately see that $\mathcal{E}^1_{\text{out}}$ does not depend on λ. On the other hand, if $1 < \gamma \leq 2$ the expression for $\mathcal{E}^1_{\text{out}}$ becomes

$$\mathcal{E}^1_{\text{out}}(r,t) = u^5_{\text{out}},$$

so that we directly get

$$\left|\mathcal{E}^1_{\text{out}}\,(1 - \chi(\frac{r}{R_0}))\right| \leq C\frac{\mu_0^{\frac{5}{2}}}{r^5}\,\mathbf{1}_{\{r > R_0^{-1}\}}\,. \tag{12.9}$$

Let us consider now $\gamma > 2$. In this case, the expression of $\mathcal{E}^1_{\text{out}}$ is a bit more involved

$$\begin{aligned}\mathcal{E}^1_{\text{out}}(r,t) &= \eta(\frac{r}{t})(u^1_{\text{out}})^5 + \left(1 - \eta(\frac{r}{t})\right)\,A\,\left[\frac{\gamma(\gamma-1)}{r^{\gamma+2}} + \frac{A^4}{r^{5\gamma}}\right] \\ &\quad + \underbrace{t^{-2}\left(u^1_{\text{out}} - u^2_{\text{out}}\right)\Delta\eta(\frac{r}{t}) + 2t^{-}\nabla\left(u^1_{\text{out}} - u^2_{\text{out}}\right)\cdot\nabla\eta(\frac{r}{t})}_{:=\bar{\mathcal{E}}_1^{out}} \\ &\quad + \underbrace{\left(u^1_{\text{out}} - u^2_{\text{out}}\right)\,t^{-2}\,\eta'(\frac{r}{t})}_{:=\hat{\mathcal{E}}_1^{out}}.\end{aligned} \tag{12.10}$$

A close analysis of each one of the terms appearing in (12.10) gives that

$$\begin{aligned}\left|\mathcal{E}^1_{\text{out}}\,(1 - \eta(\frac{r}{R_0}))\right| \leq C\Bigg\{&\frac{t^{-(\gamma-1)}}{r^3}\,\mathbf{1}_{\{r>t\}} \\ &+ \frac{t^{-2}\mu_0^{\frac{1}{2}}}{r}\,\mathbf{1}_{\{t<r<2t\}} + \frac{t^{-\frac{5}{2}}}{r^5}\,\mathbf{1}_{\{r_0\sqrt{t}<r<t\}}\Bigg\}.\end{aligned} \tag{12.11}$$

From (12.9)-(12.10) and (12.11), we obtain that

$$\left|\mathcal{E}^1_{\text{out}}\left(1 - \chi(\frac{r}{R_0})\right)\right| \lesssim \begin{cases}\mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0(\frac{r}{\sqrt{t}}) & \text{if} \quad 1 < \gamma \leq 2 \\ t^{-2} h_0(\frac{r}{\sqrt{t}}) & \text{if} \quad \gamma > 2.\end{cases}$$

Going back to (12.1), we are left with the description of $\bar{\mathcal{E}}_1 = \bar{\mathcal{E}}_1[\lambda]$ and $\hat{\mathcal{E}}_1[\lambda]$. Directly we check

$$\left|\bar{\mathcal{E}}_1(r,t)\right|,\,\left|\hat{\mathcal{E}}_1(r,t)\right| \leq CR_0^{-2}\,\frac{\mu_0^{\frac{1}{2}}}{r}\,\mathbf{1}_{\{R_0<r<2R_0\}}, \tag{12.12}$$

for some positive constant C. This gives right away

$$\left|\bar{\mathcal{E}}_1 + \hat{\mathcal{E}}_1\right| \lesssim \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0(\frac{r}{\sqrt{t}}).$$

Let us fix λ_1 and λ_2 satisfying (2.11). We write, for some $\bar{\lambda} = s\lambda_1 + (1-s)\lambda_2$, $s \in (0,1)$,

$$\bar{\mathcal{E}}_1[\lambda_1](r,t) - \bar{\mathcal{E}}_1[\lambda_2](r,t) = D_\lambda\bar{\mathcal{E}}_1[\bar{\lambda}][\lambda_1 - \lambda_2](r,t),$$

where

$$D_\lambda\bar{\mathcal{E}}_1[\bar{\lambda}] = R_0^{-2}(\partial_\lambda u_{\text{in}}[\bar{\lambda}])\Delta\eta(\frac{r}{R_0}) + 2R_0^{-1}\nabla\left((\partial_\lambda u_{\text{in}})[\bar{\lambda}]\right)\cdot\nabla\eta(\frac{r}{R_0}).$$

Since in the region we are considering

$$\partial_\lambda u_{\text{in}}[\bar\lambda] = 2\mu_0(1+\Lambda)(\partial_\mu u_{\text{in}})[\bar\lambda], \quad |(\partial_\mu u_{\text{in}})| \le c\frac{\mu_0^{-\frac12}}{r},$$

we have

$$\begin{aligned}\left|\bar{\mathcal{E}}_1[\lambda_1](r,t) - \bar{\mathcal{E}}_1[\lambda_2]\right|_{\infty, B(x,1)\times[t,t+1]} &\le \left(\mu_0(t_0)t_0^{-1}\right)\mu_0^{\frac12} t^{-\frac32} h_0(\frac{r}{\sqrt t})\|\lambda_1-\lambda_2\|_\sharp \\ &\le c_1\mu_0^{\frac12} t^{-\frac32} h_0(\frac{r}{\sqrt t})\|\lambda_1-\lambda_2\|_\sharp,\end{aligned}$$

for some constant $c_1 \in (0,1)$, provided t_0 is large enough. Furthermore, we also have, for any $t > t_0$,

$$[\bar{\mathcal{E}}_1[\lambda_1] - \bar{\mathcal{E}}_1[\lambda_2]]_{0,\sigma,[t,t+1]} \le c_1\mu_0^{\frac12} t^{-\frac32} h_0(\frac{r}{\sqrt t})\left([\lambda_1-\lambda_2]_{0,\sigma,[t,t+1]}\right),$$

with again $c_1 \in (0,1)$. Collecting all the previous estimates, we get the proof of the Lemma. □

Remark 12.1. From the proof of the result, we also get that the constants **c** in (2.50) and (2.51) can be made as small as one needs, provided that the initial time t_0 is chosen large enough.

.

13. Appendix C

Proof of Lemma 2.4. Under the assumptions (2.11) on λ, we get that, for any $r > 0$ and $t > t_0$,

$$|\mathcal{E}_{2,1}(r,t)| + [\mathcal{E}_{2,1}(r,t)]_{0,\sigma,[t,t+1]} \lesssim \mu_0^{\frac12} t^{-\frac32} h_0(\frac{r}{\sqrt t}), \tag{13.1}$$

where h_0 is given by (2.44), and also estimates similar to (2.50) and (2.51) for $\partial_\lambda\mathcal{E}_{2,1}$. These estimates follow from (2.49)-(2.50), (2.41) and from

$$\left|\frac{\alpha(t)}{\mu+r}\left(\eta(\frac{r}{R_0}) - \mathbf{1}_{\{r<2M\}}\right)\right| \le |\alpha(t)| t^{-\frac12} h_0(\frac{r}{\sqrt t}).$$

Here we use again $R_0 = r_0\sqrt t$. Furthermore, in the region where $\eta(\frac{r}{R_0}) - \mathbf{1}_{\{r<2M\}} \ne 0$, the above function is regular enough to have

$$[\frac{\alpha(t)}{\mu+r}\left(\eta(\frac{r}{R_0}) - \mathbf{1}_{\{r<2M\}}\right)]_{0,\sigma,B(x,1)\times[t,t+1]} \le |\alpha(t)| t^{-\frac12} h_0(\frac{r}{\sqrt t}), \quad r = |x|.$$

Using (2.43), we get (13.1). Let us consider now $\mathcal{E}_{22}(1-\eta_R)(r,t)$. We claim that

$$\|\mathcal{E}_{22}(1-\eta_R)(r,t)\|_* \le c_2. \tag{13.2}$$

Given $d > 1$, define $h_*(s) = \begin{cases} \frac1s & \text{for} \quad s\to 0 \\ \frac{1}{s^d} & \text{for} \quad s\to\infty\end{cases}$. Arguing as in the proof of Lemma 11.1, we get the existence of ψ_* so that

$$\partial_t\psi_* = \Delta\psi_* + \mu_0^{\frac12} t^{-\frac32} h_*\left(\frac{r}{\sqrt t}\right), \quad \text{with} \quad \psi_*(r,t) = \mu_0^{\frac12} t^{-\frac12}\varphi_*(\frac{r}{\sqrt t}), \quad \varphi(s) = \begin{cases} s & \text{for} \quad s\to 0 \\ \frac{1}{s^d} & \text{for} \quad s\to\infty\end{cases}.$$

Comparing the above equation and the equation satisfied by ϕ_0, and using the maximum principle, we obtain that, in the region where $(1-\eta_R) \ne 0$,

$$|\phi_0(x,t)| \le \|\lambda\|_\sharp \mu_0^{\frac12} t^{-\frac12}\varphi_*(\frac{r}{\sqrt t}). \tag{13.3}$$

We proceed now with the estimate of $(1-\eta_R)\mathcal{E}_{22}$. A Taylor expansion gives the existence of $s^* \in (0,1)$, so that

$$\mathcal{E}_{22}(r,t) = 5(U_1 + s^*\phi_0)^4\,\phi_0.$$

Let $\bar{M}$ be a large fixed number. From (2.38) and (2.13), we see that, if $r < \bar{M}\sqrt{t}$,

$$|(1-\eta_R)\mathcal{E}_{22}| \lesssim w_\mu^4 \phi_0 \lesssim R^{-2}\mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0(\frac{r}{\sqrt{t}}).$$

On the other hand, thanks to (13.3) we see that, for $r > \bar{M}\sqrt{t}$, we get

$$|(1-\eta_R)\mathcal{E}_{22}| \lesssim (\phi_0)^5 \lesssim \mu_0^{\frac{5}{2}} t^{-\frac{5}{2}} h_0(\frac{r}{\sqrt{t}}).$$

Thus we get the L^∞ bound in estimate (13.2). The control on the Hölder norm contained in (2.61) and (2.62) follows arguing as in the proof of (2.50)-(2.51) in the proof of Lemma 2.3, and from the assumption on λ in (2.11). We leave the details to the reader.

□

References

[1] T. Aubin, Problèmes isopérimetriques et espaces de Sobolev, *J. Differ. Geometry* 11 (1976), 573–598

[2] L.A. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.* 42 (1989), 271–297.

[3] M. Caputo, Linear model of dissipation whose Q is almost frequency independent-II, *Geophys. J. R. Astron. Soc.* 13 (5), (1967), 529-539.

[4] G. Ciraolo, A. Figalli, F. Maggi, A quantitative analysis of metrics on R^n with almost constant positive scalar curvature, with applications to fast diffusion flows *Int. Math. Res. Not. IMRN*, to appear.

[5] C. Collot, F. Merle, P. Raphael, Dynamics near the ground state for the energy critical nonlinear heat equation in large dimensions. *Comm. Math. Phys.* 352 (2017), no. 1, 215- 285

[6] C. Cortázar, M. del Pino, M. Musso, Green's function and infinite-time bubbling in the critical nonlinear heat equation. To appear in JEMS.

[7] P. Daskalopoulos, M. del Pino, N. Sesum, Type II ancient compact solutions to the Yamabe flow, *J. Reine Angew Math.*, to appear.

[8] J. Davila, M. del Pino, J. Wei, Singularity formation for the two-dimensional harmonic map flow into S^2, arXiv:1702.05801.

[9] R. Donninger, J. Krieger, Nonscattering solutions and blow-up at infinity for the critical wave equation. *Math. Ann.*, 357, 1 (2013), 89-163.

[10] T.Duyckaerts, C. E. Kenig, F. Merle, Profiles of bounded radial solutions of the focusing, energy-critical wave equation, *Geometric and Functional Analysis* 22 (2012), no. 3, 639–698.

[11] T.Duyckaerts, C.E. Kenig, F. Merle, Classification of radial solutions of the focusing, energy-critical wave equation, *Cambridge Journal of Mathematics* 1 (2013), no. 1, 75–144.

[12] T.Duyckaerts, C.E. Kenig, F. Merle, Solutions of the focusing nonradial critical wave equation with the compactness property, to appear in *Ann. Sc. Norm. Sup. Pisa*, arXiv:1402.0365.

[13] T.Duyckaerts, C.E. Kenig, F. Merle, Concentration-compactness and universal profiles for the non-radial energy critical wave equation, preprint, arXiv:1510.01750.

[14] J. Krieger, K. Nakanishi, W. Schlag. Center-stable manifold of the ground state in the energy space for the critical wave equation. *Math. Ann.* 361 (2015), no. 1-2, 150.

[15] J. Krieger, W. Schlag, D. Tataru, Slow blow-up solutions for the $H^1(R^3)$ critical focusing semilinear wave equation. *Duke Math. J.* 147 (2009), no. 1, 1-53.

[16] M. Fila, J.R. King, Grow up and slow decay in the critical Sobolev case. *Netw. Heterog. Media* 7 (2012), no. 4, 661–671.

[17] S. Filippas, M. Herrero, J. Velázquez, Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 456 (2000), no. 2004, 2957-2982.

[18] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect.* I, 13 (1966), 109–124.

[19] T-E. Ghoul, N. Masmoudi, Stability of infinite time blow up for the Patlak Keller Segel system, arXiv:1610.00456.

[20] C. Gui, W.-M. Ni, X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in Rn, *Comm. Pure Appl. Math.* 45 (1992), 1153–1181.

[21] H. Matano, F. Merle, Classification of type I and type II behaviors for a supercritical nonlinearheat equation. *J. Funct. Anal.* 256 (2009), no. 4, 992–1064.

[22] H. Matano, F. Merle, On nonexistence of type II blowup for a supercritical nonlinear heat equation. *Comm. Pure Appl. Math.* 57 (2004), no. 11, 1494–1541.

[23] H. Matano, F. Merle, Threshold and generic type I behaviors for a supercritical nonlinear heat equation. *J. Funct. Anal.* 261, no. 3, (2011), 716–748.

[24] F. Merle, P. Raphaël, I. Rodnianski, Blowup dynamics for smooth data equivariant solutions to the critical Schrodinger map problem. *Invent. Math.* 193 (2013), no. 2, 249–365.
[25] C. Ortoleva, G. Perelman, Nondispersive vanishing and blow up at infinity for the energy critical nonlinear Schrdinger equation in 3. *Algebra i Analiz* 25 (2013), no. 2, 162–192.
[26] P. Polácik, P. Quittner, A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation, *Nonlinear Analysis* 64 (2006), 1679-1689.
[27] P. Polácik, P. Quittner, Asymptotic behavior of threshold and sub-threshold solutions of a semilinear heat equation, *Asymptotic Analysis* 57 (2008), 125–141.
[28] P. Polácik, P. Quittner, Ph. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part II: Parabolic equations, *Indiana Univ. Math. J.* 56 (2007), 879-908.
[29] P. Polácik, E. Yanagida, On bounded and unbounded global solutions of a supercritical semilinear heat equation, *Math. Annalen* 327 (2003), 745–771.
[30] P. Polácik, E. Yanagida, Global unbounded solutions of the Fujita equation in the intermediate range. *Math. Ann.* 360 (2014), no. 1-2, 255-266.
[31] P. Quittner, The decay of global solutions of a semilinear heat equation, *Discrete Contin. Dynam. Systems A* 21 (2008), 307–318.
[32] P. Quitnner, P. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhauser Advanced Texts, ISBN: 978-3-7643-8441-8 (2007).
[33] P. Raphaël, I. Rodnianksi, Stable blowup dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems.*Publ. Math. Inst. Hautes Études Sci.* 115 (2012), 1–122.
[34] P. Raphaël, R. Schweyer, Stable blowup dynamics for the 1-corotational energy critical harmonic heat flow. *Comm. Pure Appl. Math.* 66 (2013), no. 3, 414–480.
[35] O. Rey, The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent.*J. Funct. Anal.* 89 (1990), no. 1, 1-52.
[36] R. Schweyer, Type II blow up for the four dimensional energy critical semilinear heat equation, *J. Funct. Anal.* 263 (2012), pp. 3922–3983.

Department of Mathematical Sciences University of Bath, Bath BA2 7AY, United Kingdom, and Departamento de Ingeniería Matemática-CMM Universidad de Chile, Santiago 837-0456, Chile
E-mail address: m.delpino@bath.ac.uk

Department of Mathematical Sciences University of Bath, Bath BA2 7AY, United Kingdom, and Departamento de Matemáticas, Universidad Católica de Chile, Macul 782-0436, Chile
E-mail address: m.musso@bath.ac.uk

Department of Mathematics University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: jcwei@math.ubc.ca