A GENERAL NONUNIQUENESS RESULT FOR YAMABE-TYPE PROBLEMS FOR CONFORMALLY VARIATIONAL RIEMANNIAN INVARIANTS

JOÃO HENRIQUE ANDRADE, JEFFREY S. CASE, PAOLO PICCIONE, AND JUNCHENG WEI

Abstract. Given a conformally variational scalar Riemannian invariant I, we identify a sufficient condition for a closed Riemannian manifold to admit finite regular coverings with many nonhomothetic conformal rescalings with I constant. We also identify a sufficient condition for the universal cover to admit infinitely many geometrically distinct periodic conformal rescalings with I constant. Using these conditions, we improve known nonuniqueness results for the Q-curvatures of orders two, four, and six, and establish nonuniqueness results for higher-order Q-curvatures and renormalized volume coefficients.

1. Introduction

The affirmative solution to the Yamabe Problem [32] asserts that if (M^n, g_0), $n \geq 3$, is a closed Riemannian manifold, then there is a unit volume conformal rescaling $g \in [g_0]$ with scalar curvature

$$Q_2^g = Y_{Q_2}(M^n, [g_0]) := \inf_{\widehat{g} \in [g_0]} \left\{ \int_M Q_2^{\widehat{g}} \, \mathrm{dV}_{\widehat{g}} : \mathrm{Vol}_{\widehat{g}}(M) = 1 \right\}.$$

Bettiol and Piccione [8] used this fact to produce numerous examples of Riemannian manifolds which admit many nonhomothetic conformal rescalings with constant scalar curvature. More precisely, if (M^n, g) is *any* closed Riemannian manifold of dimension at least three with positive Yamabe constant whose fundamental group has infinite profinite completion—for example, appropriately chosen Riemannian products with a compact symmetric space of Euclidean or negative type—then for each $\ell \in \mathbb{N}$ there is a finite regular covering $\pi \colon \widetilde{M}^n \to M^n$ for which the conformal manifold $(\widetilde{M}^n, [\pi^* g])$ admits pairwise nonhomothetic metrics $\{g_j\}_{j=1}^{\ell} \subset [\pi^* g]$ of constant scalar curvature. This gives a far-reaching generalization of Schoen's observation [41] that the number of conformal rescalings of the product metric on $S^1(L) \times S^{n-1}$ with constant scalar curvature tends to ∞ as $L \to \infty$.

Bettiol and Piccione's nonuniqueness result has since been generalized to the fourth- and sixth-order Q-curvatures [2, 9], as well as some of their fractional analogues [7]. These results require conditions which imply two key properties. First, the Yamabe-type constant

$$Y_{Q_{2k}}^{+}(M^n, [g_0]) := \inf_{\widehat{g} \in [g_0]} \left\{ \int_M Q_{2k}^{\widehat{g}} \, \mathrm{dV}_{\widehat{g}} : \mathrm{Vol}_{\widehat{g}}(M) = 1 \right\}$$

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associated to the Q-curvature [10] of order $2k$ should be positive and strictly less than that of the round n-sphere. Second, the corresponding GJMS operator [24] should have a positive Green's function. If both properties hold, then a smooth minimizer of $Y^+_{Q_{2k}}(M^n,[g_0])$ exists [33].

Notably, the aforementioned conditions must also be **geometric**; i.e. if (M^n,g) satisfies the condition and $\pi\colon \widetilde{M}^n \to M^n$ is a finite connected covering, then $(\widetilde{M}^n,\pi^*g)$ satisfies the condition. Combining this with the observation that closed manifolds whose fundamental group has infinite profinite completion admit finite regular coverings of arbitrarily large degree [8] yields nonhomothetic metrics of constant Q_{2k}-curvature; see Section 4 for a definition and discussion of closed manifolds whose fundamental group has infinite profinite completion. In the low-order cases mentioned above, one uses known geometric sufficient conditions for the estimate of the Yamabe-type constant [4,17,32,34,36,40] and for the positivity of the Green's function [15,28,29,32,37].

In this article we show that the above method for proving nonuniqueness results only requires variational structure. This observation is clearly and succinctly formulated in terms of CVIs [13,14].

Roughly speaking, a conformally variational invariant (CVI) of weight $-2k$ is a natural scalar Riemannian invariant I such that

$$\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\int_M I^{e^{2tu}g}\,\mathrm{dV}_{e^{2tu}g} = (n-2k)\int_M uI^g\,\mathrm{dV}_g$$

for all closed Riemannian manifolds (M^n,g), $n>2k$, and all $u\in C^\infty(M)$; see Section 2 for a more precise definition. Examples include the scalar curvature, the Q-curvatures [10], and the renormalized volume coefficients [22,23]. The vector space of CVIs of weight $-2k$ is classified [13] when $k\leq 3$.

Given a CVI I of weight $-2k$, there is [14] is a minimal positive integer $r\leq 2k$, called the **rank** of I, such that there is a formally self-adjoint, conformally covariant, polydifferential operator $D\colon \bigl(C^\infty(M)\bigr)^{\otimes(r-1)}\to C^\infty(M)$ for which

$$D^g(1^{\otimes(r-1)}) = \left(\frac{n-2k}{r}\right)^{r-1} I^g \tag{1.1}$$

on any Riemannian manifold (M^n,g), $n>2k$; see Section 2 for details. For example, the Q-curvatures have rank 2, and the renormalized volume coefficient v_k has maximal rank $2k$. Equation (1.1) implies that if (M^n,g) is closed and $u\in C^\infty(M)$ is positive, then

$$\left(\frac{n-2k}{r}\right)^{r-1}\int_M I^{g_u}\,\mathrm{dV}_{g_u} = \int_M uD^g(u^{\otimes(r-1)})\,\mathrm{dV}_g, \tag{1.2}$$

where $g_u := u^{\frac{2r}{n-2k}}g$.

Equation (1.2) allows one to define a Yamabe-type constant associated with a CVI I either in terms of the total I-curvature or the Dirichlet energy of the associated operator D. However, for CVIs of rank at least three, the operator $u\mapsto D(u^{\otimes(r-1)})$ is nonlinear, and so one generally must constrain the allowable representatives of the conformal class in order to obtain a finite Yamabe-type constant and to guarantee the ellipticity of the PDE satisfied by its minimizers. A **geometric cone** of weight $-2k$ and rank r is a function U which assigns to each closed Riemannian manifold (M^n,g), $n>2k$, an open subset $U^g\subset C^\infty(M)$ such that

(i) if $u\in U$, then $cu\in U$ for each constant $c>0$;
(ii) given $w\in C^\infty(M;\mathbb{R}_+)$, it holds that $u\in U^g$ if and only if $uw\in U^{g_w}$, $g_w := w^{\frac{2r}{n-2k}}g$; and
(iii) if $\pi\colon\widetilde{M}^n\to M^n$ is a finite connected covering, then $\pi^*(U^g)\subseteq U^{\pi^*g}$.

Different choices of geometric cone U may be relevant for the same CVI. For example, when studying the fourth-order Q-curvature, one can consider [27] the cones of all functions, of all positive functions, and of all positive functions which are conformal factors for a metric of positive scalar curvature.

The **(I, U)-Yamabe constant** is

$$Y_{(I,U)}(M^n, [g]) := \left(\frac{r}{n-2k}\right)^{r-1} \inf_{u \in U^g} \left\{ \int_M u D^g(u^{\otimes(r-1)}) \,\mathrm{dV}_g : \int_M |u|^{\frac{rn}{n-2k}} \,\mathrm{dV}_g = 1 \right\}. \tag{1.3}$$

When $U = C^\infty(M)$, we call this the ***I*-Yamabe constant**, denoted $Y_I(M^n, [g])$. The Yamabe constant [32] is proportional to the Q_2-Yamabe constant. The **metric (I, U)-Yamabe constant** is

$$Y^+_{(I,U)}(M^n, [g]) := \inf_{g_u \in [g]_U} \left\{ \int_M I^{g_u} \,\mathrm{dV}_{g_u} : \mathrm{Vol}_{g_u}(M) = 1 \right\}. \tag{1.4}$$

When $U = C^\infty(M)$, we call this the **metric *I*-Yamabe constant**, denoted $Y^+_I(M^n, [g])$. Here

$$[g]_U := \left\{ u^{\frac{2r}{n-2k}} g : u \in U^g \cap C^\infty(M; \mathbb{R}_+) \right\} \tag{1.5}$$

is the set of all Riemannian metrics conformal to g with a conformal factor in U. Note that

$$Y_{(I,U)}(M^n, [g]) \leq Y^+_{(I,U)}(M^n, [g]),$$

and if $U \subseteq C^\infty(M; \mathbb{R}_+)$, then equality holds. Additionally, if there is a positive minimizer of $Y_{(I,U)}(M^n, [g])$, then equality holds.

There is not a general result that guarantees the existence of a minimizer for the (I, U)-Yamabe constant. Instead, we isolate certain essential features of a "good" existence result.

Definition 1.1. Let I be a CVI of weight $-2k$ and rank r, and let U be a geometric cone. A **geometric Aubin set** for (I, U) is a set $\mathcal{A}$ of closed Riemannian n-manifolds, $n > 2k$, such that

(i) if $(M^n, g) \in \mathcal{A}$, then $0 < Y_{(I,U)}(M^n, [g]) \leq Y_{(I,U)}(S^n, [g_{\mathrm{rd}}])$ with equality if and only if (M^n, g) is conformally equivalent to (S^n, g_{rd});

(ii) if $(M^n, g) \in \mathcal{A}$, then there is a positive minimizer of the (I, U)-Yamabe constant (1.3); and

(iii) if $(M^n, g) \in \mathcal{A}$ and $\pi\colon \widetilde{M}^n \to M^n$ is a finite connected covering, then $(\widetilde{M}^n, \pi^* g) \in \mathcal{A}$.

Our terminology is inspired by Aubin's observation [4] that if $Y_{Q_2}(M^n, [g]) < Y_{Q_2}(S^n, [g_{\mathrm{rd}}])$, then minimizers of the Yamabe constant exist. Combining results of Aubin [4] and Schoen [40] implies that the set of all closed Riemannian n-manifolds, $n \geq 3$, with positive Yamabe constant is a geometric Aubin set for the pair $(I, U) = (Q_2, C^\infty(M))$. A result of Gursky, Hang, and Lin [27] implies that the set of all closed Riemannian manifolds with positive Q_2- and Q_4-Yamabe constant is a geometric Aubin set for the pair $(I, U) = (Q_4, C^\infty(M))$. In Section 3 we list all maximal examples of geometric Aubin sets known to the authors.

Our definitions capture the key ingredients of the aforementioned nonuniqueness theorems.

Theorem 1.2. *Let I be a CVI of weight $-2k$ and rank r, and let U be a geometric cone. Suppose that there is a nonempty geometric Aubin set $\mathcal{A}$ for (I, U). If $(M^n, g) \in \mathcal{A}$ and $\pi_1(M)$ has infinite profinite completion, then for each $\ell \in \mathbb{N}$ there is a finite regular covering $\pi\colon \widetilde{M}^n \to M^n$ such that $(\widetilde{M}^n, [\pi^* g])$ admits pairwise nonhomothetic representatives $\{g_j\}_{j=1}^{\ell}$ with constant I-curvature.*

Here was say that two Riemannian metrics g_1, g_2 on M are **homothetic** if there is a diffeomorphism $\Phi \in \mathrm{Diff}(M)$ and a constant $c > 0$ such that $\Phi^* g_1 = c^2 g_2$. Theorem 1.2 does not require the full strength of a geometric Aubin set; it only requires that the set $\mathcal{A}$ be closed under finite regular coverings and that there is a constant $C = C(\mathcal{A}) > 0$ such that if $(M^n, g) \in \mathcal{A}$, then $Y_{(I,U)}(M^n, [g]) > 0$ and there is a representative $\widehat{g} \in [g]_U$ such that $\mathrm{Vol}_{\widehat{g}}(M) = 1$ and $I^{\widehat{g}} \leq C$ is constant.

Theorem 1.2 implies that there are infinitely functions $u \in C^\infty(\widetilde{M};\mathbb{R}_+)$ for which $I^{\widetilde{g}_u}$ is constant in the universal cover $\widetilde{\pi}\colon \widetilde{M}^n \to M^n$, where $\widetilde{g}_u := u^{\frac{2r}{n-2k}}\widetilde{\pi}^*g$. In fact, one can find infinitely many geometrically distinct solutions:

Theorem 1.3. *Let I be a CVI of weight $-2k$ and rank r, and let U be a geometric cone. Suppose that there is a nonempty geometric Aubin set $\mathcal{A}$ for (I,U). Suppose that $(M^n,g) \in \mathcal{A}$ is such that $\pi_1(M)$ has infinite profinite completion. If the conformal universal cover of $(M^n,[g])$ is not conformal to Euclidean space, then there is a countable set $\{\widetilde{g}_j\}_{j\in\mathbb{N}}$ of pairwise nonhomothetic periodic metrics $\widetilde{g}_j \in [\widetilde{\pi}^*g]$, each of which has constant I-curvature.*

Here the **conformal universal cover** of $(M^n,[g])$ is the conformal manifold $(\widetilde{M}^n,[\widetilde{\pi}^*g])$, where $\widetilde{\pi}\colon \widetilde{M}^n \to M^n$ is the universal cover. A metric $\widetilde{g} \in [\widetilde{\pi}^*g]$ is **periodic** if it descends to a compact quotient of $\widetilde{M}^n$.

Theorem 1.3 is new even for the scalar curvature (cf. [8]) and the Q_4-curvature (cf. [9]).

While the condition that the Riemannian universal cover of a closed manifold is conformally equivalent to flat Euclidean space is quite restrictive [20], we do not presently know enough about geometric Aubin sets for general pairs (I,U) to remove the assumption on the universal cover from Theorem 1.3. However, for the two families of CVIs I for which there is a known geometric cone U such that the pair (I,U) admits a nonempty geometric Aubin set, this assumption can be removed.

The first family is that of the noncritical Q-curvatures.

Corollary 1.4. *Fix $k,m \in \mathbb{N}$. There is an $N = N(k,m) \in \mathbb{N}$ such that if*

(i) *(M_1^m,g_1) is a closed Riemannian manifold for which $\mathrm{Ric}_{g_1} = -(m-1)g_1$ and $\pi_1(M_1)$ has infinite profinite completion, and*
(ii) *(M_2^{n-m},g_2) is a closed Riemannian manifold for which $\mathrm{Ric}_{g_2} = (n-m-1)g_2$ and $n \geq N$,*

then for each $\ell \in \mathbb{N}$, there is a finite regular covering $\pi\colon \widetilde{X}^n \to M_1^m \times M_2^{n-m}$ for which there exist pairwise nonhomothetic metrics $\{\sigma_j\}_{j=1}^{\ell} \subset [\pi^*(g_1 \oplus g_2)]$ of constant Q_{2k}-curvature. Moreover, the conformal universal cover $(\widetilde{M}_1^m \times \widetilde{M}_2^{n-m}, [\widetilde{\pi}^*(g_1 \oplus g_2)])$ admits infinitely many pairwise nonhomothetic periodic representatives $\{\widetilde{\sigma}_j\}_{j=1}^{\infty}$ of constant Q_{2k}-curvature.

In general the constant N is hard to estimate [15]. However, N is known [37] when restricting to locally conformally flat manifolds:

Corollary 1.5. *Fix $k,m \in \mathbb{N}$. Let $n \geq 2k+2m-1$ be a positive integer. For every closed hyperbolic manifold (M^m,g_{hyp}) and every $\ell \in \mathbb{N}$, there is a finite regular covering $\pi\colon \widetilde{M}^m \to M^m$ such that $(\widetilde{M}^m \times S^{n-m}, [\pi^*g_{\mathrm{hyp}} \oplus g_{\mathrm{rd}}])$ admits pairwise nonhomothetic representatives $\{\sigma_j\}_{j=1}^{\ell}$ of constant Q_{2k}-curvature. Moreover, $(H^m \times S^{n-m}, [g_{\mathrm{hyp}} \oplus g_{\mathrm{rd}}])$ admits infinitely many pairwise nonhomothetic periodic conformal representatives of constant Q_{2k}-curvature.*

Since $(H^m \times S^{n-m}, [g_{\mathrm{hyp}} \oplus g_{\mathrm{rd}}])$ is [35] conformally equivalent to $(S^n \setminus S^{m-1}, [g_{\mathrm{rd}}])$, Corollary 1.5 gives a nonuniqueness result for singular Q-Yamabe metrics. In this form, our result improves known results [2, 8, 9, 41] when $k \leq 3$, and is completely new for larger values of k.

The second family is that of the noncritical renormalized volume coefficients [22, 23].

Corollary 1.6. *Fix $k,m \in \mathbb{N}$. There is an $N = N(k,m) \in \mathbb{N}$ such that if $n \geq N$, then for every closed hyperbolic manifold (M^m,g_{hyp}) and every $\ell \in \mathbb{N}$, there is a finite regular covering $\pi\colon \widetilde{M}^m \to M^m$ such that $(\widetilde{M}^m \times S^{n-m}, [\pi^*g_{\mathrm{hyp}} \oplus g_{\mathrm{rd}}])$ admits pairwise nonhomothetic representatives $\{\sigma_j\}_{j=1}^{\ell}$ of constant v_k-curvature. Moreover, $(H^m \times S^{n-m}, [g_{\mathrm{hyp}} \oplus g_{\mathrm{rd}}])$ admits infinitely many pairwise nonhomothetic periodic representatives of constant v_k-curvature.*

The key point is that the renormalized volume coefficient v_k is proportional [25] to the σ_k-curvature [44] on locally conformally flat manifolds. When n is sufficiently large, the product metric is in the Γ_k^+-cone, allowing one to apply existence results for the σ_k-curvatures [19, 26, 43]. Our convention is that S^1 is a closed one-dimensional hyperbolic manifold, so that Corollary 1.6 recovers a nonuniqueness result of Viaclovsky [44]. See Remark 4.4 for a discussion of the values $N(k,m)$.

This article is organized as follows:

In Section 2 we summarize the key definitions and facts involving CVIs and further discuss Yamabe-type constants.

In Section 3 we list all maximal geometric Aubin sets known to the authors, including a new geometric Aubin set for the higher-order Q-curvatures.

In Section 4 we recall a key property [8] of closed manifolds whose fundamental group has infinite profinite completion. We then prove Theorems 1.2 and 1.3 and Corollaries 1.4, 1.5, and 1.6.

In Appendix A we generalize the Aubin Lemma [1, 3], which shows that, when positive, the Yamabe constant of a closed manifold is strictly less than any of its nontrivial finite connected coverings, to the higher-order Q-curvatures. This result is needed to produce geometric Aubin sets for these invariants.

2. Conformally variational Riemannian invariants

This section summarizes necessary facts and definitions involving conformally variational Riemannian invariants [13, 14].

A **natural Riemannian scalar invariant** is a universal linear combination I of complete contractions of g, g^{-1}, and covariant derivatives of the Riemann curvature tensor. Universality means that the coefficients are rational functions of the dimension, so that I may be regarded as a function which assigns to each Riemannian manifold (M^n, g), $n \gg 1$, a function $I^g \in C^\infty(M)$. We say that I is **homogeneous** of **weight** $-2k$ if $I^{c^2 g} = c^{-2k} I^g$ for all constants $c > 0$. Necessarily k is a nonnegative integer. In practice, natural homogeneous scalar Riemannian invariants of weight $-2k$ are defined on all Riemannian manifolds of dimension $n \geq 2k$. When the metric g is clear from context, we write I for I^g.

A **conformally variational invariant** (CVI) is a natural homogeneous scalar Riemannian invariant I such that for all Riemannian manifolds (M^n, g) on which I is defined, the linear operator

$$C^\infty(M) \ni u \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} I^{e^{2tu}g} \in C^\infty(M)$$

is formally self-adjoint with respect to the L^2-inner product induced by the Riemannian volume element dV_g. The formal self-adjointness of this operator is equivalent [12, Lemma 2] to the existence of a natural Riemannian functional $\mathcal{F} \colon [g] \to \mathbb{R}$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}(e^{2tu}g) = \int_M u I^g \, \mathrm{dV}_g$$

for all $u \in C^\infty(M)$. When $n > 2k$, one can take $\mathcal{F}(g) = \frac{1}{n-2k}\int_M I^g \, \mathrm{dV}_g$.

A **natural $(r-1)$-differential operator**, $r \in \mathbb{N} := \{1, 2, 3, \dots\}$, is an operator

$$D \colon \left(C^\infty(M)\right)^{\otimes(r-1)} \to C^\infty(M),$$

defined on any Riemannian manifold (M^n, g), $n \gg 1$, such that $D\left(u_1 \otimes \cdots \otimes u_{r-1}\right)$ can be expressed as a universal linear combination of complete contractions of tensor products of g, g^{-1}, covariant derivatives of the Riemann curvature tensor, and covariant derivatives of the functions u_j. The tensor product is over $\mathbb{R}$; thus $(r-1)$-differential operators are multilinear operators which are differential when all but one of the factors in the tensor product is held fixed. Note that a natural 0-differential operator is a natural scalar Riemannian invariant. A **natural polydifferential operator** is an operator which is a natural $(r-1)$-differential operator for some $r \in \mathbb{N}$.

We now restrict our attention to natural polydifferential operators which are homogeneous of degree $-2k$ in g and defined on all Riemannian manifolds of dimension $n \geq 2k$.

A natural $(r-1)$-differential operator D is **conformally covariant** if for any $n \in \mathbb{N}$, there is a multi-index $a \in \mathbb{R}^{r-1}$ and a constant $b \in \mathbb{R}$ such that

$$D^{e^{2\Upsilon}g}\left(u_1 \otimes \cdots \otimes u_{r-1}\right) = e^{-b\Upsilon} D^g \left(e^{a_1\Upsilon} u_1 \otimes \cdots \otimes e^{a_{r-1}\Upsilon} u_{r-1}\right)$$

for all Riemannian manifolds (M^n, g) and all $\Upsilon, u_1, \ldots, u_{r-1} \in C^\infty(M)$. In this case D is homogeneous of degree $|a| - b$, and we call (a, b) the **bidegree** of D.

A natural $(r-1)$-differential operator D is **formally self-adjoint** if for every Riemannian manifold (M^n, g) and every $u_1, \ldots, u_r \in C^\infty(M)$ such that $u_1 \cdots u_r$ has compact support, the **Dirichlet form**

$$\mathfrak{D}\left(u_1 \otimes \cdots \otimes u_r\right) := \int_M u_1 D\left(u_2 \otimes \cdots \otimes u_r\right) \mathrm{dV}_g$$

is symmetric; i.e. $\mathfrak{D}(u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(r)}) = \mathfrak{D}(u_1 \otimes \cdots \otimes u_r)$ for all permutations σ of $\{1, \ldots, r\}$. Note that if D is formally self-adjoint, then the map $(u_1 \otimes \cdots \otimes u_{r-1}) \mapsto D\left(u_1 \otimes \cdots \otimes u_{r-1}\right)$ is symmetric. If $D\colon \left(C^\infty(M)\right)^{\otimes(r-1)} \to C^\infty(M)$ is a formally self-adjoint, conformally covariant, polydifferential operator which is homogeneous of degree $-2k$ in the metric, then [14, Lemma 3.8] it has bidegree $\left(\frac{n-2k}{r}, \frac{n(r-1)+2k}{r}\right)$. In particular, if additionally $n > 2k$, then

$$\int_M u_1 D^{g_w}(u_2 \otimes \cdots \otimes u_r) \, \mathrm{dV}_{g_w} = \int_M w u_1 D^g(w u_2 \otimes \cdots \otimes w u_r) \, \mathrm{dV}_g,$$

where $g_w := w^{\frac{2r}{n-2k}} g$.

Let I be a CVI of weight $-2k$. A natural $(r-1)$-differential operator D **recovers** I if for all Riemannian manifolds of dimension $n > 2k$, one has

$$D(1 \otimes \cdots \otimes 1) = \left(\frac{n-2k}{r}\right)^{r-1} I. \tag{2.1}$$

(There is a corresponding notion [14] for dimension $n = 2k$ which is not needed in this article.) A natural $(r-1)$-differential operator is **associated** to I if it is formally self-adjoint, conformally covariant, and recovers I. Note that if D recovers I, then Equation (2.1) is equivalent to

$$\left(\frac{n-2k}{r}\right)^{r-1} u^{\frac{n(r-1)+2k}{n-2k}} I^{g_u} = D\left(u^{\otimes(r-1)}\right) \tag{2.2}$$

for all $u \in C^\infty(M; \mathbb{R}_+)$, where again $g_u = u^{\frac{2r}{n-2k}} g$.

Case, Lin, and Yuan [14, Theorem 1.6] showed that each CVI has an associated polydifferential operator. More precisely, if I is a CVI of weight $-2k$, then there is a minimal integer $1 \leq r \leq 2k$ for which there exists a natural $(r-1)$-differential operator associated with I. This number is the rank of I defined in the introduction.

Let I be a CVI of weight $-2k$ with rank r. The ***I*-Yamabe quotient** of a closed Riemannian manifold (M^n, g), $n > 2k$, is

$$\mathcal{I}_{2k}^g := \frac{\int_M I_{2k}^g \, \mathrm{dV}_g}{\mathrm{Vol}_g(M)^{\frac{n-2k}{n}}}. \tag{2.3}$$

The transformation rule (2.2) implies that if $u \in C^\infty(M;\mathbb{R}_+)$, then

$$\mathcal{I}_{2k}^{g_u} = \left(\frac{r}{n-2k}\right)^{r-1} \frac{\int_M u D^g\left(u^{\otimes(r-1)}\right) \mathrm{dV}_g}{\left(\int_M u^{\frac{rn}{n-2k}} \mathrm{dV}_g\right)^{\frac{n-2k}{n}}}.$$

If U is a geometric cone, then the metric (I,U)-Yamabe constant (1.4) is

$$Y_{(I,U)}^+ = \inf\left\{\mathcal{I}_{2k}^{g_u} : g_u \in [g]_U\right\}. \tag{2.4}$$

where we recall that $[g]_U$ is defined by Equation (1.5).

3. Examples of geometric Aubin sets

In this section we list all known maximal geometric Aubin sets. We also identify a geometric Aubin set for the higher-order Q-curvatures. To that end, we denote by $\mathcal{M}^n$ the set of all closed Riemannian n-manifolds.

3.1. Second-order Q-curvature.

The second-order Q-curvature is proportional to the scalar curvature, the subject of the well-studied Yamabe Problem [32]. Since $Y_{Q_2}(M^n,[g]) > 0$ if and only if there exists a metric $\widehat{g} \in [g]$ with positive scalar curvature, the set

$$\mathcal{A} = \{(M^n,g) \in \mathcal{M}^n : Y_{Q_2}(M^n,[g]) > 0\}$$

is closed under finite connected coverings. The resolution of the Yamabe Problem implies that $\mathcal{A}$ is a geometric Aubin set for $(Q_2, C^\infty(M))$.

3.2. Renormalized volume coefficients.

The renormalized volume coefficients v_k, defined on any Riemannian manifold of dimension $n \geq 2k$, are CVIs for which the equation $v_k^{g_u} = c$ is in general a fully nonlinear second-order PDE in the conformal factor [13, 23]. This problem has only been solved when $k \in \{1,2\}$ or when restricted to locally conformally flat manifolds [19, 26, 43]; in these cases v_k is the σ_k-curvature [25, 44].

We reformulate the aforementioned results in terms of a geometric Aubin set. To that end, let

$$\mathsf{P} := \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{2(n-1)} g\right)$$

be the Schouten tensor of (M^n,g), regarded, using g^{-1}, as a section of $T^*M \otimes TM$. Given $j \in \mathbb{N}$, denote by $\sigma_j(\mathsf{P})$ the j-th elementary symmetric function of the eigenvalues of P. The **positive j-cone** is the set

$$\Gamma_j^+ := \{\hat{g} \in [g] : \sigma_1(\mathsf{P}), \sigma_2(\mathsf{P}), \ldots, \sigma_j(\mathsf{P}) > 0\}.$$

If $\hat{g} \in \Gamma_k^+$, then the equation $\sigma_k^{\hat{g}} = f$, expressed as a PDE in the conformal factor $\widehat{g} = e^{2u} g$, is elliptic [44, Section 6.3]. Ellipticity is key in producing a geometric Aubin set:

Proposition 3.1. *Let $k \in \mathbb{N}$. Pick an integer $n > 2k$. Then*

$$\mathcal{A} := \left\{(M^n,g) \in \mathcal{M}^n : \Gamma_k^+ \neq \emptyset, W^g = 0\right\}$$

is a geometric Aubin set for (v_k, Γ_k^+), where W^g is the Weyl tensor of g.

Proof. Since $v_1 = Q_2$, the case $k=1$ follows from the discussion in Subsection 3.1.

Suppose that $k \geq 2$. Since the conditions $\Gamma_k^+ \neq \emptyset$ and $W^g = 0$ are equivalent to the existence of a metric $\widehat{g} \in [g]$ with $\sigma_j(\mathsf{P}^{\widehat{g}}) > 0$, $1 \leq j \leq k$, and $W^{\widehat{g}} = 0$, we see that $\mathcal{A}$ is closed under finite connected coverings. Let $(M^n,g) \in \mathcal{A}$. Since $n \geq 5$ and the Weyl tensor vanishes, (M^n,g) is locally conformally flat. Therefore v_k is a nonzero multiple of the σ_k-curvature [25, Proposition 1]. A result of Guan and Wang [26, Theorem 1(A)] implies that $\mathcal{A}$ is a geometric Aubin set. □

3.3. **Fourth-order Q-curvature.** The difficulty in studying the σ_k-curvatures comes from the need to restrict to an elliptic cone. In contrast, the difficulty in studying the higher-order Q-curvatures [11] comes from the general lack of a maximum principle for higher-order operators. The most general existence results are available in the fourth-order case [27,28,30]. In particular:

Proposition 3.2. *Fix an integer $n \geq 6$. Then*
$$\mathcal{A} := \{(M^n, g) \in \mathcal{M}^n : Y_{Q_2}(M^n, [g]), Y_{Q_4}(M^n, [g]) > 0\}$$
is a geometric Aubin set for $(Q_4, C^\infty(M))$.

Remark 3.3. It is expected [27, p. 1352] that Proposition 3.2 is true for all $n \geq 5$.

Proof. The condition $Y_{Q_2}(M^n, [g]), Y_{Q_4}(M^n, [g]) > 0$ is equivalent [27, Theorem 1.1] to the existence of a metric $\widehat{g} \in [g]$ such that $Q_2^{\widehat{g}}, Q_4^{\widehat{g}} > 0$. Thus $\mathcal{A}$ is closed under finite connected coverings.

Let $(M^n, g) \in \mathcal{A}$. As noted above, we may assume that $Q_2^g, Q_4^g > 0$. Thus [28, Proposition C] the Green's function for the Paneitz operator is positive. Since $Y_{Q_2}(M^n, [g]) > 0$, we deduce [45, Theorem 1.3] that
$$Y_{Q_4}(M^n, [g]) \leq Y_{Q_4}(S^n, [g_{\mathrm{rd}}])$$
with equality if and only if $(M^n, [g])$ is conformally diffeomorphic to $(S^n, [g_{\mathrm{rd}}])$. Combining an existence result of Mazumdar [33, Theorem 3] with the fact [5, Theorem 6] that $Y_{Q_4}(S^n, [g_{\mathrm{rd}}])$ is achieved by a positive function implies that there is a positive minimizer of $Y_{Q_4}(M^n, [g_{\mathrm{rd}}])$. □

3.4. **Higher-order Q-curvatures.** The aforementioned result of Mazumdar states that if (M^n, g), $n > 2k$, is closed Riemannian n-manifold for which the GJMS operator [24] P_{2k} of order $2k$ has positive Green's function and satisfies $0 < Y_{Q_{2k}}(M^n, [g]) < Y_{Q_{2k}}(S^n, [g_{\mathrm{rd}}])$, then there is a positive minimizer of $Y_{Q_{2k}}(M^n, [g])$. The condition $Y_{Q_{2k}}(M^n, [g]) > 0$ is not geometric, nor is either condition easy to check. The most general geometric sufficient conditions for the existence of minimizers follow from the work of Case and Malchiodi [15]:

Let $m \in \mathbb{N}_0$ be a nonnegative integer. An ***m*-special Einstein product (*m*-SEP)** [21] is a Riemannian product manifold $(M_1^m \times M_2^{n-m}, g_1 \oplus g_2)$ such that $\mathrm{Ric}_{g_1} = -(m-1)\lambda g_1$ and $\mathrm{Ric}_{g_2} = (n-m-1)\lambda g_2$ for some constant $\lambda > 0$. Case and Malchiodi showed that if n is sufficiently large relative to k and m, then the Green's function of P_{2k} is positive and
$$0 < Y_{Q_{2k}}(M_1^m \times M_2^{n-m}, [g_1 \oplus g_2]) < Y_{Q_{2k}}(S^n, [g_{\mathrm{rd}}]).$$
However, the property of being conformal to an m-SEP is not preserved under pullback by finite connected coverings. Instead, we say that (X^n, g) is **virtually** conformal to an m-SEP if there is a finite connected covering $\pi\colon \widetilde{X}^n \to X^n$ such that $(\widetilde{X}^n, \pi^* g)$ is conformal to an m-SEP. This gives rise to a geometric Aubin set:

Proposition 3.4. *Fix $k, m \in \mathbb{N}$. There is an integer $N = N(k, m)$ such that if $n \geq N$, then*
$$\mathcal{A} := \{(X^n, g) \in \mathcal{M}^n : (X^n, g) \text{ is virtually conformal to an } m\text{-SEP}\}$$
is a geometric Aubin set for $(Q_{2k}, C^\infty(M))$.

Proof. Case and Malchiodi showed [15, Theorem 1.3] that there is an $N(k, m) \in \mathbb{N}$ such that if (X^n, g) is conformal to an m-SEP, then $0 < Y_{Q_{2k}}(X^n, [g]) < Y_{Q_{2k}}(S^n, [g_{\mathrm{rd}}])$ and there is a $\widehat{g} \in [g]$ which minimizes the Q_{2k}-Yamabe constant $Y_{Q_{2k}}(X^n, [g])$. Moreover, $N(k, m) \geq 2k + 2m - 1$. Set $N := N(k, m)$.

Suppose that $\pi\colon \widetilde{X}^n \to X^n$ is a finite connected covering and that $(\widetilde{X}^n, \widetilde{g})$, $\widetilde{g} := \pi^* g$, is conformal to an m-SEP. Then the GJMS operator $P_{2k}^{\widetilde{g}}$ satisfies the Strong Maximum

Principle [15, Theorem 1.2]. In particular, $\ker P_{2k}^{\widetilde{g}} = \{0\}$. Therefore, P_{2k}^{g} has a trivial kernel, and so its Green's function G_{2k}^{g} exists. Since $P_{2k}^{\widetilde{g}}(\pi^* G_{2k}^{g}) \geq 0$, we deduce that $G_{2k}^{g} > 0$. Let d denote the degree of π. On the one hand, since $\mathcal{I}_{2k}^{\pi^*\hat{g}} = d^{2k/n}\mathcal{I}_{2k}^{\hat{g}}$ for every $\hat{g} \in [g]$, we see that

$$Y_{Q_{2k}}(X^n, [g]) = \inf\left\{ \mathcal{I}_{2k}^{\hat{g}} : \hat{g} \in [g] \right\} \geq d^{-2k/n} Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]) > 0.$$

On the other hand, since there is a positive minimizer of $Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}])$, we deduce from Lemma A.1 below that

$$Y_{Q_{2k}}(X^n, [g]) < Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]) < Y_{Q_{2k}}(S^n, [g_{\mathrm{rd}}]).$$

Hence [33, Theorem 3] there is a $\widehat{g} \in [g]$ which minimizes $Y_{Q_{2k}}(X^n, [g])$.

Finally, suppose that (X^n, g) is virtually conformal to an m-SEP and that $\varphi\colon \widetilde{X}^n \to X^n$ is a finite connected covering. Let $(M_1^m \times M_2^{n-m}, g_1 \oplus g_2)$ be an m-SEP and let $\pi\colon M_1^m \times M_2^{n-m} \to X^n$ be a finite connected covering such that $g_1 \oplus g_2 \in [\pi^* g]$. Since (M_2^{n-m}, g_2) has positive Ricci curvature, $\pi_1(M_2)$ is finite. Hence, by passing to a finite cover, we can assume that $\pi_1(M_1 \times M_2) = \pi_1(M_1)$. Since π and φ are covering maps, we may regard $H := \pi_*\big(\pi_1(M_1)\big) \cap \varphi_*\big(\pi_1(\widetilde{X})\big)$ as a (necessarily finite index) subgroup of $\pi_1(M_1)$. Let $\widehat{\pi}\colon \widehat{M_1} \to M_1$ be a finite connected covering with $\widehat{\pi}_*\big(\pi_1(\widehat{M_1})\big) = H$. Then $(\widehat{M}_1^m \times M_2^{n-m}, \widehat{\pi}^* g_1 \oplus g_2)$ is an m-SEP. Since $\pi_*(\widehat{\pi} \times 1)_*(\pi_1(\widehat{M_1} \times M_2))$ is a subgroup of $\varphi_*\big(\pi_1(\widetilde{X})\big)$, the Lifting Criterion [31, Theorem 11.18] implies that $\pi \circ (\widehat{\pi} \times 1)\colon \widehat{M}_1^m \times M_2^{n-m} \to X^n$ lifts to a finite connected covering $\widehat{\varphi}\colon \widehat{M}_1^m \times M_2^{n-m} \to \widetilde{X}^n$. Therefore, $(\widetilde{X}, \varphi^* g)$ is virtually conformal to an m-SEP. □

Remark 3.5. Proposition 3.4 does not give the best expected geometric Aubin set. For example, Andrade, Piccione, and Wei [2] conjectured that if (M^n, g), $n \geq 7$, is a closed Riemannian manifold with $Y_{Q_2}(M^n, [g]), Y_{Q_4}(M^n, [g]) > 0$ and which admits a metric $\widehat{g} \in [g]$ with $Q_6^{\widehat{g}} \geq 0$ and $Q_6^{\widehat{g}} \neq 0$, then there is a positive minimizer for $Y_{Q_6}(M^n, [g])$. If this is true, then the set of all closed Riemannian n-manifolds which admit a conformal representative with positive Q_2-, Q_4-, and Q_6-curvature is a geometric Aubin set.

While it is known that $N(k, m) \geq 2k + 2m - 1$ in Proposition 3.4, the minimal value of $N(k, m)$ is not known. However, results of Qing and Raske [37] give the optimal value of N for locally conformally flat manifolds which are virtually conformal to an m-SEP.

Proposition 3.6. *Fix $k, m \in \mathbb{N}$. Let $n \geq 2k + 2m - 1$ be an integer. Then*

$$\mathcal{A} := \{(X^n, g) \in \mathcal{M}^n : W^g = 0 \textit{ and } (X^n, g) \textit{ is virtually conformal to an } m\textit{-SEP}\}$$

is a geometric Aubin set for $(Q_{2k}, C^\infty(M))$.

Proof. The case $k = 1$ follows from the discussion of Subsection 3.1.

Suppose now that $k \geq 2$. Let $(X^n, g) \in \mathcal{A}$. Then $n \geq 5$, and hence (X^n, g) is locally conformally flat. Since (X^n, g) is conformal to an m-SEP, we see [15, Theorem 1.1 and Lemma 4.3] that $0 < Y_{Q_{2k}}(X^n, [g]) < Y_{Q_{2k}}(S^n, [g_{\mathrm{rd}}])$. Hence [33, Theorem 3] there is a minimizer $\widehat{g} \in [g]$ for $Y_{Q_{2k}}(X^n, [g])$. The rest of the argument follows as in the proof of Proposition 3.4. □

4. Proofs of our main results

Let G be a group with identity e_G. We say that G is **residually finite** if for every $g \in G$ with $g \neq e_G$, there is a finite group H and a group homomorphism $\pi\colon G \to H$ such that $\pi(g) \neq e_H$. The **profinite completion** of a group G is the limit

$$\widehat{G} := \varprojlim G/N,$$

where N ranges over all finite index normal subgroups of G. It readily follows that if G is infinite and residually finite, then its profinite completion is infinite. In particular, the Selberg–Malcev Lemma [38, Section 7.6] implies that the fundamental group of a closed symmetric space of noncompact type—for example, a closed hyperbolic manifold—has infinite profinite completion. Additional examples have been discussed by Bettiol and Piccione [8, Section 3.2].

The basic idea of the proof of Theorem 1.2 is to inductively apply the following topological observation of Bettiol and Piccione [8, Lemma 3.6] to construct a tower of finite regular coverings for which minimizers of the (I,U)-Yamabe constant must be nonhomothetic.

Lemma 4.1. *Let (M^n,g) be a closed Riemannian manifold for which $\pi_1(M)$ has infinite profinite completion. Given $V\in\mathbb{R}$, there exists a finite regular covering $\pi\colon \widetilde{M}^n\to M^n$ such that $\mathrm{Vol}_{\pi^*g}(\widetilde{M})>V$.*

Lifting these metrics to the conformal universal cover produces infinitely many representatives with constant I-curvature for which no two conformal factors are constant multiples of one another. To conclude that these representatives are in fact nonhomothetic requires an application of the Ferrand–Obata Theorem [18,39] on noncompact manifolds.

Lemma 4.2. *Let I be a CVI of weight $-2k$ and rank r, and let U be a geometric cone. Suppose that there is a nonempty geometric Aubin set $\mathcal{A}$ for (I,U). Let $(M^n,g)\in\mathcal{A}$ be such that there is a sequence $\{\pi_j\colon M_j^n\to M^n\}_{j\in\mathbb{N}}$ of finite connected coverings of degree $m_j\geq j$ and a sequence $\{g_j\}_{j\in\mathbb{N}}$ of minimizers of $Y_{(I,U)}(M_j^n,[\pi_j^*g])$ such that for each $j\in\mathbb{N}$, there is a diffeomorphism $\Phi_j\in\mathrm{Diff}(\widetilde{M})$ and a constant $c_j>0$ such that $\Phi_j^*\widetilde{\pi}^*g=c_j^2\widetilde{\pi}_j^*g_j$, where $\widetilde{\pi}\colon\widetilde{M}^n\to M^n$ and $\widetilde{\pi}_j\colon\widetilde{M}^n\to M_j^n$ are the universal covers of M^n and M_j^n, respectively. Then $(\widetilde{M}^n,\widetilde{\pi}^*g)$ is conformally equivalent to flat Euclidean space.*

Proof. Since $(M^n,g)\in\mathcal{A}$ and the statement of the lemma depends only on $[g]$, we may assume without loss of generality that $I^g=Y_{(I,U)}(M^n,[g])$ and $\mathrm{Vol}_g(M)=1$. Since $m_j\to\infty$ as $j\to\infty$, we see that $\widetilde{M}$ is noncompact.

Fix $j\in\mathbb{N}$. Observe that $\widetilde{\pi}=\pi_j\circ\widetilde{\pi}_j$. Let $\mathrm{Aut}(\widetilde{\pi})$ and $\mathrm{Aut}(\widetilde{\pi}_j)$ denote the groups of deck transformations for $\widetilde{\pi}$ and $\widetilde{\pi}_j$, respectively. Let F be a connected fundamental domain for $\widetilde{\pi}$. On the one hand, since $\mathrm{Aut}(\widetilde{\pi}_j)$ is a finite index subgroup of $\mathrm{Aut}(\widetilde{\pi})$, there is a $\tau_j\in\mathrm{Aut}(\widetilde{\pi})$ such that

$$\mathrm{Vol}_{\tau_j^*\widetilde{\pi}_j^*g_j}(F)=\min\left\{\mathrm{Vol}_{\sigma^*\widetilde{\pi}_j^*g_j}(F):\sigma\in\mathrm{Aut}(\widetilde{\pi})\right\}.$$

Therefore $\mathrm{Vol}_{\tau_j^*\widetilde{\pi}_j^*g_j}(F)\leq m_j^{-1}$. On the other hand, we may pick a $\sigma_j\in\mathrm{Aut}(\widetilde{\pi})$ such that $\Psi_j:=\sigma_j\circ\Phi_j\circ\tau_j$ satisfies $\Psi_j(F)\cap F\neq\emptyset$. Note that $\Psi_j^*\widetilde{\pi}^*g=c_j^2\tau_j^*\widetilde{\pi}_j^*g_j$.

We now compute

$$\mathcal{I}_j:=\frac{\int_F I^{\Psi_j^*\widetilde{\pi}^*g}\,\mathrm{dV}_{\Psi_j^*\widetilde{\pi}^*g}}{\mathrm{Vol}_{\Psi_j^*\widetilde{\pi}^*g}(F)^{\frac{n-2k}{n}}}$$

in two ways. First, since $I^g=Y_{(I,U)}(M^n,[g])$, we directly compute that

$$\mathcal{I}_j=Y_{(I,U)}(M^n,[g])\,\mathrm{Vol}_{\widetilde{\pi}^*g}(\Psi_j(F))^{\frac{2k}{n}}. \tag{4.1}$$

Second, since I is homogeneous of degree $-2k$ with respect to constant rescalings and since $I^{g_j}=Y_{(I,U)}(M_j^n,[\pi_j^*g])$, we compute that

$$\mathcal{I}_j=\frac{\int_F I^{\tau_j^*\widetilde{\pi}_j^*g_j}\,\mathrm{dV}_{\tau_j^*\widetilde{\pi}_j^*g_j}}{\mathrm{Vol}_{\tau_j^*\widetilde{\pi}_j^*g_j}(F)^{\frac{n-2k}{n}}}=Y_{(I,U)}(M_j^n,[\pi_j^*g])\,\mathrm{Vol}_{\tau_j^*\widetilde{\pi}_j^*g_j}(F)^{\frac{2k}{n}}.$$

Our choice of τ_j and the nonnegativity of $Y_{(I,U)}(M_j^n,[\pi_j^*g])$ imply that

$$\mathcal{I}_j \leq Y_{(I,U)}(M_j^n,[\pi_j^*g])m_j^{-\frac{2k}{n}}.$$

Therefore $\limsup_{j\to\infty}\mathcal{I}_j \leq 0$. Since $Y_{(I,U)}(M^n,[g])>0$, we deduce from Equation (4.1) that no subsequence of $\{\Psi_j\}_{j=1}^\infty$ converges in the compact-open topology. Thus the conformal group of $(\widetilde{M}^n,\widetilde{\pi}^*g)$ is not proper. Since $\widetilde{M}$ is noncompact, we conclude [18, Theorem A_1; 39, Theorem 3.3] that $(\widetilde{M}^n,\widetilde{\pi}^*g)$ is conformally equivalent to flat Euclidean space $(\mathbb{R}^n,\mathrm{d}x^2)$. □

Lemmas 4.1 and 4.2 allow us to refine an argument of Bettiol and Piccione [8, Theorem B] and prove the following generalization of Theorems 1.2 and 1.3.

Theorem 4.3. *Let I be a CVI of weight $-2k$ and rank r, and let U be a geometric cone. Suppose that there is a nonempty geometric Aubin set $\mathcal{A}$ for (I,U). Let $(M^n,g)\in\mathcal{A}$ be such that $\pi_1(M)$ has infinite profinite completion. Then there is an infinite tower*

$$\cdots \xrightarrow{\pi_{k+1}} M_k^n \xrightarrow{\pi_k} \cdots \xrightarrow{\pi_3} M_2^n \xrightarrow{\pi_2} M_1^n \xrightarrow{\pi_1} M_0^n := M^n$$

of finite regular coverings and a sequence $(g_j)_{j=0}^\infty$ of minimizers of $Y_{(I,U)}(M_j^n,[\Pi_j^*g])$ such that for each integer $j\geq 0$, the set $\{(\Pi_j^\ell)^*g_\ell\}_{\ell=0}^j$ consists of pairwise nonhomothetic representatives of $[\Pi_j^*g]$ with constant I-curvature, where $\Pi_j^\ell := \pi_{\ell+1}\circ\cdots\circ\pi_j\colon M_j^n\to M_\ell^n$ and $\Pi_j := \Pi_j^0\colon M_j^n\to M^n$ are defined for all $j\geq\ell$, with the convention $\Pi_j^j=\mathrm{Id}$. Moreover, if $(\widetilde{M}^n,\widetilde{\pi}^*g)$ is not conformally equivalent to $(\mathbb{R}^n,\mathrm{d}x^2)$, then, after passing to a subtower if necessary, the metrics $\{\widetilde{\pi}_j^*g_j\}_{j=0}^\infty$ are pairwise nonhomothetic, where $\widetilde{\pi}\colon\widetilde{M}^n\to M^n$ and $\widetilde{\pi}_j\colon\widetilde{M}^n\to M_j^n$ are the universal covers of M^n and M_j^n, respectively.

Proof. To simplify notation, we denote $Y := Y_{(I,U)}(M^n,[g])$. Similarly, given a finite connected covering $\Pi_j\colon M_j^n\to M^n$, we denote $Y_j := Y_{(I,U)}(M_j^n,[\Pi^*g])$.

We first recursively construct the sequence $\{\pi_j\colon M_j^n\to M_{j-1}^n\}_{j=1}^\infty$. Set $M_0^n := M^n$. Suppose that $j\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ is such that closed manifolds $\{M_\ell^n\}_{\ell=0}^j$ and finite regular coverings $\{\pi_\ell\colon M_\ell^n\to M_{\ell-1}^n\}_{\ell=1}^j$ are given, with the convention that the latter set is empty if $j=0$. Let $V_j>0$ be such that

$$V_j > \left(\frac{Y_{(I,U)}(S^n,[g_{\mathrm{rd}}])}{Y_{(I,U)}(M_j^n,[\Pi_j^*g])}\right)^{\frac{n}{2k}}. \tag{4.2}$$

Choose, using Lemma 4.1, a finite regular covering $\pi_{j+1}\colon M_{j+1}^n\to M_j^n$ such that

$$\mathrm{Vol}_{\pi_{j+1}^*\Pi_j^*g}(M_{j+1}) > V_j.$$

We now construct the sequence $\{g_j\}_{j=1}^\infty$. Let $j\in\mathbb{N}_0$. Since $\Pi_j\colon M_j^n\to M^n$ is a finite connected covering, $(M_j^n,\Pi_j^*g)\in\mathcal{A}$. Thus we may pick a metric $g_j\in[\Pi_j^*g]$ such that $I^{g_j}=Y_{(I,U)}(M_j^n,[\Pi_j^*g])$ and $\mathrm{Vol}_{g_j}(M_j)=1$. Let $\ell\leq j$ be a nonnegative integer. On the one hand, combining Inequality (4.2) with our construction of π_ℓ yields

$$\mathcal{I}^{\pi_\ell^*g_{\ell-1}} = Y_{(I,U)}(M_{\ell-1}^n,[\Pi_{\ell-1}^*g])\,\mathrm{Vol}_{\pi_\ell^*g_{\ell-1}}(M_\ell)^{\frac{2k}{n}} > Y_{(I,U)}(S^n,[g_{\mathrm{rd}}]) \geq \mathcal{I}^{g_\ell}.$$

On the other hand, since both $g_{\ell-1}$ and g_ℓ have constant ℓ-curvature, we compute that

$$\begin{aligned}\mathcal{I}^{(\Pi_j^{\ell-1})^*g_{\ell-1}} = \mathcal{I}^{(\Pi_j^{\ell})^*\pi_\ell^* g_{\ell-1}} = \mathcal{I}^{\pi_\ell^* g_{\ell-1}} \left(\frac{\mathrm{Vol}_{(\Pi_j^\ell)^*\pi_\ell^* g_{\ell-1}}(M_j)}{\mathrm{Vol}_{\pi_\ell^* g_{\ell-1}}(M_\ell)} \right)^{\frac{2k}{n}} = \mathcal{I}^{\pi_\ell^* g_{\ell-1}} \left(\deg \Pi_j^\ell \right)^{\frac{2k}{n}} \\ > \mathcal{I}^{g_\ell} \left(\deg \Pi_j^\ell \right)^{\frac{2k}{n}} = \mathcal{I}^{g_\ell} \, \mathrm{Vol}_{(\Pi_j^\ell)^* g_\ell}(M_j)^{\frac{2k}{n}} = \mathcal{I}^{(\Pi_j^\ell)^* g_\ell}.\end{aligned}$$

In particular, $\mathcal{I}^{g_j} < \mathcal{I}^{(\Pi_j^{j-1})^* g_{j-1}} < \cdots < \mathcal{I}^{(\Pi_j^0)^* g_0}$. The scale and diffeomorphism invariance of the total I-curvature implies that these metrics are pairwise nonhomothetic.

Finally, suppose that $(\widetilde{M}^n, \widetilde{\pi}^* g)$ is not conformally equivalent to Euclidean space. Lemma 4.2 implies that no subsequence of $\{\widetilde{\pi}_j^* g_j\}_{j=0}^\infty$ consists of pairwise homothetic metrics. The conclusion readily follows. □

The existence of closed Riemannian manifolds admitting many nonhomothetic metrics of constant Q_{2k}-curvature in their conformal class follows from work of Qing and Raske [37] and of Case and Malchiodi [15].

Proof of Corollary 1.4. Let U be the geometric cone of all smooth functions. Let N and $\mathcal{A}$ be as in Proposition 3.4. Then $(M_1^m \times M_2^{n-m}, g_1 \oplus g_2) \in \mathcal{A}$ and $\pi_1(M_1 \times M_2)$ has infinite profinite completion. The conclusions now follow from Theorems 1.2 and 1.3. □

Proof of Corollary 1.5. Let U be the geometric cone of all smooth functions. Let $\mathcal{A}$ be as in Proposition 3.6. Then $(M^m \times S^{n-m}, g_{\mathrm{hyp}} \oplus g_{\mathrm{rd}}) \in \mathcal{A}$ and $\pi_1(M)$ has infinite profinite completion. The conclusions now follow from Theorem 1.2 and 1.3. □

The existence of closed Riemannian manifolds admitting many nonhomothetic metrics of constant v_k-curvature in their conformal class follows from the solution [26, 43] of the σ_k-Yamabe Problem.

Proof of Corollary 1.6. Fix $k, m \in \mathbb{N}$. Given $n \in \mathbb{N}$, the k-th elementary symmetric function of the block diagonal matrix $A = -I_m \oplus I_{n-m}$, where I_m and I_{n-m} are the $m \times m$ and $(n-m) \times (n-m)$ identity matrices, respectively, is

$$\sigma_k(A) = \sum_{j=0}^k (-1)^j \binom{m}{j} \binom{n-m}{k-j}. \tag{4.3}$$

In particular, $\sigma_k(A) = n^k/k! + \mathcal{O}(n^{k-1})$ for $n \gg 1$. It follows that there is a constant $N = N(k,m)$ such that if $n \geq N$, then $\sigma_j(A) > 0$ for all $j \in \{1, 2, \ldots, k\}$.

Now let $(M_1^m, g_{\mathrm{hyp}})$ and $(M_2^{n-m}, g_{\mathrm{rd}})$ be closed spaceforms with constant sectional curvature -1 and 1, respectively, where $n \geq N$. Then their Riemannian product $(M_1^m \times M_2^{n-m}, g_{\mathrm{hyp}} \oplus g_{\mathrm{rd}})$ is locally conformally flat [6, Example 1.167(3)]. Moreover, the Schouten tensor of the product is

$$P = \frac{1}{2}\left(-g_{\mathrm{hyp}} \oplus g_{\mathrm{rd}}\right). \tag{4.4}$$

Thus $(M_1^m \times M_2^{n-m}, g_{\mathrm{hyp}} \oplus g_{\mathrm{rd}})$ is in the geometric Aubin set of Proposition 3.1. The conclusions now follow from Theorem 1.2 and 1.3 and the Selberg–Malcev Lemma. □

Remark 4.4. In the notation of the proof of Corollary 1.6, it is easily checked that $N(1,m) = 2m+1$. Direct computation using Equation (4.3) implies that $N(n,1) = n - 2k + 1$. The dimensions m

and n for which v_2 or v_3 vanish on $(H^m \times S^{n-m})$ are known [16, Lemma 6.1]. Thus estimates for $N(2,m)$ and $N(3,m)$ are known. Indeed, Equation (4.3) implies that

$$\begin{aligned} 2v_1 &= n - 2m, \\ 4v_2 &= \frac{n^2 - (4m+1)n + 4m^2}{2}, \\ 8v_3 &= \frac{(n-2m)(n^2 - (4m+3)n + 4m^2 + 2)}{6}. \end{aligned}$$

In particular, one has

$$\begin{aligned} N(2,m) &> \frac{4m+1+\sqrt{8m+1}}{2}, \\ N(3,m) &> \frac{4m+3+\sqrt{24m+1}}{2}. \end{aligned}$$

When $k \geq 4$, there are [42] only finitely many choices of m and n for which $v_k = 0$, so sharp estimates for $N(k,m)$, $k \geq 4$, are more difficult to find.

Appendix A. The Aubin Lemma for the Q-curvature

The Aubin Lemma [3, Theorem 6] asserts that if $\pi\colon (\widetilde{X}^n, \widetilde{g}) \to (X^n, g)$ is a finite connected Riemannian covering and $Y_{Q_2}(\widetilde{X}^n, [\widetilde{g}]) > 0$, then $Y_{Q_2}(X^n, [g]) < Y_{Q_2}(\widetilde{X}^n, [\widetilde{g}])$. Aubin's proof assumes a regular covering, but this assumption can be removed [1, Lemma 3.6]. The proof of the Aubin Lemma requires only the existence of a minimizer for $Y_{Q_2}(\widetilde{X}^n, [\widetilde{g}])$ and Jensen's inequality, and as such, can be extended to the higher-order Q-curvatures.

Lemma A.1. *Let $k \in \mathbb{N}$. Let (X^n, g), $n > 2k$, be a closed Riemannian manifold. Suppose that $\pi\colon \widetilde{X}^n \to X^n$ is a finite connected covering of degree at least two such that $Y_{Q_{2k}}(\widetilde{X}^n, [\pi^* g]) > 0$ and there is a positive minimizer for $Y_{Q_{2k}}(\widetilde{X}^n, [\pi^* g])$. Then*

$$Y_{Q_{2k}}(X^n, g) < Y_{Q_{2k}}(\widetilde{X}^n, [\pi^* g]).$$

Proof. Denote $\widetilde{g} := \pi^* g$. Let $u \in C^\infty(\widetilde{X}; \mathbb{R}_+)$ be such that

$$\begin{aligned} P_{2k}^{\widetilde{g}} u &= \frac{n-2k}{2} Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]) u^{\frac{n+2k}{n-2k}}, \\ \int_{\widetilde{X}} u^{\frac{2n}{n-2k}} \, \mathrm{dV}_{\widetilde{g}} &= 1, \end{aligned}$$

where P_{2k} is the GJMS operator [24] of order $2k$.

Denote by $d \geq 2$ the degree of π. Given $x \in \widetilde{X}$, set $\{x_1, \dots, x_d\} = \pi^{-1}\bigl(\pi(x)\bigr)$. Given a constant $p > 0$, define

$$\widetilde{u}_{(p)}(x) := \sum_{j=1}^d u(x_j)^p.$$

This defines a function $\widetilde{u}_{(p)} \in C^\infty(\widetilde{X}; \mathbb{R}_+)$. Set $\widetilde{u} := \widetilde{u}_{(1)}$. Observe that if $V \subset X$ is an evenly covered open set, then

$$\widetilde{u}_{(p)} = \sum_{j=1}^d (u \circ \gamma_j)^p,$$

where $\gamma_j \colon V_1 \to V_j$ are isometries between the connected components $V_1, \dots, V_d$ of $\pi^{-1}(V)$. We may piece these isometries together to produce m distinct isometries

$$\widetilde{\gamma}_j \colon \widetilde{X} \setminus \mathcal{S} \to \widetilde{X} \setminus \mathcal{S}, \qquad j \in \{1, \dots, d\},$$

where $\mathcal{S} \subset \widetilde{X}$ is a piecewise smooth compact $(n-1)$-dimensional submanifold of $\widetilde{X}^n$ such that $\pi^{-1}\big(\pi(\mathcal{S})\big) = \mathcal{S}$. Therefore

$$\widetilde{u}_{(p)} = \sum_{j=1}^{d} \widetilde{u}_j^p,$$

where $\widetilde{u}_j := u \circ \widetilde{\gamma}_j$. Since each $\widetilde{\gamma}_j$ is an isometry, we deduce that

$$P_{2k}^{\widetilde{g}} \widetilde{u} = \frac{n-2k}{2} Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]) \sum_{j=1}^{d} \widetilde{u}_j^{\frac{n+2k}{n-2k}},$$
$$\int_{\widetilde{X}} \widetilde{u}_{\left(\frac{2n}{n-2k}\right)} \, \mathrm{dV}_{\widetilde{g}} = d.$$

Now, since $\widetilde{u}(x) = \widetilde{u}(x')$ for any $x, x' \in \widetilde{X}$ such that $\pi(x) = \pi(x')$, there is a $u_0 \in C^\infty(X; \mathbb{R}_+)$ such that $\widetilde{u} = u_0 \circ \pi$. Direct computation gives

$$\begin{aligned} \mathcal{I}_{2k}^{u_0^{\frac{4}{n-2k}} g} &= d^{-\frac{2k}{n}} \mathcal{I}_{2k}^{\widetilde{u}^{\frac{4}{n-2k}} \widetilde{g}} \\ &= d^{-\frac{2k}{n}} \left(\int_{\widetilde{X}} \widetilde{u} P_{2k}^{\widetilde{g}}(\widetilde{u}) \, \mathrm{dV}_{\widetilde{g}} \right) \left(\int_{\widetilde{X}} \widetilde{u}^{\frac{2n}{n-2k}} \, \mathrm{dV}_{\widetilde{g}} \right)^{-\frac{n-2k}{n}} . \end{aligned} \tag{A.1}$$

Since $d \geq 2$ and $u > 0$, applying Hölder's inequality and then convexity yields

$$\widetilde{u} \left(\sum_{j=1}^{d} \widetilde{u}_j^{\frac{n+2k}{n-2k}} \right) \leq \widetilde{u} \left(\sum_{j=1}^{d} \widetilde{u}_j^{\frac{2n}{n-2k}} \right)^{\frac{2k}{n}} \left(\sum_{j=1}^{d} \widetilde{u}_j^{\frac{n}{n-2k}} \right)^{\frac{n-2k}{n}} < \widetilde{u}^2 \left(\sum_{j=1}^{d} \widetilde{u}_j^{\frac{2n}{n-2k}} \right)^{\frac{2k}{n}} .$$

Since $Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]) > 0$, we deduce that

$$\begin{aligned} \int_{\widetilde{X}} \widetilde{u} P_{2k}^{\widetilde{g}}(\widetilde{u}) \, \mathrm{dV}_{\widetilde{g}} &= Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]) \int_{\widetilde{X}} \widetilde{u} \left(\sum_{j=1}^{d} \widetilde{u}_j^{\frac{n+2k}{n-2k}} \right) \mathrm{dV}_{\widetilde{g}} \\ &< Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]) \int_{\widetilde{X}} \widetilde{u}^2 \widetilde{u}_{\left(\frac{2n}{n-2k}\right)}^{\frac{2k}{n}} \, \mathrm{dV}_{\widetilde{g}} \\ &\leq Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]) \left(\int_{\widetilde{X}} \widetilde{u}^{\frac{2n}{n-2k}} \, \mathrm{dV}_{\widetilde{g}} \right)^{\frac{n-2k}{n}} \left(\int_{\widetilde{X}} \widetilde{u}_{\left(\frac{2n}{n-2k}\right)} \, \mathrm{dV}_{\widetilde{g}} \right)^{\frac{2k}{n}} \\ &= d^{\frac{2k}{n}} Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]) \left(\int_{\widetilde{X}} \widetilde{u}^{\frac{2n}{n-2k}} \, \mathrm{dV}_{\widetilde{g}} \right)^{\frac{n-2k}{n}} . \end{aligned}$$

Combining this with Equation (A.1) yields

$$Y_{Q_{2k}}(X^n, [g]) \leq \mathcal{I}_{2k}^{u_0^{\frac{4}{n-2k}} g} < Y_{Q_{2k}}(\widetilde{X}^n, [\widetilde{g}]).$$ □

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(J.H. Andrade) Department of Mathematics, University of British Columbia
Vancouver, BC V6T 1Z2, Canada
and
Institute of Mathematics and Statistics, University of São Paulo
São Paulo, SP 05508-090, Brazil
Email address: andradejh@math.ubc.ca
Email address: andradejh@ime.usp.br

(J.S. Case) Department of Mathematics, The Pennsylvania State University
State College, PA 16802, USA
Email address: jscase@psu.edu

(P. Piccione) Institute of Mathematics and Statistics, University of São Paulo
São Paulo, SP 05508-090, Brazil
Email address: piccione@ime.usp.br

(J. Wei) Department of Mathematics, University of British Columbia
Vancouver, BC V6T 1Z2, Canada
Email address: jcwei@math.ubc.ca