Lecture 20

Donaldson-Thomas theory

There are two main paradigms for presenting curves:

- Parameterized curves, i.e. image of maps $\rightsquigarrow$ GW theory
- Curves given by equations, i.e. ideal sheaves | subschemes $\rightsquigarrow$ DT theory

DT theory began as a theory which counts holomorphic bundles on a CY3 as holomorphic Chern-Simons theory. The idea is that the holomorphic forms on a CY3 $H^{0,k}(X)$ looks like $H^k_{DR}(M^3)$ DeRham cohomology of a real 3-mfd. Chern-Simons theory is a theory of 3-mfd invariants and some of the constructions in the real 3-mfd case can be imitated.

The outcome is that if $\overline{M}(X, ch)$ is a compact moduli space of vector bundles then there is $[\overline{M}(X,ch)]^{vir} \in H_0(\overline{M}(X,ch))$ a virtual class.

to construct $\overline{M}(X,ch)$ we need to fix $ch \in H^*(X)$ the chern character of the bundles and more subtly, we must fix a stability condition.

In algebraic geometry we identify a holomorphic vector bundle on X with its sheaf of sections (such sheaves are locally free sheaves of $\mathcal{O}_X$ modules).

DT theory not only works for moduli of bundles, but in fact for moduli spaces of coherent sheaves more generally.

For curve counting we consider very special kinds of sheaves, namely ideal sheaves $I_Z \subset \mathcal{O}_X$, i.e. the sheaf of functions which vanish on some subscheme $Z \subset X$. On a CY3, if a sheaf has the chern character of
(curve)

an ideal sheaf, it must actually be an ideal sheaf so there is a bijective correspondence $I_Z \subset \mathcal{O}_X \longleftrightarrow Z \subset X$ between ideal sheaves and subschemes.

In the early 1960's Grothendieck constructed the <u>Hilbert scheme</u> which is a scheme that parameterizes subschemes (of a fixed Hilbert polynomial). Moduli space of ideal sheaves can thus be identified with the Hilbert scheme.

<u>**Def'n**</u> Let X be a CY3 and $\beta \in H_2(X,\mathbb{Z})$ a curve class and $n \in \mathbb{Z}$. Let $I_n(X,\beta)$ be the Hilbert scheme parameterizing subschemes $Z \subset X$ with $[Z] = \beta$ and $\chi(\mathcal{O}_Z) = n$. Equivalently, $I_n(X,\beta)$ may be regarded as the moduli space of ideal sheaves $I_Z \subset \mathcal{O}_X$ with $ch(I_Z) = (ch_0, ch_1, ch_2, ch_3) = (1, 0, -\beta, -n)$

Since $[Z] \in H_2(X,\beta)$ Z has dim 1 (although it may have 0 dim'l components). If $Z \subset X$ is a smooth curve of genus g, then $\chi(\mathcal{O}_Z) = 1-g$.

In general, subschemes can be non-reduced and or contain embedded points.

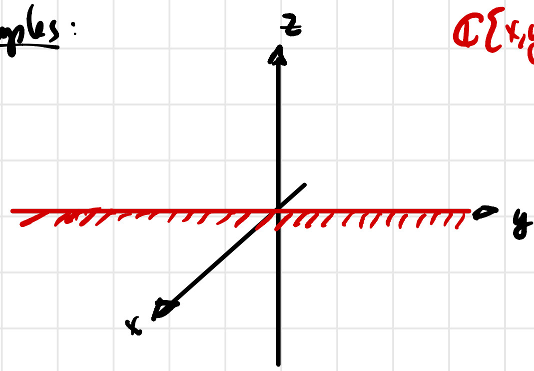
Schemes vs Varieties

closed algebraic sets $Z \subset \mathbb{A}^n \rightsquigarrow$ radical ideals $I_Z \subset \mathbb{C}[x_1 \cdots x_n]$

closed subschemes $Z \subset \mathbb{A}^n \rightsquigarrow$ ideals $I_Z \subset \mathbb{C}[x_1 \cdots x_n]$

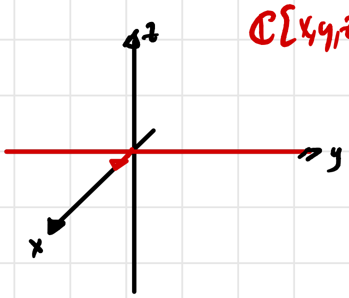
so for a subscheme $Z \subset \mathbb{A}^n$, its ring of functions $\mathcal{O}_Z = \mathbb{C}[x_1 \cdots x_n]/I_Z$ can have nilpotent elements.

examples:



$\mathbb{C}[x,y,z]/(z,x^2)$

the ideal (z, x^2) describes a subscheme which is supported on y axis $\sqrt{(z,x^2)} = (z,x)$ but is infinitesimally thickened into the xy plane. The "function" x is a nilpotent element of the functions of a subscheme.



$\mathbb{C}[x,y,z]/(z, x^2, xy)$

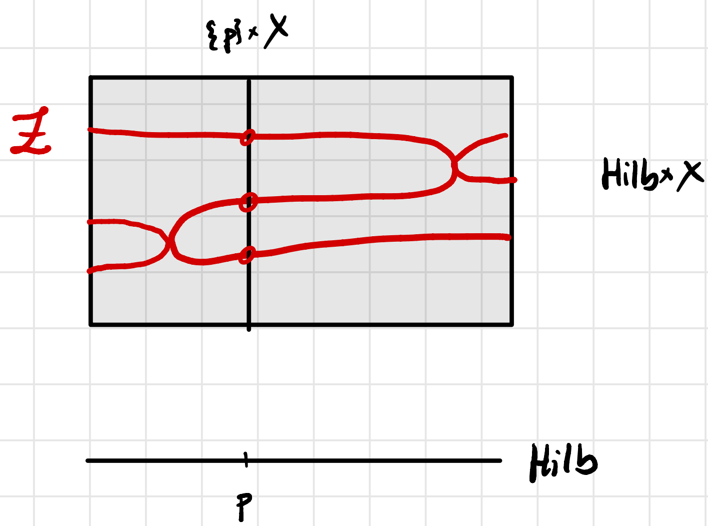
away from $x=y=z=0$ the ideal is radical, but the function x is still a nilpotent. This curve has an <u>embedded point</u> at the origin.

Lecture 21 Let $X \subset \mathbb{P}^N$ be a projective variety and let $Z \subset X$ be a subscheme. The Hilbert polynomial of Z is $P_Z(m) = \chi(\mathcal{O}_Z(m))$. For $n >> 0$,

$P_Z(m) = H^0(Z, \mathcal{O}(m)) =$ dim of space of degree N polynomials on Z. $P_Z(m)$ is a polynomial of degree $\dim Z$

Grothendieck showed in the 60's $\exists$ a projective scheme $\mathrm{Hilb}^P(X)$ whose points correspond subschemes $Z \subset X$ with $P_Z = P$. There is a universal subscheme $\mathcal{Z} \subset \mathrm{Hilb}^P(X) \times X$ such that if $\pi : \mathrm{Hilb}^P(X) \times X \longrightarrow \mathrm{Hilb}^P(X)$ and $p \in \mathrm{Hilb}^P(X)$ corresponds to $Z_p \subset X$ then $\mathcal{Z}\big|_{\pi^{-1}(p)} = Z_p$. Moreover, if $\mathcal{Y} \subset B \times X$ is a family of subschemes over B with constant Hilbert poly P, then $\mathcal{Y} \subset B \times X$ is induced from the universal family via a unique map $B \xrightarrow{f} \mathrm{Hilb}^P(X)$

$$\begin{array}{ccc} B \times X & \xrightarrow{f \times id} & \mathrm{Hilb}^P(X) \times X \\ \cup & & \cup \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & \mathrm{Hilb}^P(X) \end{array}$$



<u>Example</u> if $Z \subset X$ is 0 dim'l and consists of k (reduced) points then $P_Z(m) = k$ ← constant polynomial.

$\mathrm{Hilb}^k(X)$ = Hilbert scheme of k points. $\mathrm{Hilb}^k(X) \longrightarrow \mathrm{Sym}^k(X)$ (Hilbert - Chow morphism)

dominant birational map
only an iso when $\dim X = 1$

$\mathrm{Hilb}^k(X)$ is non-singular if $\dim X = 1$ or 2
(for $\dim X = 2$ $\mathrm{Hilb}^k(X) \to \mathrm{Sym}^k(X)$ is a resolution of singularities)

$\mathrm{Hilb}^2(X) = \mathrm{Bl}_\Delta(\mathrm{Sym}^2 X)$ ⟵ subscheme of length 2 remembers the direction 2 points come together

<u>Ex.</u> If $C \subset X$ is a curve $[C] = \beta$ then $P_C(m) = \chi(\mathcal{O}_C) + m H \cdot \beta$ ← by varying ample line bundle, equiv embedding $X \subset \mathbb{P}^N$ we can recover all of β.

β and $n = \chi(\mathcal{O}_C)$ we will call $\mathrm{Hilb}^P(X) = I_n(X, \beta)$ in this case.

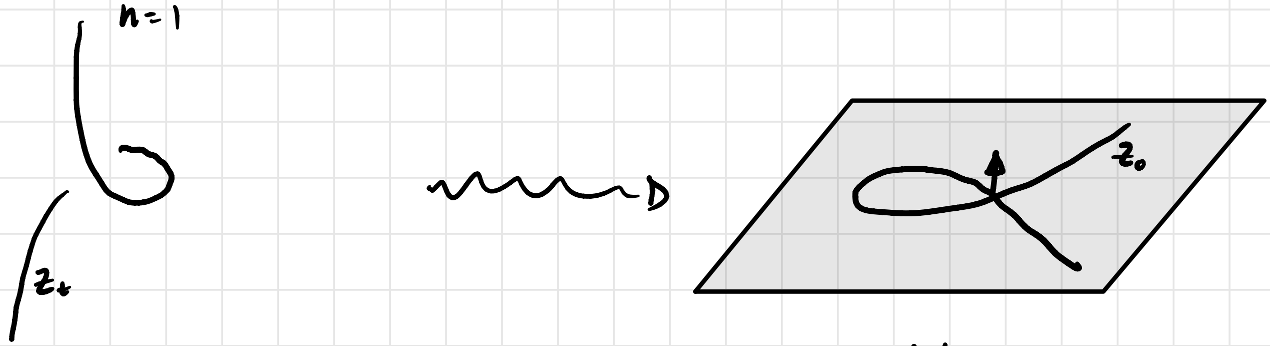
Scheme phenomena are forced on us when we consider moduli spaces of curves:

we may have a family $Z_t \subset X$ $t \in \mathbb{A}^1$ with $[Z_t] = \beta$, $\chi(\mathcal{O}_{Z_t}) = n$

where $Z_{t \neq 0}$ are smooth, reduced curves but Z_0 has an embedded point. (families of curves in a projective threefold with constant β and n are <u>flat</u>).

ex. $Z_t \subset \mathbb{P}^3$ (not CY but serves to illustrate).

family of twisted cubics degenerating to a plane cubic

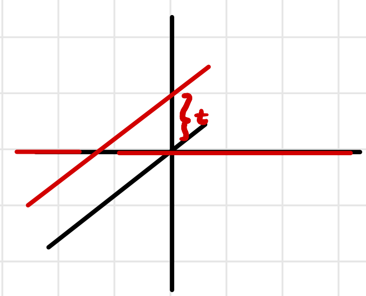


$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$(x:y) \mapsto (x^3 : x^2 y : x y^2 : y^3)$

if curve didn't have an embedded point at origin, then $n = \chi(\mathcal{O}_{Z_0}) = 0$ since arithmetic genus of plane cubic is 1.

<u>local model</u>: $Z_{t \neq 0} = \{x = z = 0\} \cup \{y = z - t = 0\} \subset \mathbb{C}^3$

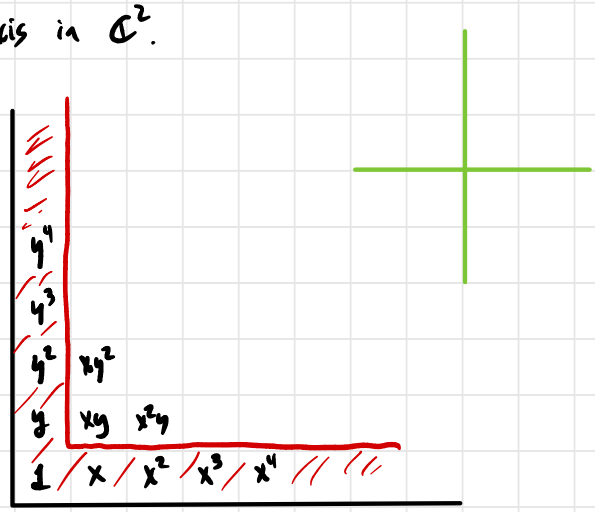


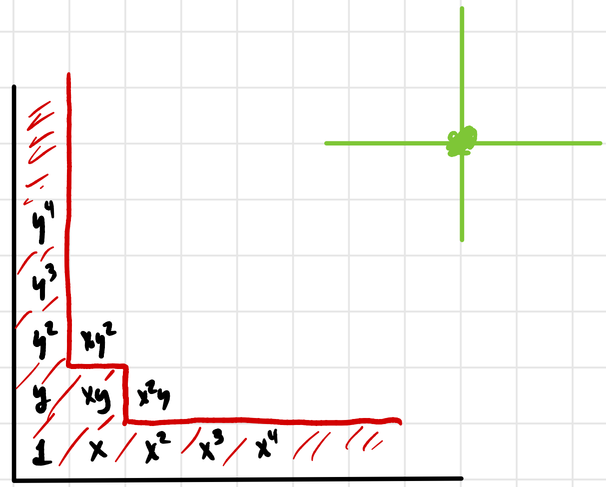
$$I_{Z_{t \neq 0}} = (x, z) \cdot (y, z - t) = (xy, zy, xz - ty, z^2 - tz)$$

$$\lim_{t \to 0} I_{Z_t} = (xy, zy, xz, z^2) \subsetneq (z, xy)$$ nilpotent elt in $\mathcal{O}_{Z_0}$ is z

Lecture 22

Here is an important way to visualize a scheme defined by an ideal generated by monomials. Let's start in dim 2 think about the union of the x and y axis in $\mathbb{C}^2$.

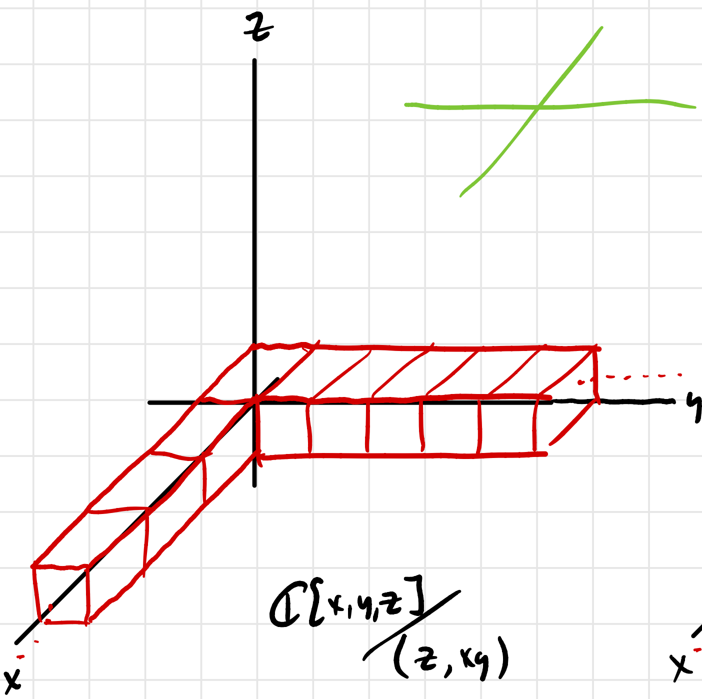




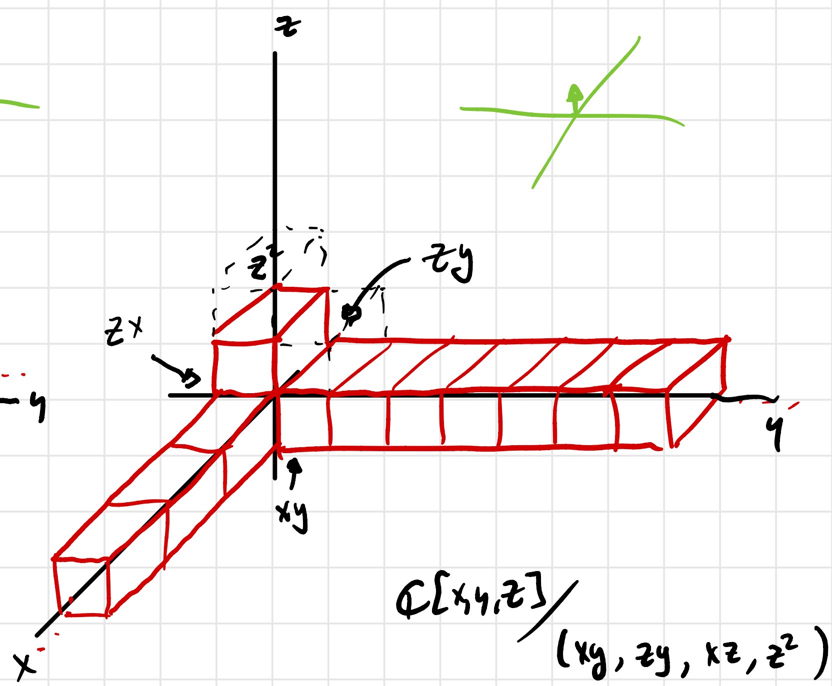
$Z = \{xy = 0\}$ $\quad \mathcal{O}_Z = \mathbb{C}[x,y]/(xy)$

$Z = \{x^2y = xy^2 = 0\}$ $\quad \mathcal{O}_Z = \mathbb{C}[x,y]/(xy^2, x^2y)$

For monomial ideals in $\mathbb{C}[x,y,z]$ we do something similar with boxes in the positive octant

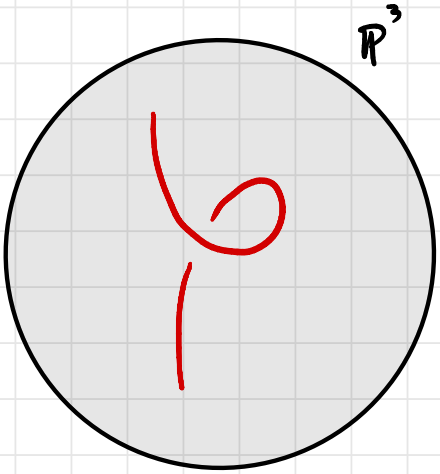


$\mathbb{C}[x,y,z]/(z, xy)$

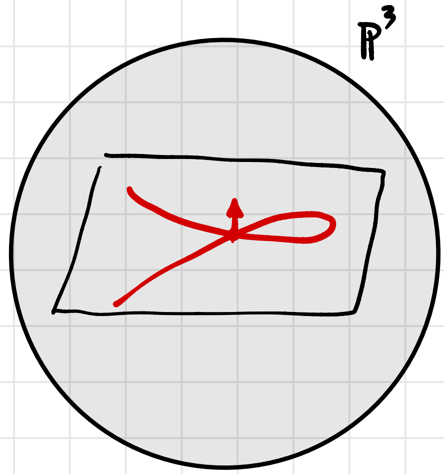


$\mathbb{C}[x,y,z]/(xy, zy, xz, z^2)$

In fact we can get flat families of subschemes like this:



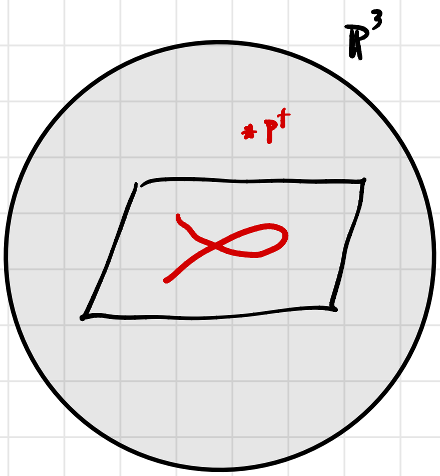
$\rightsquigarrow$



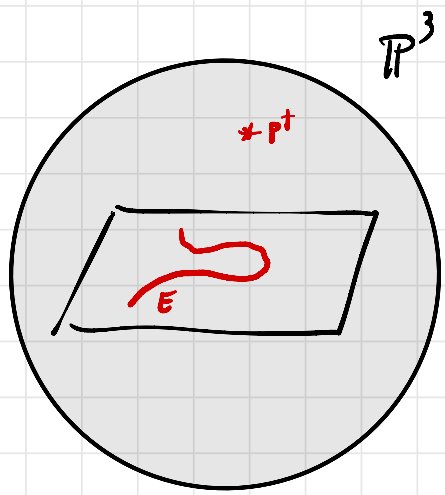
smooth genus 0 $(\chi=1)$
curve class $3[L]$

planar nodal
cubic with embedded
point

$\rightsquigarrow$



$\rightsquigarrow$



planar nodal
cubic with point
far away

smooth planar cubic
with point far away

$$\chi(E \cup pt) = \chi(E) + \chi(pt)$$
$$= 0 + 1 = 1$$

We see that with subschemes, we must allow disconnected things and we must allow points as well as curves.

We use the notation $I_n(X,\beta)$ rather than $\mathrm{Hilb}^{n+m\beta\cdot H}(X)$ because we may regard it as a moduli space of sheaves: suppose I is a coherent sheaf on a CY3 X with $ch(I) = (1,0,-\beta,-n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6$ then the canonical map $I \hookrightarrow I^{\vee\vee} \cong \mathcal{O}_X$ is injective and thus realizes I as a subsheaf of $\mathcal{O}_X$ and thus an ideal sheaf $I = I_Z$ then I is automatically stable and $[Z] = \beta$ $\chi(\mathcal{O}_Z) = n$.

Lecture 23

Like stable maps, the Hilbert schemes $I_n(X,\beta)$ can be very singular and have components of different dimension. But like stable maps they behave well because of a virtual fundamental class (they are "virtually smooth"). This ideal is codified by the notion of a perfect obstruction theory which is a gadget that keeps track of how the moduli space is locally cut out of a smooth space by equations, namely if M is some moduli space parameterizing objects E, then locally near $[E] \in M$ M is described as the zero locus of a map

Kuranishi map analytic of formal function

$$\mathrm{Def}(E) \xrightarrow{K} \mathrm{Ob}(E)$$

Infinitesimal deformations of E, i.e. $T_{[E]}M$

Space of obstructions to lifting infitesimal deformation to an actual one for example if $[f: C \to X] \in \overline{M}_g(X,\beta)$ is an embedding of a smooth curve, then $Def(f) = H^0(C, f^* N_{C/X})$ and $Ob(f) = H^1(C, f^* N_{C/X})$

For any moduli of sheaves on a CY3 we have $Def(E) = Ext^1(E,E)$ $Ob(E) = Ext^2(E,E)$. In the case where E is a bundle then

$$Ext^i(E,E) = H^i(X, E^* \otimes E) = H^i(X, End\, E)$$

Serre duality says that for any smooth X of dim d

$$Ext^i(F,G) \cong Ext^{d-i}(G, F \otimes K_X)$$

In particular, for X a CY3 $Ext^1(E,E) \cong Ext^2(E,E)^\vee$ i.e.

<u>Key fact</u> Deformations are dual to obstructions

So the Hilbert scheme / moduli of ideal sheaves $I_n(X,\beta)$ is locally near I_Z given by $\kappa^{-1}(0) \subset Def(I_Z)$ where

$$\begin{array}{ccc} Def(I_Z) & \xrightarrow{\kappa} & Ob(I_Z) \\ \| & & \| \\ Ext^1(I_Z,I_Z) & & Ext^2(I_Z,I_Z) \cong Ext^1(I_Z,I_Z)^\vee \end{array}$$

so κ is a section of $T^* Def(I)$, namely a differential 1-form on Def.

Since Def is just a vector space, every 1-form is exact so $\kappa = df$ where

$f : Ext^1(I_Z,I_Z) \longrightarrow \mathbb{C}$ (Jargon f is the local superpotential)

Locally at a subscheme $Z \subset X$ $[Z] \in I_n(X,\beta)$ $I_n(X,\beta)$ is given by $\{df = 0\}$, i.e. it is the critical locus of a function $f : \mathrm{Ext}^1(I_Z, I_Z) \to \mathbb{C}$

- Since $\dim \mathrm{Def} = \dim \mathrm{Ob}$ $\mathrm{vdim} = 0$ and we get $[I_n(X,\beta)]^{vir} \in H_0(I_n(X,\beta); \mathbb{Z})$

<u>**Def'n**</u> $N^{DT}_{n,\beta}(X) = \int_{[I_n(X,\beta)]^{vir}} 1 \quad \in \mathbb{Z}$

Recall that one property of the virtual class is the following: if M is a moduli space with a virtual class $[M]^{vir}$ and M is smooth but not of the expected dimension, then $[M]^{vir} = [M] \cap c_{TOP}(\mathrm{Ob})$. In particular, if $\mathrm{vdim} = 0$ then
$$\int_{[M]^{vir}} 1 = \int_{[M]} c_{TOP}(\mathrm{Ob})$$
where $\mathrm{Ob} \to M$ is the obstruction bundle

For DT theory $\mathrm{Ob} = \mathrm{Def}^\vee = T^*M$ so
$$\int_{[M]^{vir}} 1 = \int_{[M]} c_{TOP}(T^*M) = (-1)^{\dim M} e(M)$$
topological euler char.

Amazingly, a formula like the above holds even if M is singular

Theorem (Behrend) Let $M = I_n(X,\beta)$ or more generally any moduli space of sheaves on a CY3, then

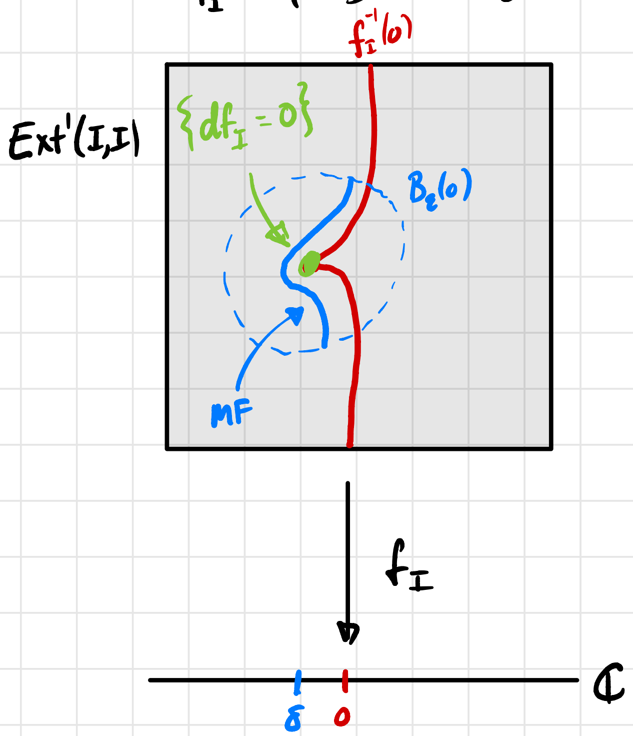
$$\int_{[M]^{vir}} 1 = e_{vir}(M) := \sum_{k\in\mathbb{Z}}{}' \, k \cdot e(\nu_M^{-1}(k))$$

defined for any scheme $/\mathbb{C}$

where $\nu_M : M \longrightarrow \mathbb{Z}$ is the Behrend function a constructible function defined by $\nu_M([I]) = (-1)^{\dim \mathrm{Ext}^1(I,I)}\left(1 - e(MF_{f_I})\right)$ where MF_{f_I} is the Milnor fiber of $f_I : \mathrm{Ext}^1(I,I) \longrightarrow \mathbb{C}$ the local superpotential at $[I] \in M$.

The Milnor fiber is a classical invariant in singularity theory.

$$MF_{f_I} = \{ f_I^{-1}(\delta) \cap B_\varepsilon(0) : 0 < \delta \ll \varepsilon \ll 1 \}$$



Example If M is smooth then $f = 0$ and $MF_f = \emptyset$ so $\nu_M([I]) = (-1)^{\dim \mathrm{Ext}^1(I,I)} = (-1)^{\dim M}$

In general, ν_M weights singularities and non-reduced structure.

<u>example</u>: $M = \mathrm{Spec}\left(\mathbb{C}[x]/x^n\right)$, i.e. it is a "fat" point of length n.

The tangent space $T_{pt}M = \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$, $x \longmapsto x^{n+1}$

since $\{df = 0\} \Leftrightarrow x^n = 0$

$$\nu_M(pt) = (-1)(1 - e(\underbrace{\{x^{n+1} = \delta\} \cap B_\varepsilon(0)}_{\text{scales to } \{x^{n+1}=1\} = n+1 \text{ pts}}))$$

so $\nu_M(pt) = (-1)(1-(n+1)) = n$

so while $e(M) = 1$ $\quad e_{vir}(M) = n$

Lecture 24

Compare Paradigms:

$$N^{DT}_{n,\beta}(X) = \int_{[I_n(X,\beta)]^{vir}} 1 = \underbrace{\sum_{k \in \mathbb{Z}}{}' \, k\, e(\nu^{-1}(k))}$$

Integral $\qquad\qquad$ weighted Euler char

Advantages of right hand side:

- Can be computed strata by strata. Euler char is <u>motivic</u>:

$e(X) = e(X - Z) + e(Z)$ for Z closed $\qquad e(X \times Y) = e(X) \cdot e(Y)$.

- Doesn't require compactness. If moduli space is non-compact (for example if X is non-compact) we can <u>define</u> $N^{DT}_{n,\beta}(X)$ by RHS

- If M has a $\mathbb{C}^\times$ action, then $e(M) = e(M^{\mathbb{C}^\times})$ fixed point locus

- Suggests the existence of <u>Categorified DT Invariants</u>:

ordinary Euler char $\quad e(M) = \sum_i (-1)^k \dim H^k(M)$

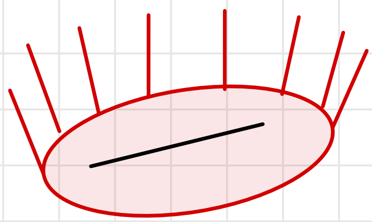
is there some cohomology $\tilde{H}^*(M)$ so that $e_{vir}(M) = \sum_i (-1)^k \dim \tilde{H}^k(M)$?

(yes!). Such a thing $\tilde{H}^*(M)$ is the <u>categorified</u> DT invariant associated to M

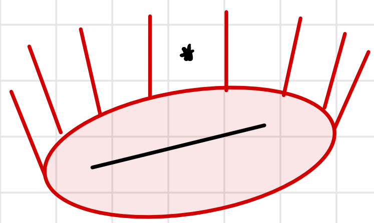
$$\text{Numbers} \underset{\text{euler char}}{\leftrightsquigarrow} \text{Graded Vector spaces}$$

(set) $\qquad\qquad$ (category)

<u>Computations</u> $\quad X = \text{total}(\mathcal{O}(-3) \to \mathbb{P}^2) = \text{"local } \mathbb{P}^2\text{"}$

$I_1(X, [\mathbb{P}^1]) = \Big\{$  $\Big\} = \{\text{line in } \mathbb{P}^2\} = \mathbb{P}^2$

so $N^{DT}_{1,[\mathbb{P}^1]} = e_{vir}(\mathbb{P}^2) = (-1)^{\dim \mathbb{P}^2} e(\mathbb{P}^2) = 3$

$I_2(X, [\mathbb{P}^1]) = \Big\{$  $\Big\} = \left\{ \begin{array}{l} \text{line in } \mathbb{P}^2 \text{ with a point} \\ \text{in } X \text{ (when point is on} \\ \text{line, it has the structure} \\ \text{of an embedded point)} \end{array} \right\}$

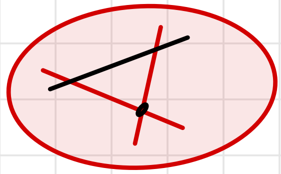
5 dim'l and smooth

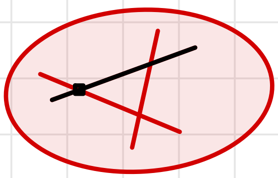
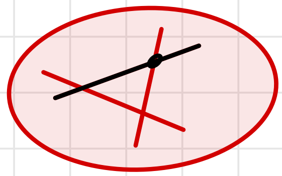
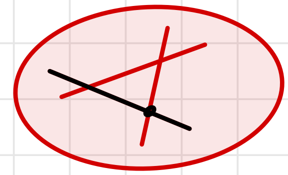
need to check local str. at embedded points

$$N^{DT}_{2,[\mathbb{P}^1]}(X) = e_{vir}(I_2(X,[\mathbb{P}^1])) = (-1)^5 e(I_2(X,[\mathbb{P}^1])) = -\,e(I_2(X,[\mathbb{P}^1])^{\mathbb{C}^\times})$$

subschemes fixed by

$\mathbb{C}^\times \subset (\mathbb{C}^\times)^3$ ← torus acting on toric X.

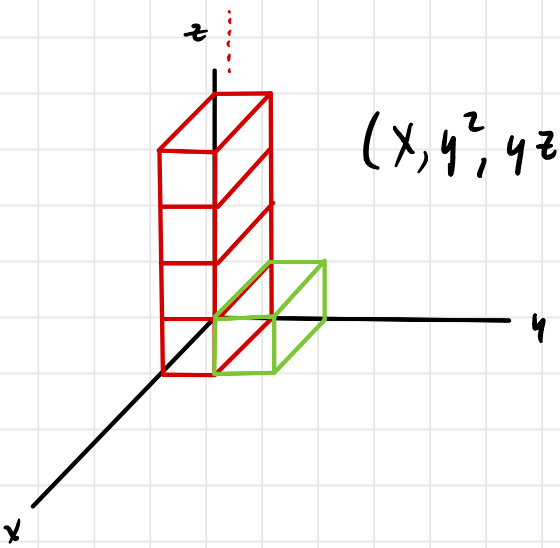
$$I_2(X,[\mathbb{P}^1])^{\mathbb{C}^\times} = \Big\{$$  $$\Big\} \cup \cdots$$

$$\Big\{$$  $$\Big\} \cup \Big\{$$  $$\Big\} \cup \cdots \cup \Big\{$$  $$\Big\}$$

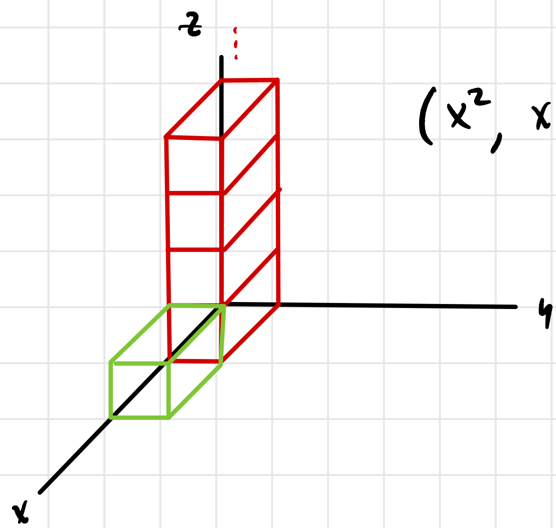
so $e\left(I_2(X,[\mathbb{P}^1])^{\mathbb{C}^\times}\right) =$

$$3 + 6 \cdot \#\left\{ Z \subset \mathbb{C}^3,\ \mathbb{C}^\times \text{ invariant, supported on } z\text{-axis, embedded point at origin} \right\}$$

$$= 3 + 6 \cdot \#\left\{ I \subset \mathbb{C}[x,y,z],\ I \text{ generated by monomials}, \sqrt{I} = (x,y) \text{ and } \dim \sqrt{I}/I = 1 \right\}$$



(x, y^2, yz)



(x^2, xz, y)

So $N_2(X,[\mathbb{P}^1]) = -15$. Can compute by Box counting! We will return to this when we discuss the topological vertex.

Lecture 25

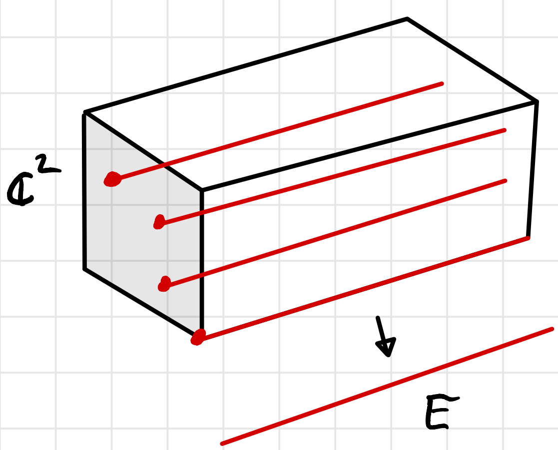
example Local elliptic curve $X = \mathrm{Tot}(L \oplus L^{-1} \to E)$ L is generic degree 0 line bundle. In GW theory we chose L to be generic so $E \subset X$ is super rigid: E doesn't deform and no multiple of E deforms. In DT theory we are less concerned about non-compactness. We first compute for $\tilde{X} = \mathrm{Tot}(\mathcal{O}_E \oplus \mathcal{O}_E \to E)$

$\tilde{X} = \mathbb{C}^2 \times E$

$$N^{DT}_{n,d[E]} = e_{vir}(I_n(\tilde{X}, d[E]))$$ the group $(\mathbb{C}^\times)^2 \times E$ acts on $I_n(X, d[E])$ and $e_{vir}(I_n(\tilde{X}, d[E])) = e_{vir}\left(I_n(\tilde{X}, d[E])^{(\mathbb{C}^\times)^2 \times E}\right)$

What subschemes $Z \subset \mathbb{C}^2 \times E$ are preserved by $(\mathbb{C}^\times)^2 \times E$?

First just consider $I_n(\tilde{X}, d)^E \leftarrow$ no embedded points and such subschemes $Z \subset \mathbb{C}^2 \times E$ are determined by their restriction to $Z|_{\mathbb{C}^2 \times p} \leftarrow$ length d zero dim'l subscheme



$$I_n(\tilde{X}, d[E])^E = \begin{cases} \emptyset & n \neq 0 \\ \mathrm{Hilb}^d(\mathbb{C}^2) & n = 0 \end{cases}$$

$$N^{DT}_{n,d[E]}(\tilde{X}) = \begin{cases} e_{vir}(\mathrm{Hilb}^d(\mathbb{C}^2)) & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$= \#$ partions of d $= p(d)$.

$$e_{vir}(\mathrm{Hilb}^d(\mathbb{C}^2)) = (-1)^{2d} e(\mathrm{Hilb}^d(\mathbb{C}^2)) = e\left(\mathrm{Hilb}^d(\mathbb{C}^2)^{(\mathbb{C}^\times)^2}\right) = \#\left\{ I \subset \mathbb{C}[x,y],\ I \text{ generated by monomials},\ \dim \mathbb{C}[x,y]/I = d \right\}$$

GW/DT Correspondence

Recall the GW potentials and partition function

$F_g^{GW}(x) = \sum_\beta' N_{g,\beta}^{GW} v^\beta$ genus g potential

$F^{GW}(x) = \sum_g' F_g \lambda^{2g-2}$ all genus potential

$F'_{GW} = F^{GW} - F^{GW}|_{v=0}$ ← F'_{GW} doesn't include $\beta = 0$ invariants

$Z_{GW} = \exp(F_{GW})$ ← GW partition function, generating function for possibly disconnected invariants

$Z'_{GW} = \exp(F'_{GW}) = \dfrac{Z_{GW}}{Z_{GW}|_{v=0}}$ ← generating fnc for possibly disconnected invariants with no collapsing connected components

Def'n $Z_{DT}(x) = \sum_{n,\beta}' N_{n,\beta}^{DT}(x) \, v^\beta (-q)^n$ ← DT partition function generating function for DT invs (with a sign $(-1)^n$ for convenience).

DT theory is inherently disconnected and includes point contributions so it is most closely analogous to Z_{GW}, however we prefer Z'_{GW} to Z_{GW} (no ill defined terms e.g.).

For DT theory we remove degree zero contributions formally:

Def'n $Z'_{DT} = \dfrac{Z_{DT}}{Z_{DT}|_{v=0}}$ ← a priori, it's not clear what this is the generating for geometrically (turns out to be PT theory)

Lecture 30

GW/DT correspondence conjectured in 2003 MNOP, proven by Pardon in 2023:

$$\boxed{Z'_{DT}(x) = Z'_{GW}(x) \quad \text{after the change of variables} \quad q = e^{i\lambda}}$$

Same function, GW invariants are taylor coef's, DT invs are Fourier coefs.

The variable change $q = e^{i\lambda}$ is strange. For this change of variables to even make sense requires the following property (conj by MNOP 2003, proven by Bridgeland ~2010):

thm The coefficient of v^β in Z'_{DT} is the Laurent expansion of a rational function in q invariant under $q \longleftrightarrow q^{-1}$.

i.e. a palendromic Laurent polynomial $3q^{-2} + 7q^{-1} + 2 + 7q + 3q^2$ or

something like $q + 2q^2 + 3q^3 + \cdots = \dfrac{q}{(1-q)^2} \longleftrightarrow \dfrac{q^{-1}}{(1-q^{-1})^2} \cdot \dfrac{q^2}{q^2} = \dfrac{q}{(1-q)^2}$

Correspondence makes sense for fixed β (can compare v^β terms of Z'_{DT} and Z'_{GW} separately), but for fixed β one must know all g to get a single n and vise-versa. Physicists call this a non-parturbative duality.

Z'_{GW} is an expansion for small string coupling constant λ

Z'_{DT} " " " " " $q \iff \lambda \to i\infty$ (large string coupling constant)

S-duality between A-model and B-model.

Example local elliptic curve $X = \text{total}(L \oplus L^{-1} \to E)$ $\quad N^{DT}_{n,d[E]} = \begin{cases} p(d) & n=0 \\ 0 & \text{otherwise} \end{cases}$

$$Z_{DT} = \sum_{d,n}' N^{DT}_{n,d} (-q)^n v^d = \sum_d p(d) v^d = \prod_{k=1}^{\infty} (1-v^k)^{-1} \qquad Z\big|_{v=0} = 1 \text{ so}$$

$Z_{DT} = Z'_{DT}$. Recall $N^{GW}_{\substack{g,d[E] \\ d\neq 0}} = \begin{cases} 0 & g \neq 1 \\ \frac{1}{d}\sigma(d) & g=1 \end{cases}$

$$F'_{GW} = \sum_{g=0}^{\infty}{}' \sum_{d=1}^{\infty}{}' N^{GW}_{g,d[E]} \lambda^{2g-2} v^d$$

$$= \sum_{d=1}^{\infty}{}' v^d \frac{1}{d}\sigma(d) = \sum_{d>0}' \sum_{k|d}' \frac{k v^d}{d} \qquad d = k\cdot m$$

$$= \sum_{k,m>0}' \frac{1}{m} v^{km} = \sum_{k>0} -\log(1-v^k) = \log \prod_{k=1}^{\infty} \frac{1}{1-v^k} \implies Z'_{GW} = \prod_{k=1}^{\infty} (1-v^k)^{-1}$$

not so interesting since there is no dependence on λ / q.

Example $X = \text{total}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1)$

recall $F'_{GW} = \sum_{d>0}' \frac{v^d}{d} \left(2\sin\left(\frac{d\lambda}{2}\right)\right)^{-2}$ (recall our discussion of GV invariants!)

$$\text{so } F'_{GW} = \sum_{d>0} \frac{v^d}{d} \left(\frac{1}{i}\left(e^{id\lambda/2} - e^{-id\lambda/2}\right)\right)^{-2}$$

$$= \sum_{d>0}' -\frac{v^d}{d} \left(e^{-id\lambda/2}\left(e^{id\lambda}-1\right)\right)^{-2}$$

$$= \sum_{d>0} -\frac{v^d}{d} \frac{q^d}{(1-q^d)^2} = \sum_{d,m>0}' -\frac{v^d}{d} m q^{dm}$$

$$= \sum_{m>0} \sum_{d>0}' -\frac{(vq^m)^d}{d} = \sum_{m>0}' m \log(1-vq^m) = \log\left(\prod_{m=1}^{\infty} (1-vq^m)^m\right)$$

$$\text{so } Z'_{GW} = Z'_{DT} = \prod_{m=1}^{\infty} (1-vq^m)^m$$

already surprising that this is an integer series

Suppose that X is a toric CY3: it has a $T=(\mathbb{C}^\times)^3$ action with an orbit as a dense open set. E.g. $\mathrm{Total}(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1)$ or $\mathrm{Tot}(\mathcal{O}(-2,-2)\to\mathbb{P}^1\times\mathbb{P}^1)$

T acts on X with isolated fixed points and each fixed point is the origin of an affine coord chart $\mathbb{C}^3\subset X$. The induced action of T on $I_n(X,\beta)$ has fixed points given by ideal sheaves generated by monomials in each coordinate patch. So T-fixed subschemes $\longleftrightarrow$ configurations of boxes in each coord. patch.

How can we handle the Behrend function? There is a 2-dim'l torus $T_{CY}\subset T$ which acts trivially on the fibers of $K_X\cong\mathbb{C}\times X$. In a $\mathbb{C}^3$ coordinate patch of X, with $(t_1,t_2,t_3)\in T$ acting by (t_1x,t_2y,t_3z) $T_{CY}=\{(t_1,t_2,t_3): t_1t_2t_3=1\}$

you can check that T_{CY} fixed ideals $I\subset\mathbb{C}[x,y,z]$ are still those generated by monomials.

If $[I]\in I_n(X,\beta)^{T_{CY}}$ then T_{CY} acts on $\mathrm{Ext}^i(I,I)$ and the Kuranishi map $\mathrm{Ext}^1(I,I)\longrightarrow\mathrm{Ext}^2(I,I)$ is T_{CY} equivariant. Moreover Serre duality

$\mathrm{Ext}^2(I,I)\xrightarrow{\cong}\mathrm{Ext}^1(I,I)^*$ is T_{CY} equivariant (but not T equiv!)

and so the superpotential $f:\mathrm{Ext}^1(I,I)\longrightarrow\mathbb{C}$ is T_{CY} invariant

T_{CY} acts trivially here.

T_{CY} acts, 0 is only fixed point (if there was a fixed linear subspace, then $[I]\in I_n(X,\beta)$ would not be an isolated fixed point).

$\Rightarrow \; e(MF_f) = 0$ since $MF_f = \{ f^{-1}(\delta) \cap B_\varepsilon(0) \}$ has a free S^1 action

We've shown that $\nu([I]) = (-1)^{\dim Ext^1(I,I)} (1 - e(MF_{f_I})) = (-1)^{\dim Ext^1(I,I)}$

Lecture 31 |

Behrend function is ± 1 and all we have to compute is $\dim Ext^1(I,I)$. mod 2

Prop (MNOP) Let $[I] \in I_n(X,\beta)^{T_{CY}}$ and let $\sigma(\beta,n) = \dim Ext^1(I,I)$ mod 2.

Then $\sigma(\beta, n+k) = k + \sigma(\beta,n)$ mod 2.

So this means for X a toric CY3, we get

$$Z^{DT}(X) = \sum_{n,\beta}' N^{DT}_{n,\beta}(X)\, v^\beta (-q)^n = \sum_{n,\beta}' e_{vir}(I_n(X,\beta))\, v^\beta (-q)^n$$

$$= \sum_{n,\beta} (-1)^{\sigma(\beta,n)} e(I_n(X,\beta)^{T_{CY}})\, v^\beta (-q)^n$$

$$= \sum_\beta v^\beta (-1)^{\sigma(\beta,0)} \underbrace{\sum_n' \#\{ I_n(X,\beta)^T \}\, q^n}_{\substack{\text{generating function} \\ \text{for box counting.}}}$$

Let $Z^{DT}_\beta(X)$ be the v^β coef. of $Z^{DT}(X)$ then

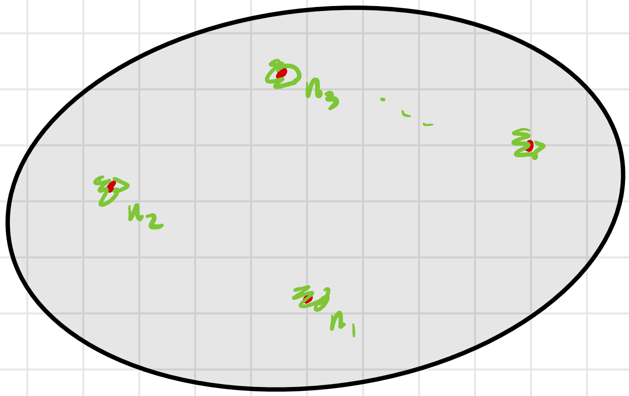
$$Z^{DT}_\beta(X) = \pm \sum_n' \#\{ I_n(X,\beta)^T \}\, q^n$$

completely combinatorial problem solved by topological vertex.

<u>example</u>: $\beta = 0$ $\quad Z_0^{DT}(X) = \pm \sum_n' \#\{\mathrm{Hilb}^n(X)^T\} q^n$

sign must be + since $\mathrm{Hilb}^0(X) = pt$

X



T invariant 0-dim'l subschemes are all supported at $X^T = \{p_1, \dots, p_F\}$

$F = \#$ of fixed points $= e(X)$

$$Z_0^{DT}(X) = \sum_n' \sum_{n_1 + \dots + n_F = n}' \prod_{i=1}^{F} \#\{\mathrm{Hilb}^{n_i}_{p_i}(\mathbb{C}^3_{p_i})^T\} q^{n_i}$$

affine chart around p_i

$$= \prod_{i=1}^{F} \sum_{n_i=0}^{\infty}{}' \#\{\mathrm{Hilb}^{n_i}(\mathbb{C}^3)^T\} q^{n_i} = \left(\sum_{n=0}^{\infty}{}' \#\{\mathrm{Hilb}^n(\mathbb{C}^3)^T\} q^n\right)^{e(X)}$$

number of 3D partitions of n = number of ways of stacking n boxes into a corner $= P_{3D}(n)$

1909 MacMahon showed $\displaystyle\sum_{n=0}^{\infty}{}' P_{3D}(n) = \prod_{k=1}^{\infty}(1-q^k)^{-k} = M(q) = 1 + q + 3q^2 + 6q^3 + 13q^4 + \dots$

So $Z_0(X) = M(q)^{e(X)}$ X toric. In fact,

<u>Thm</u> $Z_0(X) = M(q)^{e(X)}$ holds for all X (even if X is not toric).

Recall that for $X = \mathrm{tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1)$, the GW/DT correspondence predicts

$Z'_{DT}(X) = \displaystyle\prod_{m=1}^{\infty}(1 - v q^m)^m$ since $Z_0(X) = M(q)^2$ and $Z' = \dfrac{Z}{Z_0}$ we see that

$$Z_{DT}(X) = M(q)^2 \prod_{m=1}^{\infty}(1-vq^m)^m = M(q)^2 \prod_{m=1}^{\infty}(1 - mvq^m + O(v^2)) = M(q)^2\left(1 - \left(\sum mq^m\right)v + O(v^2)\right)$$

so $Z^{DT}_{[\mathbb{P}^1]} = \mathrm{coef}_{v^1} Z = -M(q)^2 \dfrac{q}{(1-q)^2}$

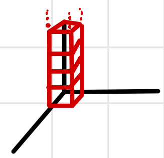
on the other hand

$$Z_{[\mathbb{P}^1]} = \sum_n{}' e_{vir}\left(I_n(X,[\mathbb{P}^1])\right)(-q)^n$$

$$\pm \sum_n{}' \#\{I_n(X,[\mathbb{P}^1])^T\}\, q^n$$

If subscheme has no embedded points, it is just the zero section $\mathbb{P}^1 \subset X$ and $n = \chi(\mathcal{O}_{\mathbb{P}^1}) = 1$. So sum starts at $n=1$ and overall sign is negative.

$$Z_{\mathbb{P}^1} = -\sum_{n=1}^{\infty} q^n \sum_{n_1+n_2=n-1}{}' \{\#\text{ of ways of adding } n_1 \text{ boxes to}$$



$$\}\cdot\underbrace{\{\#\text{ of adding } n_2 \text{ boxes}\cdots\}}_{\text{call this } b(n_2)}$$

$$= -q\left(\sum_{n=0}^{\infty} b(n)q^n\right)^2$$

GW/DT is thus predicting that

$$\left(\sum_{n=0}^{\infty} b(n)q^n\right) = M(q)\frac{1}{1-q} = (1+q+3q^2+6q^3+13q^4+\cdots)\cdot(1+q+q^2+q^3+q^4+\cdots)$$

$$= 1+2q+5q^2+12q^3+24q^4+\cdots$$

number of ways of stacking boxes outside a column □.

number of ways of stacking boxes in an empty room

number of ways of stacking inside the column.

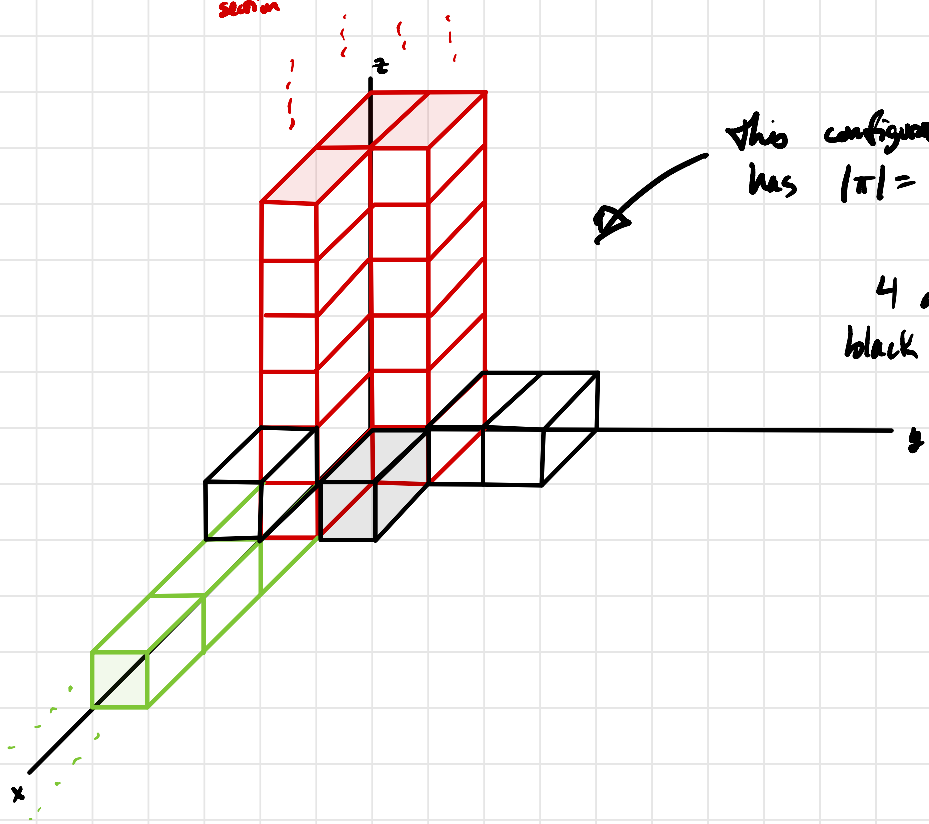
Lecture 32 The full computation for $X = \mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$. First application of the topological vertex. The topological vertex is a box counting generating function:

Def'n Let (μ, ν, λ) be triple of 2D partitions viewed as Young diagrams:

Let $V_{\mu\nu\lambda}(q) = \sum'_{\substack{\pi \text{ 3D partitions} \\ \text{asymptotic to} \\ \mu\nu\lambda}} q^{|\pi|}$ where $|\pi|$ = number of boxes in π with each box counted by $1 - \#$ of legs box is contained in.

example $V_{\square \phi \boxplus}(q)$ counts things like this:

x-axis cross-section, y-axis cross-section, z-axis cross section



this configuration π has $|\pi| = 4 - 2 = 2$

4 added black boxes

the two boxes in both legs count negative

Such π correspond to monomial ideals $I \subset \mathbb{C}[x,y,z]$ so that

in $\mathbb{C}[x,y,z,x^{-1}]$ $I = (y,z)$, in $\mathbb{C}[x,y,z,y^{-1}]$ $I = (1)$, and in $\mathbb{C}[x,y,z,z^{-1}]$ $I = (x^2, xy, y^2)$

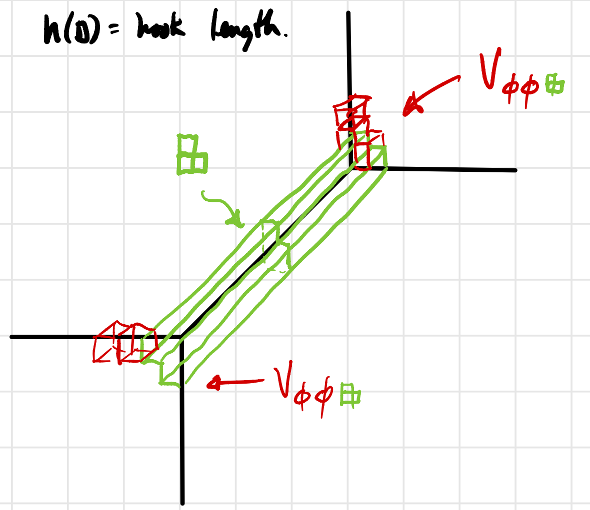
Okounkov-Reshitikin-Vafa gave a formula for $V_{\mu\nu\lambda}(q)$ in terms of Schur functions for example:

Thm $V_{\phi\phi\lambda}(q) = M(q)\, q^{-\binom{\lambda}{2}} s_{\lambda^t}(1,q,q^2,\cdots)$ where $M(q) = \prod_{m=1}^{\infty}(1-q^m)^{-m}$ $(=V_{\phi\phi\phi})$

$\binom{\lambda}{2} = \sum_i \binom{\lambda_i}{2}$ $\quad s_{\lambda^t}(x_1,x_2,\cdots)$ schur symmetric function labelled by λ^t conjugate partition.

example $s_\square(x_1,x_2,\cdots) = x_1+x_2+\cdots$ so $s_\square(1,q,q^2,\cdots) = 1+q+q^2+\cdots = \frac{1}{1-q}$

In fact, $q^{-\binom{\lambda}{2}} s_{\lambda^t}(1,q,q^2,\cdots) = \prod_{\square\in\lambda} \frac{1}{1-q^{h(\square)}}$ $\quad h(\square)$ = hook length.



$$Z^{DT}(X) = \sum_{d=0}^{\infty}{}' \sum_n{}' N^{DT}_{n,d[\mathbb{P}^1]}(X)\, v^d (-q)^n$$

$$= \sum_{d=0}^{\infty} v^d (-1)^{\sigma(d)} \sum_n{}' \#\{I_n(X,d[\mathbb{P}^1])^T\}\, q^n$$

$$= \sum_{d=0}^{\infty}{}' v^d (-1)^{\sigma(d)} \sum_{\lambda\vdash d} V_{\phi\phi\lambda}(q) V_{\phi\phi\lambda'}(q)\, q^{\chi(\mathcal{O}_{C_\lambda})}$$

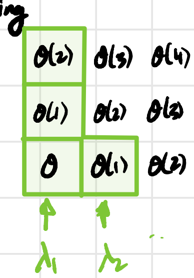
$C_\lambda \subset X$ is T-invariant subscheme with cross section λ and no embedded points.

$$= \sum_{d=0}^{\infty}{}' v^d (-1)^{\sigma(d)} \sum_{\lambda\vdash d} q^{-\binom{\lambda}{2}-\binom{\lambda'}{2}} M(q)^2 s_\lambda(1,q,\cdots) s_{\lambda'}(1,q,q^2,\cdots)\, q^{\chi(\mathcal{O}_{C_\lambda})}$$

The only two things we don't know in the above is $\chi(\mathcal{O}_{C_\lambda})$ and $\sigma(d) = \dim \mathrm{Ext}^1(I_{C_\lambda}, I_{C_\lambda}) \bmod 2$

MNOP gives general formulas for this but we can also compute directly using $\pi: X \to \mathbb{P}^1$. For example $\chi(\mathcal{O}_{C_\lambda}) = \chi(\overbrace{\pi_*\mathcal{O}_{C_\lambda}}^{\text{rk } d \text{ bundle on } \mathbb{P}^1})$

$\pi_*\mathcal{O}_{C_\lambda} = \bigoplus_{\square\in\lambda}$



$$\pi_*\mathcal{O}_{C_\lambda} = \bigoplus_{i=1}^{\ell(\lambda)} \bigoplus_{j=1}^{\lambda_i} \mathcal{O}(i+j-2) \quad \text{so}$$

$$\chi(\pi_* \mathcal{O}_{C_\lambda}) = \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} j+i-1 = \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} j + \sum_{j=1}^{\ell(\lambda')} \sum_{i=1}^{\lambda'_j} i \quad - |\lambda|$$

$$\sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} j = \sum_{i=1}^{\ell(\lambda)} \binom{\lambda_i+1}{2} = \sum_{i=1}^{\ell(\lambda)} \binom{\lambda_i}{2} + \lambda_i = \binom{\lambda}{2} + |\lambda| \qquad \text{so}$$

$$\chi(\mathcal{O}_{C_\lambda}) = \chi(\pi_* \mathcal{O}_{C_\lambda}) = \binom{\lambda}{2} + \binom{\lambda'}{2} + |\lambda|$$

turns out $\sigma(d) = d \mod 2$

$$Z^{DT}(X) = \sum_{d=0}^{\infty} v^d (-1)^{\sigma(d)} \sum_{\lambda \vdash d} q^{-\binom{\lambda}{2} - \binom{\lambda'}{2}} M(q)^2 s_\lambda(1,q,\dots) s_{\lambda'}(1,q,q^2,\dots)\, q^{\chi(\mathcal{O}_{C_\lambda})}$$

$$= \sum_{d=0}^{\infty} (-v)^d \sum_{\lambda \vdash d} M(q)^2 s_\lambda(1,q,\dots) s_{\lambda'}(1,q,q^2\dots)\, q^{|\lambda|}$$

$$= M(q)^2 \sum_{\lambda} s_\lambda(1,q,\dots) s_{\lambda'}(1,q,q^2\dots) (-qv)^{|\lambda|}$$

$$= M(q)^2 \sum_{\lambda} s_\lambda(1,q,q^2,\dots) s_{\lambda'}(-qv,-q^2v,-q^3v,\dots)$$

finally: orthogonality of Schur functions

$$\sum_{\lambda} s_\lambda(x_1,x_2,\dots) s_{\lambda'}(y_1,y_2\dots) = \prod_{i,j} (1+x_i y_j) \qquad \text{so}$$

$$Z^{DT}(X) = M(q)^2 \prod_{i,j \geq 1} (1 - v q^{i+j-1}) \qquad m = i+j-1$$

$$= M(q)^2 \prod_{m=1}^{\infty} (1 - v q^m)^m$$