Homework 3b

Math 615

March 25, 2024

In this problem set, you will compute some Donaldson-Thomas invariants of "local $\mathbb{P}^1 \times \mathbb{P}^1$" and then use the GW/DT correspondence and the Gopakumar-Vafa conjecture to compute various Gromov-Witten invariants and Gopakumar-Vafa invariants.

Let $X = \mathrm{Total}(K_{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathbb{P}^1 \times \mathbb{P}^1)$ be the total space of the canonical line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. The cone of effective classes in $H_2(X, \mathbb{Z})$ is generated by the curves $C_1 = \mathbb{P}^1 \times \{\text{point}\}$ and $C_2 = \{\text{point}\} \times \mathbb{P}^1$ and we abreviate the class $\beta = d_1[C_1] + d_2[C_2]$ by simply (d_1, d_2).

It will be helpful in the below problems to recall from class the theorem that the value of the Behrend function at torus fixed points of the moduli space is $(-1)^{s(\beta)+n}$ for some function $s(\beta)$. The value $(-1)^{s(\beta)}$ can easily be determined in the cases we consider by finding some value of n such that $I_n(X, \beta)$ is smooth.

Problem 1. Invariants in the class $\beta = (1, 0)$.

1. Use box counting techniques to compute the Donaldson-Thomas partition function in the class $\beta = (1, 0)$:
$$Z^{DT}_{(1,0)}(X) = \sum_n e_{vir}(I_n(X, (1, 0)))\, q^n.$$
Recall from class that the generating function for counting boxes added onto a infinite bar with asymptotic shape $\square$ is given by
$$V_{\emptyset,\emptyset,\square}(q) = V_{\emptyset,\emptyset,\emptyset}(q) \cdot \frac{1}{1-q}$$
where
$$V_{\emptyset,\emptyset,\emptyset}(q) = M(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-m}.$$
2. Use the Gromov-Witten/Donaldson-Thomas correspondence to get a prediction for the Gromov-Witten potential in the class $(1, 0)$:
$$F^{GW}_{(1,0)}(\lambda) = \sum_g N^{GW}_{g,(1,0)}(X) \lambda^{2g-2}.$$
What is the value of $N^{GW}_{0,(1,0)}$? What is the value of $N^{GW}_{1,(1,0)}$?
3. Use the Gopakumar-Vafa conjecture to find the predicted values of the Gopakumar-Vafa invariants $n_{g,(1,0)}(X)$ for all g.

Problem 2. Invariants in the class $\beta = (1,1)$.

1. Use box counting techniques to compute the Donaldson-Thomas partition function in the class $\beta = (1,1)$:

$$Z^{DT}_{(1,1)}(X) = \sum_n e_{vir}(I_n(X,(1,1)))\, q^n.$$

   The generating function for counting boxes added onto two infinite bars (along say the x and y axis), each with asymptotic shape $\square$, is given by

$$V_{\square,\square,\emptyset}(q) = V_{\emptyset,\emptyset,\emptyset}(q) \cdot \left(q^{-1} + \frac{1}{(1-q)^2} \right).$$

2. Use the Gromov-Witten/Donaldson-Thomas correspondence to get a prediction for the Gromov-Witten potential in the class $(1,1)$:

$$F^{GW}_{(1,1)}(\lambda) = \sum_g N^{GW}_{g,(1,1)}(X)\lambda^{2g-2}.$$

   Warning! Beware that the GW/DT correspondence determines $Z^{GW}_{(1,1)}(X)'$, the generating function for the *possibly disconnected* GW invariants. To determine $F^{GW}_{(1,0)}(X)$, you must pass between the disconnected and the connected invariants. This issue did not arise for the class $(1,0)$ because invariants in that class are automatically connected.

   What is the value of $N^{GW}_{0,(1,1)}$? What is the value of $N^{GW}_{1,(1,1)}$?

3. Use the Gopakumar-Vafa conjecture to find the predicted values of the Gopakumar-Vafa invariants $n_{g,(1,1)}(X)$ for all g.

Problem 3. Invariants in the class $\beta = (2,0)$.

1. Use box counting techniques to compute the Donaldson-Thomas partition function in the class $\beta = (2,0)$:

$$Z^{DT}_{(2,0)}(X) = \sum_n e_{vir}(I_n(X,(2,0)))\, q^n.$$

   The generating function for counting boxes added onto an infinite bar with asymptotic shape $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, is given by

$$V_{\emptyset,\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix}}(q) = V_{\emptyset,\emptyset,\emptyset}(q) \cdot \frac{1}{(1-q)(1-q^2)}.$$

2. Use the Gromov-Witten/Donaldson-Thomas correspondence to determine the Gromov-Witten potential in the class $(2,0)$:

$$F^{GW}_{(2,0)}(\lambda) = \sum_g N^{GW}_{g,(2,0)}(X)\lambda^{2g-2}.$$

Warning! You must pass from disconnected to connected invariants (see previous warning).

What is the value of $N^{GW}_{0,(2,0)}$? What is the value of $N^{GW}_{1,(2,0)}$?

3. Use the Gopakumar-Vafa conjecture to find the predicted values of the Gopakumar-Vafa invariants $n_{g,(2,0)}(X)$ for all g. **Warning:** because the class $(2, 0)$ is divisible, using the Gopakumar-Vafa formula is a little trickier in this case.