Old Idaho Usual Here

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N. Kildonan [1] raised the following problem. Take an arbitrary word using at most 10 distinct letters, such as $DONALD\,COXETER$. Can one subsititute distinct digits for the 10 letters so as to make the resulting base 10 number divisible by d? The answer depends on d. If d has 100 digits then the answer is clearly NO. If $d = 100$ then the answer is again NO, since ER cannot be 00. If d is 2, the answer is clearly YES: just let the units digit be even; $d = 5$ or $d = 10$ are just as easy.

Kildonan proved that divisibility by $d = 3$ can always be achieved and R. Israel and R. I. Hess extended this to $d = 9$; the case of $d = 7$ was left unresolved. In this note we settle all cases. The reader interested in an immediate challenge should try to prove that divisibility by $d = 45$ is always possible. This appears to be the hardest case.

To phrase things precisely, a *word* is a string made from 10 or fewer distinct letters; for each word and each possible substitution of distinct digits for the letters, there is an associated *value*: the base 10 number one gets after making the substitution. If all substitutions yield a value for the word w that is not divisible by d, then w is called a *blocker* for d. If any word ending (on the right) with w fails to be divisible by d, then w is called a *strong blocker* for d. An integer d is called *attainable* if the value of every sufficiently long word can be made divisible by d by some substitution of distinct digits for letters. Thus d is not attainable if there exist arbitrarily long blockers. The use of arbitrarily long strings is important because, for example, AB is a blocker for 101, but only because it is too short. An integer d is *strongly attainable* if the value of every word can be made divisible by d by an appropriate substitution.

In this paper we will find all attainable integers; moreover, they are all strongly attainable. Note that any divisor of an attainable number is attainable.

Some cases, such as $d = 2$, $d = 5$, or $d = 10$ are extremely easy to attain, and it is just about as easy to attain $d = 4$ or $d = 8$. It takes a little work to show that $d = 3$ and $d = 9$ are attainable (proofs given below). Our main theorem resolves the attainability status of all integers.

Theorem 1 An integer is attainable if and only if it divides one of the integers 18, 24, 45, 50, 60, or 80.

To start, we discuss the cases of $d = 3$ and $d = 9$ for completeness and to introduce the ideas needed later. We use the well known fact that when d is 3 or 9, then d divides a number if and only if d divides the sum of its digits.

The number $d = 3$ is attainable (Kildonan [1]). Given a word, let the 10 letters be grouped as A_i, B_i, and C_i, where each A_i has a multiplicity (perhaps 0) that is divisible by 3, each B_i has a multiplicity of the form $3k+1$, and each C_i has a multiplicity of the form $3k+2$. Look for one, two, or three pairs among the B_i and replace them with digits 1 and 2, and 4 and 5 for the second pair, and 7 and 8 for the third pair. Then look for pairs of the C_i and replace them with digits in any of the still-available pairs among $(1, 2)$, $(4, 5)$, and $(7, 8)$. These substitutions take care of $B_i \cup C_i$ except possibly four letters (since we used three pairs) and we can substitute 0, 3, 6, and 9 for them. The letters A_i can be assigned the remaining digits in any order. Thus, the final number has a digit sum divisible by $d = 3$.

The number $d = 9$ is attainable (Solution II by Israel and Hess [1]). Suppose a word has length n. Suppose some letter occurs k times, where $n - k$ is not divisible by 3. Assign 9 to this letter and assign 0 to 8 arbitrarily to the other letters. Let the value of the resulting number be $v \pmod 9$. Now replace each digit from 0 to 8 by the next higher digit, wrapping back to 0 in the case of 8. This adds $n - k$ to the value modulo 9. But $n - k$ is relatively prime to 9, so we can do this $-v/(n - k)$ times, where the division uses the inverse of $n - k$ modulo 9, in order to obtain the value 0 modulo 9.

The other case is that every letter has a multiplicity $k \equiv n \pmod 3$. If in fact every multiplicity is congruent to $n \pmod 9$, then any assignment will yield a value congruent to $n(0 + 1 + \cdots + 9) = 45n \equiv 0 \pmod 9$. Otherwise there is a multiplicity $k \equiv n \pmod 3$ but $k \not\equiv n \pmod 9$, and then we proceed as in the first half of the proof: assign 9 to this letter, 0 to 8 to the other letters, and then cyclically permute the values 0 to 8. Each permutation adds $n - k$ modulo 9 and this will eventually transform the value v, which is divisible by 3, to a value divisible by 9, because 3 divides $n - k$ but 9 does not.

Now to the proof of Theorem 1, which follows from these four lemmas.

Lemma 1 Any integer divisible by a prime greater than 5 is not attainable.

Lemma 2 The largest attainable powers of 2, 3, and 5 are 16, 9, and 25, respectively.

Lemma 3 The numbers 36, 48, 75, 90, 100, and 120 are not attainable.

Lemma 4 The numbers 18, 24, 45, 50, 60, and 80 are attainable.

The ordering of these lemmas indicates how Theorem 1 was found. First the cases of $d = 7$ and $d = 11$ were settled and that led to the general result of Lemma 1. It followed that the only candidates for attainability had the form $2^a 3^b 5^c$. Once the powers of 2, 3, and 5 were resolved (Lemma 2), the candidate list was reduced to the 45 divisors of $3600 = 16 \cdot 9 \cdot 25$. Resolving the situation for those divisors, with some computer help, led to Lemmas 3 and 4. Finally, the computer searches were eliminated and the whole thing was redone by hand. Theorem 1 follows from the lemmas because Lemmas 3 and 4 settle the status of all 45 divisors of 3600.

A key idea is that the ten digits sum to 45. So we begin with Lemma 3, which shows how unattainability is proved. We use the fact that an integer is congruent modulo 9 (hence modulo 3) to the sum of its digits. Let A, B, C, D, E, F, G, H, J, K be the ten letters and let w^g be the concatenation of g copies of word w. The table at right lists the blockers needed for Lemmas 2 and 3; most were found by a computer search.

d	blocker
27	AAB
32	$ABBAB$
36	$(ABCDEFGHJ)^5J^4K$
48	$(ABCDEFGH)^2KKJK$
75	$AABA$
90	$A^6(BCDEFGHJ)^7JK$
100	AB
120	$ABCDEFGHJJJK$
125	BBA

We show that the words in the table are blockers. The easiest case is $d = 100$, since the value of any word ending in AB is not divisible by 100.

Case 1. The number $d = 36$ can be blocked. We have

$$(ABCDEFGHJ)^5J^4K \equiv K + 4J + 5(45 - K) \equiv 4J - 4K \pmod 9 .$$

The only way $4(J-K)$ is divisible by 9 is if JK is either 90 or 09, and neither is divisible by 4. Extension on the left by A^{9i} preserves the value modulo 36, because 111111111 is divisible by 9.

Case 2. The number $d = 48$ can be blocked. The rightmost 4 digits of the word $(ABCDEFGH)^2KKJK$ must be one of 0080, 2272, 4464, 6656, or 8848, as these are the only words of the form $KKJK$ that are divisible by 16. However, now the value of the word modulo 3 is one of the entries below, where we work with vectors and ignore K which occurs three times:

$$\begin{aligned} &2\big((45,45,45,45,45) - (8,7,6,5,4) - (0,2,4,6,8)\big) + (8,7,6,5,4) \\ &\quad = (82,79,76,73,70), \end{aligned}$$

and no entry is divisible by 3. Left extension by A^{3i} preserves the value modulo 48.

Case 3. The number $d = 75$ can be blocked. Here we have the congruence $AABA \equiv 51A + 10B \pmod{75}$. Multiplying by 53 transforms the congruence to $3A + 5B \equiv 0 \pmod{75}$. However, $3 \leq 3A + 5B \leq 69$, so the congruence is never satisfied. Left extension by A^{3i} preserves the value modulo 75.

Case 4. The number $d = 90$ can be blocked. We have

$$A^6(BCDEFGHJ)^7JK \equiv 6A + 7(45 - A) + J \pmod 9 ,$$

because K must be 0. The expression simplifies to $J - A$ modulo 9, which cannot be divisible by 9 because 0 is already assigned to K. Left extension by A^{9i} preserves the value modulo 9.

Case 5. The number $d = 120$ can be blocked. We have

$$ABCDEFGHJJJK \equiv 45 + 2J \equiv 2J \pmod{3} \; .$$

However, JJK must be either 440 or 880 to obtain divisibility by 40, therefore, $2J$ is either 8 or 16, and so is not divisible by 3. Left extension by A^{3i} preserves the value modulo 120.

Case 6. The number $d = 32$ can be blocked. Any word ending in $ABBAB$ has a value satisfying $10010A + 1101B \equiv 26A + 13B \pmod{32}$. If this is congruent to 0 modulo 32, then we may cancel 13, leaving $2A+B$. However, this sum is between 1 and $18 + 8 = 26$, so it is not divisible by 32.

Case 7. The number $d = 125$ can be blocked. A number is divisible by 125 if and only if it ends in 125, 250, 375, 500, 625, 750, 875, or 000. Thus, BBA is a strong blocker for 125.

Case 8. The number $d = 27$ can be blocked. The value of AAB satisfies the congruence $110A + B \equiv 2A + B \pmod{27}$. However, $1 \leq 2A + B \leq 26$, which is not divisible by 27. This shows nonattainability, because we can add the prefix A^{27i}, which leaves the value modulo 27 unchanged.

Next we prove Lemma 1. Our first proof of this was a little complicated (see the Proposition that follows), but when we focused on words involving two letters only we discovered Theorem 2, which yields Lemma 1 in all cases except $d = 7$. Recall Euler's theorem, that $a^{\phi(d)} \equiv 1 \pmod{d}$ when $\gcd(a, d) = 1$. It follows that if d is coprime to 10, then there is a smallest positive integer, denoted by $\mathrm{ord}_d(10)$, such that $10^{\mathrm{ord}_d(10)} \equiv 1 \pmod{d}$.

Theorem 2 Let d be coprime to 10 and greater than 10 with $e = \mathrm{ord}_d(10)$. Then $w = A^{ke-1}B$ is a blocker for d for any positive integer k.

Proof: Assume first that $k = 1$ so that w is just $A^{e-1}B$. If 3 does not divide d then the value of w satisfies the congruence

$$B + A\sum_{i=1}^{e-1} 10^i \;=\; B - A + A\frac{10^e - 1}{9} \equiv B - A \pmod{d} \; .$$

Since $d > 10$, d cannot divide $B - A$. Now suppose that 3 divides d and $d > 81$. Suppose the value of w, in the formula just given, is a multiple of d. Then multiplying by 9 yields $9(B - A) + A\,(10^e - 1) = 9Kd$, and hence d divides $9(B - A)$. However, $A \neq B$ and $-81 \leq 9(B - A) \leq 81$, so $d > 81$ cannot divide $9(B - A)$, a contradiction.

There remain the cases where 3 divides d and $11 \leq d \leq 81$, namely $d \in \{21, 27, 33, 39, 51, 57, 63, 69, 81\}$. Suppose that d is one of these but $d \neq 21, 27, 81$; then $\mathrm{ord}_d(10) = \mathrm{ord}_{3d}(10)$. This means that from $B - A + A\,(10^e - 1)\,/9 = Kd$, we have $9(B - A) + A\,(10^e - 1) = 3K(3d)$, whence $3d$ divides $9(B - A)$. Thus, d divides $3(B - A)$, which means that $d \leq 27$, a contradiction.

For $d = 21$ the value of w modulo 21 is $B - A$, which is not divisible by 21. For $d = 27$ the value of w modulo 27 is $2A + B$ and $1 \leq 2A + B \leq 26$, so the value is not divisible by d. For $d = 81$ the value of w modulo 81 is $8A + B$ and $1 \leq 8A + B \leq 80$, so the value is not divisible by d.

The extension to the case of general k is straightforward. ∎

The preceding result blocks all primes greater than 10. We need to deal also with $d = 7$. One can give an alternate construction in the general case that includes $d = 7$, and we give the following without proof.

Proposition Suppose that d is coprime to 10 and d does not divide 9. Let

$$w = KJK^eHK^eGK^eFK^eEK^eDK^eCK^eBK^eAK^e,$$

where $e = \text{ord}_d(10) - 1$. Then after any substitution the value of w is congruent to $\frac{9(d+1)}{2} \pmod d$, and so is not divisible by d. ∎

For $d = 7$ the word w of the Proposition has length 55. A different approach led to the much shorter example $OLD\,IDAHO\,USUAL\,HERE$, its value is always $3 \cdot 45 \pmod 7$. This 17-character word is thus a blocker for $d = 7$, and it can be made arbitrarily long by prepending E^6.

On to Lemma 2. The positive results for $d = 16$ and $d = 25$ are not difficult, but they are omitted as they follow from the cases of $d = 80$ and $d = 50$ (proved below); the case of $d = 9$ was discussed earlier, as were the negative results for $d = 32$, 27, and 125. It remains only to prove Lemma 4.

Case 1. The number $d = 50$ is strongly attainable. Just use 00 or 50 for the rightmost two digits.

Case 2. The number $d = 80$ is strongly attainable. Assign 0 to the rightmost letter; 8 to the next new letter that occurs reading from the right, 4 to the next one, and 2 to the next one after that. The value is then divisible by 16 and also by 5; divisibility by 80 is only affected by the four rightmost digits.

Case 3. The number $d = 18$ is strongly attainable. Given a word, find an assignment that makes it divisible by 9. If the rightmost digit is even, we are done. Otherwise, replace this digit y with $9 - y$. This preserves divisibility by 9 and makes the rightmost digit even.

Case 4. The number $d = 60$ is strongly attainable. The word ends in either AA or BA. In either case, assign 0 to A and 6 to B. Let the eight remaining letters be grouped as A_i, B_i, and C_i as in the proof for $d = 3$. Use the pairs $(1, 2)$, $(4, 5)$, and $(7, 8)$ on whatever pairs of letters can be found within B_i or within C_i. This leaves at most two single letters in the B and C groups. Use 3 and 9 for the two singletons, and any remaining digits for the A group. The final value is then divisible by 3, 4, and 5.

Case 5. The number $d = 24$ is strongly attainable. The idea is to modify the proof for $d = 3$ so as to guarantee divisibility by 8. Recalling the proof that $d = 3$ is strongly attainable, call two letters *matched* if they are replaced in that proof by 1 and 2, or by 4 and 5, or by 7 and 8. If the word ends in AAA just make sure A is either 0 or 8. The remaining cases are that the word ends in one of the patterns ABC, ABB, BBA, or ABA.

If the ending is ABC with A and C matched, then use 152 or 192, depending on whether B is part of a matched pair or not. If A and C are unmatched use 320 or 360 according as B is part of a matched pair or not.

If the ending is ABB with A and B matched, then use 488, since the matching can use 4 and 8 as well as 1 and 2. If both A and B are unmatched, then use 600. If A is matched and B is not, use 800. If B is matched and A is not, use 088.

If the ending is BBA, then proceed as if the ending was ABB, but use instead 448, 336, 008, and 880 for the four subcases.

If the ending is ABA, then proceed similarly, using 848, 696, 808, and 080 for the four subcases.

Case 6. The number $d = 45$ is strongly attainable. Let $m(X)$ denote the multiplicity of the letter X in the given word reduced modulo 9; let $\hat{X}$ denote the digit assigned to X. Let A be the rightmost letter and assign 0 to it, thus ensuring divisibility by 5.

Assume first that the multiplicities of at least eight of the nine remaining digits are all mutually congruent modulo 3, and assign 9 to the other letter. Let the m-values of the eight letters be $3a_i + c$, where each a_i is a non-negative integer and $c \in \{0, 1, 2\}$. Let L_i be the digits assigned to these eight letters. Since $\sum L_i = 36$, which 9 divides, the value modulo 9 of the word is $3\sum a_i L_i$. So we want $\sum a_i L_i$ to be divisible by 3. Assign the pairs $(1, 2)$, $(4, 5)$, and $(7, 8)$ to pairs of letters with equal a_i. Assign 3 and 6 to the remaining two. The total is then divisible by 9 and therefore by 45.

In the other case we can choose a letter, K say, with $m(K)$ not congruent modulo 3 to the length of the word; therefore S, the sum of the multiplicities of the nine letters other than K, is not divisible by 3. Let $\hat{K} = 9$.

Case 6a. There is a letter, say B, with $m(B) = m(A)$. Then let $\hat{B} = 1$ and assign the remaining digits arbitrarily.

Case 6b. There is no letter as in Case 6a. Then we can find two letters among B, C, D, E, F, G, H, J, say C and D, with $m(C) \not\equiv m(D) \pmod 3$. Consider B; we know $m(B) \neq m(A)$. Set $\hat{B}$ to be the non-negative residue modulo 9 of $m(D) - m(C)$ and note that $\hat{B} - \hat{A} \equiv m(D) - m(C) \pmod 9$ which is not divisible by 3. Assign unused digits to the letters C and D so that $\hat{C} - \hat{D} \equiv m(B) - m(A) \pmod 9$; there are enough digits left for this to be possible. Assign the remaining digits arbitrarily.

Now we can treat both cases to get the result. The assignment produces some total value, reduced modulo 9 to v. If $v \neq 0$ then replace each digit between 0 and 8 by the next higher digit, wrapping back to 0 in the case of 8. This adds S to the value modulo 9 and does not alter $\hat{B} - \hat{A}$ or $\hat{C} - \hat{D}$ modulo 9. But S is relatively prime to 9, so we can do this $-v/S$ times, where the division uses the inverse modulo 9 of S, in order to achieve divisibility by 9.

If $\hat{A}$ is 0 or 5, then we are done.

If $\hat{A}$ is 3 or 6, then switch digits of A and B, where we know that $\hat{B}$ is not divisible by 3; this is because the value modulo 3 of $\hat{B} - \hat{A}$, which starts out nonzero, does not change in the translational step. If we are in Case 6b, then also switch $\hat{C}$ and $\hat{D}$; the net change is

$$\begin{aligned} & (\hat{B} - \hat{A})(m(A) - m(B)) + (\hat{C} - \hat{D})(m(D) - m(C)) \\ \equiv \; & (\hat{B} - \hat{A})(\hat{D} - \hat{C}) + (\hat{C} - \hat{D})(\hat{B} - \hat{A}) \;\equiv\; 0 \pmod{9} \;, \end{aligned}$$

so divisibility by 9 is preserved.

As $\hat{A}$ is now not divisible by 3, we can multiply each digit less than 9 by $5/\hat{A} \pmod{9}$. This preserves divisibility by 9 and makes $\hat{A} = 5$. The total is now divisible by 45. The proof that $d = 45$ is strongly attainable is complete, as is the proof Lemma 4, and thus the proof of Theorem 1 is complete. ∎

There are several variations to this problem that one might consider, such as using bases other than 10. Another variant is to restrict the alphabet to the two letters A and B. We use the terms 2-*attainable* and 2-*blocker* in this context. Using techniques similar to those presented, we obtained the following result.

Theorem 3 A number is 2-attainable if and only if it divides one of 24, 50, 60, 70, 80, or 90.

The negative part of the proof required finding a 2-blocker for each $d \in \{28, 36, 48, 120, 175\}$. The reader might enjoy finding them; they are all short, of length at most 7.

Acknowledgment

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References

[1] N. Kildonan, Problem 1859 (*Solution*), ***Crux Mathematicorum***, 20:6, June 1994, pp. 168–170.

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