Tauberian Theorems and Prime Densities.

by Klaus Hoechsmann

1. Our aim is to derive a version of the so-called Tauberian Theorem by Wiener-Ikehara and to demonstrate some of its uses in analytic number theory. This presentation does not strive to be particularly brief or "elementary", but it will make a serious attempt at being clear and at placing its subject into the context of standard undergraduate mathematics. Since it is neither a text on complex variables nor a treatise on harmonic analysis, it can allow itself to use the familiar rudiments of both theories; since it is not intended for the expert number theorist, it can use more ordinary prose than is customary in this science.

For the statement of the main theorem, the following jargon will be useful. Two functions $F(s)$ and $G(s)$ of a complex parameter s will be called *analytically similar on* the right half-plane $R_a = \{s|\Re(s) > a\}$, if their difference is holomorphic in R_a and can be extended to a continuous function on the closure thereof.Two functions $f(u), g(u)$ of the real variable u will be said to be *asymptotically similar* if the limits $\lim_{u\to\infty} f(u)$ and $\lim_{u\to\infty} g(u)$ either both fail to exist or are equal.

THEOREM I : Let $\{a_n\}$ and $\{b_n\}$ be non-negative sequences with partial sums $A(x) = \sum_{n\le x} a_n$ and $B(x) = \sum_{n\le x} b_n$, respectively. Suppose that the functions $\sum_n a_n n^{-s}$ and $\sum_n b_n n^{-s}$ exist and are analytically similar on some R_a, with $a > 0$. Then the functions $A(x)x^{-a}$ and $B(x)x^{-a}$ are asymptotically similar. In particular, $\lim_{x\to\infty} B(x)x^{-a} = c \neq 0$ implies $\lim_{x\to\infty} A(x)/B(x) = 1$, i.e. $A(x) \sim B(x)$ in the usual notation.

For example, if $b_n = 1$, for all n, the function $\sum_n b_n n^{-s}$ is Riemann's $\zeta(s)$. Because of its obvious proximity to the integral

$$\frac{1}{s-1} = \int_1^\infty x^{-s} dx,$$

it is not surprising that this function is holomorphic in R_1 and extends meromorphically to R_0 with a simple pole (residue 1) at $s = 1$ and no other poles.

In view of Theorem I, the "real reason" behind the famous prime number theorem is Euler's version of unique factorization in **Z**, namely

$$\sum_n n^{-s} = \prod_p (1 - p^{-s})^{-1},$$

$\Re(s) > 1$, the product running over all natural primes. Indeed this relation shows that the logarithmic derivative of $\zeta(s)$ is essentially ,i.e. up to a summand which is holomorphic in R_0 , equal to $-\sum_p \log p/p^s$. Of course, being the logarithmic derivative of a meromorphic function, it has only simple poles, whose residues betray the orders of zeroes and poles of its parent. Since $\zeta(s)$ visibly has no zeroes in R_1 and — as we shall see below — not even on its closure, it follows that $\zeta(s)$ and $\sum_p \log p/p^s$ are analytically similar on R_1, whence

$$\sum_{p\le x} \log p \sim x,$$

by Theorem I. This is the prime number theorem. If this version of does not ring any bells, the reader need only reflect on the excruciating slowness of the log which entails , by an elementary estimate, that $\sum_{p\leq x}\log p \sim \pi(x)\log x$, whence the jazzier form

$$\pi(x) \sim x/\log x$$

of the theorem. It *is* surely one of the most amazing results in all of mathematics. On the left you have $\pi(x)$, the number of primes up to x, a function whose computation a la Eratosthenes can be safely entrusted to any Neanderthaler, on the right you have Napier's log — not just any log, *the* log.

We still need to explain, why $\zeta(s)$ has no zeroes on the line $\{\Re(s) = 1\}$, the boundary of R_1. In the evolution of the prime number theorem this remained the missing link for several decades, although it does not seem very difficult in retrospect. A few years after it was first supplied by Hadmard and de la Vallée Poussin, Mertens found a surprisingly simple proof, which will be presented here in stylish modern garb by Deligne.

The fact is that $-\zeta'(s)/\zeta(s)$ would have a hard time living up to its *a priori* constraints as well as producing an extra pole somewhere. Imagine a function $\lambda(s) = \sum_n k_n n^{-s}$, with $s \in R_a$, which has a meromorphic extension to the closure of R_a. Let v_t be its residue at $s = a + it$, and suppose that v_t is a *non-positive integer* for all $t \neq 0$, while $v_0 = 1$. If you now further suppose that $k_n \geq 0$, for all n, you make it impossible for any v_t to be negative — and here is why.

First think of t as a function $\tau(n) = n^{-it}$ from the natural numbers to the unit circle. You can then consider integral linear combinations ρ of such τ and define for them

$$\lambda(\rho, s) = \sum_n \rho(n)k_n n^{-s},$$

letting $v(\rho)$ be the residue of $\lambda(\rho, s)$ at $s = a$. Note that $\lambda(1, s)$ is our old $\lambda(s)$ and that $v(\tau)$ is our old v_t if $\tau(n) = n^{-it}$. Since the k_n are real, we always have $v(\bar{\rho}) = v(\rho)$; since they are non-negative , $\rho \geq 0$ implies $v(\rho) \geq 0$. This last statement is the clincher and the reason for considering linear combinations of τ's. It holds because $v(\rho)(s-a)^{-1}$ is the principal part of $\lambda(\rho, s)$, which *must* be non- negative for $s = a + \epsilon$ since every term $\rho(n)k_n n^{-s}$ is. Now take $\rho = (1 + \tau + \bar{\tau})^2 = 3 + 2\tau + 2\bar{\tau} + \tau^2 + \bar{\tau}^2$. Then obviously $\rho \geq 0$, and hence

$$0 \leq v(\rho) = 3 + 4v(\tau) + 2v(\tau^2),$$

which shows that $v(\tau) < 0$ is impossble since $v(\tau^2) \leq 0$ anyway.

2. As a first step toward proving Theorem I, let us bring the partial sums $A(x)$ and $B(x)$ explicitly into the game. Summation by parts easily yields

$$\sum_{n\leq N} a_n n^{-s} = A(N)N^{-s} + s\int_1^N A(x)x^{-1-s}dx$$

without any hypothesis on a_n or s. Now let $a \geq 0$ and suppose that $\sum_n a_n n^{-s}$ converges absolutely for every $a + \epsilon > a$. Then the above equation shows that $A(N)/N^{a+\epsilon}$ is bounded for $N \to \infty$ (replace a_n by $|a_n|$ and drop the then non-negative integral). Since this implies that $A(N)/N^{a+2\epsilon} \to 0$ as $N \to \infty$, we obtain the formula

$$\sum_n a_n n^{-s} = s\int_1^\infty A(x)x^{-1-s}dx.$$

The substitution $x = e^u$ immediately shows the integral occurring here to be the Laplace transform

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(u)e^{-us}du$$

of the function $f(u) = A(e^u)$, making Theorem I a special case of the following result, which we state using the notation $f_a(u) = f(u)e^{-au}$.

THEOREM II : Let f and g be non-negative, measurable functions on $\mathbf{R}$. If the Laplace transforms $\mathcal{L}\{f\}$ and $\mathcal{L}\{g\}$ exist and are analytically similar on R_a, then f_a and g_a are asymptotically similar.

At this point, we may cautiously proceed to perform two simplifying manoeuvres: to get rid of the subscript a and to replace the pair f, g by $\phi = f - g$. The condition $a > 0$ has shielded the analytic behavior of $\mathcal{L}\{f\}$ on the boudary of R_a from being affected by the factor s in the formula linking $\mathcal{L}\{f\}$ to the original Dirichlet series; this is no longer necessary. Moreover, the distinction between f and f_a has facilitated the statement of Theorem II. If we abolish it – writing f but thinking f_a and, incidentally, replacing R_a by R_0 – we will have to reformulate the the monotonicity hypothesis on f and g; this can and will be done shortly. Let us therefore take $\phi = f - g$ and hope that, for a certain kind of function ϕ, the analytic triviality of $\mathcal{L}\{\phi\}$ on R_0 implies the asymptotic triviality of ϕ. Reasonable though it seems, this is too optimistic in the present context: the integral $\mathcal{L}\{\phi\}$ would obviously mask any sufficiently slender irregularities on ϕ and hence cannot reveal its actual pointwise behavior. However, it does permit conclusions about a kind of *average* asymptotic tendency, namely the limits, as $u \to \infty$, of the convolutions $k * \phi$ with a family $\mathcal{F}$ of "averaging functions" k. If all these limits are zero, and if $\mathcal{F}$ is large enough, we shall say that ϕ is *asymptotically quasi-trivial*. By "large enough", we mean that $\mathcal{F}$ contain a Dirac sequence, i.e. a sequence of probability densities increasingly concentrated around 0.

To understand the significance of this, it is helpful to envisage k as a sort of "blip" around 0, i.e. a non-negative function with large values near 0, small values elsewhere, and L^1-norm (integral over all of $\mathbf{R}$) equal to 1. The value at u of the convolution,

$$(k * \phi)(u) = \int_{\mathbf{R}} k(v-u)\phi(v)dv,$$

is clearly a k-weighted average of the values $\phi(v)$ around $v = u$. As we move along a Dirac sequence, k gets taller and slimmer, thus concentrating the averaging process more and more tightly around $v = u$.

Here, then, is our main lemma.

LEMMA A : Let ϕ be a measurable, real-valued function on $\mathbf{R}$. Suppose that ϕ is bounded below and that $\Phi(s) = \int_0^\infty \phi(u)e^{-us}du$ converges absolutely for $s \in R_0$. If $\Phi(s)$ can be extended continuously to the closure of R_0, then ϕ is asymptotically quasi-trivial.

The condition of boundedness is technical and could be relaxed in several ways. About the functions mentioned in Theorem II, Lemma A says this: if $g_a(u)$ tends to some finite limit as $u \to \infty$, then $(k * f_a)(u)$ tends to the same limit, for all k in some Dirac sequence. Theorem II now follows by means of another, easier lemma. It requires a "steady" function, i.e. one which does not slip back too much. A precise definition will be given in paragraph 4; for now suffice it to say that products of non-decreasing functions and exponential functions are steady.

LEMMA B : Suppose that $f : \mathbf{R}_+ \to \mathbf{R}_+$ is such that $\lim_{u\to\infty}(k * f)(u) = c$ for all members k of a Dirac sequence. Then $\lim_{u\to\infty} f(u) = c$, provided that f is steady.

Once steadiness has been defined, the proof of Lemma B is a straightforward exercise in elementary real analysis – cf. appendix B. The proof of Lemma A – cf. appendix A – consists in performing most of the derivation of the "Fourier inversion formula" but finishing with a different cadenza.

3. The time has come to look at some applications of Theorem I beyond the proof of the prime number theorem. Its principal use is in the transition from the relatively tractable *analytic* density to the more difficult *natural* density of a set of prime numbers (or ideals). By *natural density* of a set $M \subset \mathbf{N}$ of natural primes p, we mean the limit

$$D(M) = \lim_{x\to\infty} \frac{\#\{p \le x,\ p \in M\}}{\#\{p \le x\}},$$

which may or may not exist for a particular set M. The notion of *analytic density*, though strange at first sight is more germane to Dirichlet series. It is given by

$$d(M) = \lim_{s\to 1^+} \frac{\lambda_M(s)}{\lambda_{\mathbf{N}}(s)},$$

where $\lambda_M(s)$ is a function defined in R_1 by a suitable series, e.g. $\sum_{p\in M} p^{-s}\log p$. Admittedly this is a slight (?) deviation from Dirichlet's definition, which did not include the coefficients $\log p$ — cf. Remarks below — but, even so, $d(M)$ has a better chance of existing than $D(M)$. Theorem I yields a way of going from the former to the latter, as follows.

$d(M)$ can also be obtained as $\lim_{s\to 1^+}(s-1)\lambda_M(s)$ because $-\lambda_{\mathbf{N}}$ is essentially the logarithmic derivative of $\zeta(s)$, hence equals $-(s-1)^{-1}$ plus a function holomorphic at $s=1$. This shows that $d(M)$ is the residue of λ_M at $s=1$, provided there *is* such a thing. Therefore, IF λ_M is known to be meromorphic on the closure of R_1 with at most a simple pole at $s=1$, it follows that λ_M and $d(M)\lambda_{\mathbf{N}}$ are analytically similar on R_1. Setting $M(x) = \{p \in M,\ p \le x\}$, we then conclude via Theorem I that $x^{-1}\sum_{p\in M(x)}\log p$ and $d(M)x^{-1}\sum_{p\le x}\log p$ behave similarly as $x\to\infty$. By the prime number theorem, the second of these expressions tends to $d(M)$. Therefore

$$\sum_{p\in M(x)} \log p \sim d(M)x \sim d(M)\sum_{p\le x}\log p,$$

whence, by elementary means (cf. Remarks)

$$\#M(x) \sim d(M)\cdot \#\{p \le x\},$$

i.e. $D(M)$ exists and equals $d(M)$.

In the classical equidistribution theorems, the crucial "IF" of this argument is satisfied because of the kind intervention of L-functions.

The logaritmic derivatives of these functions have two main virtues: firstly they are meromorphic on R_1 with no poles except possibly at $s=1$, and secondly their linear span — whose inhabitants are of course equally well behaved — contains the functions λ_M (modulo holomorphic summands) for certain interesting sets M. Although this essay cannot give a more detailed exposition of L-functions, it can at least indicate where these virtues come from. The second flows directly from the orthogonality relations between the group characters involved in the definition of the L-function. The first is obtained by a modification of the argument given in paragraph 2 for the good behavior of $-\zeta'(s)/\zeta(s)$: however, one now has to work with integral linear combinations of $\tau(n) = \chi(n)n^{-it}$, where χ is a character of the group belonging to the L-function in question.

REMARKS :

1) The customary definition of analytic density involves sums over p^{-s} rather than $p^{-s}\log p$, that is, logarithms rather than logaritmic dervatives. That simplfies the notation but complicates the analysis by introducing non-meromorphic functions.

2) The elementary lemma used in removing a factor $\log n$ from a series says that $\sum_{n\le N} c_n \sim cN$ implies $\sum_{n\le N} c_n/\log n \sim cN/\log N$.

A. Before we can prove Lemma A, we first need to recall a few basic facts about Fourier transforms and secondly to define the family $\mathcal{F}$ which determines its meaning .

A function $g : \mathbf{R} \to \mathbf{C}$ is said to be in $L^1(\mathbf{R})$ if it is integrable on finite intervals and $\|g\|_1 = \int_{\mathbf{R}} |g(t)|dt$ is finite. For every such g we can define the *Fourier transform* by

$$\hat{g}(u) = \int_{\mathbf{R}} g(t)e^{iut}dt.$$

FACTS : Let $g, h \in L^1(\mathbf{R})$. Then

1) $\hat{g}$ is uniformly continuous on $\mathbf{R}$ and bounded by $\|g\|_1$.
2) $\hat{g}h$ has the same integral over $\mathbf{R}$ as $g\hat{h}$.
3) Setting $\check{g}(u) = \hat{g}(-u)$, we have $(h \cdot \check{g})\hat{} = \hat{h} * g$.
4) $\hat{g}$ vanishes at infinity.
5) Setting $h_n(t) = h(t/n)$, we have $\hat{h}_n(u) = n \cdot \hat{h}(nu)$.

REASONS : (1) is obtained by an elementary estimate; its role is to prepare the way for (2), which then results from a straightforward change of order of integration; (3) is a variant of (2) obtained by change of variable; it will be referred to as the "weak inversion formula" — cf. Remarks below. (4), the so-called Riemann-Lebesgue Lemma will be needed only for g with compact support, where it comes from a simple integration by parts , if $g \in C^1$, and more generally from setting $g = f + h$ with $f \in C^1$ and h uniformly small. (5) is immediate from the definition.

The convoluting family $\mathcal{F}$ to be used in the formulation of Lemma A will consist of all non- negative functions in $L^1(\mathbf{R})$ which happen to be Fourier transforms of continuous functions with compact support. The purpose of recording the rather obvious fact (5) is to show that any element $\hat{h} \in \mathcal{F}$ with $\|\hat{h}\|_1 = 1$ spawns a Dirac sequence in $\mathcal{F}$ such as is needed for Lemma B. But *are* there any non-zero elements of $\mathcal{F}$?

Before exhibiting one , let us note that if h is even and real-valued then so is $\hat{h}$, because the imaginary part $h(t) \sin ut$ of the integrand is odd. For such h we therefore have

$$\hat{h}(u) = 2\int_0^\infty h(t) \cos ut dt.$$

Taking $h(t) = 1 - |t|$ on the interval $(-1, 1)$ and zero elsewhere, we see that

$$\hat{h}(u) = 2\int_0^1 (1-t) \cos ut dt = u^{-2}(1 - \cos u),$$

an easy integration by parts. Hence $\hat{h} \in \mathcal{F}$.

Now the way is clear for the proof of Lemma A . As in paragraph 2, we write $\phi_c(u)$ for $\phi(u)e^{-cu}$.Then, replacing ϕ by ϕ_a, we may assume, without loss of generality, that $a = 0$.

The question is : does $\hat{h} * \phi$ vanish at infinity for all $\hat{h} \in \mathcal{F}$? The answer seems to be : sure, just put $g = \phi$ in the weak inversion formula (3) and use Riemann-Lebesgue (4), right ?

Wrong! For nothing assures us that $\phi \in L^1$. Instead we have some hypothesis on its Laplace transform Φ, which we had better use. In fact, $\Phi(\epsilon + it) = \int_0^\infty \phi(u)e^{-\epsilon u - iut}du$ is nothing other than $\check{\phi}_\epsilon(t)$, which by hypothesis is therefore continuous for $(\epsilon, t) \in [0, 1] \times \mathbf{R}$. So, to salvage our idea let us back off a little and replace ϕ by ϕ_ϵ , which by assumption *is* in L^1. We are now staring at the correct formula

$$(h \cdot \check{\phi}_\epsilon)\hat{} = \hat{h} * \phi_\epsilon,$$

eager to see what happens as $\epsilon \to 0$. What we would like to find is $(h \cdot \psi)\hat{} = \hat{h} * \phi$, where $\psi = \lim_{\epsilon \to 0} \check{\phi}_\epsilon$, and then jump to the conclusion as planned.

There is no problem with the left hand side : since h has compact support , $(h\cdot\check{\phi}_\epsilon)\hat{}$ is an integral over $[-N,N]$ for N suitably large , and the integrands are uniformly continuous for $(\epsilon,t)\in[0,1]\times[-N,N]$.

As to the right hand side, we have

$$(\hat{h}*\phi_\epsilon)(u)=\int_0^\infty \hat{h}(u-v)\phi(v)e^{-\epsilon v}dv$$

whose convergence as desired would be guaranteed by the monotone convergence theorem *if* ϕ where non-negative. As it is , we can at least find some $c\geq 0$ such that $\phi(v)+c\geq 0$. Putting $K_r(u)=\int_0^\infty \hat{h}(u-v)e^{-rv}dv$, we can now add cK_ϵ to both sides to our weak inversion identity before letting ϵ go to 0. The result is

$$(h\cdot\psi)\hat{}+cK_0=\hat{h}*\phi+cK_0,$$

which proves what we want.

REMARKS :

1) In this last dodge , making the integrands non-negative , a function $c(v)$ could have been used instead of the constant c, thus relaxing the boundedness condition of the theorem. Any $c(v)$, vanishing for $v<0$ and such that $\phi(v)+c(v)\geq 0$ would do the trick, provided that $(\hat{h}*c)(u)$ is finite for all large u. If we restricted $\mathcal{F}$ to the sequence $\mathcal{H}=\{\hat{h}_n\}$ with $h_1(t)=1-|t|$ as above , we could allow any $c(v)=O(v^r)$ with $r<1$.

2) Fact (3) recalled and used above is not usually singled out for special attention , although it *is* a kind of embryonic version of the very important *inversion formula :* $(\check{g})\hat{}=2\pi g$, which holds if (for instance) g is continuous with both g and $\hat{g}$ in L^1. It is obtained by using a suitable sequence h_n for h in (3) and letting $n\to\infty$.

The arguments given here for Lemmas A and B are a variation on this general pattern. This is in line with the intuitive idea that inversion must be used in order to retrieve information about a function from conditions on its Laplace transform. Since Laplace inversion involves integrating along vertical lines in **C**, this idea may well have prompted Riemann's move of turning s into a *complex* parameter. Viewed this way, the "intrusion" of harmonic analysis into analytic number theory appears very natural .

3) Consider the set $\mathcal{K}_\phi$ of *all* $k\in L^1$ such that $k*\phi$ vanishes at infinity. How big is it? What has been proved so far can be summarized by saying that $\mathcal{K}_\phi$ contains $\mathcal{F}$, provided that ϕ satisfies the hypotheses of Lemma A. In particular, the Fourier transforms of members of $\mathcal{K}_\phi$ have no common zeroes — this being already the case for the subset $\mathcal{H}$, by the inversion formula. Now, if ϕ is bounded above as well as below — which it *is* in the context of Theorem I — $\mathcal{K}_\phi$ is easily shown to be a *closed ideal* in L^1. Then a famous theorem of Wiener's says that it must be *all* of L^1, because every *proper* closed ideal is annihilated by some continuous homomorphism $L^1(\mathbf{R})\to\mathbf{C}$, i.e. evaluation of Fourier transforms at one particular point .

Thus , at the price of boundedness (which we would willingly pay) , Wiener's theorem would afford a considerable strengthening of Theorem A , justifying also the use of the unspecific word "quasi" in its statement. However, since it is difficult and not absolutely necessary, we shall pass it up and go on to the proof of Lemma B.

B. In preparation for the proof of Lemma B , two notions need to be clarified.

For $\delta > 0$, let $I(\delta)$ and $J(\delta)$ denote the closure and the complement , respectively , of the interval $(-\delta, \delta)$. Consider a sequence p_n of non- negative L^1 functions of norm 1 and let $P_n(\delta)$ and $Q_n(\delta)$ stand for the integral of p_n over $I(\delta)$ and $J(\delta)$, respectively . p_n is called a *Dirac sequence* if , for all $\delta > 0$, $\lim_{n\to\infty} P_n(\delta) = 1$ or — equivalently — $\lim_{n\to\infty} Q_n(\delta) = 0$. One way to make such a sequence is to set $p_n(u) = n \cdot p_1(nu)$, starting from any admissible p_1.

Secondly , consider a non-negative , measurable function $f(u)$. We shall say that f is *steady* if , for every $\alpha > 1$, there exists a $\delta > 0$ such that

$$0 < r < 2\delta \Rightarrow f(u + r) \geq \alpha^{-1} f(u).$$

In other words , the factor by which f is allowed to decrease over any interval of length 2δ can be made to be arbitrarily close to 1 by controlling the size of δ . This a very relaxed kind of monotonicity. Observe that all non-decreasing functions and all exponential functions , as well as all sums and products of steady functions are steady. The number 2 occurring in the definition is a harmless technicality designed to smoothen the calculations below.

Now let f be a steady function vanishing on the negative reals , and let p_n be a Dirac sequence . The problem is to show :

$$\lim_{u\to\infty} (p_n * f)(u) = c \quad \text{for all } n \qquad \Rightarrow \qquad \lim_{u\to\infty} f(u) = c,$$

the main issue being the *existence* of this limit. To simplify notation , put $f_n = p_n * f$.Then , with α and δ as in the definition of steadiness , we have

$$f_n(u + \delta) \geq \int_{I(\delta)} f(u + \delta - v) p_n(v) dv \geq \alpha^{-1} f(u) P_n(\delta),$$

whence

$$c \geq \alpha^{-1} P_n(\delta) \limsup_{u\to\infty} f(u) \qquad \text{and thus} \qquad \alpha c \geq \limsup_{u\to\infty} f(u),$$

by letting $n \to \infty$ at the end.

According to the last-stated inequality , $f(u)$ must be bounded on some interval of the form $[b, \infty)$; in fact, since $f(b - Nr) \leq \alpha^N f(b)$, it must be bounded on *any* such interval. Remembering that $f(u) = 0$ for $u < 0$, we see that f is bounded, say $\leq B$, on all of $\mathbf{R}$.

Replacing $f(u)$ by B on $J(\delta)$, steadiness yields the estimate

$$f_n(u - \delta) \leq \int_{I(\delta)} f(u - \delta - v) p_n(v) dv + BQ_n(\delta) \leq \alpha f(u) P_n(\delta) + BQ_n(\delta),$$

whence

$$c \leq \alpha \liminf_{u\to\infty} f(u) P_n(\delta) + BQ_n(\delta) \qquad \text{and thus} \qquad \alpha^{-1} c \leq \liminf_{u\to\infty} f(u),$$

again letting $n \to \infty$ at the end.

With $\alpha > 1$ arbitrarily close to 1, the inequalities $\alpha^{-1} c \leq \liminf f(u) \leq \limsup f(u) \leq \alpha c$ finish the proof.

REMARKS :

1) With trivial modifications this proof would work also if steadiness were defined *additively*, as follows: for every $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < r < 2\delta$ implies $f(u + r) \geq f(u) - \epsilon$.

2) Lemma B and the condition of steadiness are both entitled to be called "Tauberian", although this label tends to be used fairly indiscriminantly to describe context rather than content. It is commonly affixed, for instance, to Theorem I and to the famous theorem by Wiener mentioned in the last paragraph.

The original Tauber's Theorem of 1897 placed conditions on the coefficients of a power-series to force its summability on the boundary of its disc of convergence. The name is most appropriate in situations where the existence of a stricter limit is deduced from that of a more general one.