Worksheet Review 1: Sets, indexed collections, functions, images and preimages

1. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be function. Recall the formal definition of what it means that $\lim_{x\to+\infty} f(x) = +\infty$. Write it down (it has to have quantifiers).

2. For $N \in \mathbb{N}$, let $A_N = \{x \in \mathbb{R} : f(x) > N\}$.

   (a) By definition, $A_N = f^{-1}((N, +\infty))$ – in words, A_N is the preimage of the ray $(N, +\infty)$ uder f. Make sure you understand this statement. Make an example of a function f such that $f^{-1}((1, +\infty)) = (10, +\infty)$ (and prove that your example works).

   (b) Prove that $\cap_{N\in\mathbb{N}} A_N = \emptyset$.

   (c) Prove that the following are equivalent:

      i. $\lim_{x\to+\infty} f(x) = +\infty$

      ii. For every N, there exists $m > 0$ such that $A_N \supseteq (m, +\infty)$.

      iii. $\forall N \; \exists m : f((m, +\infty)) \subset (N, +\infty)$.

      iv. $\forall N \; \exists m : f^{-1}((N, +\infty)) \supset (m, +\infty)$.

   (d) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(\mathbb{N}) = \{0\}$ but $\forall N \in \mathbb{N} \; \exists x \in \mathbb{R} : f(x) > N$. Can such a function satisfy $\lim_{x\to+\infty} f(x) = +\infty$?