Worksheet 20. Cardinality 3: Countable and uncountable sets

1. Let $A_1, \dots A_n$ be countable sets. Prove that $A_1 \times \cdots \times A_n$ is countable.
   Hint. use induction.

2. Prove that if there exists an injective function $f : A \to \mathbb{N}$, then A is countable.

   Solution: see notes for the last lecture.

3. Prove that if there exists a surjective function $f : \mathbb{N} \to A$, then A is countable.

   Solution: see notes for the last lecture.

4. Prove that if A_n is countable for all $n \in \mathbb{N}$, then $A = \cup_{n=1}^{\infty} A_n$ is also countable.
   Hint. Try to arrange the elements of A in a table.

   Solution: see notes for the last lecture.

5. Let A be a countably infinite set, and let $f : B \to A$ be a surjective function such that $f^{-1}(x)$ is a countable set for every $x \in A$. Prove that B is countably infinite.
   Hint. Use the previous problem.

   Solution: see notes for the last lecture.

6. Find a bijective function between $[0, 2\pi)$ and the unit circle.

   Solution. Let S be the unit circle and let $f : [0, 2\pi) \to S$ be defined by the formula $f(x) = (\cos(t), \sin(t))$. It is an easy exercise (in trig) to prove that f is bijective.

7. Prove that $|(0,1)| = |[0,1)|$.
   Hint. Choose a countable subset A of $(0,1)$. Then make a bijection between A and $A \cup \{0\}$. Then define your function on the rest of the interval.

   Solution.

Lemma. If $f : A \to B$ and $g : C \to D$ are bijections where $A \cap C = B \cap D = \emptyset$, then $h : A \cup C \to B \cup D$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in C. \end{cases}$$

is a bijection.

Proof of Lemma. Assume $h(x) = h(x') = y$. Assume $y \in B$. Then x, x' must be in A since h maps C into the disjoint set D. Therefore $f(x) = f(x')$ and so $x = x'$ as f is injective. A similar argument works if $y \in D$. Hence h is injective. Now let $y \in B$. Then there is an $x \in A$ so that $h(x) = f(x) = y$. Therefore $B \subset \text{Range}(h)$. Similarly $D \subset \text{Range}(h)$. We are done.

Proof of Question 7, continued. Let $a_n = (n+1)^{-1} \in (0,1)$ for $n \in \mathbb{N}$ and $A = \{a_n : n \in \mathbb{N}\}$. Define $f : A \to A \cup \{0\}$ by

$$f(a_1) = 0 \text{ and } f(a_n) = a_{n-1} \text{ if } n \geq 2.$$

Assume $f(a_i) = f(a_j) = y$. If $y = 0$, then $a_i = a_j = a_1$. If $y = a_k \in A$, then $a_i = a_j = a_{k+1}$. Hence f is injective. For any a_n, $a_n = f(a_{n+1}) \in \text{ran}(f)$ and $0 = f(a_1)$, so f is onto. Hence f is bijective. Now define let g be the identity function on $(0,1) - A$ (clearly a bijection). Defining $h : (0,1) \to [0,1)$ as in the Lemma above we see that h is a bijection (by the Lemma) and so $|(0,1)| = |[0,1)|$.

8. Prove that if A is a countable set, then $|\mathbb{R} - A| = |\mathbb{R}|$.
(In particular, $|\mathbb{I}| = |\mathbb{R}|$). **Hint.** Use the previous problem.

Solution As in the Hint, let us choose $b_n \in (n, n+1) - A$ for all $n \in \mathbb{N}$. Such b_n exists for every n, since if it didn't exist for some value $n = k$, it would mean that A contains the interval $(k, k+1)$; then A would be uncountable since the interval is uncountable – a contradiction. Thus, we have $B = \{b_n : n \in \mathbb{N}\}$, which is a denumerable set of reals disjoint from A. Now choose a bijection $f : A \cup B \to B$. (Given any two denumerable sets, there exists a bijection between them: suppose $g_1 : \mathbb{N} \to B$ is a bijection; suppose $g_2 : \mathbb{N} \to A \cup B$ is a bijection; then $g_1 \circ g_2^{-1} : A \cup B \to B$ is a bijection). Now define $g(x) : \mathbb{R} \to \mathbb{R} - A$ by:

$$g(x) := \begin{cases} x & \text{if } x \notin A \cup B \\ f(x) & \text{if } x \in A \cup B. \end{cases}$$

By the Lemma from the solution of Problem 7, applied to the identity function from $\mathbb{R} - A \cup B$ to $\mathbb{R} - A \cup B$ and the function f from $A \cup B$ to B, the function g we defined is a bijection from $\mathbb{R} = (\mathbb{R} - (A \cup B)) \cup (A \cup B)$ to $\mathbb{R} - A = (\mathbb{R} - (A \cup B)) \cup B$, and we are done.

Remark. In particular, note that this statement says that in terms of cardinality, "there are more irrational numbers than rational numbers": we have $|\mathbb{Q}| = |\mathbb{N}| = \aleph_0$, and we just proved that $|\mathbb{I}| = |\mathbb{R}| = c$, and we proved that $\aleph_0 < c$.

9. Let A be the set of all possible sequences of 0s and 1s. Prove that A is uncountable.

   Solution. It was proved in Homework 12 that the set of such sequences is in bijection with $\mathcal{P}(\mathbb{N})$, and we proved in class that for any set A, the cardinality of $\mathcal{P}(A)$ is strictly greater than the cardinality of A. Applying the statement to $\mathbb{N}$, we get the result.

   Alternatively we can run Cantor's diagonal argument for these sequences: suppose, for the sake of contradiction, that we arranged these sequences in a list indexed by the natural numbers: $s_1, s_2, ...s_n, ...$. Now we construct a new sequence, call it $c = \{c_n\}$ (for Cantor): to define c_1, look at the first term. of the sequence s_1. If it is 0, let $c_1 = 1$, if it is 1, let $c_1 = 0$. Then to construct c_2, use the second term of s_2, and make c_2 its opposite: if it was 1, let $c_2 = 0$, if it was 0, let $c_2 = 1$, and so on. The resulting sequence c cannot coincide with any of the sequences $s_1, ...s_n, ..$, because its nth term is different from the nth term of s_n, for every n. Thus we constructed a sequence that is not on the list – a contradiction with the assumption that we listed them all.

10. Prove that the cardinality of the set from the previous problem is, in fact, continuum. (*This problem is optional and a bit outside the scope of the course*).

    Solution. We can use binary system instead of the decimal system to establish a bijective function between the real numbers and sequences of 0s and 1s (again, in fact we have to throw away countably many such sequences first, to remove ambiguity of representing the rational numbers of the form $a/2^k$ by such sequences of digits (they can be represented either with a finite sequence of 0s and 1s in binary, or with a sequence with all 1s from some point (e.g. $1 = 0.111111111.... = 1.0000000...$ in binary, in the same way as $1 = 0.99999....$ in the decimal system). Anyway, this is a countable set and throwing it away doesn't change cardinality by Problem 8.

11. Prove that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.

    Solution. First, use the fact that we know $|\mathbb{R}| = |[0, 1]|$, and let us prove that $|[0, 1] \times [0, 1]| = |[0, 1]|$. Let us use the decimal expansions: represent the numbers by sequences of digits (again, to be precise, to have a bijective function we need to prohibit sequences with a tail of 9s). Now make a function $F : [0, 1] \times [0, 1] \to [0, 1]$ by "mixing their digits", namely, let f be defined by: $f(a,b) = 0.a_0b_0a_1b_1a_2b_2....$, where $a = 0.a_0a_1....$ and $b = 0.b_0b_1.....$ Exercise (easy): prove that this function is bijective.