Worksheet 17: Rational and irrational numbers. Proof by contradiction.

Let $\mathbb{Q} = \{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\}$ be the set of rational numbers, and let $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers.

1. Prove by any method that if $x, y \in \mathbb{Q}$, then $x + y \in \mathbb{Q}$ and $xy \in \mathbb{Q}$. Is it true that if $x, y \in \mathbb{I}$ then $x + y \in \mathbb{I}$? Is it true that then $xy \in \mathbb{I}$?

   Solution. 1. Direct proof: if $x, y \in \mathbb{Q}$, then there exist $a_1, a_2 \in \mathbb{Z}$, $b_1, b_2 \in \mathbb{N}$ such that $x = a_1/b_1$, $y = a_2/b_2$. Then $xy = (a_1 a_2)/(b_1 b_2)$ and $x + y = \frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1 b_2 + a_2 b_1}{b_1 b_2}$.

   For $x, y \in \mathbb{I}$, both of these statements are false. Counterexample for the product: take $x = y = \sqrt{2}$. Then $xy = 2 \in \mathbb{Q}$. For the sum, take $x = \sqrt{2}$, $y = -\sqrt{2}$. Then $x + y = 0 \in \mathbb{Q}$.

2. Prove that if $x \in \mathbb{Q}$, $y \in \mathbb{I}$, then $x + y \in \mathbb{I}$. Prove also that if $x \neq 0$, then $xy \in \mathbb{I}$.

   Solution. Let us use the proof by contradiction. Suppose $x \in \mathbb{Q}$, $y \in \mathbb{I}$, and $z := x + y \in \mathbb{Q}$. But then we have $x = z - y$, and we proved in (1) that sum of rational numbers is rational, so this makes y rational – a contradiction.

   How can you improve this proof by making it contrapositive instead of contradiction?

3. Assume that π is irrational. Is the number $\pi - 3.141592$ rational or irrational?

   Solution. The number 3.141592 is rational (any finite or periodic decimal is rational). Then by the previous problem, $\pi - 3.141592$ is irrational.

4. Prove that $\sqrt{3}$ is irrational.

   Solution. We prove it the same way as we did for $\sqrt{2}$: by contradiction. Suppose $\sqrt{3}$ was rational; then let us write it as $\sqrt{3} = \frac{a}{b}$ with $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Then we have $3 = \frac{a^2}{b^2}$. Then $3b^2 = a^2$, so $3|a^2$. Now we need a Lemma:

   Lemma. If $3|a^2$ then $3|a$.

   Proof of Lemma. Contrapositive: suppose 3 does not divide a. Then either $a \equiv 1 \mod 3$ or $a \equiv 2 \mod 3$. In the first case, $a^2 \equiv 1^2 \equiv 1$

mod 3; in the second case, $a^2 \equiv 2^2 \equiv 1 \mod 3$. In both cases, we obtain $a^2 \equiv 1 \mod 3$, in particular, 3 does not divide a^2.

Now we can use the Lemma to complete the proof: since $3|a^2$, by the Lemma $3|a$, so there exists $k \in \mathbb{N}$ such that $a = 3k$. Then we get: $3b^2 = a^2 = (3k)^2 = 9k^2$. Now we can cancel 3: get $b^2 = 3k^2$. Then by the same argument, $3|b$. We obtain a contradiction with the assumption that $\gcd(a, b) = 1$.

5. Prove that $\sqrt{6}$ is irrational.

   Solution. We try the same strategy as before: suppose $\sqrt{6} = \frac{a}{b}$ with $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Then we get: $6 = \frac{a^2}{b^2}$. Then $6b^2 = a^2$. In particular, $3|a^2$; then $3|a$ by the above Lemma. Then $a = 3k$ for some $k \in \mathbb{N}$, and we get $6b^2 = 9k^2$, so $2b^2 = 3k^2$. As above, we are trying to prove that $3|b$, to obtain a contradiction with the assumption that $\gcd(a, b) = 1$. So now we need another Lemma:

   Lemma. If $3|2n$, then $3|n$. (We proved this lemma earlier in the class; proof is left as an exercise).

   Now apply the Lemma to $n = b^2$, and we obtain that since $3|2b^2$, then $3|b^2$, and then as in the previous problem, $3|b$, which completes the proof.

   Exercise: Construct a proof exploiting divisibility by 2 *instead of by* 3.

6. Let $A = \{a+b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ (as in the homework). Prove that $\sqrt{3} \notin A$.

   Solution. By contradiction: suppose $\sqrt{3} \in A$. This would mean that there exist $a, b \in \mathbb{Q}$ such that $\sqrt{3} = a+b\sqrt{2}$. Then $3 = a^2+2ab\sqrt{2}+2b^2$. Then $ab\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} \in \mathbb{Q}$ or $ab = 0$. Since we know $\sqrt{2}$ is irrational, we get that $ab = 0$ so a or $b = 0$. If $a = 0$, we get: $\sqrt{3} = b\sqrt{2}$, then $3 = 2b^2$, which is impossible since 3 is odd. If $b = 0$, we get $\sqrt{3}$ is rational, which we know is false, too.

   Variation: prove that $\sqrt{3} + \sqrt{2}$ *is irrational.*

7. Prove that if an integer c satisfies $c \equiv 3 \mod 4$, then c cannot be represented as a sum of two perfect squares.

   Solution. Suppose we had $c = a^2 + b^2$ with $a, b \in \mathbb{Z}$. We prove a Lemma:

   Lemma. For any $a \in \mathbb{Z}$, either $a^2 \equiv 0$ or $a^2 \equiv 1 \mod 4$.

   Proof of Lemma. By cases: consider the cases $a \equiv 0, 1, 2, 3 \mod 4$, and see that in every case we get 0 or 1.

   Then $a^2 + b^2$ can be only congruent to 0, 1 or 2 mod 4, and $c \equiv 3 \mod 4$, then c cannot equal $a^2 + b^2$.

   How can you write the same proof better?

8. Prove that $\sqrt{3^k}$ is irrational if $k \in \mathbb{N}$ is odd.

   Solution. We proved in the midterm that if k is odd, then $3^k \equiv 3 \mod 4$ (make sure you know how to do that!)

   Now suppose $\sqrt{3^k} = \frac{a}{b}$ with $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Then we have $3^k b^2 = a^2$, and in particular, $3b^2 \equiv a^2 \mod 4$. The only way this can happen is when a and b are even, so that $a^2 \equiv 0 \mod 4$, but that contradicts the assumption that $\gcd(a, b) = 1$.

9. Prove that $\log_2(3)$ is irrational.

   Solution. Suppose we had $\log_2(3) = \frac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{N}$ and $\gcd(a, b) = 1$. First observe that $3 > 1$, hence $\log_2(3)$ is positive, and therefore $a \in \mathbb{N}$ as well. Then we have $3 = 2^{a/b}$ (by definition of $\log_2$), and then $3^b = 2^a$, with both $a, b \in \mathbb{N}$. Then 3^b is even, a contradiction.