Worksheet 16: Inverse functions. Images and preimages of sets.

Recall the definition (from 12.6): if $f : A \to B$ is a *bijective* function, then there is a well-defined *inverse function* $f^{-1} : B \to A$.

1. (from the last worksheet). Prove that if $f : A \to B$ is injective and $g : B \to C$ is injective, then $g \circ f : A \to C$ is injective. Is the converse statement true?

   Solution. Let $a_1, a_2 \in A$, $a_1 \neq a_2$. Then since f is injective, $f(a_1) \neq f(a_2)$ (both are elements of B). Since g is injective, $g(f(a_1)) \neq g(f(a_2))$, QED.

   The converse is false: Let $A = \{1\}$, $B = \{a, b, c\}$, $C = \{\alpha, \beta\}$. Let $f : \{1\} \to \{a, b, c\}$ be defined as $f(1) = a$; let $g : \{a, b, c\} \to \{\alpha, \beta\}$ be defined by $g(a) = \alpha$, $g(b) = g(c) = \beta$. Then $g \circ f : A \to C$ is defined by $g \circ f(1) = \alpha$ and is injective (this is 'vacuously true' – nothing to prove since A has just one element.

   However, g is not injective.

   What is true about the converse is: If $g \circ f$ is injective, then f is injective; and if f is also surjective, then g has to be injective as well. (Make sure you understand why!)

2. Does the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \tan(x)$ have an inverse? How can we modify this function to make the inverse function well-defined?

   Solution. No because it is not injective. To make it have an inverse, we can change the domain: for example if we let $A = (-\pi/2, \pi/2)$, then $f : A \to \mathbb{R}$ is bijective and therefore has an inverse.

3. True or false: $\arcsin(\sin(x)) = x$ for all $x \in \mathbb{R}$? What about $\sin(\arcsin(x)) = x$ for all $x \in [-1, 1]$? Write both statements in terms of composition of functions.

   Solution. The statement $\sin(\arcsin(x)) = x$ is true for all $x \in \mathbb{R}$, but $\arcsin(\sin(x)) = x$ is true *only if* $x \in [-\pi/2, \pi/2]$. In terms of compositions, we have: $\sin : \mathbb{R} \to [-1, 1]$ (surjective, not injective), and $\arcsin : [-1, 1] \to \mathbb{R}$, but if we write it this way, it is not surjective ($\mathbb{R}$ is a valid codomain, but it does not coincide with the range). We can write $\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$, then it is bijective, and has an inverse function (namely, $\sin : [-\pi/2, \pi/2] \to [-1, 1]$). So we have:

$$\arcsin \circ \sin : [-\pi/2, \pi/2] \to [-\pi/2, \pi/2]$$

is the identity function (on the set $[-\pi/2, \pi/2]$), but NOT the identity function on $\mathbb{R}$. And

$$\sin \circ \arcsin : [-1, 1] \to [-1, 1]$$

is the identity function.

4. Prove every statement in Theorem 12.4 on p. 243 (see also the posted notes). **We did it last lecture, and most of it is in the notes for the last class.**