Worksheet 15: Review functions: injective, surjective, bijective functions. Range.

1. Determine the range of the functions $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

   (a) $f(x) = \dfrac{x^2}{1+x^2}$

   (b) $f(x) = \dfrac{x}{1+|x|}$

   Solution.

   a) $f(x) = \dfrac{x^2}{1+x^2}$

   Claim: $f(\mathbb{R}) = [0, 1)$

   Proof:

   ($\subseteq$) For any real number $r \in \mathbb{R}$, we have that $0 \leq r^2 < 1 + r^2$. So, dividing the inequality by $1 + r^2$ (which is non-zero), we get $0 \leq \frac{r^2}{1+r^2} = f(r) < 1$. Hence, $f(r) \in [0, 1)$, and $f(\mathbb{R}) \subseteq [0, 1)$.

   ($\supseteq$) For every $s \in [0, 1)$, which means that $0 \leq s < 1$. We have that $1 - s > 0$, so $\frac{s}{1-s} \geq 0$ and we may take the square root. Let $r = \sqrt{\frac{s}{1-s}} \in \mathbb{R}$. Then,

   $$f(r) = \frac{r^2}{r^2+1} = \frac{\frac{s}{1-s}}{\frac{s}{1-s}+1} = \frac{\frac{s}{1-s}}{\frac{1}{1-s}} = s$$

   So $s \in f(\mathbb{R})$, and hence $[0, 1) \subseteq f(\mathbb{R})$, which completes the claim.

   b) $f(x) = \dfrac{x}{1+|x|}$

   Claim: $\text{ran}(f) = (-1, 1)$

   Proof:

   ($\subseteq$) Let $x \in \mathbb{R}$. We consider two cases:

   - If $0 \leq x$, then $|x| = x$ and so, $0 \leq f(x) = \frac{x}{1+x} < 1$. Hence, $f(x) \in [0, 1)$.
   - If $0 > x$, then $|x| = -x > 0$. So, $0 < -x < 1 - x$. Dividing the whole inequality by $1 - x$, we get $0 < -\frac{x}{1-x} < 1$, which is equivalent to $-1 < \frac{x}{1-x} = f(x) < 0$. Thus, $f(x) \in (-1, 0)$.

   Thus, $\text{ran}(f) \subseteq (-1, 0) \cup [0, 1) = (-1, 1)$.

   ($\supseteq$) Let $y \in (-1, 1)$. We consider two cases:

- If $y \in [0,1)$, then $1-y > 0$ and $y \geq 0$. So, consider $x = \frac{y}{1-y} \in [0,\infty)$. In particular, $|x| = x$. Then,

$$f(x) = \frac{x}{1+|x|} = \frac{x}{1+x} = \frac{\frac{y}{1-y}}{\frac{y}{1-y}+1} = \frac{\frac{y}{1-y}}{\frac{1}{1-y}} = y$$

  Hence, $y \in \text{ran}(f)$.
- If $y \in (-1,0)$, then $y+1 > 0$ and $y < 0$. Hence, $x = \frac{y}{y+1} \in (-\infty, 0)$, which means $|x| = -x$. Then,

$$f(x) = \frac{x}{1+|x|} = \frac{x}{1-x} = \frac{\frac{y}{y+1}}{1-\frac{y}{y+1}} = \frac{\frac{y}{y+1}}{\frac{1}{y+1}} = y,$$

  and $y \in \text{ran}(f)$.

So, $[0,1) \cup (-1,0) \subseteq \text{ran}(f)$, which is equivalent to $(-1,1) \subseteq \text{ran}(f)$, and we are done.

2. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by

$$f(a,b) = \frac{(a+1)(a+2b)}{2}$$

Show that the image of f is contained in $\mathbb{N}$, so that $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a well-defined function.

Solution. We need to prove that for every pair $(a,b) \in \mathbb{N} \times \mathbb{N}$, the number $f(a,b) = \frac{(a+1)(a+2b)}{2}$ is in $\mathbb{N}$. Clearly if $a, b > 0$ then $f(a,b) > 0$, so we just need to show that $f(a,b)$ is always an integer, which is equivalent to shwoing that for any $a, b \in N$, the number $(a+1)(a+2b)$ is even. To show this, note that $a+2b \equiv a \mod 2$, so $(a+1)(a+2b) \equiv a(a+1) \mod 2$, and the latter number is always even since one of a and $a+1$ has to be even, which completes the proof.

3. Explain why multiplication by 2 defines a bijection from $\mathbb{R}$ to $\mathbb{R}$, but not from $\mathbb{Z}$ to $\mathbb{Z}$.

Solution.

<u>Claim:</u> The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x$ is a bijection.
<u>Proof:</u>
To show that f is surjective, let $b \in \mathbb{R}$. Consider $a = \frac{b}{2}$. Since $b \in \mathbb{R}$, we have that $a \in \mathbb{R}$, and $f(a) = 2a = 2\left(\frac{b}{2}\right) = b$. So, f is surjective.
To show that f is injective, let $a_1, a_2 \in \mathbb{R}$ be such that $f(a_1) = f(a_2)$. Then, by definition of f, we get that $2a_1 = 2a_2$, which means $a_1 = a_2$. Thus, f is injective.
Therefore, f is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

Claim: The function $g : \mathbb{Z} \to \mathbb{Z}$ where $g(x) = 2x$ is not a bijection.
Proof:
To show that g is not a bijection, it suffices to prove that g is not surjective, that is, to prove that there exists $b \in \mathbb{Z}$ such that for every $a \in \mathbb{Z}$, $g(a) \neq b$. Let $b = 3 \in \mathbb{Z}$. For every $a \in \mathbb{Z}$, we have that $g(a) = 2a$ from definition, so $g(a)$ is even. It follows from $b = 3$ being odd that $g(a) \neq b$ for any $a \in \mathbb{Z}$ because of different parity. Hence, g is not surjective, and therefore, not a bijection.

Remark: Even though f and g are defined by the same formula (multiplication by 2), they are different functions because their domains and codomains are different.

4. Write four different bijections $f : \mathbb{N} \to \mathbb{N}$.

Solution

- $f_1 : \mathbb{N} \to \mathbb{N}$ where $f_1(n) = n$. We are fixing each number as its own value under f_1.
- $f_2 : \mathbb{N} \to \mathbb{N}$ where $f_2(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases}$. Loosely speaking, we are swapping consecutive pairs of numbers (i.e. $f(1) = 2$ while $f(2) = 1$; $f(3) = 4$ while $f(4) = 3$, and so on...)
- $f_3 : \mathbb{N} \to \mathbb{N}$ where $f_3(n) = \begin{cases} 1 & \text{if } n = 2 \\ 2 & \text{if } n = 1 \\ n & \text{if } n \geq 3 \end{cases}$. Loosely speaking, we are swapping the first pair of numbers, and fixing the rest.
- $f_4 : \mathbb{N} \to \mathbb{N}$ where given a natural number n, f_4 reverses the order of the digits of n except for the left most zeroes (i.e. $f(450216) = 612054$, $f(470) = 740$, $f(900) = 900$, and so on..) In this case, f_4 is its own inverse!

5. *Final Exam - Dec 2010* Prove that the following function is bijective

$$f : \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\} \text{ defined by } f(x) = \frac{x+1}{x+2}$$

Solution.

To show that f is injective, suppose that $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{R} - \{-2\}$, that is,

$$\begin{aligned} \frac{x_1+1}{x_1+2} &= \frac{x_2+1}{x_2+2} \\ (x_1+1)(x_2+2) &= (x_2+1)(x_1+2) \\ x_1x_2 + x_2 + 2x_1 + 2 &= x_1x_2 + x_1 + 2x_2 + 2 \\ x_1 &= x_2 \end{aligned}$$

To show that f is surjective, let $b \in \mathbb{R} - \{1\}$. Consider $a = \frac{2b-1}{1-b} \in \mathbb{R}$. We first show that $a \neq -2$ by contradiction. Suppose that $a = -2$, that is,

$$-2 = \frac{2b-1}{1-b} \Leftrightarrow -2 + 2b = 2b - 1 \Leftrightarrow -2 = -1,$$

which is impossible. Thus, $a \in \mathbb{R} - \{-2\}$, and

$$f(a) = \frac{a+1}{a+2} = \frac{\frac{2b-1}{1-b} + 1}{\frac{2b-1}{1-b} + 2} = \left(\frac{b}{1-b}\right)\left(\frac{1-b}{1}\right) = b.$$

Hence, f is also surjective, which means f is a bijection.