Worksheet 13: Congruence of integers; functions.

1. Let $d \in \mathbb{N}$. Prove that

$$(a \equiv b \mod d) \wedge (a' \equiv b' \mod d) \Rightarrow aa' \equiv bb' \mod d.$$

2. Prove that if an integer a is written with the digits $a_n, \dots, a_0$, then a and $a_0 + \dots + a_n$ are in the same congruence class $\mod 9$.

3. Prove that for any integers a and b, the sum $a^2 + b^2$ lies in one of the classes $[1]$, $[0]$, or $[2] \mod 4$. Deduce that the number 1000535 cannot be represented as a sum of two squares.

4. Prove that there do not exist integers a, b and c such that

$$12345678910111213 = a^2 + 25b^2 + 5c^2.$$

5. Let $A = \{1, 2, 3\}$, and let $B = \{a, b, c, d\}$. Let $R = \{(1, a), (2, b), (2, c), (3, a), (3, d)\}$ - a relation from A to B. Draw a diagram representing this relation.

6. Represent the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ as a relation.

7. Represent the sequence $a_n = 1/n$ as a relation; think of it as a function from $\mathbb{N}$ to $\mathbb{R}$.

8. Give an example of a function that is injective but not surjective.