Math 220
Lecture 2
Thursday Jan 9

Worksheet 1: Sets and Logic

1. Which of the following sentences are *statements* in the mathematical sense? For the ones that are statements, can you decide whether they are True or False?

   (a) It is sunny outside right now.

   Statement. True.

   (b) Tomorrow the weather will be nice. – Non-statement (can be made into a statement if you define 'nice').

   in fact, False. (c) The 100th digit of the decimal expansion of π is 7.

   statement. We don't know T/F, but it has a True or False value.

   (d) The digits of π encode the meaning of the Universe.

   non-statement.

   (e) This statement is False. – NOT ambiguous. (self-referential, but ok – try it: – see explanation next pages. (see p.3)

   This is a non-statement!

   (f) This statement is True.

   True statement? – true works. False: also works.

   Also a non-statement

   So far, non-statements happened because of language ambiguity.

   Suppose: define 'unprovable' etc. (g) For any consistent system of axioms there exists a statement about natural numbers that is true, but unprovable from these axioms.

   Gödel's incompleteness Theorem (True statement) (statement once you define all ingredients, True).

   Proof by example (h) For some prime numbers p, the number $p+2$ is also prime.

   True statement. Proof: take $p=3$, $p+2=5$

   Proof by counterexample (i) For all prime numbers p, the number $p+2$ is also prime.

   False. Take $p=7$, $p+2=9=3\cdot 3$ not prime.

   (j) There exist infinitely many primes p such that the number $p+2$ is also prime.

   see next pages.

   will do these later.

This is a statement, called 'Twin prime conjecture' We do not know if it is TRUE or FALSE.

– read about it on Wikipedia :)

Discussion for the worksheet answers:

Recall: Statement: sentence that has "truth value": True or False.
(not 'sometimes', or 'maybe').
(sometimes we don't know True or False, but we know it is one or the other — e.g. (c) in the worksheet).

Note: Objection to some of the problems 12–14 on p.42 in the book:
they are asking to use formal logic to express
"All happy families are alike, all unhappy families are different" — more or less a statement.
"human beings want to be good, but not too good, not all the time" — more or less a statement
"A man should look for what is, not what he thinks should be" ← we understand what it means
I agree with it.
[But it is a non-statement, I think.

Discussion of worksheet, continued:

(e) "This statement is False":

If true: then we believe it, then it's false.

If False, then ~~[illegible]~~ we believe the opposite: it is True.

Contradiction both ways.

So it is not a statement because it cannot be True or False.

We will learn Proof by contradiction.
Many proofs eventualy boil down to a construction of this sort, to get the contradiction.

1) There are infinitely many prime numbers.

True (will 'prove' in the course)

about parts (h), (i), (j)

Now let's consider statement (j)

we have: example: $p = 29$ $p+2 = 31$ - works

we don't see examples that are larger. So it seems the conjecture might be false?

But can we prove it?

a suggestion by a student on why false:

why false: as numbers get larger, there are more factors, try to see how many numbers can be prime at all, unlikely to get infinitely many pairs so close

good idea. But not proof.

This is TWIN PRIME conjecture. - No-one can prove (yet) whether it is True or FALSE.

Upshot: one way of proving things about primes is to check every one of them
- but only works for finitely many.
- so we can make arguments proving things about infinitely many objects.

we will talk more about it in later classes.

- Read about twin prime conjecture.

Empty set: Set with no elements.

2. Are the following sets empty or not? When not empty, draw the set.

(a) The set of all $x \in \mathbb{R}$ such that $x^2 > 4$ and $x < 0$.

(b) The set of all $x \in \mathbb{R}$ such that $x^2 > 4$ and $|x| < 2$.

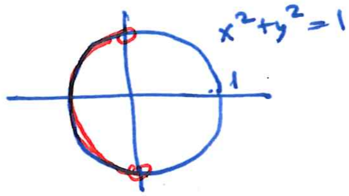
$\emptyset$ – empty set.

(c) The set of all $x \in \mathbb{R}$ such that $x^2 \geq 4$ and $|x| \leq 2$.

$S = \{2, -2\}$

← the plane

(d) The set of all $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 = 1$ and $x < 0$.



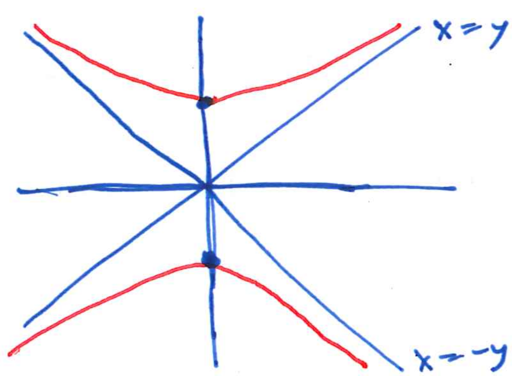
$\{(x,y) \in \mathbb{R}^2 : (x^2+y^2=1) \wedge (x<0)\}$

(e) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = -1\}$. – this set is empty.

$= \emptyset$

because $x^2 \geq 0$, $y^2 \geq 0$, so $x^2+y^2 \geq 0$, so $x^2+y^2 \neq -1$.

(f) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = -1\}$.



hyperbola

Note about notation: I used slightly different terminology in class compared to our book:

Class:		book
$\neg p$ negation	$\sim p$ negation	

p True $\Rightarrow$ $\neg p$ False

False $\neg p$ True

predicate	open sentence
$P(x,y)$ - statement whose truth depends on x, y	(what I called "predicate" the book calls "open sentence")
ex: $x^2+y^2=1$	

Notation: A - set

$|A|$ = cardinality of the set A

= number of its elements if A is finite

and "infinite" if not.

- $a, b \in \mathbb{Z}$, $a \neq 0$ — we will use $a \mid b$ "a divides b" if there is $k \in \mathbb{Z}$ such that $b = ka$.

Example $3 \mid 36$ - True ($k=12$)

$2 \nmid 41$ (b/c there is no such integer k that $41 = 2k$).

Our next thing: Conditional statements

(implication)

Read 2.3

P, Q - statements.

we defined: $P \wedge Q$, $P \vee Q$, $\sim P = \neg P$

Now: conditional statement: $P \Rightarrow Q$

Def:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

"P implies Q".   Will discuss this next class.