Today's worksheet (March 12, 2020)

Worksheet 14: Injective and surjective functions; composition.

1. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective. Can you make such a function from a finite set to itself?

2. Prove that the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(a, b) = 3a + 7b$ is surjective. Is this function injective?

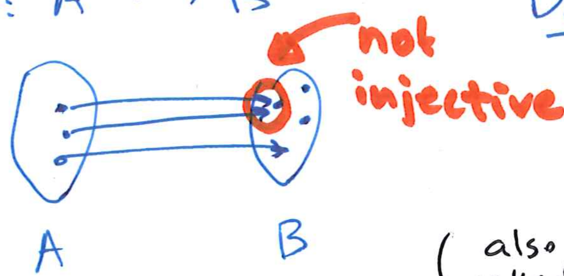
3. Prove that among any six distinct integers, there are two whose difference is divisible by 5.

4. Let $f : \{1, 2, 3\} \to \{a, b, c, d\}$ be defined by $f(1) = a$, $f(2) = c$, $f(3) = d$. Let $g : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ be defined by $g(a) = 2$, $g(b) = 1$, $g(c) = 4$, $g(d) = 5$. Find the composition $g \circ f$.

5. Prove that if $f : A \to B$ is injective and $g : B \to C$ is injective, then $g \circ f : A \to C$ is injective. Is the converse statement true?

Today: Continue Functions

$f: A \to B$



Def $f: A \to B$ is called <u>injective</u> if $f(a_1) \neq f(a_2)$ for any $a_1, a_2 \in A$ s.t. $a_1 \neq a_2$.

(also called one-to-one sometimes)

Def: $f: A \to B$ is called <u>surjective</u> (also called <u>onto</u>) if $\forall\, b \in B\ \exists\, a \in A$ s.t. $f(a) = b$

(equivalently, $\text{Range}(f) = B$)

Def: $f: A \to B$ is called <u>bijective</u> if it is both injective and surjective (one-to-one is often reserved for this)

Examples: ① $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ — not injective because $x^2 = (-x)^2$ for all x

(in fact, enough just to say: $1^2 = (-1)^2$)

Not surjective ($x^2 \geq 0$ for all x, so for example, $-1 \notin \text{Range}(f)$)

Then $\text{Range}(f) \neq \mathbb{R}$.

② $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 1/x$ } Not actually a function <u>from $\mathbb{R}$</u> to $\mathbb{R}$!

Not defined at 0!

so we have $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$

It is <u>injective</u>: <u>pf</u>: contrapositive:

suppose $f(x_1) = f(x_2)$

Then $\frac{1}{x_1} = \frac{1}{x_2}$ (and $x_1, x_2 \in \mathbb{R} \setminus \{0\}$)

Then $x_1 = x_2$. This proves f is injective

This function is <u>not</u> surjective: $f(x) \neq 0$ $\forall x \in \mathbb{R} - \{0\}$.

(3) $f: \mathbb{R} \to \mathbb{R}$ defined by: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ a, & x = 0 \end{cases}$

If we want f to be injective,
then a has to be zero!

If we take $a = 0$, then f becomes <u>bijective</u>

(4) $f(x) = e^x$

strictly increasing $\Rightarrow$ injective

(Direct proof: Let $x_1 \neq x_2$. Want to prove: $f(x_1) \neq f(x_2)$.
Given: f is strictly increasing.
WLOG, $x_1 < x_2$. Then since f is strictly increasing, $f(x_1) < f(x_2)$. So $f(x_1) \neq f(x_2)$ $\blacksquare$)

$f(x) > 0$ $\forall x \in \mathbb{R}$, so Range$(f) \subseteq (0, +\infty) \neq \mathbb{R}$.

<u>Worksheet question 2</u>

To prove that f is surjective:

we need to show: $\forall c \in \mathbb{Z}$, $\exists a, b$ s.t. $3a + 7b = c$

Many <u>ways to prove it</u>:

<u>Way 1</u>: first, take $c = 1$. If $a = -2$, $b = 1$,
we get: $3a + 7b = -6 + 7 = 1$.

Then for any $c \in \mathbb{Z}$, take $a = -2c$, $b = c$
and we get $c \cdot 1 = c \cdot \underbrace{(3 \cdot (-2) + 7 \cdot 1)}_{=1} = 3a + 7b$. $\blacksquare$

<u>Way 2</u>: Bezout's identity:

$1 = \gcd(3, 7)$ - greatest common divisor.

We proved: $\exists \underset{\mathbb{Z}}{x, y}$ s.t. $1 = 3x + 7y$.
(in fact, $x = -2$ $y = 1$) works.

Then proceed as in Way 1.

Way 3: By cases: $c \equiv 0$, $c \equiv 1$, $c \equiv 2$ mod 3.

1) if $c \equiv 0 \mod 3$, then $c = 3k$, take $a = k, b = 0$ done.

2) if $c \equiv 1 \mod 3$, then: 1) represent 1:

$$1 = 3 \cdot (-2) + 7$$

Then $c = 3k + 1 = 3k + 3 \cdot (-2) + 7$ – done

3) if $c \equiv 2 \mod 3$, same thing, represent 2:

$$2 = 3 \cdot (-4) + 7 \cdot 2$$

Proceed as in case 2.

How to come up with this: $c = 3a + 7b$ ← want.

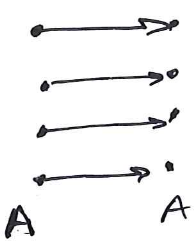
So: $c - 7b$ needs to be divisible by 3.

Now it's natural to consider cases mod 3.

Back to question 1: Let A be a finite set.

Can there be a function $f: A \to A$ that is injective but not surjective?

No! = Pigeonhole principle



Question 3 :

"rabbits"
"cages" : if you try to put $n+1$ or more rabbits into n cages, then at least two will wind up in the same cage.

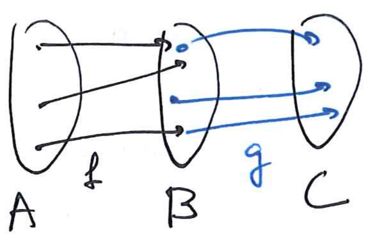
"rabbits" : the 6 given integers

"cages" congruence classes mod 5:

$[0], [1], [2], [3], [4]$ ← 5 of them.

Then of my 6 integers, two have to end up in the same congruence class.

Composition of functions



Let $f: A \to B$
$g: B \to C$ - functions.

Then their composition denoted by ~~f~~ $g \circ f$, is a function

$g \circ f : A \to C$

defined by: $g(f(a))$, for $a \in A$.

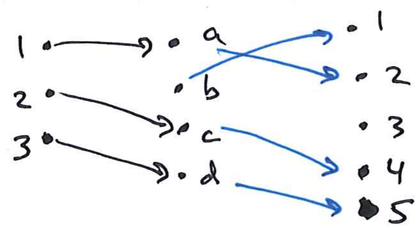
$= g \circ f(a)$

(before: $g(f(x))$)
$= g \circ f(x)$

$\boxed{\text{\circ}}$ in TeX

Question 4 from Worksheet

$A = \{1,2,3\}$



$g \circ f : A \to C$
$g \circ f : \{1,2,3\} \to \{1,2,3,4,5\}$

$g \circ f = \{(1,2), (2,4), (3,5)\}$
$\subseteq$
$A \times C$