Solution to <u>4</u> : <u>Reflexive</u>: NOT reflexive, because does not contain $(y,y)$ and $(z,z)$.

- <u>symmetric</u>: <u>Yes</u>: $(x,y)$ and $(y,x)$ appear
  $(y,z)$ and $(z,y)$ appear.

- <u>transitive</u>: NOT transitive: $(x,y) \in R$, $(y,z) \in R$ but $(x,z) \notin R$!

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<u>Def</u>: Suppose $A$ is a set, $R$ is an equivalence relation on $A$. Then we say that $R$ partitions $A$ into <u>equivalence classes</u>:
~~each~~ ~~a~~ the equivalence class of an element $a \in A$ is the set $E_a = \{ b \in A \mid aRb \}$.

# Worksheet 12: Cartesian product, indexed collections, relations

1. Let $A_n = \left[0, \frac{1}{n}\right] \times [0, n]$. Draw the picture representing $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$. (Both should be subsets of $\mathbb{R}^2$).

2. Let $R$ be the relation $a \equiv b \mod 3$ on the set $A = \{0, 1, 2, 3, 4, 5\}$. Write this relation as a subset of $A \times A$.

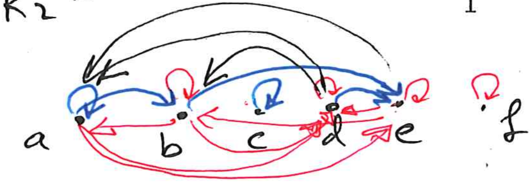
3. Let $A = \{1, 2, 3\}$, and let $B = \{a, b, c, d\}$. Let $R = \{(1, a), (2, b), (2, c), (3, a), (3, d)\}$ - a relation from $A$ to $B$. Draw a diagram representing this relation.

4. Let $A = \{x, y, z, w\}$ and let $R = \{(x, x), (x, y), (y, z), (w, w), (y, x), (z, y)\}$. Is this relation symmetric? Is it reflexive? is it transitive?

5. Let $A = \{a, b, c, d, e, f\}$ and let $R_1 = \{(a, a), (a, b), (b, e), (c, c), (d, e)\} \subset A \times A$. Find $R_2$ - the smallest relation containing $R_1$ that is an equivalence relation. Then find the partition of $A$ into equivalence classes according to $R_2$.

$R_2 =$

$x \to y$ - in $R_1$.
$x R_1 y$.





add: $(b,b), (d,d), (e,e), (f,f)$ - becomes reflexive.
$(b,a), (e,b), (e,d)$ - becomes symmetric
$(b,d), (a,e), (a,d)$ - becomes transitive - not symmetric anymore.
$(e,a), (d,a), (d,b)$ - now fine.

Recall: Relation on a set $A$ : subset of $A \times A$

which means, a set of pairs $(a_1, a_2)$

$a_1, a_2 \in A$

$a_1 R a_2$ means $(a_1, a_2) \in R$

our relation.

Equivalence relation:

- $R$ - reflexive

  $(x, x) \in R$ for every $x \in R$

- $R$ is symmetric: $(x, y) \in R \Rightarrow (y, x) \in R$
- $R$ transitive: $(x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$.

Equivalence classes: $\{a, b, d, e\}$ — equivalence classes.
$\{c\}$, $\{f\}$

Example: cities in North America
equivalence relation: connected by rail
Equivalence classes:

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collection of socks.
relation: being of the same colour.

red    blue    green    white    black

striped

Partition: 1) The union of all equivalence classes is $A$
2) Two equivalence classes are either the same or have empty intersection.

Each equivalence class is a subset of $A$.

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Let $R$ be an equivalence relation on $A$
For every $a \in A$, let $E_a$ be the equivalence class of $a$.

Then $A = \bigcup_{a \in A} E_a$ and $E_a \cap E_b = \emptyset$ if $E_a \neq E_b$.

$R$ is an equivalence relation, so:

$a \in E_a$ (reflexive)
and the $\boxed{\text{reason that } E_a \cap E_b = \emptyset \text{ if } E_a \neq E_b}$ is that $R$ is symmetric and transitive.

Prove: if $R$ is an equivalence relation then if $E_a \neq E_b$, then $E_a \cap E_b = \emptyset$.

Example: in our question 5, we had:

$A = \{a,b,c,d,e,f\} = \{a,b,d,e\} \cup \{c\} \cup \{f\}$

here: $E_a = E_b = E_d = E_e = \{a,b,d,e\}$

$E_c = \{c\}$

$E_f = \{f\}$

Proving our statement: try contrapositive

Suppose $E_a \cap E_b \neq \emptyset$. Then there exists $y \in E_a \cap E_b$.

This means, $y R a$ (because $y \in E_a$)

and $y R b$ (because $y \in E_b$)

Now, because $R$ is symmetric, we also have $b R y$.

Now: $b R y$ and $y R a$

Since $R$ is transitive, we must have $b R a$.

Then $b \in E_a$.

Now let $x \in E_b$. Then $x R b$ but $\underbrace{b R a}_{\text{we proved}}$

Then $x R a$.

We proved: $E_b \subseteq E_a$.

WLOG, this proves $E_a \subseteq E_b$ as well.

could rename $a$ and $b$.

Then $E_a = E_b$.

Congruence of integers (section 11.4)

Proposition: Let $d \in \mathbb{N}$.
Congruence mod $d$ is an equivalence relation on $\mathbb{Z}$.

Proof: Need to prove: reflexive, symmetric, transitive.

1) $a \equiv a \mod d$ for any $a \in \mathbb{Z}$
(because $d \mid 0$)

2) $a \equiv b \Rightarrow b \equiv a$ (because $d \mid a-b \Rightarrow d \mid b-a$)

3) Transitive: $\left\{ \begin{array}{l} a \equiv b \mod d \\ b \equiv c \mod d \end{array} \right. \Rightarrow a \equiv c \mod d.$

$a - c = (a-b) + (b-c)$
Then if $d \mid a-b$ and $d \mid b-c$, then $d \mid a-c$.

Congruence classes: $[a]_d$ - class of $a$ mod $d$
$= \{ b \in \mathbb{Z} \mid b \equiv a \mod d \}$

Example: $d = 5$
$[3]_5 = \{ \ldots -7, -2, 3, 8, 13, 18, \ldots \}$
$= \{ n \in \mathbb{Z} \mid n \equiv 3 \mod 5 \}$

Def: $\mathbb{Z}_d$ = set of congruence classes mod $d$
$= \{ [0], [1], [2], \ldots, [d-1] \}.$

Arithmetic operations work on congruence classes,
if $a_1, a_2$ are in the same class mod $d$
$b_1, b_2$ — / ——
Then $a_1 b_1$ and $a_2 b_2$ are in the same class.