Today's & next class:

- sets
- subsets, power sets
- operations on sets
- proofs about sets.

Note: this is parallel to logic.

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subset (relation between sets)

1) Definition We say $A \subseteq B$ "A a subset of B" if every element of A is also an element of B

relation between an element and a set.

(in quantifiers: $\forall x \in A,\ x \boxed{\in} B$

or: $\boxed{x \in A \Rightarrow x \in B}$)

Example: Let $A = \{1, \boxed{\{1,2\}}, \{1\}\}$.

Which statements are true:

- $1 \in A$ T
- $1 \boxed{\subseteq} A$ ← False. Need a set on the left ($\subset$ or $\subseteq$: same thing ↑ indicates they could be equal)
- $2 \in A$ ← 2 is an element of a set, which itself is an element of A.
- $2 \subseteq A$ ← False same as above for $1 \subseteq A$: need a set on the left.
- T. $\{1\} \in A$ see circled $\{1\}$.
- $\{1\} \subseteq A$: T because: 1 is the element of $\{1\}$, is also an element of A.
- $\{2\} \subseteq A$: F - 2 is not an element of A.
- $\{1,2\} \in A$ ← True see $\boxed{\{1,2\}}$
- $\{1,2\} \subseteq A$ - F : 2 is not an element of A.
- $\{\{1\}\} \subseteq A$ - true : $\{1\}$ is an element of A

  $\{\{1\}\} \in A$ - False : it is not on the list.

Main point: $1, \{1\}, \{\{1\}\}, \{\{\{1\}\}\}$ are NOT the same:

1: number

$\{1\}$: a set that consists of the number 1, call this set B

$\{\{1\}\}$: set whose only element is the set B, call this set C

$\{\{\{1\}\}\}$: the set whose only element is the set C. Call it D.

We have:

$1 \in B$

$B \in C$

$C \in D$

but NOT:

$1 \in C$

$B \subseteq C$

$C \subseteq D$

$B \subseteq D$

Def: Let A be a set.

$\mathcal{P}(A)$ = set whose elements are all the subsets of A

$$\mathcal{P}(A) = \{ B : B \subseteq A \}$$

power set of A.

so for any set A,

$\mathcal{P}(A) \ni \emptyset \iff \emptyset \in \mathcal{P}(A)$

$\mathcal{P}(A) \ni A \qquad A \in \mathcal{P}(A)$

Worksheet 10: Sets

In all problems, A, B, C are sets (subsets of a universal set U).

1. Write down (using set notation) the power set of the set $A = \{\emptyset, \{\emptyset\}, 1\}$.

$$\mathcal{P}(A) = \{ \emptyset, \underbrace{\{\emptyset\}, \{\{\emptyset\}\}, \{1\}}_{\text{1-element subsets}}, \underbrace{\{\emptyset, \{\emptyset\}\}, \{\emptyset, 1\}, \{\{\emptyset\}, 1\}}_{\text{2-element subsets}},$$

$$\text{3-elements } \{\emptyset, \{\emptyset\}, 1\} \} \quad \text{(8 elements)}$$

2. Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ (one of the DeMorgan laws for sets).

3. Prove or disprove: $(A \cup B) \setminus B = A$.

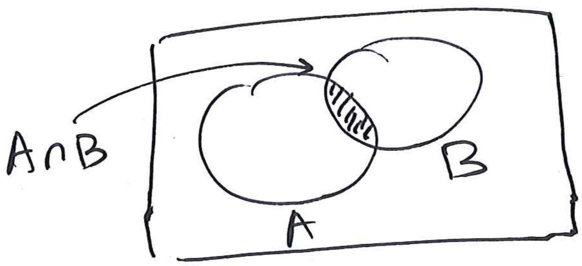
In general, if $|A| = n$ (this means, A has n elements)

then $|P(A)| = 2^n$

<u>why</u>: as we are forming different subsets, we can $\underbrace{\text{take/not take}}_{\text{making this decision } n \text{ times.}}$ every given element.

Each decision doubles the number of outcomes.

Unions and intersections of sets:



Venn diagram - very useful illustration. But <u>NOT proof</u>.

← represents U: universal set.

Usually, all sets we talk about in a particular problem, are subsets of some big set (called the <u>universe</u>)

(often $U = \mathbb{R}$ or $U = \mathbb{Z}$, ...)

(U ← universe)

"set of all sets" is an impossible thing.

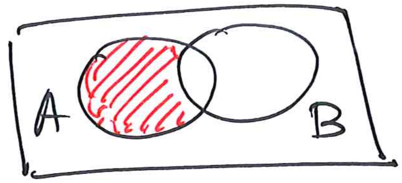
Question: is this set its own element? has no answer: it's like "I am lying" statement.

Note: <u>Textbook</u>: Chapter 1 — all
chapter 8: 8.1–8.3 ; HW6 due next Tuesday.
HW7 due Thursday.

Def: Set difference: $A - B$ or $A \setminus B$

"setminus" in TeX

$$A - B = A \setminus B = \{ x \in A \mid x \notin B \}$$



Proving things about sets:

often you need to prove: $A \subseteq B$ where A, B are given sets

or abstract sets.

(e.g. question 2 in worksheet)

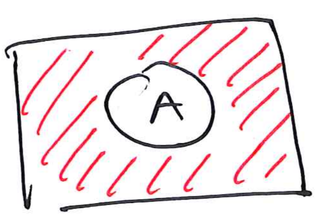
To prove it, always start with:

Let $x \in A$. Then argue about this x and somehow show it is also an element of B.

To prove $A = B$: you need $x \in A \iff x \in B$.
(proving a biconditional)

Def set complement: $\overline{A} = U \setminus A$

universe



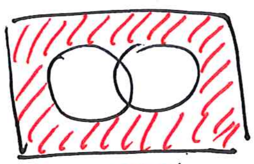
Worksheet 10: Sets

In all problems, A, B, C are sets (subsets of a universal set U).

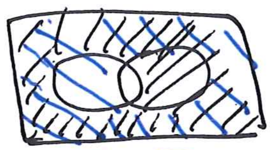
1. Write down (using set notation) the power set of the set $A = \{\emptyset, \{\emptyset\}, 1\}$.

Solution to #2 :

2. Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ (one of the DeMorgan laws for sets).



$\overline{A \cup B}$



$\overline{A} \cap \overline{B}$

- looks true.

Proof: Need to prove: $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

and $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$

one set contains the other.

~~3. Prove or disprove: $(A \cup B) \setminus B = A$.~~

Proving: $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$:

Let $x \in \overline{A \cup B}$

Then by definition of complement, $x \notin A \cup B$ $\Leftrightarrow$

Then by def. of union: $\sim (x \in A \vee x \in B)$ $\Leftrightarrow$

Then (by De Morgan) $(x \notin A) \wedge (x \notin B)$ $\Leftrightarrow$

Then $x \in \overline{A} \cap \overline{B}$ by def. of complement and intersection.

(in fact, all my statements are equivalent to each other).

Worksheet 10: Sets

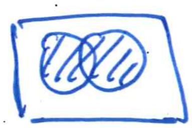
In all problems, A, B, C are sets (subsets of a universal set U).

1. Write down (using set notation) the power set of the set $A = \{\emptyset, \{\emptyset\}, 1\}$.

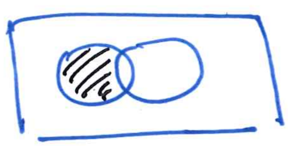
2. Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ (one of the DeMorgan laws for sets).

Solution to # 3:

3. Prove or disprove: $(A \cup B) \setminus B = A$.



$A \cup B$



$(A \cup B) \setminus B$ – doesn't look like A!

This suggests the statement is False.
We need a counterexample.
(The picture helps us make it!)
Let $A = \{1, 2\}$ Let $B = \{2\}$.
Then $A \cup B = \{1, 2\}$ and $(A \cup B) \setminus B = \{1\}$
$\neq A$.

Connection between sets and logic:

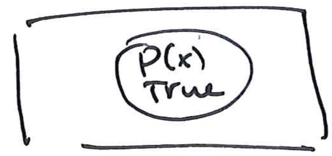
Let U be the universal set.

Let $P(x)$ be an open sentence, with $x \in U$

(example: $U = \mathbb{R}$, $P(x) =$ "$x > 0$".)

To $P(x)$ we can associate a set:

$$A_P = \{ \underset{\underset{U.}{\uparrow}}{x} : P(x) \text{ is True} \}$$



Conversely, if $A \subset U$, can make an open sentence $P_A(x) =$ '$x \in A$'.

Now: sets $\qquad$ statements

1) $A \cup B \quad \leftrightsquigarrow \quad P_A(x) \vee P_B(x)$

$A \cap B \quad \leftrightsquigarrow \quad P_A(x) \wedge P_B(x)$

2) $\overline{A} \quad \leftrightsquigarrow \quad \sim P_A(x)$

Quantifiers: often you have a collection of sets:

$A_n = \{ b \in \mathbb{Z} : n \mid b \}$ ← set that depends on a parameter.

Can make $\bigcup\limits_{n \in \mathbb{N}} A_n$ = (infinite) union of the sets A_n

↑ union of them all: $A_1 \cup A_2 \cup A_3 \cup \cdots$

$= \{ b \in \mathbb{Z} : \exists\, n$ such that $b \in A_n \}$

– the set of integers that belongs to at least one of the sets A_n.