Images of sets

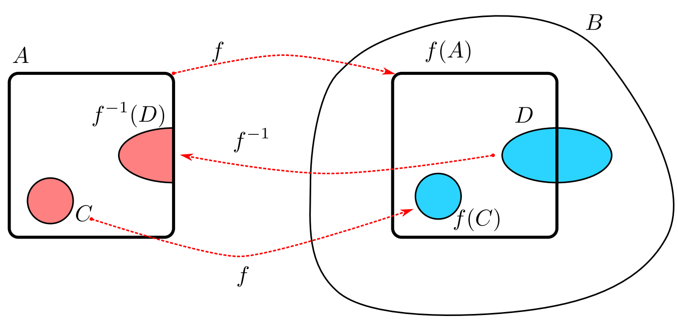
Of course, we don't have to just think of the image of a single point. It also makes sense to think of the image of a set

Definition (12.9 Image and pre-image)**.** Let $f : A \to B$, and let $C \subseteq A$ and let $D \subseteq B$

- The set $f(C) = \{f(x) : x \in C\}$ is the "image of A in B". This is a subset of B.
- The set $f^{-1}(D) = \{x \in A : f(x) \in D\}$ is the "preimage of D in A". This is a subset of A.

Do not think of f^{-1} as the inverse function. The inverse function exists only when f is *bijective*. Think of f as the "pre-image". If f is bijective and the function f^{-1} exists, then the notations agree: $f^{-1}(\{y\}) = \{f^{-1}(y)\}$ for all $y \in B$ (Note that we like to talk about pre-images of *sets*, and this is why we are starting with a 1-element set $\{y\}$ here). More generally, when f is bijective and so the inverse function f^{-1} exists, then the pre-image of a set $D \subseteq B$ under f is the same as the image of D under f^{-1}.

Here is a picture (made by Prof. Rechnitzer) illustrating the images/preimages.



Exercise. For illustrating all the proofs below, it is the easiest to think of a very simple function: let $A = \{a, b, c\}$, let $B = \{1, 2, 3\}$, and define $f(a) = f(b) = 1$, $f(c) = 3$. This function is neither injective nor surjective, and it is a nice source of counter-examples. Draw an arrows diagram for this function as practice.

Let us explore some of the relationships between subsets and their images and preimages. For example — it is clearly the case that if $C_1 \subseteq C_2 \subseteq A$ then $f(C_1) \subseteq f(C_2)$. Similarly if $D_1 \subseteq D_2 \subseteq B$ then $f^{-1}(D_1) \subseteq f^{-1}(D_2)$.

Theorem (12.4)**.** *Let $f : A \to B$, and let $C \subseteq A$ and $D \subseteq B$. Further, let C_1, C_2 be subsets of A and let D_1, D_2 be subsets of B. The following are true*

(a) $C \subseteq f^{-1}(f(C))$ *— not necessarily equal.*
The equality holds for all subsets C of A if and only if f is injective.

(b) $f(f^{-1}(D)) \subseteq D$ *— not necessarily equal.*
The equality holds for all subsets D of B if and only if f is surjective.

(c) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$ *— not necessarily equal.*
The equality holds for all C_1, C_2 of A if and only if f is injective.

(d) $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$ *— is equal*

(e) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$

(f) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

So what does this theorem tell us? It says that preimages are well-behaved:

- The preimage of the intersection is the intersection of the preimages
- The preimage of the union is the union of the preimages

It also tells us that images are nearly well-behaved:

- The image of the union is the union of the images
- The image of the intersection is *a subset of* the intersection of the images.

Proof. (a) Let $x \in C$. We need to show that $x \in f^{-1}(f(C))$. So what is this set — by the definition it is $\{a \in A : f(a) \in f(C)\}$. Since $x \in C$ we have $f(x) \in f(C)$. Since $f(x) \in f(C)$ it follows that $x \in f^{-1}(f(C))$.

Note that the converse is generally false — see the example in the Exercise above the theorem. In this example, take $C = \{a\}$. Then $f(C) = \{1\}$, and

$$f^{-1}(f(C)) = f^{-1}(\{1\}) = \{a, b\} \neq \{a\}.$$

This is exactly the idea behind proving the "only if" part: if f is not injective, we can always construct the set C for which this happens (make sure you can do it!).

Also, check that if f is injective, then $f^{-1}(f(C))$ does equal C for all subsets C of the domain.

(b) Very similar to (a).

(c) We need to show that if an element is in the set on the left then it is in the set on the right. Let $b \in f(C_1 \cap C_2)$. Hence $\exists a \in C_1 \cap C_2$ such that $f(a) = b$. This means that $a \in C_1$ and $a \in C_2$. It follows that $f(a) = b \in f(C_1)$ and $f(a) = b \in f(C_2)$, and hence $b \in f(C_1) \cap f(C_2)$. The converse is false — again, use the example from the exercise to see why, and argue about injectivity similarly to part (a).

(f) What do we have to show? Let $a \in f^{-1}(D_1 \cup D_2)$ and so $f(a) \in D_1 \cup D_2$. This means that $f(a) \in D_1$ or $f(a) \in D_2$. If $f(a) \in D_1$ it follows that $a \in f^{-1}(D_1)$. Similarly if $f(a) \in D_2$ then $a \in f^{-1}(D_2)$. Since a lies in one of these two sets, it follows that $a \in f^{-1}(D_1) \cup f^{-1}(D_2)$.

Let $a \in f^{-1}(D_1) \cup f^{-1}(D_2)$. Then a is an element of one of these two sets. If $a \in f^{-1}(D_1)$, then $f(a) \in D_1$. Similarly if $a \in f^{-1}(D_2)$ then $f(a) \in D_2$. In either case $f(a) \in D_1 \cup D_2$ and so $a \in f^{-1}(D_1 \cup D_2)$.

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