4 marks 1. Construct the converse, contrapositive, and the negation of the statement

"If it is 5 o'clock, Her Majesty is having tea."

4 marks 2. Using any method you like, prove that the following statement is a tautology (that is, it is true for any truth values of the statements P, Q, and R):

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R).$$

4 marks 3. (a) For the following statement, write the negation both in symbols and in words:

$$\exists (x, y) \in \mathbb{R} \times \mathbb{R}, \text{ s.t. } x^2 + y^2 = 1 \text{ and } x = y.$$

(b) Sketch the set of points (x, y) that satisfy the condition from part (a) (if this set is not empty).

6 marks 4. Prove or disprove:

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy \geq 0$.

(b) $\exists y \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}, xy < 0$.

(c) $\forall x, y \in \mathbb{R}$, $(x \geq y) \Rightarrow (x^2 \geq y^2)$.

4 marks 5. Prove or disprove the following statement

Let $n \in \mathbb{Z}$. Then n is even if and only if $5n^2 - 2n + 3$ is odd.

6. Prove or disprove

If n is an odd prime then $n^2 \equiv 1 \mod 8$.

7. Express each of the following statements as a conditional statement in "if-then" form. For (a),(b) and (c) also write the negation (without phrases like "it is false that"), converse and contrapositive. Your final answers should use clear English, not logical symbols.

(a) Every odd number is prime.

(b) Passing the test requires solving all the problems.

(c) Being first in line guarantees getting a good seat.

(d) I get mad whenever you do that.

(e) I won't say that unless I mean it.

8. Show that $(P \wedge R) \wedge (Q \vee R) \equiv (P \wedge R)$. (We have several ways of doing this including using known equivalences, proving the corresponding biconditional or writing out a truth table.)

9. The statement

For all integers m and n, either $m \leq n$ or $m^2 \geq n^2$.

can be expressed using quantifiers as:

$$\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \; m \leq n \text{ or } m^2 \geq n^2$$

or if you prefer as

$$\forall m, n \in \mathbb{Z}, \; m \leq n \text{ or } m^2 \geq n^2.$$

Consider the following two statements:

(a) There exist integers a and b such that both $ab < 0$ and $a + b > 0$.

(b) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

- Using quantifiers, express in symbols the negations of the statements in both (a) and (b).
- Express in words the negations of the statements in (a) and (b).
- Decide which is true in each case, the statement or its negation.

10. Given a real number x,

- let $A(x)$ be the statement "$\frac{1}{2} < x < \frac{5}{2}$",
- let $B(x)$ be the statement "$x \in \mathbb{Z}$",
- let $C(x)$ be the statement "$x^2 = 1$", and

Which statements below are true for all $x \in \mathbb{R}$?

(a) $A(x) \Rightarrow C(x)$

(b) $C(x) \Rightarrow B(x)$

(c) $(A(x) \wedge B(x)) \Rightarrow C(x)$

(d) $C(x) \Rightarrow (A(x) \wedge B(x))$

(e) $(A(x) \vee C(x)) \Rightarrow B(x)$

11. Consider the following two statements:

(a) For all $w \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $w < x$.

(b) There exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, $y < z$

One of the statements is true, and the other one is false. Determine which is which and prove your answers (both of them). *(Final exam 2005)*

12. (a) Prove that $3|2n \Leftrightarrow 3|n$.
The contrapositive will really help in one direction.

(b) Prove that if $2|n$ and $3|n$ then $6|n$. *(Consider n modulo 6.)*

(c) Prove that the product of any three consecutive natural numbers is divisible by 6.

13. Is it true that if a natural number is divisible by 4 and by 6, then it must be divisible by $4 \times 6 = 24$?

14. Let $a, b, c, n \in \mathbb{Z}$, where $n \geq 2$. Prove that if $a \equiv b \mod n$ and $a \equiv c \mod n$, then $b \equiv c \mod n$.

15. Find the smallest natural number a such that $2012^{2013} \equiv a \mod 5$.