4 marks 1. Construct the converse, contrapositive, and the negation of the statement

"If it is 5 o'clock, Her Majesty is having tea."

Solution: Remark on marking: converse and contrapositive were 1 point each, negation was 2 points.

Converse: If H.M. is having tea, it is 5 o'clock.

Contrapositive: If H.M. is not having tea, it is not 5 o'clock.

Negation: It is 5 o'clock, and H.M. is not having tea.

Most popular negation mistake: "If it is 5 o'clock, H.M. is not having tea". I feel compelled to include a discussion of why this is really wrong (apart from referring to the rules we talked about in class): "P does not imply Q" is not at all the same as "P implies not Q" – consider, for example, the statement "If I do well on the exam, it will snow in Vancouver tomorrow". This statement is clearly false. So its negation must be true. But if you negate it in this fashion, you will get "If I do well on the exam, it will not snow in Vancouver tomorrow", which is nowhere closer to true than the original statement. If you made any other mistake, also try the same with the sentence about snow in Vancouver, and see why it does not make sense.

4 marks

2. Using any method you like, prove that the following statement is a tautology (that is, it is true for any truth values of the statements P, Q, and R):

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R).$$

Solution: The most popular (accepted) solution was by truth tables. I will not reproduce it here. Note that the truth table had to have 8 rows, one for every possible combination of T/F values for P, Q, and R.

My preferred solution:

In fact, the statements $(P \Rightarrow (Q \Rightarrow R))$ and $(P \wedge Q \Rightarrow R)$ are logically equivalent, and in particular, it is a tautology that one implies the other. Let us prove their logical equivalence.

$$\begin{aligned} P \Rightarrow (Q \Rightarrow R) &\equiv {\sim}P \vee (Q \Rightarrow R) \equiv {\sim}P \vee ({\sim}Q \vee R) \\ &\equiv ({\sim}P \vee {\sim}Q) \vee R \text{ (by associativity law)} \\ &\equiv {\sim}(P \wedge Q) \vee R \text{ (by DeMorgan's law)} \\ &\equiv (P \wedge Q) \Rightarrow R. \end{aligned}$$

□

One more good solution (possibly my favourite): we have to show that the implication $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$ is true regardless of the truth values of P, Q, and R. How can this implication be false? By definition of the implication, it would be false if $P \Rightarrow (Q \Rightarrow R)$ was true and $(P \wedge Q) \Rightarrow R$ was false. Now, $(P \wedge Q) \Rightarrow R$ is false only when $P \wedge Q$ is true, but R is false. But then P is true, Q is true, and R is false. Let us now look at the left-hand side. If P and Q are true, and R is false, then the implication $Q \Rightarrow R$ is false (by definition of the implication); and then $P \Rightarrow (Q \Rightarrow R)$ is also false. Thus, we have shown that it is impossible for the left-hand side to be true, but the right-hand side to be false, and so the implication is a tautology.

4 marks

3. (a) For the following statement, write the negation both in symbols and in words:

$$\exists (x, y) \in \mathbb{R} \times \mathbb{R}, \text{ s.t. } x^2 + y^2 = 1 \text{ and } x = y.$$

Solution: In symbols: $\forall (x, y) \in \mathbb{R} \times \mathbb{R}, x^2 + y^2 \neq 1$ or $x \neq y$.

In words: every point on the plane either does not lie on the unit circle $x^2 + y^2 = 1$, or does not lie on the line $x = y$.

Most popular mistake: using "such that" (or "s.t.") in the negation, in combination with the quantifier $\forall$. If you made this mistake, please read the handouts on quantifiers that are posted on the website.

(b) Sketch the set of points (x, y) that satisfy the condition from part (a) (if this set is not empty).

Solution: This set is the intersection of the unit circle with the line $x = y$. (it consists of two points: $(\sqrt{2}/2, \sqrt{2}/2)$ and $(-\sqrt{2}/2, -\sqrt{2}/2)$).

6 marks 4. Prove or disprove:

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy \geq 0$.

Solution: This statement is true. There are two reasonable proofs.

Proof 1. Let x be any real number. We need to show that there exists a real number y such that the product xy is non-negative. There are 3 cases: $x > 0$, $x < 0$, and $x = 0$. In the case $x > 0$, take $y = 1$. Then $xy = x > 0$. In the case $x < 0$, take $y = -1$. Then $xy = -x > 0$. In the case $x = 0$, take $y = 0$ (any number y would work), then $xy = 0 \geq 0$.

Proof 2. Let x be any real number. Take $y = 0$. Then $xy = 0 \geq 0$. (So we showed that the number $y = 0$ "works" for every real number x, and so the statement is true).

Very important remark: Even though here it turned out that the same number y "works" for all x, one should not confuse the two statements:

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy \geq 0$. (that's the statement we were asked about), and

$\exists y \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}, xy \geq 0$.

Even though here both statements turn out to be true, these are very different statements. If not sure about this, please read the handouts on quantifiers on the website.

(b) $\exists y \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}, xy < 0$.

Solution: Note that this statement is exactly the negation of the statement in part (a). Since the statement in (a) is true, this statement is false.

(c) $\forall x, y \in \mathbb{R}$, $(x \geq y) \Rightarrow (x^2 \geq y^2)$.

Solution: This statement is false, and to show that it is false, we just need to provide a counterexample – that is, find a pair of real numbers x and y such that $x \geq y$ and $x^2 < y^2$. Take $x = 0$, $y = 1$, for example.

4 marks 5. Prove or disprove the following statement

Let $n \in \mathbb{Z}$. Then n is even if and only if $5n^2 - 2n + 3$ is odd.

Solution: The implication "$\Rightarrow$". Let n be even. Then there exists $k \in \mathbb{Z}$ such that $n = 2k$. Then

$$5n^2 - 2n + 3 = 5 \cdot 4k^2 - 2 \cdot 2k + 3 = 2(10k^2 - 2k + 1) + 1,$$

so $5n^2 - 2n + 3$ is odd.

The implication "$\Leftarrow$". Proof by contrapositive. Let n be odd. Then $n = 2k + 1$ for some integer $k \in \mathbb{Z}$. Then

$$5n^2 - 2n + 3 = 5 \cdot (4k^2 + 4k + 1) - 2 \cdot (2k + 1) + 3 = 20k^2 + 20k + 5 - 4k - 2 + 3 = 2(10k^2 + 8k + 4),$$

so $5n^2 - 2n + 3$ is even.

6. Prove or disprove

If n is an odd prime then $n^2 \equiv 1 \mod 8$.

Solution: We can prove the result without needing the primality of n. Indeed, we only need that it is odd.

Proof. Let n be an odd integer. Then $n = 2l + 1$ for some $l \in \mathbb{Z}$. However we need to test for congruence mod 8, so we will write n as $8k + j$ where $j = 1, 3, 5, 7$. So we now have 4 cases to check.

- If $n = 8k + 1$ then $n^2 = 64k^2 + 16k + 1$, so $n^2 \equiv 1 \mod 8$.
- If $n = 8k + 3$ then $n^2 = 64k^2 + 48k + 9$, so $n^2 \equiv 1 \mod 8$.
- If $n = 8k + 5$ then $n^2 = 64k^2 + 80k + 25$, so $n^2 \equiv 1 \mod 8$.
- If $n = 8k + 7$ then $n^2 = 64k^2 + 112k + 49$, so $n^2 \equiv 1 \mod 8$.

In all cases, $n^2 \equiv 1 \mod 8$.

Hence providing n is odd, then $n^2 \equiv 1 \mod 8$. The primality of n does not matter. □

7. Express each of the following statements as a conditional statement in "if-then" form. For (a),(b) and (c) also write the negation (without phrases like "it is false that"), converse and contrapositive. Your final answers should use clear English, not logical symbols.

(a) Every odd number is prime.

(b) Passing the test requires solving all the problems.

(c) Being first in line guarantees getting a good seat.

(d) I get mad whenever you do that.

(e) I won't say that unless I mean it.

Solution:

(a) Statement: Every odd number is prime.
"If-then" form: If a number is odd, then it is prime.
Negation: There exists an odd number that is not prime.
Converse: If a number is prime, then it is odd.
Contrapositive: If a number is not prime, then it is not odd.

(b) Statement: Passing the test requires solving all the problems.
"If-then" form: If one passes the test, one has to have solved all the problems.
Negation: One can pass the test without solving all the problems.
Converse: If one solves all the problems, then one passes the test.
Contrapositive: If one has not solved all the problems, then one does not pass the test.

(c) Statement: Being first in line guarantees getting a good seat.
"If-then" form: If one is first in line, then one gets a good seat.
Negation: One can be first in line without getting a good seat.
Converse: If one gets a good seat, then one must be first in line.
Contrapositive: If one does not a get a good seat, then one must not have been first in line.

(d) Statement: I get mad whenever you do that.
"If-then" form: If you do that, then I get mad.

(e) Statement: I won't say that unless I mean it.
"If-then" form: If I say that, then I mean it.

8. Show that $(P \wedge R) \wedge (Q \vee R) \equiv (P \wedge R)$. (We have several ways of doing this including using known equivalences, proving the corresponding biconditional or writing out a truth table.)

Solution:

We could do this by writing out the truth table. Unlike the last Homework, in this case it doesn't take long to prove the biconditional:

$$(P \wedge R) \wedge (Q \vee R) \Leftrightarrow (P \wedge R). \tag{1}$$

Assume $(P \wedge R) \wedge (Q \vee R)$ is T. Then by definition $P \wedge R$ is T. This proves $\Rightarrow$ in (1).
Assume $P \wedge R$ is T. Then R is T and hence so is $Q \vee R$. We therefore see that both $P \wedge R$ and $Q \vee R$ are T and hence so is $(P \wedge R) \wedge (Q \vee R)$. This proves $\Leftarrow$ in (1) and so completes the proof of (1).

9. The statement

$$\text{For all integers } m \text{ and } n, \text{ either } m \leq n \text{ or } m^2 \geq n^2.$$

can be expressed using quantifiers as:

$$\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \;\; m \leq n \text{ or } m^2 \geq n^2$$

or if you prefer as

$$\forall m, n \in \mathbb{Z}, \;\; m \leq n \text{ or } m^2 \geq n^2.$$

Consider the following two statements:

(a) There exist integers a and b such that both $ab < 0$ and $a + b > 0$.

(b) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

- Using quantifiers, express in symbols the negations of the statements in both (a) and (b).
- Express in words the negations of the statements in (a) and (b).
- Decide which is true in each case, the statement or its negation.

Solution:

(a) $\exists a \in \mathbb{Z} \, \exists b \in \mathbb{Z}$ s.t. $(ab < 0)$ and $(a + b > 0)$
Negation is —$\forall a \in \mathbb{Z} \, \forall b \in \mathbb{Z}$, $(ab \geq 0)$ or $(a + b \leq 0)$ or in words:
For all integers a, b either $ab \geq 0$ or $a + b \leq 0$.
The original statement is T. Take $a = 2$ and $b = -1$.

(b) $\forall x \in \mathbb{R} \, \forall y \in \mathbb{R}$, $x \neq y \Rightarrow x^2 + y^2 > 0$
Negation is — $\exists x \in \mathbb{R} \, \exists y \in \mathbb{R}$ s.t. $x = y$ and $x^2 + y^2 \leq 0$ or in words:
There are real numbers x, y so that $x \neq y$ and $x^2 + y^2 \leq 0$.
The original statement is T. If $x \neq y$ then one of the values is non-zero, say $x \neq 0$. Therefore $x^2 + y^2 \geq x^2 > 0$. Think about this solution. To be very careful we should note that the other possibility is that $y \neq 0$. In this case $x^2 + y^2 \geq y^2 > 0$ and we obtain the same result. It may be safer to spell out the argument in both cases. This is up to you but be careful–sometimes the other cases are not so obvious.

10. Given a real number x,

- let $A(x)$ be the statement "$\frac{1}{2} < x < \frac{5}{2}$",
- let $B(x)$ be the statement "$x \in \mathbb{Z}$",
- let $C(x)$ be the statement "$x^2 = 1$", and

Which statements below are true for all $x \in \mathbb{R}$?

(a) $A(x) \Rightarrow C(x)$
(b) $C(x) \Rightarrow B(x)$
(c) $(A(x) \wedge B(x)) \Rightarrow C(x)$
(d) $C(x) \Rightarrow (A(x) \wedge B(x))$
(e) $(A(x) \vee C(x)) \Rightarrow B(x)$

Solution:

(a) $\forall x \in \mathbb{R}\ A(x) \Rightarrow C(x)$ is false, since for $x = 2$, $A(2)$ is true ($\frac{1}{2} < 2 < \frac{5}{2}$), but $C(2)$ is false ($2^2 = 4 \neq 1$),

(b) $\forall x \in \mathbb{R}\ C(x) \Rightarrow B(x)$ is true, since if $x^2 = 1$, then x is 1 or -1 and hence is an integer. We have shown for every $x \in \mathbb{R}$ $C(x)$ implies $B(x)$.

(c) $\forall x \in \mathbb{R}\ (A(x) \wedge B(x)) \Rightarrow C(x)$ is false, since for $x = 2$, $(A(2) \wedge B(2))$ is true (both $A(2)$ and $B(2)$ are true), but $C(2)$ is false (as shown in (a)),

(d) $\forall x \in \mathbb{R}\ C(x) \Rightarrow (A(x) \wedge B(x))$ is false, since for $x = -1$, $C(-1)$ is true ($(-1)^2 = 1$), but $A(-1)$ is false ($-1 < \frac{1}{2}$), and so $A(-1) \wedge B(-1)$ is false,

(e) $\forall x \in \mathbb{R}\ (A(x) \vee C(x)) \Rightarrow B(x)$ is false, since for $x = \frac{3}{2}$, $A(\frac{3}{2})$ is true ($\frac{1}{2} < \frac{3}{2} < \frac{5}{2}$), which makes $A(\frac{3}{2}) \vee C(\frac{3}{2})$ true, but $B(\frac{3}{2})$ is false ($\frac{3}{2} \notin \mathbb{Z}$).

11. Consider the following two statements:

(a) For all $w \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $w < x$.
(b) There exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, $y < z$

One of the statements is true, and the other one is false. Determine which is which and prove your answers (both of them). *(Final exam 2005)*

Solution:

(a) The first statement is true. No matter which real w we choose, we can pick $x = w + 1$ which is a real number. Then $w < x$.

(b) The second statement is false — we show the negation is true. The negation is "For all $y \in \mathbb{R}$, there exists $z \in \mathbb{R}$ so that $y \geq z$. Now, no matter which y we choose, we can pick $z = y - 1$ and then $y > z$. Since the negation is true, the original statement is false.

Solution: Quiz: Consider the statement: $\forall x \in \mathbb{R}\, \exists y \in \mathbb{Z}, |x - y| < 0.1$. State this statement in English, and then form its negation, both symbolically and in English.

Solution: The original statement: For every real number x there exists an integer y such that the distance between x and y is less than 0.1 (in other words, every real number can be approximated by an integer with approximation error less than 0.1.)

Negation: $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{Z}$, $|x - y| > 0.1$. There exists a real number x such that the distance from x to any integer is at least 0.1 (this would say that such a number x *cannot* be approximated by any integer with error at least 0.1). This is true: for example, take $x = 0.5$ – the distance from x to the nearest integer is greater than 0.1. Thus the original statement is false.

12. (a) Prove that $3|2n \Leftrightarrow 3|n$.
The contrapositive will really help in one direction.

Solution:

Proof. ($\Leftarrow$) Let $3|n$. Then, there exists $k \in \mathbb{N}$ such that $n = 3k$. So, $2n = 3(2k)$. Hence, $3|2n$.
($\Rightarrow$) We instead prove the contrapositive: "If $3 \nmid n$ then $3 \nmid 2n$."

Assume $3 \nmid n$ then $n = 3k + 1$ or $n = 3k + 2$ for some $k \in \mathbb{Z}$.

- If $n = 3k + 1$ then $2n = 6k + 2 = 3(2k) + 2$. This is clearly not a multiple of 3 and so $3 \nmid 2n$.
- If $n = 3k + 2$ then $2n = 6k + 4 = 3(2k + 1) + 1$. This is clearly not a multiple of 3 and so $3 \nmid 2n$.

In either case $3 \nmid 2n$. □

(b) Prove that if $2|n$ and $3|n$ then $6|n$. *(Consider n modulo 6.)*

Solution: Since $2|n$, there exists an integer k such that $n = 2k$. Then we have $3|2k$. Then by part (a), $3|k$. Then there exists an integer m, such that $k = 3m$. Then $n = 2 \cdot 3m = 6m$, so $6|n$.

(c) Prove that the product of any three consecutive natural numbers is divisible by 6.

Solution: Let $N = a(a+1)(a+2)$ be the product of three consecutive numbers. Note that multiplying it all out does not help! We will prove, instead, that $2|N$ and $3|N$, and then by Part (b) conclude that $6|N$. Why does 2 divide N: note that among any two consecutive numbers, one is even and one is odd. Then there is at least one even number among any three consecutive numbers. Then the whole product is even (because when you multiply an even number by an integer, you still get an even number – make sure you know a formal proof of this!). Next, $3|N$ for a similar reason: since every third number is divisible by

3, there has to be a number divisible by 3 among any three consecutive integers (we could do a formal proof of this fact by cases: the number a has to be of the form $a = 3k$, $a = 3k + 1$, or $a = 3k + 2$ for some $k \in \mathbb{Z}$. In the first case, a is divisible by 3; in the second case, $a + 2$ is divisible by 3, and in the third case, $a + 1$ is divisible by 3). Then N is a product of a number divisible by 3 and two other integers, so N is divisible by 3.

Solution: Here is an alternative solution:

Lemma. If $r \in \{0, 1, 2\}$, then $r(r + 1)(r + 2) \equiv 0 \pmod 3$.

Proof. If $r = 0$, this is obvious. If $r = 1$, then $r + 2 \equiv 0 \pmod 3$ and so the same is true of the product. Finally if $r = 2$, then $r + 1 \equiv 0 \pmod 3$ and so the same is true of the product. □

Let $k \in \mathbb{N}$. Choose $r \in \{0, 1, 2\}$ so that $k \equiv r \pmod 3$. Then $k(k + 1)(k + 2) \equiv r(r+1)(r+2) \equiv 0 \pmod 3$ by the Lemma. Therefore $3|k(k+1)(k+2)$. Since k or $k+1$ is even, we also have $2|k(k+1)(k+2)$. Therefore by (b), $6|k(k+1)(k+2)$. □

13. Is it true that if a natural number is divisible by 4 and by 6, then it must be divisible by $4 \times 6 = 24$?

Solution: It is false. Let $n = 12$. Then, n is divisible by 4 and 6 being $n = 4(3) = 6(2)$, but 12 is not divisible by $4 \times 6 = 24$ (being less than 24).

14. Let a, b, c, $n \in \mathbb{Z}$, where $n \geq 2$. Prove that if $a \equiv b \mod n$ and $a \equiv c \mod n$, then $b \equiv c \mod n$.

Solution: Let $a\, b$, c, $n \in \mathbb{Z}$ where $n \geq 2$, where $a \equiv b \mod n$ and $a \equiv c \mod n$. It follows from $a \equiv b \mod n$ that there exists $k \in \mathbb{Z}$ such that $a = kn + b$. Similarly, $a \equiv c \mod n$ implies that there exists $l \in \mathbb{Z}$ such that $a = ln+c$. So, $kn+b = ln+c$, or equivalently, $b = (l - k)n + c$. Hence, $b \equiv c \mod n$, as wanted.

15. Find the smallest natural number a such that $2012^{2013} \equiv a \mod 5$.

Solution: Recall that we proved in class that for any number d, if $a \equiv b \mod d$, then $a^n \equiv b^n$ for any positive integer n. Take $a = 2012$, $d = 5$, and $n = 2013$. We need a small number b (between 0 and 4) such that $2012 \equiv b \mod 5$ (such b is called the *remainder* of 2012 modulo 5). We note that this number b is 2 (indeed, $2012 - 2$ is divisible by 5 since it is clearly divisible by 10). Then we get: $2012^{2013} \equiv 2^{2013} \mod 5$. So, now we need to find the remainder of 2^{2013} modulo 5. Consider the first few powers of 2:

$$2^1 \equiv 2 \mod 5$$
$$2^2 = 4 \equiv 4 \mod 5$$
$$2^3 = 8 \equiv 3 \mod 5$$
$$2^4 = 16 \equiv 1 \mod 5.$$

Now we stop, because we found a power of 2, namely, 2^4, that is congruent to 1 modulo 5. Then, by the fact about congruences that we already used above, any power of 16 will be congruent to the same power of 1, that is, to 1, modulo 5. Next, note that $2013 = 2012 + 1 = 4 \cdot 503 + 1$. Then

$$2^{2013} = 2^{2012} \cdot 2 = (2^4)^{503} \cdot 2 = 16^{503} \cdot 2 \equiv 1^{503} \cdot 2 \equiv 2 \mod 5.$$

So, the answer is 2.