*This midterm has **5 questions** on **10 pages**, for a total of 50 points.*

Duration: 80 minutes

- Write your name on **every** page.
- **you need to show enough work to justify your answers.**
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- Unless a problem states otherwise, you **do not** have to simplify algebraic expressions to the shortest possible form, and do not have to evaluate long numerical expressions.

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

Section number: ______________________________

Name of the instructor: ______________________________

Question:	1	2	3	4	5	Total
Points:	14	11	7	9	9	50
Score:						

1. Consider the points $A = (1, 2, 3)$, $B = (1, 5, 1)$, and $C = (-1, 2, 0)$, respectively.

2 marks (a) Find the dot product of the vectors $\overline{AB}$ and $\overline{AC}$.

3 marks (b) Find symmetric equations for the line L passing through A and B.

3 marks (c) Find the area of the triangle ABC.

3 marks (d) Find the angle between the sides AB and AC of the triangle ABC. You may express your answer in terms of arccos.

3 marks (e) A boat is travelling East at 10km/hr. To a man on the boat, who measures the wind, it appears that the wind is blowing from the North at 10km/hr. Find the actual direction and speed of the wind.

Solution:

(a) $\overline{AB} = \langle 1, 5, 1\rangle - \langle 1, 2, 3\rangle = \langle 0, 3, -2\rangle$.

$\overline{AC} = \langle -1, 2, 0\rangle - \langle 1, 2, 3\rangle = \langle -2, 0, -3\rangle$.

$\overline{AB} \cdot \overline{AC} = \langle 0, 3, -2\rangle \cdot \langle -2, 0, -3\rangle = 0 + 0 + 6 = 6$.

(b) The line L passing through A and B is:

$$\begin{cases} x = 1 \\ y = 2 + 3t \\ z = 3 - 2t \end{cases}$$

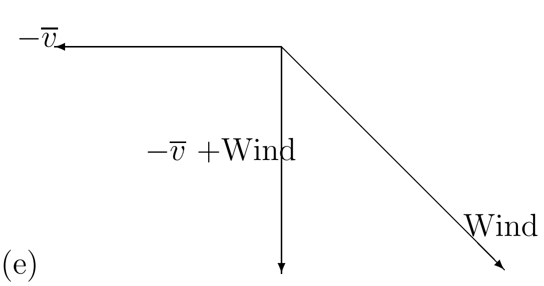
The symmetric equation is:

$$\frac{y-2}{3} = \frac{z-3}{-2}, \; x = 1.$$

(c) The area of the triangle ABC is:

$$\tfrac{1}{2}|\overline{AB}||\overline{AC}|\sin\theta = \tfrac{1}{2}|\overline{AB} \times \overline{AC}| = \tfrac{1}{2}\begin{vmatrix} i & j & k \\ 0 & 3 & -2 \\ -2 & 0 & -3 \end{vmatrix} = \tfrac{1}{2}|\langle -9, 4, 6\rangle| = \tfrac{1}{2}\sqrt{133}.$$

(d) $\theta = \arccos \dfrac{\overline{AB} \cdot \overline{AC}}{|\overline{AB}||\overline{AC}|} = \arccos \dfrac{6}{\sqrt{9+4}\sqrt{4+9}} = \arccos \dfrac{6}{13}$



(e)

If there is no wind, when the boat is traveling East in $10km/h$, the guy on the boat should feel wind blowing West in $10km/h$, shown as $-\overline{v}$ (where $\overline{v}$ is the velocity of the boat). Hence $-\overline{v} = \langle -10, 0\rangle$. Let the velocity of the wind be $\overline{w} = \langle w_1, w_2\rangle$;

Then we have $w + (-\overline{v}) = \langle 0, -10 \rangle$.

Then $\langle w_1, w_2 \rangle = \langle 0, -10 \rangle - \langle -10, 0 \rangle = \langle 10, -10 \rangle$.

$|\overline{w}| = \sqrt{100 + 100} = 10\sqrt{2}$.

The wind is blowing in the direction of SE and at the speed of $10\sqrt{2}km/h$.

2. Consider the point $A = (4, 1, 3)$ and the plane P given by the equation $x + 2y - 3z = 2$.

2 marks (a) Find the plane which passes through the point A and is parallel to the plane P.

3 marks (b) Find the distance between the two parallel planes from part (a).

3 marks (c) Find the parametric equation of the line L of intersection of the plane P with the xz-plane.

3 marks (d) Find the distance from the point B with the coordinates $(0, 3, -4)$ to the line L from Part (c) above.

Solution:

(a) Since the plane is parallel to P, its normal vector is parallel to $\langle 1, 2, -3\rangle$. Then the equation of the plane is

$$(x-4) + 2(y-1) - 3(z-3) = 0, \text{ or } x + 2y - 3z = -3$$

(b) The distance between the two planes is the same as the distance between A and P. It is easy to check that the point $M = (0, 1, 0)$ is on the plane P.

$\overline{AM} = \langle -4, 0, -3\rangle$.

Distance $d = |\text{comp}_{\mathbf{n}}\overline{AM}| = \dfrac{|\overline{AM}\cdot\langle 1,2,3\rangle|}{|\langle 1,2,3\rangle|} = \dfrac{|\langle -4,0,-3\rangle\cdot\langle 1,2,-3\rangle|}{\sqrt{1+4+9}} = \dfrac{5\sqrt{14}}{14}$.

(c) First, we need to find a point that lies on L. Since the line L is the intersection of the plane P with the $xz - plane$, every point on L satisfies $y = 0$, and the equation of P. Plugging in $y = 0$ into the equation of P, we get $x - 3z = -3$.

Let $z = 0$, then we get $x = 2$. Thus the point $(2, 0, 0)$ is on L.

The direction vector of L is perpendicular to both $\langle 0, 1, 0\rangle$ and $\langle 1, 2, -3\rangle$.

Then a direction vector of L can be found by taking the cross product $\langle 0, 1, 0\rangle \times \langle 1, 2, -3\rangle = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 2 & -3 \end{vmatrix} = \langle -3, 0, -1\rangle$.

Then we get a parametric equation of L:

$$\begin{cases} x = 2 - 3t \\ y = 0 \\ z = -t \end{cases}$$

(d) Let $A = (-3, 0, 0)$ be a point on the line L as above. Then the distance from B to L is the magnitude of the orthogonal projection of the vector $\overline{A}B$ onto L. That is, $d = |\overline{AB} - \text{Proj}_v\overline{AB}|$, where v is a direction vector of L. We have $\overline{A}B = \langle -2, 3, -4\rangle$. Then we get:

$$\text{Proj}_v\overline{A}B = \frac{\overline{AB}\cdot v}{|v|}\frac{v}{|v|} = \frac{6+4}{10}v = \langle -3, 0, -1\rangle.$$

Then $d = |\langle -2, 3, -4\rangle - \langle -3, 0, -1\rangle| = |\langle 1, 3, -3\rangle| = \sqrt{19}$.

3. Consider the function $g(x,y) = \sqrt{4 - x^2 - 2y^2}$.

4 marks (a) Determine the domain and range of the function g.

3 marks (b) Sketch the level curves $g(x,y) = k$ for the constants $k = 0, 1, 2$.

Solution:

(a) Domain:

$4 - x^2 - 2y^2 \geq 0$. Equivalently, $x^2 + 2y^2 \leq 4$. Therefore, the domain is

$$\{(x,y) \in \mathbb{R}^2 | x^2 + 2y^2 \leq 4\}.$$

It is an ellipse with its interior.

Range:

Since $x^2 \geq 0$, and $2y^2 \geq 0$, we always have $x^2 + 2y^2 \geq 0$, and so $4 - x^2 - 2y^2 \leq 4$. Thus, $Range(g) = [0,2]$ (rigorous proof not required here).

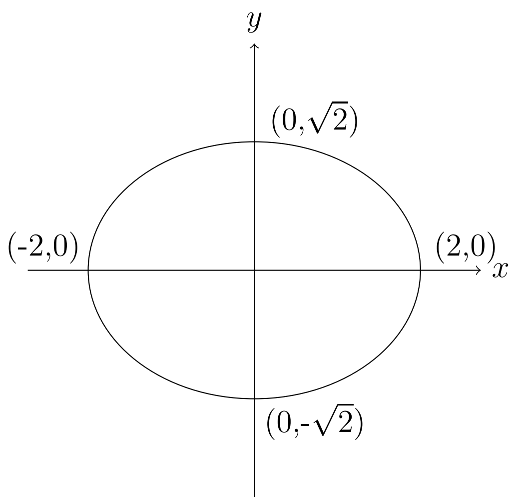
(b) Case 1: $k = 0$

$g(x,y) = \sqrt{4 - x^2 - 2y^2} = 0$

$4 - x^2 - 2y^2 = 0$

$x^2 + 2y^2 = 4$

$\frac{x^2}{4} + \frac{y^2}{2} = 1$



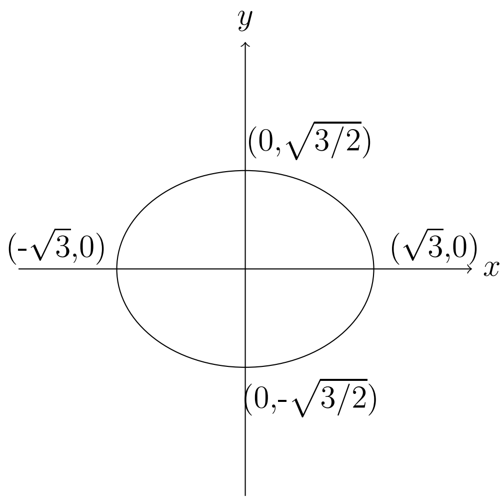
Case 2: $k = 1$

$g(x,y) = \sqrt{4 - x^2 - 2y^2} = 1$

$4 - x^2 - 2y^2 = 1$

$x^2 + 2y^2 = 3$

$\frac{x^2}{3} + \frac{y^2}{3/2} = 1$



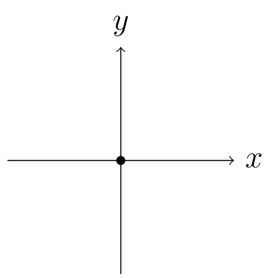
Case 2: $k = 2$

$g(x, y) = \sqrt{4 - x^2 - 2y^2} = 2$

$4 - x^2 - 2y^2 = 4$

$x^2 + 2y^2 = 0$

$x = 0$ and $y = 0$



4. Let $u(x, y) = F(x^2 + ay^2)$ for some arbitrary differentiable function F, where $a > 0$ is a constant.

4 marks (a) If $F(t) = \ln(t)$, sketch the level curves (the contour plot) of the function $u(x, y)$ in the three cases: $a > 0$, $a = 0$, $a < 0$.

2 marks (b) If $F(t) = \ln(t)$, find u_x and u_y.

3 marks (c) Let the function F be arbitrary. Find the value of a such that

$$5yu_x = xu_y.$$

Solution:

(a) $u(x, y) = F(x^2 + ay^2)$

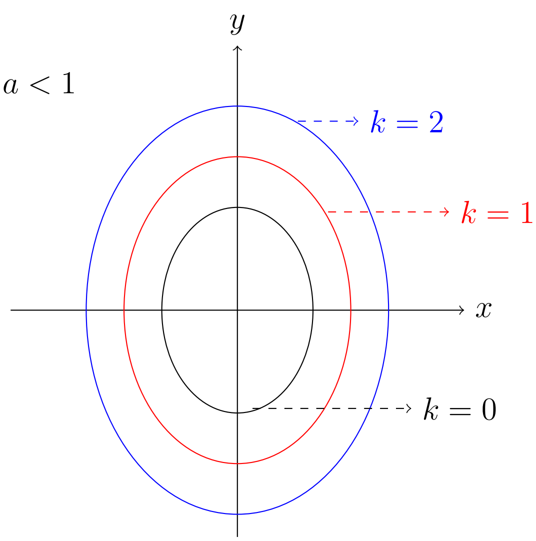
$u(x, y) = \ln(x^2 + ay^2) = k$ for some $k \in \mathbb{R}$.

$x^2 + ay^2 = e^k$

$$\frac{x^2}{e^k} + \frac{y^2}{e^k/a} = 1.$$

Case 1: $a > 0$:

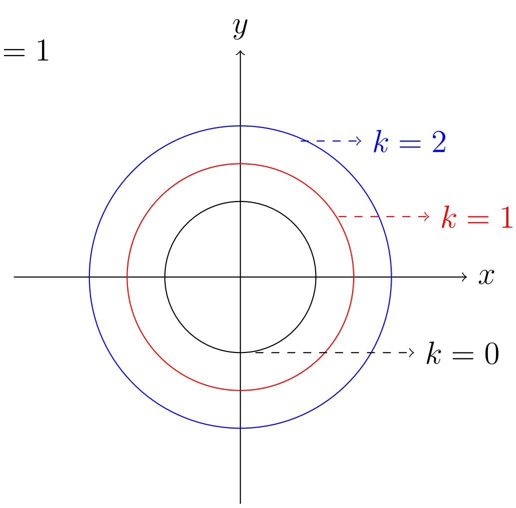
$0 < a < 1$



$x-$intercepts are $(\pm\sqrt{e^k}, 0)$

$y-$intercetps are $(0, \pm\sqrt{e^k/a})$

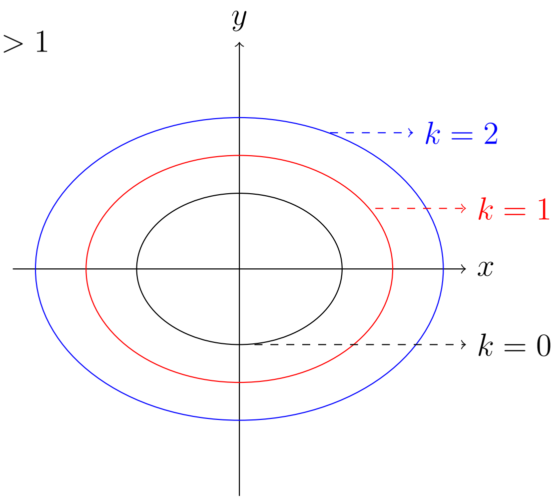
$a = 1$



$x-$intercepts are $(\pm\sqrt{e^k}, 0)$

$y-$intercetps are $(0, \pm\sqrt{e^k})$

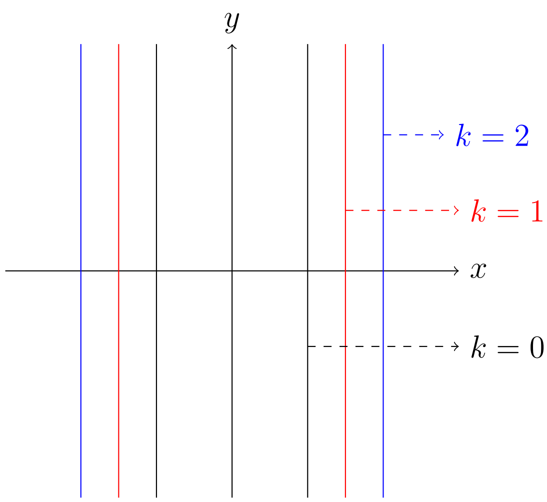
$a > 1$



$x-$intercepts are $(\pm\sqrt{e^k}, 0)$

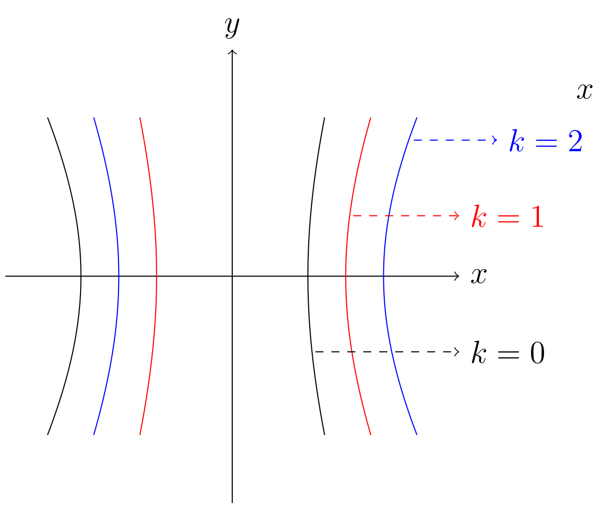
$y-$intercetps are $(0, \pm\sqrt{e^k/a})$

Case 2: $a = 0$



$x-$intercepts are $(\pm\sqrt{e^k}, 0)$

Case 3: $a < 0$:



$x-$intercepts are $(\pm\sqrt{e^k}, 0)$

Please note that all the drawings here represent the general shape of the level curves well, but the distances between level curves increase as k increases (as you move away from the origin).

(b) $u(x, y) = \ln(x^2 + ay^2)$

$$u_x = \frac{2x}{x^2 + ay^2},$$

$$u_y = \frac{2ay}{x^2 + ay^2}.$$

(c) $u_x = F'(x^2 + ay^2) \cdot 2x,$

$u_y = F'(x^2 + ay^2) \cdot 2ay.$

We want to have $5yu_x = xu_y$

That is, $10xyF'(x^2 + ay^2) = 2axyF'(x^2 + ay^2)$.

If $F' \neq 0$, $2a = 10$. Hence $a = 5$.

If $F' = 0$, then a can be any real number.

2 marks 5. (a) Let $(x, y, 0)$ be an arbitrary point in the xy-plane. What is the distance from this point to the plane $z = 4$?

2 marks (b) In $\mathbb{R}^2$, find an equation for, and draw the set of points that are equidistant from the point $(0, 0)$ and the line $y = 2$.

3 marks (c) Find the equation of the surface S consisting of all points in $\mathbb{R}^3$ that are equidistant from the point $(0, 0, 0)$ and the plane $z = 4$.

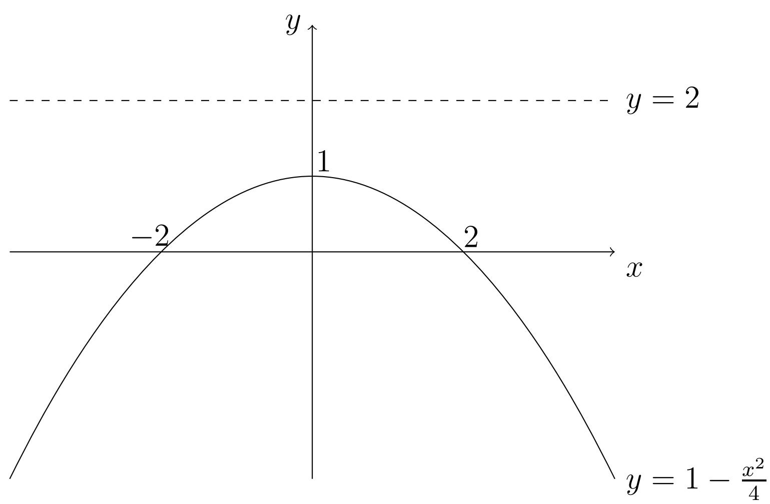
2 marks (d) Classify the surface S from part (c) as an ellipsoid, a paraboloid, or a hyperboloid of either 1 or 2 sheets.

Solution:

(a) The distance d between $(x, y, 0)$ and the plane $z = 4$ equals the distance between the two parallel planes: the xy-plane and the plane $z = 4$, so it is 4.

(b) Let $P = (x, y)$ be a point that is equidistant from the point $(0, 0)$ and the line $y = 2$.

Then we have $\sqrt{x^2 + y^2} = |y - 2|$; squaring (note that the both sides are non-negative), we get $x^2 + y^2 = y^2 - 4y + 4$. Thus, the answer is $y = 1 - \frac{x^2}{4}$; it is a parabola.



(c) Let $Q = (x, y, z)$ be a point that is equidistant from the point $(0, 0, 0)$ and the line $z = 4$.

$\sqrt{x^2 + y^2 + z^2} = |z - 4|$

$x^2 + y^2 + z^2 = z^2 - 8z + 16$

$S : x^2 + y^2 + 8z = 16.$

(d) For fixed $z = z_0 < 2$, the level curve on the plane $z = z_0$ is a circle.

For fixed $x = x_0$, the level curve on the plane $x = x_0$ is a parabola.

For fixed $y = y_0$, the level curve on the plane $y = y_0$ is a parabola.

Therefore, the surface S is a paraboloid.