Math 200 Midterm I (October 11, 2012)
Sections 107. Instructor: Julia Gordon

Problem 1: All parts of this problem refer to four points A, B, C, D, where $A(0,0,0)$ is the origin, and the coordinates of the other three are: $B(0,1,0)$, $C(1,0,1)$, $D(3,0,-3)$.

(a) [*2 points*] Find the component of the vector $\overline{BC}$ along the vector $\overline{BD}$.

(b) [*2 points*] Find the area of the triangle ABC.

(c) [*3 points*] Find the equation of the plane containing the points A, B, and C.

(d) [*3 points*] Find the distance from the point D to the plane containing A, B, and C.

(e) [*3 points*] Find the volume of the parallelepiped spanned by the vectors $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$.

(f) [*4 points*] Find the parametric equation for the line of intersection of the plane $x = z$ and the plane $x + 2y + z = 1$.

Solutions

(a) $\overline{BC} = \langle 1, -1, 1 \rangle$

$\overline{BD} = \langle 3, -1, 3 \rangle$

$$\text{comp}_{\overline{BD}} \overline{BC} = \overline{BC} \cdot \frac{\overline{BD}}{|\overline{BD}|} = \frac{\langle 1,-1,1\rangle \cdot \langle 3,-1,-3\rangle}{\sqrt{9+9+1}} = \boxed{\frac{1}{\sqrt{19}}}$$

(b) $\overline{AB} = \langle 0, 1, 0 \rangle$

$\overline{AC} = \langle 1, 0, 1 \rangle$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \bar{i} - \bar{k}$$

$$\text{Area} = \frac{1}{2} |\overline{AB} \times \overline{AC}| = \frac{1}{2} |\bar{i} - \bar{k}| = \frac{1}{2}\sqrt{1+1} = \boxed{\frac{1}{2}\sqrt{2}}$$

(c) A normal vector to this plane is $\overline{AB} \times \overline{AC}$, which we just found in (b) — it is $\bar{i} - \bar{k} = \langle 1, 0, -1 \rangle$. We can use the point A as "the point on the plane".

Answer: $\boxed{x - z = 0}$

(d) Distance from D to the plane is the absolute value of the component ~~of~~ of the vector $\overline{AD}$ along $\bar{n} = \langle 1, 0, -1 \rangle$.

Answer: $|\text{comp}_{\bar{n}} \overline{AD}| = \left| \frac{\langle 1,0,-1\rangle \cdot \langle 3,0,-3\rangle}{\sqrt{2}} \right| = \boxed{\frac{6}{\sqrt{2}}}$

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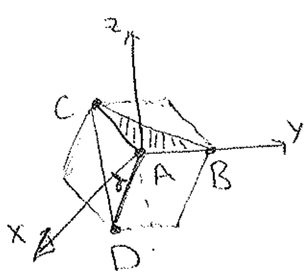
(e) : Can use the scalar triple product:

$$V = \overline{AD} \cdot (\overline{AB} \times \overline{AC}) = \left| \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & -3 \end{matrix} \right| = \boxed{6}$$

Another way is to notice that this volume equals the distance from D to the plane (ABC) times the area of the parallelogram spanned by $\overline{AB}$ and $\overline{AC}$, which is twice the area of the triangle from (b).



So, we get

$$V = 2 \cdot (\text{answer in (b)}) \cdot (\text{answer in (d)})$$

$$= 2 \cdot \frac{1}{2}\sqrt{2} \cdot \frac{6}{\sqrt{2}} = \boxed{6}$$

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(f): The direction vector of this line is parallel to the both planes, so we can obtain it by taking the cross product of the normal vectors.

So, $\bar{v} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 2i - 2j + 2k$

Common point: $\begin{cases} x = z \\ x + 2y + z = 1 \end{cases}$

can take $x = z = 0$, then $y = \frac{1}{2}$

Answer: $\begin{cases} x(t) = 2t \\ y(t) = \frac{1}{2} - 2t \\ z(t) = 2t \end{cases}$

(There are many equivalent answers).

Problem 2:

(a) [*3 points*] Find the parametric equation of the line through the points $(1, 1, 2)$ and $(0, 1, 3)$.

(b) [*2 points*] Find the symmetric equation of the same line.

(c) [*3 points*] Find the angle between this line and the plane $y = x$.

(d) [*3 points*] Find the point of intersection of your line with the plane $3x + y + 2z = 6$.

(a) $\bar{v}$ = vector from $(1,1,2)$ to $(0,1,3)$ $= \langle -1, 0, 1 \rangle$

Then $\begin{cases} x(t) = 1-t \\ y(t) = 1 \\ z(t) = 2+t \end{cases}$ (there are many equivalent correct answers)

(b) $\boxed{y = 1, \quad x - 1 = z - 2}$

(c) First, find the angle β between the line and the normal vector to the plane. (Remember, we want an acute angle). The normal vector is $\bar{n} = \langle 1, -1, 0 \rangle$.

$$\cos\beta = \left| \frac{\bar{v} \cdot \bar{n}}{|\bar{v}|\,|\bar{n}|} \right| = \left| \frac{\langle 1, 0, 1 \rangle \cdot \langle 1, -1, 0 \rangle}{\sqrt{2} \cdot \sqrt{2}} \right| = \frac{1}{2}$$

Then $\beta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Now, the angle α that we want is $\frac{\pi}{2} - \beta = \frac{\pi}{6}$.

Answer: $\boxed{\alpha = \frac{\pi}{6}}$

(d) We need the point $(x(t), y(t), z(t))$ on the line to satisfy the equation of the plane. Plug it in and solve for t:

$$3(1-t) + 1 + 2(2+t) = 6$$

$$3 - 3t + 1 + 4 + 2t = 6$$

$$8 - t = 6$$

$$t = 2.$$

Then $(x, y, z) = (\underset{1-t}{1-2}, \underset{1}{1}, \underset{2+t}{2+2}) = \boxed{(-1, 1, 4)}$

Problem 3: All parts of this problem are about the surface $z = 9 - x^2 - 4y^2$.

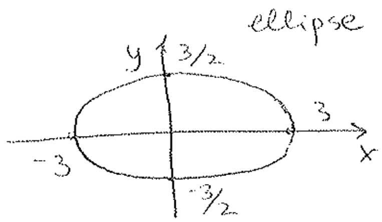
(a) *[3 points]* Sketch the traces of this surface on the planes $z = 0$, and $x = 1$. Label the axes in your plots, and label as many features of the plots as possible.

(b) *[2 points]* Classify this surface and sketch it.

(c) *[4 points]* Find the equation of the tangent plane to this surface at the point $(1, 1, 4)$.

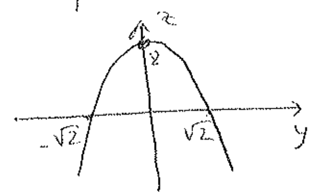
(d) *[4 points]* Find the point (a, b, c) on the surface such that the tangent plane at (a, b, c) is parallel to the plane $z = -x - 3y$.

(a)



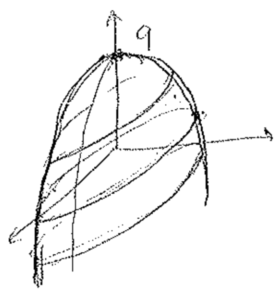
$z = 0$

The plane $x = 1$:



(drawing not to scale ☺)

(b) It is an elliptic paraboloid



(c) $z = 9 - x^2 - 4y^2 = f(x, y)$

The tangent plane has the equation $z = L(x, y)$, where $L(x,y)$ is the linearization of $f(x,y)$ at the point $(1,1)$.

$f_x = -2x \qquad f_x(1,1) = -2$

$f_y = -8y \qquad f_y(1,1) = -8$

Then $L(x,y) = 4 - 2(x-1) - 8(y-1)$, and the answer is

$$\boxed{z = 4 - 2(x-1) - 8(y-1)}$$

(d) In general, if (a,b,c) is a point on our paraboloid, then $c = 9 - a^2 - 4b^2$

Then we need to find a and b.

The tangent plane at (a,b,c) will have the equation

$$z = c + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Then the normal vector to the tangent plane is $\langle f_x(a,b), f_y(a,b), -1 \rangle$.

We need it to be proportional to the vector

$\langle -1, -3, -1 \rangle$ (the normal of the given plane)

So, we have

$$f_x(a,b) = k \cdot (-1)$$
$$f_y(a,b) = k \cdot (-3) \quad \text{for some } k.$$
$$-1 = k \cdot (-1)$$

Then $k = 1$. So we need to find a, b such that

$$f_x(a,b) = -1$$
$$f_y(a,b) = -3.$$

We know: $f_x(a,b) = -2a$

$f_y(a,b) = -8b.$

Then $-2a = -1$

$-8b = -3$

So, $\boxed{a = 1/2, \quad b = 3/8, \quad c = 9 - \left(\frac{1}{2}\right)^2 - 4 \cdot \left(\frac{3}{8}\right)^2}$

Problem 4:

(a) [*2 points*] Let $g(x, y) = e^x \cos y - y^2 x$. Find $\frac{\partial g}{\partial y}$.

(b) [*3 points*] Is there a function $f(x, y)$ that satisfies:

$$f_x = e^x \cos y - y^2 x$$
$$f_y = e^x \sin y + x^2 y.$$

Explain your reasoning.

(a) $\frac{\partial g}{\partial y} = -e^x \sin y - 2xy$

(b) Let us compare $\frac{\partial}{\partial y}(e^x \cos y - y^2 x)$

and $\frac{\partial}{\partial x}(e^x \sin y + x^2 y)$.

By Clairaut's Theorem, for any function with continuous second partial derivatives, we must have $f_{xy} = f_{yx}$. So if such an f existed, these expressions have to be the same.

We have:

$$\frac{\partial}{\partial y}(e^x \cos y - y^2 x) = -e^x \sin y - 2xy$$

$$\frac{\partial}{\partial x}(e^x \sin y + x^2 y) = e^x \sin y + 2xy.$$

and these two are not equal. (but are continuous)

Then f cannot exist!