Error Behaviour - A Trivial Example

In these notes, we look at the error that results when various numerical methods are used to generate approximate solutions to the initial value problem

$$\begin{aligned} y' &= y \\ y(0) &= 1 \end{aligned}$$

This problem is so simple that we can determine exactly both the real solution

$$\phi(t) = e^t$$

and the approximate solution generated by Euler, Improved Euler and Runge-Kutta.

Euler's Method

Since $f(t,y) = y$, Euler's method with step size h says

$$y_{n+1} = y_n + hf(t_n, y_n) = y_n + hy_n = (1+h)y_n$$

Hence

$$\begin{aligned} y_0 &= 1 \\ y_1 &= (1+h)y_0 = 1+h \\ y_2 &= (1+h)y_1 = (1+h)^2 \\ &\vdots \\ y_n &= (1+h)^n \end{aligned}$$

The number y_n is the value of the approximate solution at $t = nh$, so the error $E(h,t)$ at time t with step size h (assuming that t is an integer multiple of h) is

$$\begin{aligned} E(h,t) &= \phi(t) - y_{t/h} \\ &= e^t - (1+h)^{t/h} \\ &= e^t - e^{\frac{t}{h}\ln(1+h)} \end{aligned}$$

Assuming that $h \ll 1$, then by Taylor's expansion

$$\tfrac{1}{h}\ln(1+h) = \tfrac{1}{h}\big(h - \tfrac{1}{2}h^2 + O(h^3)\big) = 1 - \tfrac{1}{2}h + O(h^2)$$

The notation $O(h^m)$ is used to denote a function whose absolute value is bounded by a constant times h^m when h is sufficiently small. Subbing in

$$E(h,t) = e^t - e^{t - th/2 + O(h^2 t)}$$

If $th \ll 1$ (or equivalently $t^2/n \ll 1$ or $n \gg t^2$) in addition to $h \ll 1$

$$\begin{aligned} E(h,t) &= e^t - e^{t - th/2 + O(h^2 t)} \\ &= e^t \left(1 - e^{-th/2 + O(h^2 t)}\right) \\ &= e^t \left(1 - 1 + th/2 + O(h^2 t) + O(t^2 h^2)\right) \end{aligned}$$

We have just substituted in $e^x = 1 + x + O(x^2)$ and used $h^2 t \le ht$ to simplify the $O(x^2)$ term. So the error

$$E(h,t) = \tfrac{t}{2} e^t h\big(1 + O(h)\big)$$

grows linearly with h and exponentially with t.

Improved Euler Method

Since $f(t,y) = y$, the Improved Euler method with step size h says

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right) \\ &= y_n + \tfrac{h}{2}\left(y_n + y_n + hf(t_n, y_n)\right) \\ &= y_n + \tfrac{h}{2}\left(y_n + y_n + hy_n\right) \\ &= y_n \left[1 + \tfrac{h}{2}(2 + h)\right] \\ &= y_n \left[1 + h + \tfrac{h^2}{2}\right] \end{aligned}$$

Iterating

$$y_n = \left(1 + h + \tfrac{h^2}{2}\right)^n = \left(1 + h + \tfrac{h^2}{2}\right)^{t/h}$$

so that the error is

$$\begin{aligned} E(h,t) &= \phi(t) - y_{t/h} \\ &= e^t - \left(1 + h + \tfrac{h^2}{2}\right)^{t/h} \\ &= e^t - e^{\frac{t}{h}\ln(1 + h + h^2/2)} \end{aligned}$$

Taylor expanding

$$\begin{aligned} \ln(1 + h + h^2/2) &= h + \tfrac{1}{2}h^2 - \tfrac{1}{2}\left(h + \tfrac{1}{2}h^2\right)^2 + \tfrac{1}{3}\left(h + \tfrac{1}{2}h^2\right)^3 + O(h^4) \\ &= h + \tfrac{1}{2}h^2 - \tfrac{1}{2}h^2 - \tfrac{1}{2}h^3 + O(h^4) + \tfrac{1}{3}h^3 + O(h^4) \\ &= h - \tfrac{1}{6}h^3 + O(h^4) \end{aligned}$$

and Taylor expanding again

$$\begin{aligned} E(h,t) &= e^t - e^{\frac{t}{h}(h - h^3/6 + O(h^4))} = e^t - e^{t(1 - h^2/6 + O(h^3))} \\ &= e^t \left(1 - e^{-th^2/6 + O(th^3)}\right) \\ &= e^t \left(1 - 1 - [-th^2/6 + O(th^3)] + O(t^2 h^4)\right) \\ &= \tfrac{1}{6} t e^t h^2 \left(1 + O(h) + O(th^2)\right) \end{aligned}$$

This time the error is proportional to h^2.

Runge Kutta

Since $f(t,y)=y$, $4^{\rm th}$ order Runge Kutta with step size h says

$$\begin{aligned}
k_{n1} &= f(t_n, y_n) = y_n \\
k_{n2} &= f\big(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_{n1}\big) = y_n + \tfrac{h}{2}k_{n1} = y_n\left(1+\tfrac{h}{2}\right) \\
k_{n3} &= f\big(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_{n2}\big) = y_n + \tfrac{h}{2}k_{n2} = y_n\left(1+\tfrac{h}{2}+\tfrac{h^2}{4}\right) \\
k_{n4} &= f\big(t_n + h, y_n + hk_{n3}\big) = y_n + hk_{n3} = y_n\left(1+h+\tfrac{h^2}{2}+\tfrac{h^3}{4}\right) \\
y_{n+1} &= y_n + \tfrac{h}{6}(k_{n1}+2k_{n2}+2k_{n3}+k_{n4})
\end{aligned}$$

Since

$$\begin{aligned}
\tfrac{h}{6}(k_{n1}+2k_{n2}+2k_{n3}+k_{n4}) &= \tfrac{h}{6}y_n\left(1+2+h+2+h+\tfrac{h^2}{2}+1+h+\tfrac{h^2}{2}+\tfrac{h^3}{4}\right) \\
&= \tfrac{h}{6}y_n\left(6+3h+h^2+\tfrac{h^3}{4}\right)
\end{aligned}$$

we iterate

$$y_{n+1} = y_n\left(1+h+\tfrac{1}{2}h^2+\tfrac{1}{6}h^3+\tfrac{1}{24}h^4\right)$$

to yield

$$y_n = \left(1+h+\tfrac{1}{2}h^2+\tfrac{1}{6}h^3+\tfrac{1}{24}h^4\right)^n = \left(1+h+\tfrac{1}{2}h^2+\tfrac{1}{6}h^3+\tfrac{1}{24}h^4\right)^{t/h}$$

so that the error is

$$\begin{aligned}
E(h,t) &= \phi(t) - y_{t/h} \\
&= e^t - \left(1+h+\tfrac{1}{2}h^2+\tfrac{1}{6}h^3+\tfrac{1}{24}h^4\right)^{t/h}
\end{aligned}$$

We can of course expand in powers of h just like we did with Euler and Improved Euler. But it is easier to use a computer algebra package like maple. The maple commands

$err\ :=\ (h,t)\ ->\ e\hat{\ }(-t)*(1+h+h\hat{\ }2/2+h\hat{\ }3/6+h\hat{\ }4/24)\hat{\ }(t/h);$

$series(err(h,t), h=0, 6);$

first defines the function $err(h,t)$ and then computes its Taylor expansion in h about $h=0$ to order 6. The result is

$$1 - \tfrac{1}{120}th^4 + \tfrac{1}{144}th^5 + O(h^6)$$

so that the error

$$E(h,t) = \tfrac{1}{120}te^th^4(1+O(h))$$

is of order h^4.