Sobolev Background

These notes provide some background concerning Sobolev spaces that is used in Uhlmann's notes. Here Ω is a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary. We shall use $D^\alpha f(x)$ with $\alpha = (\alpha_1, \cdots, \alpha_n)$ to denote the partial derivative $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f(x)$. The order of this partial dertivative is $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition 1 ($H^\ell(\Omega)$, $\ell \geq 0$, integer) Define, for each integer $\ell \geq 0$, the norm $\|f\|_{\ell,\Omega}$ (we sometimes drop the Ω from the notation) by

$$\|f\|_{\ell,\Omega}^2 = \sum_{|\alpha| \leq \ell} \int_\Omega \big|D^\alpha f(x)\big|^2 \, d^n x$$

Then $H^\ell(\Omega)$ is the completion of $C^\infty(\bar{\Omega})$ under the norm $\| \cdot \|_{\ell,\Omega}$.

Definition 2 ($H^\ell(\partial\Omega)$, $\ell \geq 0$, integer) We want to define the space $H^\ell(\partial\Omega)$ in the same way as we defined the space $H^\ell(\Omega)$, but for functions that are only defined on $\partial\Omega$. To do so, we need a measure on $\partial\Omega$. The easy way to create such a measure is to use local coordinates. Of course we may need more than one coordinate patch to cover all of $\partial\Omega$. So we first use a partition of unity to write $f = \sum_m f\zeta_m$ with each $f\zeta_m$ supported in a single coordinate patch. Then we write $f\zeta_m$ in local coordinates and proceed as in the last definition. Of course this can give many different norms, through different choices of partition of unity and local coordinates. But that doesn't matter because they are all equivalent. That is, if $\| \cdot \|$ and $\| \cdot \|'$ are two such norms, then there are constants c and c' such that

$$\|f\| \leq c\|f\|' \qquad \|f\|' \leq c'\|f\|$$

So a sequence of functions converges with respect to one of the norms if and only if it converges with respect to the other.

Definition 3 ($H^t(\Omega)$, $t \geq 0$, real) If t happens to be an integer, we use the above definition. Otherwise, write $t = \ell + \sigma$ with ℓ a nonnegative integer and $0 < \sigma < 1$. We define the norm $\|f\|_{t,\Omega}$ by

$$\|f\|_{t,\Omega}^2 = \|f\|_{\ell,\Omega}^2 + \sum_{|\alpha|=\ell} \int_\Omega \int_\Omega \left| \tfrac{D^\alpha f(x) - D^\alpha f(y)}{|x-y|^\sigma} \right|^2 \tfrac{d^n x\, d^n y}{|x-y|^n}$$

Note that, for smooth f, the integrand

$$\tfrac{|D^\alpha f(x) - D^\alpha f(y)|^2}{|x-y|^{2\sigma+n}} \leq \text{const}\, \tfrac{|x-y|^2}{|x-y|^{2\sigma+n}} = \text{const}\, \tfrac{1}{|x-y|^{2\sigma+n-2}}$$

is locally integrable (since $2\sigma+n-2<n$) so the integral converges. Then $H^t(\Omega)$ is the completion of $C^\infty(\bar\Omega)$ under the norm $\|\ \cdot\ \|_{t,\Omega}$.

You should think of $H^t(\Omega)$ as consisting of functions all of whose derivatives up to order t (including "fractional order" derivatives) are L^2. I will now try to provide some motivation for this. The first observation to make is that, in the double integral over x,y, only those points with $x-y$ very small make any difference to the norm. In fact if, in the definition of the norm, we were to restrict the integration to $|x-y|\ge\varepsilon$ for some fixed $\varepsilon>0$, then

$$\begin{aligned}
\|f\|_{\ell,\Omega}^2\le\|f\|_{t,\Omega,\varepsilon}^2 &\equiv \|f\|_{\ell,\Omega}^2+\sum_{|\alpha|=\ell}\iint_{\substack{x,y\in\Omega\\|x-y|\ge\varepsilon}}\left|\tfrac{D^\alpha f(x)-D^\alpha f(y)}{|x-y|^\sigma}\right|^2\tfrac{d^nx\,d^ny}{|x-y|^n}\\
&\le \|f\|_{\ell,\Omega}^2+\sum_{|\alpha|=\ell}\iint_{\substack{x,y\in\Omega\\|x-y|\ge\varepsilon}}\tfrac{2|D^\alpha f(x)|^2+2|D^\alpha f(y)|^2}{|x-y|^{2\sigma+n}}d^nx\,d^ny\\
&= \|f\|_{\ell,\Omega}^2+\sum_{|\alpha|=\ell}4\iint_{\substack{x,y\in\Omega\\|x-y|\ge\varepsilon}}\tfrac{|D^\alpha f(x)|^2}{|x-y|^{2\sigma+n}}d^nx\,d^ny\\
&\le \|f\|_{\ell,\Omega}^2+\sum_{|\alpha|=\ell}4\int_\Omega|D^\alpha f(x)|^2\ d^nx\ \int_{|z|\ge\varepsilon}\tfrac{1}{|z|^{2\sigma+n}}d^nz\\
&\le C\|f\|_{\ell,\Omega}^2
\end{aligned}$$

for some constant C, because the integral $\int_{|z|\ge\varepsilon}\frac{1}{|z|^{2\sigma+n}}d^nz$ converges. The second observation is that it is possible to rewrite the integral in a more transparent form in the special case that $\Omega=\mathrm{I\!R}^n$. Making the change of variables $y=x+z$,

$$\begin{aligned}
&\int_{\mathrm{I\!R}^n}\int_{\mathrm{I\!R}^n}\left|\tfrac{D^\alpha f(x)-D^\alpha f(y)}{|x-y|^\sigma}\right|^2\tfrac{d^nx\,d^ny}{|x-y|^n}\\
&\qquad=\int d^nz\,\tfrac{1}{|z|^{2\sigma+n}}\int d^nx\ |D^\alpha f(x)-D^\alpha f(x+z)|^2\\
&\qquad=\int d^nz\,\tfrac{1}{|z|^{2\sigma+n}}\int d^nk\ |k^\alpha\tilde f(k)-k^\alpha e^{ik\cdot z}\tilde f(k)|^2
\end{aligned}$$

Here we have used:

(a) The L^2 norm of a function is the same as the L^2 norm of its Fourier transform. (Actually there may be some factors of 2π that come in here, depending on your Fourier transform conventions. I shall consistently omit all unimportant factors of 2π that arise from Fourier transform operations.)

(b) The Fourier transform of $\frac{\partial}{\partial x_m}g(x)$ is $ik_m\tilde g(k)$ so that the Fourier transform of $D^\alpha f(x)$ is $i^{|\alpha|}k^\alpha\tilde f(k)$ where $k^\alpha=k_1^{\alpha_1}\cdots k_n^{\alpha_n}$.

(c) the Fourier transform (with respect to x) of the translate $g(x+z)$ is $e^{ik\cdot z}\tilde g(k)$.

Now exchange the order of integration

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\tfrac{D^\alpha f(x)-D^\alpha f(y)}{|x-y|^\sigma}\right|^2\tfrac{d^nx\,d^ny}{|x-y|^n} = \int d^nk\ |k^\alpha\tilde f(k)|^2\int d^nz\,\tfrac{1}{|z|^{2\sigma+n}}|1-e^{ik\cdot z}|^2$$

and, for each k, make the change of variables $z = |k|^{-1}Rw$ where R is a rotation matrix chosen so that $k = |k|R\hat e_1$ with $\hat e_1$ a unit vector along the first coordinate axis. This gives

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\tfrac{D^\alpha f(x)-D^\alpha f(y)}{|x-y|^\sigma}\right|^2\tfrac{d^nx\,d^ny}{|x-y|^n}\\
&\qquad = \int d^nk\ |k^\alpha\tilde f(k)|^2|k|^{-n+2\sigma+n}\int d^nw\,\tfrac{1}{|w|^{2\sigma+n}}|1-e^{i\hat e_1\cdot w}|^2\\
&\qquad = c_{\sigma,n}\|k^\alpha|k|^\sigma\tilde f\|_{L^2}^2
\end{aligned}$$

Note that the constant $c_{\sigma,n} = \int d^nw\,\frac{1}{|w|^{2\sigma+n}}|1-e^{i\hat e_1\cdot w}|^2$ is finite because the integrand is bounded by $\frac{4}{|w|^{2\sigma+n}}$ for large $|w|$ and by $\frac{\text{const}}{|w|^{2\sigma+n-2}}$ for small $|w|$. Fourier transforming converts differentiation with respect to x into multiplication by k and it converts our difference quotient into a factor of $|k|^\sigma$. So it acts like a fractional derivative.

Definition 4 ($H^t(\partial\Omega)$, $t\ge 0$, real) Again introduce a partition of unity and local coordinates into a definition like the last one.

Definition 5 ($H_0^t(\Omega)$, $t\ge 0$, real) This space is the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{t,\Omega}$.

Example 6 Consider $n=1$, $\Omega=(0,1)$. Start with $f\in C_0^\infty(\Omega)$. Since f is supported in Ω, we can think of it as being defined on all $\mathbb{R}$, taking value zero everywhere except in $(0,1)$. In particular $f^{(j)}(0)=0$. So

$$\begin{aligned}
\big|f^{(j)}(x)\big| = \big|f^{(j)}(x)-f^{(j)}(0)\big| &= \Big|\int_0^x f^{(j+1)}(t)\ dt\Big|\\
&= \Big|\int_0^1 \chi_{[0,x]}(t)f^{(j+1)}(t)\ dt\Big|\\
&\le \Big[\int_0^1 \chi_{[0,x]}(t)^2\ dt\Big]^{1/2}\Big[\int_0^1 \big|f^{(j+1)}(t)\big|^2\ dt\Big]^{1/2}\\
&= \sqrt{x}\Big[\int_0^1 \big|f^{(j+1)}(t)\big|^2\ dt\Big]^{1/2}\\
&\le \sqrt{x}\|f\|_t
\end{aligned}$$

provided $t\ge j+1$. Now suppose that $f\in H_0^t(\Omega)$. It is a limit, in $H^t(\Omega)$, of functions $f_\ell\in C_0^\infty(\Omega)$. We can always choose the sequence so that $\|f_\ell\|_t\le 2\|f\|_t$ for every ℓ. Thus, if $j\le t-1$, $\big|f_\ell^{(j)}(x)\big|\le\sqrt{x}\|f_\ell\|_t\le 2\sqrt{x}\|f\|_t$ for all ℓ and all $0<x<1$. So all of the first $t-1$ derivatives of every $f\in H_0^t(\Omega)$ vanish on the boundary of Ω.

Theorem 7 *Let $t > \frac{1}{2}$. Define the restriction map*

$$r : C^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega)$$
$$u \mapsto u \restriction \partial\Omega$$

There exists a unique map

$$R : H^t(\Omega) \to H^{t-1/2}(\partial\Omega)$$

and constants C, C' such that

(i) R extends r. That is, $Ru = ru$ for all u in the domain, $C^\infty(\bar{\Omega})$, of r.

(ii) R is bounded. That is, $\|Ru\|_{t-1/2,\partial\Omega} \le C\|u\|_{t,\Omega}$.

(iii) R is surjective (onto).

(iv) R has kernel $\begin{cases} H_0^t(\Omega) & \text{if } t \le 1 \\ H_0^1(\Omega) \cap H^t(\Omega) & \text{if } t \ge 1 \end{cases}$

(v) For each $f \in H^{t-1/2}(\partial\Omega)$, there is a $u \in H^t(\Omega)$ such that

$$Ru = f \quad \text{and} \qquad \|u\|_{t,\Omega} \le C'\|f\|_{t-1/2,\partial\Omega}$$

That is, R has a bounded right inverse.

We shall not give a complete proof. It can be found in Adam's Book. See Theorem 7.53. But we shall try to motivate most of it and give detailed proofs for parts of it. The first part of the proof is to show that r is bounded. Once this is done, we can extend r by continuity to all of $H^t(\Omega)$ (since the domain of r is dense in $H^t(\Omega)$) and call the result R. Parts (i) and (ii) of the Theorem and also the uniqueness of R are then automatic. Since r vanishes on $C_0^\infty(\Omega)$, which is dense in $H_0^t(\Omega)$, the boundedness of r will also imply that the kernel of R contains $H_0^t(\Omega)$. This is part, but not all, of (iv). While we are on the subject of (iv), note that, when t is large, many derivatives of any $f \in H_0^t(\Omega)$ must vanish on $\partial\Omega$. On the other hand, to be in the kernel of R only f itself — not its derivatives — must vanish on $\partial\Omega$. That's why the kernel is $H_0^1(\Omega) \cap H^t(\Omega)$ rather than $H_0^t(\Omega)$ when $t > 1$. We shall not discuss (iv) further. For a readable proof in the special case of $H^1(\Omega)$, see Theorem 2 in §5.5 of Evans' book.

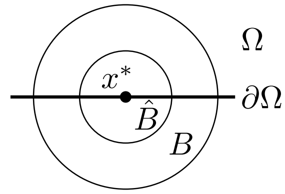
To get a first look at why r is bounded, we prove that r is bounded from (smooth functions in) $H^1(\Omega)$ to $H^0(\partial\Omega) = L^2(\partial\Omega)$. That is $\|ru\|_{0,\partial\Omega} \le C\|u\|_{1,\Omega}$. When $t = 1$ we really want the stronger result that $\|ru\|_{1/2,\partial\Omega} \le C\|u\|_{1,\Omega}$. — i.e. that, in restricting to the boundary, we only loose half a derivative. We'll discuss that after proving

Lemma 8 *There is a constant C (depending only on Ω) such that, for all $u \in C^\infty(\bar{\Omega})$,*

$$\|ru\|_{0,\partial\Omega} \le C\|u\|_{1,\Omega}$$

Proof: We may assume that u is real valued.

Step 1. Let $x^* \in \partial\Omega$. Assume that $\partial\Omega$ is flat and lying in $x_n = 0$ near x^*. Let $\hat{B} \subset B$ be concentric balls centred on x^* with $B \cap \partial\Omega \subset \{x_n = 0\}$ and $B \cap \Omega = B \cap \{x_n > 0\} \equiv B_+$.



Let $\zeta \in C_0^\infty(B)$ obey $\zeta \geq 0$ and $\zeta \restriction \hat{B} = 1$. Denote $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$. Then

$$\int_{\partial\Omega \cap \hat{B}} (ru)^2 \; d^{n-1}x' \leq \int_{B \cap \{x_n=0\}} \zeta u^2 \; d^{n-1}x'$$

Now use that $\int_0^r D_n f(x', x_n) \, dx_n = f(x', r) - f(x', 0) = -f(x', 0)$ if $f(x', r) = 0$, to give

$$\begin{aligned}
\int_{\partial\Omega \cap \hat{B}} (ru)^2 \; d^{n-1}x' &\leq -\int_{B+} D_n\big(\zeta u^2\big) \; d^n x \\
&= -\int_{B+} \big[u^2 D_n\zeta + 2\zeta u D_n u\big] \; d^n x \\
&\leq \tfrac{1}{2} C \int_{\Omega} \big[u^2 + |u D_n u|\big] \; d^n x \text{ where } C = 2(1 + \sup |D_n \zeta|) \\
&\leq \tfrac{1}{2} C \big[\|u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)} \; \|D_n u\|_{L^2(\Omega)}\big] \\
&\leq C \big[\|u\|_{L^2(\Omega)}^2 + \|D_n u\|_{L^2(\Omega)}^2\big] \text{ since } ab \leq \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 \\
&\leq C \|u\|_{1,\Omega}^2
\end{aligned}$$

Step 2. If $x^* \in \partial\Omega$ but $\partial\Omega$ is not flat and oriented correctly near x^*, make a change of variables to straighten it out and reorient it. Since $\partial\Omega$ is smooth, the Jacobean is bounded and so just changes the value of the constant C of step 1.

Step 3. Since $\partial\Omega$ is compact, we can cover it by finitely many $\hat{B}$'s as in step 1. This completes the proof.

■

The next Lemma illustrates the loss of only one half of a derivative as well as the requirement that $t > \frac{1}{2}$ in part (ii) of Theorem 7. In the illustration, we replace Ω by $\mathbb{R}^n$ and $\partial\Omega$ by $\{x_n = 0\}$, which we identify with $\mathbb{R}^{n-1}$. As in the last proof, we write $x' = (x_1, \cdots, x_{n-1})$.

Lemma 9 *Define* $r : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{n-1})$ *by*

$$(ru)(x') = u(x', 0)$$

If $t > \frac{1}{2}$, then there is a constant C (depending only on n and t) such that

$$\|ru\|_{t-\frac{1}{2},\mathbb{R}^{n-1}} \le C\|u\|_{t,\mathbb{R}^n}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Proof: If $k \in \mathbb{R}^n$, write $k = (k', k_n)$ with $k' = (k_1, \cdots, k_{n-1})$. In terms of the Fourier transform of u, the (square of the) norm of u is

$$\|u\|_{t,\mathbb{R}^n}^2 = \int D_{n,t}(k)|\tilde{u}(k)|^2 \ d^n k$$

where, using $[t]$ to denote the integer part of t,

$$D_{n,t}(k) = \sum_{|\alpha|\le[t]} k^{2\alpha} + \begin{cases} 0 & \text{if } t = [t] \\ c_{\sigma,n}|k|^{2\sigma}\sum_{|\alpha|=[t]} k^{2\alpha} & \text{if } \sigma = t - [t] > 0 \end{cases}$$

Since $D_{n,t}(k)$ is bounded above and below by constants (depending on n and t) times $(1+k^2)^t$, we may redefine $\|u\|_{t,\mathbb{R}^n}$ to be the equivalent norm

$$\|u\|_{t,\mathbb{R}^n} = \Big[\int (1+k^2)^t|\tilde{u}(k)|^2 \ d^n k\Big]^{1/2}$$

Similarly, we may redefine $\|ru\|_{t-\frac{1}{2},\mathbb{R}^{n-1}}$ to be the equivalent norm

$$\|ru\|_{t-\frac{1}{2},\mathbb{R}^{n-1}} = \Big[\int (1+k'^2)^{t-\frac{1}{2}}|\widetilde{ru}(k')|^2 \ d^{n-1}k'\Big]^{1/2}$$

The definition of the Fourier transform gives

$$\int e^{ik'\cdot x'}(\widetilde{ru})(k') \ d^{n-1}k' = (ru)(x') = u(x',0) = \int e^{ik'\cdot x'}\tilde{u}(k',k_n) \ d^{n-1}k' dk_n$$

so that

$$(\widetilde{ru})(k') = \int \tilde{u}(k',k_n) \ dk_n$$

Thus, by Cauchy–Schwarz,

$$\begin{aligned}
\big|(\widetilde{ru})(k')\big|^2 &\le \Big|\int \tilde{u}(k',k_n)\ \big(1+k'^2+k_n^2\big)^{t/2}\big(1+k'^2+k_n^2\big)^{-t/2} \ dk_n\Big|^2 \\
&\le \Big[\int \tfrac{1}{(1+k'^2+k_n^2)^t} \ dk_n\Big]\Big[\int \big|\tilde{u}(k',k_n)\big|^2 \big(1+k'^2+k_n^2\big)^t \ dk_n\Big] \\
&= \Big[\tfrac{1}{(1+k'^2)^{t-1/2}}\int \tfrac{1}{(1+p^2)^t} \ dp\Big]\Big[\int \big|\tilde{u}(k',k_n)\big|^2\big(1+k'^2+k_n^2\big)^t \ dk_n\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{where } k_n = p\sqrt{1+k'^2} \\
&= \tfrac{c}{(1+k'^2)^{t-1/2}}\int \big|\tilde{u}(k',k_n)\big|^2 \ \big(1+k'^2+k_n^2\big)^t \ dk_n
\end{aligned}$$

where the constant $c = \int \frac{1}{(1+p^2)^t}\ dp$ depends only on t and is finite because $t > \frac{1}{2}$. Hence

$$\begin{aligned}
\|ru\|^2_{t-\frac{1}{2},\mathbb{R}^{n-1}} &= \int (1+k'^2)^{t-\frac{1}{2}} |\widetilde{ru}(k')|^2\ d^{n-1}k' \\
&\le c \int \big|\tilde{u}(k',k_n)\big|^2\ \big(1+k'^2+k_n^2\big)^t\ d^n k \\
&= c\|u\|_{t,\mathbb{R}^n}
\end{aligned}$$

■

With a little more work, Lemma 9 can be plugged into step 1 of the proof of Lemma 8 to give the boundedness of r required for the proof of part (ii) of Theorem 7. This completes our discussion of parts (i) and (ii) of Theorem 7. We now discuss part (v), which implies part (iii). Part (v) says that, each $f \in H^{t-1/2}(\partial\Omega)$ can be extended to a $u \in H^t(\Omega)$ in a bounded way. To illustrate why, we again replace Ω by $\mathbb{R}^n$ and $\partial\Omega$ by $\{x_n = 0\}$. We use the same notation as in the last Lemma.

Lemma 10 *Let $t \ge \frac{1}{2}$. Then there is a constant C (depending only on n and t) such that for each $f \in H^{t-1/2}(\mathbb{R}^{n-1})$ there is an $F \in H^t(\mathbb{R}^n)$ obeying $F(x',0) = f(x')$ and*

$$\|F\|^2_{t,\mathbb{R}^n} \le C\|f\|^2_{t-\frac{1}{2},\mathbb{R}^{n-1}}$$

Proof: As in Lemma 9, we use the norms

$$\begin{aligned}
\|F\|_{t,\mathbb{R}^n} &= \Big[\int (1+k^2)^t |\tilde{F}(k)|^2\ d^n k\Big]^{1/2} \\
\|f\|_{t-\frac{1}{2},\mathbb{R}^{n-1}} &= \Big[\int (1+k'^2)^{t-\frac{1}{2}} |\tilde{f}(k')|^2\ d^{n-1}k'\Big]^{1/2}
\end{aligned}$$

in place of the originally defined (equivalent) norms $\|F\|_{t,\mathbb{R}^n}$, $\|f\|_{t-\frac{1}{2},\mathbb{R}^{n-1}}$. Try

$$\tilde{F}(k',k_n) = \tilde{f}(k') A \tfrac{(1+k'^2)^\mu}{(1+k'^2+k_n^2)^\nu}$$

To satisfy $F(x',0) = f(x')$, we need

$$\begin{aligned}
\int \tilde{F}(k',k_n)\ dk_n = \tilde{f}(k') \qquad \Longleftrightarrow \qquad 1 &= \int A \tfrac{(1+k'^2)^\mu}{(1+k'^2+k_n^2)^\nu}\ dk_n \\
&= A(1+k'^2)^{\mu-\nu+1/2} \int \tfrac{1}{(1+p^2)^\nu}\ dp
\end{aligned}$$

where we made the change of variables $k_n = p\sqrt{1+k'^2}$. The requirement $F(x',0) = f(x')$ is satisfied if we choose $\nu > \frac{1}{2}$ (so that the integral converges), $\mu = \nu - \frac{1}{2}$ and $A = \Big[\int \frac{1}{(1+p^2)^\nu}\Big]^{-1}$. With those choices

$$\begin{aligned}
\|F\|^2_{t,\mathbb{R}^n} &= \int (1+k^2)^t |\tilde{F}(k)|^2 \; d^n k \\
&= A^2 \int (1+k^2)^t \tfrac{(1+k'^2)^{2\mu}}{(1+k'^2+k_n^2)^{2\nu}} |\tilde{f}(k')|^2 \; d^{n-1}k' dk_n \\
&= A^2 \Big[\int \tfrac{1}{(1+p^2)^{2\nu-t}} \; dp\Big] \int (1+k'^2)^{2\mu+t-2\nu+1/2} |\tilde{f}(k')|^2 \; d^{n-1}k' \\
&= A^2 \Big[\int \tfrac{1}{(1+p^2)^{2\nu-t}} \; dp\Big] \; \|f\|^2_{2\mu+t-2\nu+1/2} \\
&= A^2 \Big[\int \tfrac{1}{(1+p^2)^{2\nu-t}} \; dp\Big] \; \|f\|^2_{t-1/2}
\end{aligned}$$

so it suffices to choose ν large enough that $2\nu - t > \frac{1}{2}$ and $\nu > \frac{1}{2}$. ■

Theorem 10 *There is a constant C such that for all $u \in H^1(\Omega)$*

$$\|u\|_{L^2(\Omega)} \le C\Big(\sum_{i=1}^n \|D_i u\|^2_{L^2(\Omega)} + \|Ru\|^2_{L^2(\partial\Omega)}\Big)^{1/2}$$

Proof: By Theorem 7 and the denseness of $C^\infty(\bar{\Omega})$ in $H^1(\Omega)$, it is sufficient to consider $u \in C^\infty(\bar{\Omega})$.

Step 1. Let $x^* \in \partial\Omega$. Assume that $\partial\Omega$ is flat and lying in $x_n = 0$ near x^*. Let D be any open neighbourhood of x^* in $\partial\Omega \cap \{x_n = 0\}$ and let $I \subset \mathbb{R}$ be any interval such that $D \times I \subset \Omega$. Denote $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$. Then, by the fundamental theorem of calculus and Cauchy–Schwarz,

$$\begin{aligned}
&\int_{D\times I} \big|u(x',x_n) - u(x',0)\big|^2 \; d^{n-1}x' \, dx_n \\
&\qquad = \int_D d^{n-1}x' \int_I dx_n \; \Big|\int_0^{x_n} D_n u(x',t)\, dt\Big|^2 \\
&\qquad \le \int_D d^{n-1}x' \int_I dx_n \; |x_n| \int_I |D_n u(x',t)|^2 \, dt \\
&\qquad \le c \int_D d^{n-1}x' \int_I |D_n u(x',t)|^2 \, dt
\end{aligned}$$

Hence

$$\begin{aligned}\int_{D\times I} \big|u(x)\big|^2\ d^n x &= \int_{D\times I} \big|u(x',x_n) - u(x',0) + u(x',0)\big|^2\ d^{n-1}x'\,dx_n \\ &\le 2\int_{D\times I} \big|u(x',x_n) - u(x',0)\big|^2\ d^{n-1}x'\,dx_n + 2\int_{D\times I} \big|u(x',0)\big|^2\ d^{n-1}x'\,dx_n \\ &\le 2c\int_{D\times I} d^n x\ |D_n u(x)|^2 + 2|I|\int_D \big|(Ru)(x')\big|^2\ d^{n-1}x'\end{aligned}$$

Step 2. Patch together pieces and straighten out edges to give the desired bound. ■

References

- Robert Adams, **Sobolev Spaces**, Academic Press, 1975.
- Lawrence C. Evans, **Partial Differential Equations**, AMS, 1998.
- Elliott Lieb and Michael Loss, **Analysis**, AMS, 1997.
- Gunther Uhlmann, **The Dirichlet to Neumann Map and Inverse Problems**, preprint.

