X–ray Tomography

Suppose that you shine a light beam (or beam of x-rays) down the length of a long rod. Suppose further that as the light passes through an infinitesmal hunk of rod it loses a fraction of its intensity that is proportional to the mass of the hunk. If we denote by $I(x)$ the intensity of the light at x and by $\rho(x)$ the mass density of the rod at x, then

$$\tfrac{I(x+dx)-I(x)}{I(x)} = -\gamma\,\rho(x)\,dx$$

where γ is the (positive) constant of proportionality. Dividing across by dx and taking the limit as $dx \to 0$ gives

$$\tfrac{I'(x)}{I(x)} = -\gamma\rho(x) \qquad \Longrightarrow \qquad \tfrac{d}{dx}\ln I(x) = -\gamma\rho(x)$$

By integrating, we see that the (natural logarithm of the) fraction of the light intensity that survives the trip down the rod is

$$\ln \tfrac{I(\infty)}{I(-\infty)} = -\gamma \int_{-\infty}^{\infty} \rho(x)\,dx$$

In x–ray tomography, you shine thin beams through different parts of a body and wish to recover the density $\rho(x)$ of the body from the various values of $\int_{\text{beam path}} \rho(x)\,dx$ measured.

We now formulate a mathematical statement of this problem. It is no harder to consider general dimensions, so we do so. Let $\mathbb{P}^n$ denote the set of all hyperplanes in $\mathbb{R}^n$. By definition, all hyperplanes in $\mathbb{R}^n$ are $(n-1)$–dimensional. A hyperplane may be specified by giving its direction (say a vector perpendicular to it) and one point on it. Thus each hyperplane can be written

$$\xi = \left\{\ \mathbf{x} \in \mathbb{R}^n\ \middle|\ \hat{\boldsymbol{\omega}} \cdot \mathbf{x} = p\ \right\}$$

where $\hat{\boldsymbol{\omega}}$ is a unit vector in $\mathbb{R}^n$ and $p \in \mathbb{R}$. Of course, if $\hat{\boldsymbol{\omega}}$ is normal to a hyperplane, so is $-\hat{\boldsymbol{\omega}}$ and indeed

- $(\hat{\boldsymbol{\omega}}, p)$ and $(\hat{\boldsymbol{\omega}}', p')$ give the same hyperplane if and only if $(\hat{\boldsymbol{\omega}}, p) = \pm(\hat{\boldsymbol{\omega}}', p')$
- Thus the map $(\hat{\boldsymbol{\omega}}, p) \in S^{n-1} \times \mathbb{R} \mapsto \xi$ is a double cover of $\mathbb{P}^n$ and can be used to define coordinate patches on $\mathbb{P}^n$, turning it into a manifold.
- Hence a function φ on $\mathbb{P}^n$ can be identified with a function on $S^{n-1} \times \mathbb{R}$ obeying $\varphi(\hat{\boldsymbol{\omega}}, p) = \varphi(-\hat{\boldsymbol{\omega}}, -p)$. We shall typically use the same symbol (for example, φ) to denote both the function on $\mathbb{P}^n$ and the corresponding function on $S^{n-1} \times \mathbb{R}$.

Definition (Radon) The Radon transform associates to each function $f : \mathbb{R}^n \to \mathbb{R}$ the function $\hat{f} : \mathbb{P}^n \to \mathbb{R}$ defined by

$$\hat{f}(\xi) = \int_\xi f(\mathbf{x})\, dm(\mathbf{x})$$

where $dm(\mathbf{x})$ is the standard Euclidean measure on ξ.

We are not going to worry about what regularity should be required of $f(\mathbf{x})$ and what regularity the resulting $\hat{f}(\xi)$ has. These details are dealt with in the book by Helgason listed in the references. We now pose the

Question: Given $\hat{f}$, can we determine f? The short answer is

Answer: Yes.

We shall justify this short answer by deriving two separate algorithms for determining f. Before doing so, note that once we know how to determine f from its integrals over linear spaces of codimension one, we also know how to determine f from its integrals over linear spaces of higher codimension. For example, the algorithms with $n = 2$ allow us how to recover a function defined on a planar region from its integrals over lines. As any three dimensional body is a union of planar regions, we can also recover a function on a three dimensional body from its integrals over lines, just by treating planar slices of the body individually.

The first algorithm is based on the following Lemma, which shows us how to compute the conventional Fourier transform $\tilde{f}$ of f from its Radon transform $\hat{f}$. Of course, once $\tilde{f}$ is known, you find f by applying the inverse Fourier transform to $\tilde{f}$.

Lemma. *For all $s \in \mathbb{R}$ and $\hat{\boldsymbol{\omega}} \in S^{n-1}$.*

$$\tilde{f}(s\hat{\boldsymbol{\omega}}) = \int_{-\infty}^{\infty} \hat{f}(\hat{\boldsymbol{\omega}}, r) e^{-isr}\, dr$$

Proof: By definition

$$\begin{aligned}\tilde{f}(s\hat{\boldsymbol{\omega}}) &= \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-is\hat{\boldsymbol{\omega}}\cdot\mathbf{x}} d^n\mathbf{x} \\ &= \int_{-\infty}^{\infty} dr \int_{\mathbf{x}\cdot\hat{\boldsymbol{\omega}}=r} dm(\mathbf{x})\ f(\mathbf{x}) e^{-is\hat{\boldsymbol{\omega}}\cdot\mathbf{x}}\end{aligned}$$

We have just rewritten the integral over $\mathbb{R}^n$ as an iterated integral. For, example, if $n = 3$ and $\hat{\boldsymbol{\omega}} = \hat{\mathbf{k}}$, then $dr = dz$ and $dm(\mathbf{x}) = dx\, dy$. Subbing in the definition of $\hat{f}$,

$$\begin{aligned}\tilde{f}(s\hat{\boldsymbol{\omega}}) &= \int_{-\infty}^{\infty} dr\ e^{-isr} \int_{\mathbf{x}\cdot\hat{\boldsymbol{\omega}}=r} dm(\mathbf{x})\ f(\mathbf{x}) \\ &= \int_{-\infty}^{\infty} e^{-isr} \hat{f}(\hat{\boldsymbol{\omega}}, r)\, dr\end{aligned}$$

■

In preparation for the second algorithm we define a dual transform.

Definition (Dual) The dual Radon transform associates to each function $\varphi : \mathbb{P}^n \to \mathbb{R}$ the function $\check{\varphi} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\check{\varphi}(\mathbf{x}) = \int_{\mathbf{x}\in\xi} \varphi(\xi)\, d\mu(\xi)$$

where $d\mu(\xi)$ is the unique measure on $\{\ \xi \in \mathbb{P}^n \bigm| \mathbf{x} \in \xi\ \}$ which is invariant under rotations around $\mathbf{x}$ and has mass one.

Let's take a closer look at the measure $d\mu(\xi)$. By translating, it suffices to consider $\mathbf{x} = \mathbf{0}$. Then $\{\ \xi \in \mathbb{P}^n \bigm| \mathbf{0} \in \xi\ \}$ is just the sphere S^{n-1} with antipodal points identified and a function ψ on $\{\ \xi \in \mathbb{P}^n \bigm| \mathbf{0} \in \xi\ \}$ is identified with an even function $\psi(\hat{\boldsymbol{\omega}})$ on S^{n-1}. Then, using Ω_n to denote the surface area of a unit sphere in $\mathbb{R}^n$,

$$\begin{aligned}\int_{\mathbf{0}\in\xi} \psi(\xi)\, d\mu(\xi) &= \tfrac{1}{\Omega_n} \int_{S^{n-1}} \psi(\hat{\boldsymbol{\omega}})\, d\hat{\boldsymbol{\omega}} \\ &= \int_{SO(n)} \psi(R\hat{\boldsymbol{\omega}}_0)\, dR\end{aligned} \tag{$*$}$$

where $d\hat{\boldsymbol{\omega}}$ is the standard Euclidean measure on S^{n-1}, $SO(n)$ is the group of rotations in $\mathbb{R}^n$, $\hat{\boldsymbol{\omega}}_0$ is any fixed unit vector in $\mathbb{R}^n$ and dR is Haar measure on $SO(n)$. When $n = 2$,

- S^{n-1} is just the unit circle in $\mathbb{R}^2$ centred on the origin,
- $d\hat{\boldsymbol{\omega}} = d\theta$, where θ is the usual polar angle,
- $\Omega_2 = 2\pi$,
- the elements of $SO(2)$ are matrices of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $0 \le \theta < 2\pi$ and may be labelled by the angle of rotation θ.
- $dR = \frac{d\theta}{2\pi}$.

To prove that the two right hand sides of $(*)$ are equal, it suffices to observe that both of the maps $\psi \mapsto \frac{1}{\Omega_n} \int_{S^{n-1}} \psi(\hat{\boldsymbol{\omega}})\, d\hat{\boldsymbol{\omega}}$ and $\psi \mapsto \int_{SO(n)} \psi(R\hat{\boldsymbol{\omega}}_0)\, dR$ are rotation invariant, mass one linear functionals on the space, $C(S^{n-1})$, of continuous functions on S^{n-1}. (To rotate a function, replace its argument $\mathbf{x}$ by $R^{-1}\mathbf{x}$ for some $R \in SO(n)$.) There exists only one such linear functional.

Lemma.

$$\big(\hat{f}\big)\check{} = \tfrac{\Omega_{n-1}}{\Omega_n} \int_{\mathbb{R}^n} |\mathbf{x} - \mathbf{y}|^{-1} f(\mathbf{y})\, d\mathbf{y}$$

Proof: Subbing in the definitions of the Radon transform and the dual transform and then translating and applying the second part of $(*)$,

$$\begin{aligned}
(\hat{f})\check{}(\mathbf{x}) &= \int_{\mathbf{x}\in\xi} \hat{f}(\xi)\, d\mu(\xi) \\
&= \int_{\mathbf{x}\in\xi} \Big(\int_\xi f(\mathbf{y})\, dm(\mathbf{y})\Big)\, d\mu(\xi) \\
&= \int_{\mathbf{0}\in\xi} \Big(\int_\xi f(\mathbf{x}+\mathbf{y})\, dm(\mathbf{y})\Big)\, d\mu(\xi) \\
&= \int_{SO(n)} \Big(\int_{\xi_0} f(\mathbf{x}+R\mathbf{y})\, dm(\mathbf{y})\Big)\, dR
\end{aligned}$$

where ξ_0 is any fixed hyperplane through the origin. Observe that as R runs over $SO(n)$, $\mathbf{x}+R\mathbf{y}$ runs over the surface of the sphere of radius $|\mathbf{y}|$ centered on $\mathbf{x}$. The equality of the two right hand sides of $(*)$ says that the integral $\int_{SO(n)} \psi(R\hat{\boldsymbol{\omega}}_0)\, dR$ is just the average value of ψ over the sphere S^{n-1}. Hence,

$$\begin{aligned}
(\hat{f})\check{}(\mathbf{x}) &= \int_{\xi_0} \Big(\int_{SO(n)} f(\mathbf{x}+R\mathbf{y})\, dR\Big)\, dm(\mathbf{y}) \\
&= \int_{\xi_0} \Big(\tfrac{1}{\Omega_n}\int_{S^{n-1}} f(\mathbf{x}+|\mathbf{y}|\hat{\boldsymbol{\omega}})\, d\hat{\boldsymbol{\omega}}\Big)\, dm(\mathbf{y})
\end{aligned}$$

Choose ξ_0 to be the hyperplane in $\mathbb{R}^n$ containing all points whose last coordinate is zero. Call this $\mathbb{R}^{n-1}$. Going to spherical coordinates on $\mathbb{R}^{n-1}$ and observing that $\int_{S^{n-1}} f(\mathbf{x}+|\mathbf{y}|\hat{\boldsymbol{\omega}})\, d\hat{\boldsymbol{\omega}}$ is a function only of the length of $\mathbf{y}$,

$$\begin{aligned}
(\hat{f})\check{}(\mathbf{x}) &= \int_{\mathbb{R}^{n-1}} \Big(\tfrac{1}{\Omega_n}\int_{S^{n-1}} f(\mathbf{x}+|\mathbf{y}|\hat{\boldsymbol{\omega}})\, d\hat{\boldsymbol{\omega}}\Big)\, d\mathbf{y} \\
&= \int_0^\infty dr\ r^{n-2}\Omega_{n-1}\Big(\tfrac{1}{\Omega_n}\int_{S^{n-1}} f(\mathbf{x}+r\hat{\boldsymbol{\omega}})\, d\hat{\boldsymbol{\omega}}\Big) \\
&= \tfrac{\Omega_{n-1}}{\Omega_n}\int_0^\infty dr \int_{S^{n-1}} d\hat{\boldsymbol{\omega}}\ r^{n-2} f(\mathbf{x}+r\hat{\boldsymbol{\omega}})
\end{aligned}$$

This is almost spherical coordinates on $\mathbb{R}^n$. Only the power of r is wrong. So

$$\begin{aligned}
(\hat{f})\check{}(\mathbf{x}) &= \tfrac{\Omega_{n-1}}{\Omega_n}\int_0^\infty dr \int_{S^{n-1}} d\hat{\boldsymbol{\omega}}\ r^{n-1}\ \tfrac{1}{r} f(\mathbf{x}+r\hat{\boldsymbol{\omega}}) \\
&= \tfrac{\Omega_{n-1}}{\Omega_n}\int_{\mathbb{R}^n} \tfrac{1}{|\mathbf{y}|} f(\mathbf{x}+\mathbf{y})\ d\mathbf{y}
\end{aligned}$$

The change of variables $\mathbf{y}\to\mathbf{y}-\mathbf{x}$ does it. ■

Lemma. *If* $g(\mathbf{x}) = \frac{\Omega_{n-1}}{\Omega_n}\int_{\mathbb{R}^n} |\mathbf{x}-\mathbf{y}|^{-1} f(\mathbf{y})\,d\mathbf{y}$*, then*

$$cf = \big(-\Delta\big)^{\frac{n-1}{2}} g$$

where $c = (-4\pi)^{\frac{n-1}{2}}\frac{\Gamma(n/2)}{\Gamma(1/2)}$.

"Proof": I have put in quotes because we are not going to find the value of the constant c and we are also going to ignore the possible divergence of some integrals. On the other hand, we'll define what is meant by applying a fractional power of the Laplacian to a function. Note that this is not a problem when the dimension n is odd. In fact, when $n = 3$,

$$\big(-\Delta\big)^{\frac{n-1}{2}} g = -\Delta g$$

That is, applying the Laplacian is, up to a constant, the inverse operation for convolving with $\frac{1}{|\mathbf{x}|}$. This should not be a surprise. It is just another way of saying that $\frac{1}{|\mathbf{x}-\mathbf{y}|}$ is, up to a constant, the Green's function for Laplacian in dimension 3.

Note that the claim "$cf(\mathbf{x}) = \big(-\Delta\big)^{\frac{n-1}{2}} g(\mathbf{x})$" is equivalent, by Fourier transforming, to the claim "$c\tilde{f}(\mathbf{k}) = |\mathbf{k}|^{n-1}\tilde{g}(\mathbf{k})$". (So, to apply $(-\Delta)^{\alpha}$, you Fourier transform, multiply by $|\mathbf{k}|^{2\alpha}$ and Fourier transform back.) So we want to show that the Fourier transform of $\frac{1}{|\mathbf{x}|}$ is $\frac{c}{|\mathbf{k}|^{n-1}}\frac{\Omega_n}{\Omega_{n-1}}$. So we compute

$$\int_{\mathbb{R}^n} \frac{1}{|\mathbf{x}|} e^{-i\mathbf{x}\cdot\mathbf{k}} d^n\mathbf{x} = \int_{\mathbb{R}^n} \frac{1}{|\mathbf{x}|} e^{-ix_1|\mathbf{k}|} d^n\mathbf{x}$$

where x_1 is the first coordinate of $\mathbf{x}$. Here we have made a change of variables $\mathbf{x} \to R\mathbf{x}$, where the rotation R is chosen so that $R\mathbf{k}$ is in the direction of the first coordinate axis. Now making the change of variables $\mathbf{x} = \frac{1}{|\mathbf{k}|}\mathbf{y}$

$$\int_{\mathbb{R}^n} \frac{1}{|\mathbf{x}|} e^{-i\mathbf{x}\cdot\mathbf{k}} d^n\mathbf{x} = \frac{1}{|\mathbf{k}|^{n-1}}\int_{\mathbb{R}^n} \frac{1}{|\mathbf{y}|} e^{-iy_1} d^n\mathbf{y}$$

Since $\int_{\mathbb{R}^n} \frac{1}{|\mathbf{y}|} e^{-iy_1} d^n\mathbf{y}$ is a constant, we are done, up to evaluation of the constant. ■

References

- Sirgurdur Helgason, **The Radon Transform**, Birkhäuser, 1980.

