Poisson's Equation

In these notes we shall find a formula for the solution of Poisson's equation

$$\vec{\nabla}^2\varphi = 4\pi\rho$$

Here ρ is a given (smooth) function and φ is the unknown function. In electrostatics, ρ is the charge density and φ is the electric potential. The main step in finding this formula will be to consider an

arbitrary (smooth) function φ and an
arbitrary (smooth) region V in $\mathbb{R}^3$ and an
arbitrary point $\vec{\mathbf{r}}_0$ in the interior of V

and to find a formula which expresses $\varphi(\vec{\mathbf{r}}_0)$ in terms of

$\vec{\nabla}^2\varphi(\vec{\mathbf{r}})$, with $\vec{\mathbf{r}}$ running over V and
$\vec{\nabla}\varphi(\vec{\mathbf{r}})$ and $\varphi(\vec{\mathbf{r}})$, with $\vec{\mathbf{r}}$ running only over ∂V.

This formula is

$$\varphi(\vec{\mathbf{r}}_0) = -\tfrac{1}{4\pi}\Big\{ \iiint_V \tfrac{\vec{\nabla}^2\varphi(\vec{\mathbf{r}})}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\ d^3\vec{\mathbf{r}} - \iint_{\partial V} \varphi(\vec{\mathbf{r}})\tfrac{\vec{\mathbf{r}}-\vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|^3}\cdot\hat{\mathbf{n}}\ dS - \iint_{\partial V} \tfrac{\vec{\nabla}\varphi(\vec{\mathbf{r}})}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\cdot\hat{\mathbf{n}}\ dS\Big\} \qquad (V)$$

When we take the limit as V expands to fill all of $\mathbb{R}^3$ (assuming that φ and $\vec{\nabla}\varphi$ go to zero sufficiently quickly at ∞), we will end up with the formula

$$\varphi(\vec{\mathbf{r}}_0) = -\tfrac{1}{4\pi}\iiint_V \tfrac{\vec{\nabla}^2\varphi(\vec{\mathbf{r}})}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\ d^3\vec{\mathbf{r}}$$

that expresses φ evaluated at an arbitrary point, $\vec{\mathbf{r}}_0$, of $\mathbb{R}^3$ in terms of $\vec{\nabla}^2\varphi(\vec{\mathbf{r}})$, with $\vec{\mathbf{r}}$ running over $\mathbb{R}^3$, which is exactly what we want.

Let

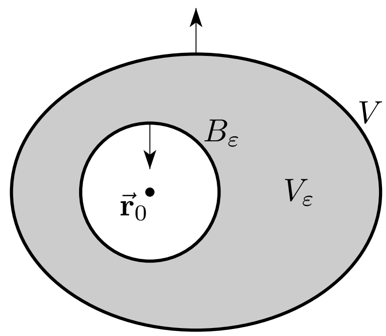
$$\vec{\mathbf{r}} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$$
$$\vec{\mathbf{r}}_0 = x_0\,\hat{\boldsymbol{\imath}} + y_0\,\hat{\boldsymbol{\jmath}} + z_0\,\hat{\mathbf{k}}$$

We shall exploit several properties of the function $\frac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}$. The first two properties are

$$\vec{\nabla}\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|} = -\tfrac{\vec{\mathbf{r}}-\vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|^3}$$
$$\vec{\nabla}^2\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|} = -\vec{\nabla}\cdot\tfrac{\vec{\mathbf{r}}-\vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|^3} = 0$$

and are valid for all $\vec{\mathbf{r}} \neq \vec{\mathbf{r}}_0$. Verifying these properties are simple two line computations. The other property of $\frac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}$ that we shall use is the following. Let B_ε be the sphere of radius ε centered on $\vec{\mathbf{r}}_0$. Then, for any continuous function $\psi(\vec{\mathbf{r}})$,

$$\lim_{\varepsilon\to0+}\iint_{B_\varepsilon}\tfrac{\psi(\vec{\mathbf{r}})}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|^p}\ dS = \lim_{\varepsilon\to0+}\iint_{B_\varepsilon}\tfrac{\psi(\vec{\mathbf{r}}_0)}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|^p}\ dS = \lim_{\varepsilon\to0+}\tfrac{\psi(\vec{\mathbf{r}}_0)}{\varepsilon^p}\iint_{B_\varepsilon} dS = \lim_{\varepsilon\to0+}\tfrac{\psi(\vec{\mathbf{r}}_0)}{\varepsilon^p}4\pi\varepsilon^2$$
$$= \begin{cases} 4\pi\psi(\vec{\mathbf{r}}_0) & \text{if } p=2 \\ 0 & \text{if } p<2 \end{cases} \qquad \text{(B)}$$



Here is the derivation of (V). Let V_ε be the part of V outside of B_ε. Note that the boundary ∂V_ε of V_ε consists of two parts — the boundary ∂V of V and the sphere B_ε — and that the unit outward normal to ∂V_ε on B_ε is $-\frac{\vec{\mathbf{r}}-\vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}$. By the divergence theorem

$$\iiint_{V_\varepsilon} \vec{\nabla}\cdot\big(\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\vec{\nabla}\varphi - \varphi\vec{\nabla}\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\big)dV = \iint_{\partial V}\big(\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\vec{\nabla}\varphi - \varphi\vec{\nabla}\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\big)\cdot\hat{\mathbf{n}}\;dS + \iint_{B_\varepsilon}\big(\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\vec{\nabla}\varphi - \varphi\vec{\nabla}\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\big)\cdot\big(-\tfrac{\vec{\mathbf{r}}-\vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\big)dS \tag{M}$$

Subbing in $\vec{\nabla}\frac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|} = -\frac{\vec{\mathbf{r}}-\vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|^3}$ and applying (B)

$$\begin{aligned}\lim_{\varepsilon\to0+}\iint_{B_\varepsilon}\big(\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\vec{\nabla}\varphi - \varphi\vec{\nabla}\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\big)\cdot\big(-\tfrac{\vec{\mathbf{r}}-\vec{\mathbf{r}}_0}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\big)dS &= -\lim_{\varepsilon\to0+}\iint_{B_\varepsilon}\big(\vec{\nabla}\varphi\cdot(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0)+\varphi\big)\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|^2}dS\\ &= -4\pi\Big[\vec{\nabla}\varphi\cdot(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0)+\varphi\Big]_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}\\ &= -4\pi\varphi(\vec{\mathbf{r}}_0)\end{aligned} \tag{R}$$

Applying $\vec{\nabla}\cdot\big(f\vec{\mathbf{F}}\big) = \vec{\nabla}f\cdot\vec{\mathbf{F}} + f\vec{\nabla}\cdot\vec{\mathbf{F}}$, twice, we see that the integrand of the left hand side is

$$\begin{aligned}\vec{\nabla}\cdot\big(\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\vec{\nabla}\varphi - \varphi\vec{\nabla}\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\big) &= \vec{\nabla}\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\cdot\vec{\nabla}\varphi + \tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\vec{\nabla}^2\varphi - \vec{\nabla}\varphi\cdot\vec{\nabla}\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|} - \varphi\vec{\nabla}^2\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\\ &= \tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\vec{\nabla}^2\varphi\end{aligned} \tag{L}$$

since $\vec{\nabla}^2\frac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|} = 0$ on V_ε. So applying $\lim_{\varepsilon\to0+}$ to (M) and applying (L) and (R) gives

$$\iiint_V \tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\vec{\nabla}^2\varphi\;dV = \iint_{\partial V}\big(\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\vec{\nabla}\varphi - \varphi\vec{\nabla}\tfrac{1}{\|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0\|}\big)\cdot\hat{\mathbf{n}}\;dS - 4\pi\varphi(\vec{\mathbf{r}}_0)$$

which is exactly equation (V).