Dirichlet to Neumann Problems

Consider a wire $0 \le x \le \ell$ with voltage $u(x)$ at x. By Ohm's law

$$u(x+dx) - u(x) = -I\rho(x)dx$$

where I is the current flowing through the wire and $\rho(x)dx$ is the resistance between x and $x+dx$. The resistance density $\rho(x)$ is called the resistivity. Dividing across by dx and taking the limit $dx \to 0$

$$u'(x) = -I\rho(x)$$

Assuming that charge is not allowed to accumulate inside the wire, I is a consant and we may eliminate it from the equation just by dividing $\rho(x)$ across and differentiating. If $\gamma(x) = \frac{1}{\rho(x)}$ is the conductivity

$$\gamma(x)u'(x) = -I \quad \Longrightarrow \quad \big(\gamma(x)u'(x)\big)' = 0 \tag{$*$}$$

Now suppose that we may only measure the voltages and currents at the ends of the wire. That is, we may only measure $u(0), u(\ell)$, $\gamma(0)u'(0)$ and $\gamma(\ell)u'(\ell)$. By $(*)$, $\gamma(x)u'(x)$ is a constant and so takes the value $\gamma(0)u'(0)$ everywhere. Thus

$$u'(x) = \gamma(0)u'(0)\tfrac{1}{\gamma(x)} \quad \Longrightarrow \quad u(\ell) - u(0) = \gamma(0)u'(0)\int_0^\ell \tfrac{dx}{\gamma(x)}$$

The only property of the wire that you can determine by measurements at the ends of the wire is the total resistance $\int_0^\ell \frac{dx}{\gamma(x)}$.

In $\mathbb{R}^n$, $n \ge 2$, the current $\mathbf{i}(\mathbf{x})$ is a vector and Ohm's Law is

$$\mathbf{i}(\mathbf{x}) = -\gamma(\mathbf{x})\boldsymbol{\nabla} u(\mathbf{x})$$

Assuming that charge is not allowed to accumulate, the net rate of charge flow across the boundary ∂V of any region V must vanish, so that

$$\int_{\partial V} \mathbf{i}(\mathbf{x}) \cdot \hat{\mathbf{n}} dS = 0$$

By the divergence theorem

$$\boldsymbol{\nabla} \cdot \mathbf{i}(\mathbf{x}) = 0 \quad \Longrightarrow \quad \boldsymbol{\nabla} \cdot \big(\gamma(\mathbf{x})\boldsymbol{\nabla} u(\mathbf{x})\big) = 0$$

Suppose now that we have a conductor filling a region Ω and that we apply a voltage f on the boundary $\partial\Omega$ of Ω and measure the current that then flows out of the region. By measuring the current exiting various parts of $\partial\Omega$, we are measuring the current flux on $\partial\Omega$,

© Joel Feldman. 2002. All rights reserved.

which determines $\gamma(\mathbf{x})\frac{\partial u}{\partial \nu}(\mathbf{x})$ on $\partial\Omega$, where $\frac{\partial u}{\partial \nu}$ is the normal derivative. For a given γ and f, the boundary value problem

$$\boldsymbol{\nabla}\cdot\big(\gamma(\mathbf{x})\boldsymbol{\nabla}u(\mathbf{x})\big)=0 \text{ in } \Omega \qquad u=f \text{ on } \partial\Omega$$

determines u on Ω and hence $k(\mathbf{x})=\gamma(\mathbf{x})\frac{\partial u}{\partial \nu}(\mathbf{x}) \restriction \partial\Omega$. Let $\Lambda_\gamma(f)$ be the k that results from a given γ and f. Clearly $\Lambda_\gamma(f)$ depends linearly on f. The map

$$\Lambda_\gamma : C^\infty(\partial\Omega)\to C^\infty(\partial\Omega)$$

is called the Dirichlet to Neumann Map. Because Λ_γ is a linear map on $C^\infty(\partial\Omega)$, it has a distributional kernel

$$\Lambda_\gamma(f)=\int_{\partial\Omega}\lambda_\gamma(x,y)f(y)\,dS(y)$$

where dS is the surface measure on $\partial\Omega$. If we measure the current k that results from all applied surface voltages f, we know $\lambda_\gamma(\mathbf{x},\mathbf{y})$ for all $\mathbf{x},\mathbf{y}\in\partial\Omega$. This is a function of $2(n-1)$ variables. The conductivity $\gamma(\mathbf{x})$ is a function of n variables. So for $n=1$, $\gamma(\mathbf{x})$ is a function of more variables than $\lambda_\gamma(\mathbf{x},\mathbf{y})$. We have already seen that, for $n=1$, $\lambda_\gamma(\mathbf{x},\mathbf{y})$ cannot possibly determine $\gamma(\mathbf{x})$. For $n=2$ ($n>2$), $\gamma(\mathbf{x})$ is a function of the same number of variables as (fewer variables than) $\lambda_\gamma(\mathbf{x},\mathbf{y})$.

In general, $\gamma(\mathbf{x})$ is a positive definite, symmetric, $n\times n$ matrix. If $\gamma(\mathbf{x})$ is scalar (that is, a multiple of the identity matrix), the medium is called isotropic. Otherwise it is called anisotropic. In Uhlmann's notes, it is proven that, for $n\ge 2$, Λ_γ does indeed determine an isotropic conductivity. However, it cannot possibly determine anistropic conductivities for the following obvious reason. Let $\Psi:\bar\Omega\to\bar\Omega$ be a diffeomorphism with $\Psi\restriction\partial\Omega$ being the identity map. Given any u,γ, set

$$\tilde\gamma=\tfrac{1}{|\det(D\Psi)|}(D\Psi)\gamma(D\Psi)^t\circ\Psi^{-1} \qquad \tilde u=u\circ\Psi^{-1}$$

where $D\Psi$ is the Jacobian (matrix of first partial derivatives) of Ψ. Then

$$\boldsymbol{\nabla}\cdot\big(\gamma(\mathbf{x})\boldsymbol{\nabla}u(\mathbf{x})\big)=0 \text{ in } \Omega \quad u=f \text{ on } \partial\Omega \quad\implies\quad \boldsymbol{\nabla}\cdot\big(\tilde\gamma(\mathbf{x})\boldsymbol{\nabla}\tilde u(\mathbf{x})\big)=0 \text{ in } \Omega \quad \tilde u=f \text{ on } \partial\Omega$$

Thus $\Lambda_\gamma=\Lambda_{\tilde\gamma}$. In Uhlmann's notes, it is proven that, for $n=2$, Λ_γ determines anisotropic conductivities up to diffeomorphisms like this. He conjectures that this is also true for $n>2$.

Example. Here is a carefully rigged example in which an isotropic conductivity is computed from a Dirichlet to Neumann map. The region $\Omega=[0,1]^2$ is square. We assume that we know

$$\begin{aligned}
&(1)\quad \boldsymbol{\nabla}\cdot\big(\gamma(x_1)\boldsymbol{\nabla}u(\mathbf{x})\big)=0 && \text{in } \Omega\\
&(2)\quad u(0,x_2)=u(1,x_2)=\sin\pi x_2 && \text{for all } 0\le x_2\le 1\\
&(3)\quad u(x_1,0)=u(x_1,1)=0 && \text{for all } 0\le x_1\le 1
\end{aligned}$$

© Joel Feldman. 2002. All rights reserved.

Note that we are assuming that the conductivity is isotropic and also is a function of x_1 only. Motivated by (1) and (2), we look for a solution of the form $u(x_1,x_2) = a(x_1)\sin(\pi x_2)$. Condition (3) is satisfied for all $a(x_1)$. Condition (2) is satisfied if and only if $a(0) = a(1) = 1$. Condition (1) is satisfied if and only if

$$\begin{aligned} 0 &= \boldsymbol{\nabla}\cdot\big(\gamma(x_1)a'(x_1)\sin\pi x_2, \gamma(x_1)a(x_1)\pi\cos\pi x_2\big) \\ &= \sin\pi x_2\ \big[\big(\gamma(x_1)a'(x_1)\big)' - \pi^2\gamma(x_1)a(x_1)\big] \end{aligned}$$

which is the case if and only if

$$(4)\quad \big(\gamma(x_1)a'(x_1)\big)' - \pi^2\gamma(x_1)a(x_1) = 0 \qquad \text{for all } 0 \le x_1 \le 1$$

We imagine that we have measured

$$k(x_1) = \gamma(x_1)\tfrac{\partial u}{\partial x_2}\big|_{x_2=0} = \gamma(x_1)\pi a(x_1)\cos\pi x_2\ \big|_{x_2=0} = \pi\gamma(x_1)a(x_1)$$

and that we wish to determine $\gamma(x_1)$. We can do so by subbing $\gamma(x_1) = \frac{k(x_1)}{\pi a(x_1)}$ into (4) and solving for a.

$$\begin{aligned} \big(k(x_1)\tfrac{a'(x_1)}{a(x_1}\big)' = \pi^2 k(x_1) \quad &\implies \quad \tfrac{d}{dx_1}\big[k(x_1)\tfrac{d}{dx_1}\ln a(x_1)\big] = \pi^2 k(x_1) \\ &\implies \quad k(x_1)\tfrac{d}{dx_1}\ln a(x_1) = \pi^2\int_0^{x_1} k(t)\ dt - \pi^2 C \\ &\implies \quad \ln a(x_1) = \pi^2\int_0^{x_1}\tfrac{1}{k(s)}\Big[\int_0^s k(t)\ dt - C\Big]\ ds + D \end{aligned}$$

To satisfy the boundary condition $a(0) = 1$, we need $D = 0$ and to satisfy $a(1) = 1$, we need

$$C = \Big[\int_0^1 \tfrac{ds}{k(s)}\Big]^{-1}\Big[\int_0^1 \tfrac{ds}{k(s)}\int_0^s k(t)\ dt\Big]$$

This determines[(1)] a and hence $\gamma = \frac{k}{\pi a}$.

References

- Gunther Uhlmann, **The Dirichlet to Neumann Map and Inverse Problems**, preprint.

---

(1) If you are worried about dividing by k in the integrals, you shouldn't be. We know that $0 \le u \le 1$ on $\partial\Omega$. By the maximum principle, this implies that $0 < u < 1$ in the interior of Ω. This in turn forces $\frac{\partial u}{\partial x_2} \ge 0$ when $x_2 = 0$. In fact, by the strong maximum principle, $\frac{\partial u}{\partial x_2} > 0$ for $x_2 = 0$, which ensures that $k(x_1) > 0$ for all $0 \le x_1 \le 1$.

© Joel Feldman. 2002. All rights reserved.