Mathematics 512

Spectral Theory of Schrödinger Operators

Prerequisites

You will need a very rudimentary knowledge of L^2 spaces and bounded operators on Hilbert spaces. Certainly a first course in Functional analysis, like Math 421 or Math 510, is more than adequate. If you have not taken such a course, reading up on the following should also suffice.

- Lebesgue measure and integration [RS1 §I.3]
- Measure theory [RS1 §I.4]
- The definition of a Hilbert space and simple examples of Hilbert spaces [RS1 §II.1]
- The Riesz representation theorem [RS1 §II.2]
- Orthonormal bases [RS1 §II.3]
- The definition of a Banach space and simple examples of Banach spaces [RS1 §III.1]
- Bounded operators, operator norm [RS1 §VI.1]
- Adjoints [RS1 §VI.2]
- Projection operators [RS1 §II.2, VI.2]

The notation [RS1] refers to

M. Reed and B. Simon, **Methods of Modern Mathematical Physics, I: Functional Analysis**, Academic Press, 1972.

Self–test

If you are concerned that you might not have the background for this course try doing the following problems. You should be able to handle most of them. Do not be upset if you cannot do all of them. There are a few “more interesting” problems in the list.

Problem 1.

a) Prove that for any open set A in $[0,1]$, χ_A is an L^1 limit of continuous functions.

b) Let B be a Borel set in $[0,1]$. Prove that χ_B is an L^1 limit of functions χ_A with A open. (Hint: use the regularity of Lebesgue measure.)

c) Prove that $C[a,b]$ is L^1 dense in $L^1[a,b]$.

Problem 2.

a) Prove that the inner product can be recovered from the norm by the **polarization identity**

$$(x,y) = \tfrac{1}{4}\Big\{\big(\|x+y\|^2 - \|x-y\|^2\big) - i\big(\|x+iy\|^2 - \|x-iy\|^2\big)\Big\}$$

b) Prove that a normed linear space is an inner product space if and only if the norm satisfies the **parallelogram law**

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + \|y\|^2$$

Problem 3.

Let V be an inner product space and let $\{x_n\}_{n=1}^N$ be an orthonormal set. Prove that

$$\left\| x - \sum_{n=1}^{N} c_n x_n \right\|$$

is minimized by choosing $c_n = (x_n, x)$.

Problem 4.

Let S be any linear subspace of a Hilbert space. Prove that $S^\perp$ is a closed linear subspace and that $\bar{S} = \left(S^\perp\right)^\perp$.

Problem 5.

Let S be any linear subspace of a Hilbert space $\mathcal{H}$. Let $f : S \to \mathbb{C}$ be a bounded linear functional on S with bound C. Prove that there is a unique extension of f to a bounded linear functional on $\mathcal{H}$ with the same bound.

Problem 6.

Let f be a C^1 function that is periodic with period 2π. Let $(\ \cdot\ ,\ \cdot\ )$ denote the inner product on $L^2([0, 2\pi])$. Set $c_n = \frac{1}{\sqrt{2\pi}}\left(e^{inx}, f\right)$, $b_n = \frac{1}{\sqrt{2\pi}}\left(e^{inx}, f'\right)$ and

$$(S_M f)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n=-M}^{M} c_n e^{in\theta}$$

a) Prove that $\sum |b_n|^2 < \infty$ and $\sum n^2 |c_n|^2 < \infty$.

b) Prove that $\sum |c_n| < \infty$.

c) Prove that $S_M f$ converges uniformly as $M \to \infty$.

d) Prove that $S_M f(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta + x) \frac{\sin(M+1/2)x}{\sin(x/2)} dx$.

e) Prove that $S_M f(\theta)$ converges to $f(\theta)$ as $M \to \infty$.

Problem 7.

Let T_n be a sequence of bounded operators on a Hilbert space. Let T be another bounded operator on the Hilbert space.

a) Prove that if the sequence converges in norm to T, then it converges strongly to T.

b) Prove that if the sequence converges strongly to T, then it converges weakly to T.

Problem 8.

a) Let T_t be the operator on $L^2(\mathbb{R})$ that maps $\phi(x)$ to $\phi(x + t)$. What is the norm of T_t? To what operator does T_t converge as $t \to \infty$ and in what topology?

b) Answer the same question with $L^2(\mathbb{R})$ replaced by $L^2(\mathbb{R}, e^{-x^2} dx)$.