Lattices and Periodic Functions

Definition L.1 Let $f(\mathbf{x})$ be a function on $\mathbb{R}^d$.

a) The vector $\boldsymbol{\gamma} \in \mathbb{R}^d$ is said to be a period for f if

$$f(\mathbf{x}+\boldsymbol{\gamma}) = f(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathbb{R}^d$$

b) Set

$$\mathcal{P}_f = \left\{ \boldsymbol{\gamma} \in \mathbb{R}^d \;\middle|\; \boldsymbol{\gamma} \text{ is a period for } f \right\}$$

If $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathcal{P}_f$ then $\boldsymbol{\gamma}+\boldsymbol{\gamma}' \in \mathcal{P}_f$ and if $\boldsymbol{\gamma} \in \mathcal{P}_f$ then $-\boldsymbol{\gamma} \in \mathcal{P}_f$ (sub $\mathbf{x} = \mathbf{z}-\boldsymbol{\gamma}$ into $f(\mathbf{x}+\boldsymbol{\gamma}) = f(\mathbf{x})$). Furthermore, the zero vector $\mathbf{0} \in \mathbb{R}^d$ is always in $\mathcal{P}_f$. Thus $\mathcal{P}_f$ is a (commutative) group under addition and

$$\boldsymbol{\gamma}_1, \ \cdots, \ \boldsymbol{\gamma}_p \in \mathcal{P}_f \Rightarrow \ n_1\boldsymbol{\gamma}_1 + \cdots + n_p\boldsymbol{\gamma}_p \in \mathcal{P}_f \qquad \text{for all } p \in \mathbb{N} \text{ and } n_1, \cdots, n_p \in \mathbb{Z}$$

Example L.2

a) If $f(x,y) = \sin\left(\frac{2\pi x}{\ell_1}\right)\cos\left(\frac{2\pi y}{\ell_2}\right)$, then $\mathcal{P}_f = \left\{ (m\ell_1, n\ell_2) \;\middle|\; m,n \in \mathbb{Z} \right\}$.

b) If $f(x,y) = \sin\left(\frac{2\pi x}{\ell_1}\right)$, then $\mathcal{P}_f = \left\{ (m\ell_1, y) \;\middle|\; m \in \mathbb{Z},\ y \in \mathbb{R} \right\}$.

c) If $f(x,y) = \sin\left(\frac{2\pi x}{\ell_1}\right)\sinh y$, then $\mathcal{P}_f = \left\{ (m\ell_1, 0) \;\middle|\; m \in \mathbb{Z} \right\}$.

To exclude functions, as in Example L.2.b, that are constant in some direction, it suffices to require that $\mathbf{0}$ be an isolated point of $\mathcal{P}_f$. That is, to require that there be a number $r > 0$ such that every nonzero $\boldsymbol{\gamma} \in \mathcal{P}_f$ obeys $|\boldsymbol{\gamma}| \ge r$.

Proposition L.3 *If $\mathcal{P}$ is an additive subgroup of $\mathbb{R}^d$ and $\mathbf{0}$ is an isolated point of $\mathcal{P}$, then there are $d' \le d$ and independent vectors $\boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_{d'} \in \mathbb{R}^d$ such that*

$$\mathcal{P} = \left\{ n_1\boldsymbol{\gamma}_1 + \cdots + n_{d'}\boldsymbol{\gamma}_{d'} \;\middle|\; n_1, \cdots, n_{d'} \in \mathbb{Z} \right\}$$

Proof:

Claim 1. $\mathcal{P}$ has a shortest nonzero element.

Proof of Claim 1: Define $r = \inf\left\{\ |\boldsymbol{\gamma}|\ \middle|\ \boldsymbol{\gamma} \in \mathcal{P},\ \boldsymbol{\gamma} \neq \mathbf{0}\ \right\}$. If there were no shortest element, there would be a sequence of vectors $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \cdots$ in $\mathcal{P}$ with $\lim_{i\to\infty} |\boldsymbol{\beta}_i| = r$ and $r < |\boldsymbol{\beta}_i| \le 2r$ for every $i = 1, 2, \cdots$. Because the closed ball of radius $2r$ is compact, the sequence has a limit point and hence has a Cauchy subsequence. In particular, there are $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$ in the sequence, with $\boldsymbol{\beta}_i \neq \boldsymbol{\beta}_j$ with $|\boldsymbol{\beta}_i - \boldsymbol{\beta}_j| < \frac{r}{2}$. But this is impossible, because $\boldsymbol{\beta}_i - \boldsymbol{\beta}_j$ would be a nonzero element of $\mathcal{P}$ with length smaller than r.

Claim 2. Let $\boldsymbol{\gamma}_1$ be a shortest nonzero element of $\mathcal{P}$ and set $\mathcal{P}_1 = \left\{\ \boldsymbol{\gamma} \in \mathcal{P}\ \middle|\ \boldsymbol{\gamma} \parallel \boldsymbol{\gamma}_1\ \right\}$. Then $\mathcal{P}_1 = \left\{\ n\boldsymbol{\gamma}_1\ \middle|\ n \in \mathbb{Z}\ \right\}$.

Proof of Claim 2: If $x\boldsymbol{\gamma}_1 \in \mathcal{P}$ with x not an integer, then $(x - [x])\boldsymbol{\gamma}_1$ (where $[\ \cdot\ ]$ denotes integer part) is a nonzero element of $\mathcal{P}$ with length strictly smaller than the length of $\boldsymbol{\gamma}_1$.

If $\mathcal{P} = \mathcal{P}_1$, we have finished. Otherwise continue with

Claim 3. Denote by $\mathbb{P}_1$ orthogonal projection in $\mathbb{R}^d$ onto the line $\left\{\ x\boldsymbol{\gamma}_1\ \middle|\ x \in \mathbb{R}\ \right\}$ and by $\mathbb{P}_1^\perp = \mathbb{1} - \mathbb{P}_1$ orthogonal projection perpendicular to the line $\left\{\ x\boldsymbol{\gamma}_1\ \middle|\ x \in \mathbb{R}\ \right\}$. Then $\mathcal{P} \setminus \mathcal{P}_1$ has an element whose distance from the line $\left\{\ x\boldsymbol{\gamma}_1\ \middle|\ x \in \mathbb{R}\ \right\}$ is a minimum, i.e. that minimizes $|\mathbb{P}_1^\perp \boldsymbol{\gamma}|$.

Proof of Claim 3: Define $r_1 = \inf\left\{\ |\mathbb{P}_1^\perp \boldsymbol{\gamma}|\ \middle|\ \boldsymbol{\gamma} \in \mathcal{P} \setminus \mathcal{P}_1\ \right\}$. If there were no minimizing element, there would be a sequence of vectors $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \cdots$ in $\mathcal{P}$ with

$$2r_1 \ge |\mathbb{P}_1^\perp \boldsymbol{\beta}_1| > |\mathbb{P}_1^\perp \boldsymbol{\beta}_2| > |\mathbb{P}_1^\perp \boldsymbol{\beta}_3| > \ \cdots\ > r_1$$

Because $|\mathbb{P}_1^\perp \boldsymbol{\beta}_i| = |\mathbb{P}_1^\perp(\boldsymbol{\beta}_i + n\boldsymbol{\gamma}_1)|$ for all n, we may assume, without loss of generality, that $|\mathbb{P}_1 \boldsymbol{\beta}_i| \le |\boldsymbol{\gamma}_1|$ for every i. Because

$$\left\{\ \mathbf{x} \in \mathbb{R}^d\ \middle|\ |\mathbb{P}_1^\perp \mathbf{x}| \le 2r_1,\ |\mathbb{P}_1 \mathbf{x}| \le |\boldsymbol{\gamma}_1|\ \right\}$$

is compact, the sequence has a limit point and hence has a Cauchy subsequence. In particular, there are $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$ in the sequence, with $\boldsymbol{\beta}_i \neq \boldsymbol{\beta}_j$ with $|\boldsymbol{\beta}_i - \boldsymbol{\beta}_j| < \frac{r}{2}$. But this is impossible, because $\boldsymbol{\beta}_i - \boldsymbol{\beta}_j$ would be a nonzero element of $\mathcal{P}$ with length smaller than r.

Claim 4. Let $\boldsymbol{\gamma}_2$ be an element of $\mathcal{P} \setminus \mathcal{P}_1$ that minimizes $|\mathbb{P}_1^\perp \boldsymbol{\gamma}|$ and set

$$\mathcal{P}_2 = \mathcal{P} \cap \left\{\ x_1\boldsymbol{\gamma}_1 + x_2\boldsymbol{\gamma}_2\ \middle|\ x_1, x_2 \in \mathbb{R}\ \right\}$$

Then $\mathcal{P}_2 = \left\{\ n_1\boldsymbol{\gamma}_1 + n_2\boldsymbol{\gamma}_2\ \middle|\ n_1, n_2 \in \mathbb{Z}\ \right\}$.

Proof of Claim 4: If $x_1\boldsymbol{\gamma}_1+x_2\boldsymbol{\gamma}_2 \in \mathcal{P}$ with x_2 not an integer, then $\boldsymbol{\gamma}' = x_1\boldsymbol{\gamma}_1+(x_2-[x_2])\boldsymbol{\gamma}_2$ is an element of of $\mathcal{P}\setminus\mathcal{P}_1$ with $|\mathbb{P}_1^\perp\boldsymbol{\gamma}'| = |x_2-[x_2]|\,|\mathbb{P}_1^\perp\boldsymbol{\gamma}_2| < |\mathbb{P}_1^\perp\boldsymbol{\gamma}_2|$. So x_2 must be an integer. But then $(x_1\boldsymbol{\gamma}_1+x_2\boldsymbol{\gamma}_2)-x_2\boldsymbol{\gamma}_2 = x_1\boldsymbol{\gamma}_1 \in \mathcal{P}$ and, by Claim 2, x_1 must be an integer as well.

If $\mathcal{P}=\mathcal{P}_2$, we have finished. Otherwise continue with $\ldots$.

■

To exclude functions, as in Example L.2.c, that are "mixed periodic/non–periodic", we shall assume that $d'=d$. Let $\boldsymbol{\gamma}_1,\ \cdots,\ \boldsymbol{\gamma}_d \in \mathbb{R}^d$ be d linearly independent vectors and set

$$\Gamma = \left\{\ n_1\boldsymbol{\gamma}_1+\cdots+n_d\boldsymbol{\gamma}_d \ \middle|\ n_1,\cdots,n_d\in\mathbb{Z}\ \right\}$$

Γ is called the lattice generated by $\boldsymbol{\gamma}_1,\ \cdots,\ \boldsymbol{\gamma}_d$.

Problem L.1 The set of generators for a lattice are not uniquely determined. Let Γ be generated by d linearly independent vectors $\boldsymbol{\gamma}_1,\ \cdots,\ \boldsymbol{\gamma}_d \in \mathbb{R}^d$. Let Γ' be generated by d linearly independent vectors $\boldsymbol{\gamma}_1',\ \cdots,\ \boldsymbol{\gamma}_d' \in \mathbb{R}^d$. Prove that $\Gamma=\Gamma'$ if and only there is a $d\times d$ matrix A with integer matrix elements and $|\det A|=1$ such that $\boldsymbol{\gamma}_i' = \sum_{j=1}^d A_{i,j}\boldsymbol{\gamma}_j$.

Problem L.2 Let $\boldsymbol{\gamma}_1,\ \cdots,\ \boldsymbol{\gamma}_d \in \mathbb{R}^d$ be d linearly independent vectors. Prove that there are two constants C and c, depending only on $\boldsymbol{\gamma}_1,\ \cdots,\ \boldsymbol{\gamma}_d$ such that

$$c|\mathbf{x}| \le \big|x_1\boldsymbol{\gamma}_1 +\ \cdots + x_d\boldsymbol{\gamma}_d\big| \le C|\mathbf{x}|$$

for all $\mathbf{x}\in\mathbb{R}^d$.

We'll now find a bunch of functions that are periodic with respect to Γ. Consider $f(\mathbf{x}) = e^{i\mathbf{b}\cdot\mathbf{x}}$. This function has period $\boldsymbol{\gamma}$ if and only if $e^{i\mathbf{b}\cdot(\mathbf{x}+\boldsymbol{\gamma})} = e^{i\mathbf{b}\cdot\mathbf{x}}$ for all $\mathbf{x}\in\mathbb{R}^d$. This is the case if and only if $e^{i\mathbf{b}\cdot\boldsymbol{\gamma}}=1$ and this is the case if and only if $\mathbf{b}\cdot\boldsymbol{\gamma}\in 2\pi\mathbb{Z}$.

Definition L.4 Let Γ be a lattice in $\mathbb{R}^d$. The dual lattice for Γ is

$$\Gamma^\# = \left\{\ \mathbf{b}\in\mathbb{R}^d \ \middle|\ \mathbf{b}\cdot\boldsymbol{\gamma}\in 2\pi\mathbb{Z} \text{ for all } \boldsymbol{\gamma}\in\Gamma\ \right\}$$

Remark L.5 Let $\boldsymbol{\gamma}_1,\ \cdots,\ \boldsymbol{\gamma}_d \in \mathbb{R}^d$ be linearly independent and denote by Γ the lattice that they generate. A vector $\mathbf{b}\in\mathbb{R}^d$ is an element of $\Gamma^\#$ if and only if

$$\mathbf{b}\cdot\boldsymbol{\gamma}_j \in 2\pi\mathbb{Z} \qquad\qquad \text{for all } 1\le j\le d$$

Example L.6 Let $\mathbf{e}_1, \cdots, \mathbf{e}_d$ be the standard basis of $\mathbb{R}^d$. That is, $\mathbf{e}_j$ has all components zero, except for the j^{th}, which is one. Choosing $\ell_1, \cdots, \ell_d > 0$ and $\boldsymbol{\gamma}_j = \ell_j \mathbf{e}_j$,

$$\Gamma = \left\{ \ (n_1\ell_1, \cdots, n_d\ell_d) \ \middle|\ n_1, \cdots, n_d \in \mathbb{Z} \ \right\}$$

Then $(x_1, \cdots, x_d)$ is in $\Gamma^\#$ if and only if

$$(x_1, \cdots, x_d) \cdot \boldsymbol{\gamma}_j = \ell_j x_j \in 2\pi\mathbb{Z} \iff x_j \in \tfrac{2\pi}{\ell_j}\mathbb{Z}$$

Thus

$$\Gamma^\# = \left\{ \ \left(n_1 \tfrac{2\pi}{\ell_1}, \cdots, n_d \tfrac{2\pi}{\ell_d}\right) \ \middle|\ n_1, \cdots, n_d \in \mathbb{Z} \ \right\}$$

Example L.7 Let

$$\Gamma = \left\{ \ n(1,0) + m(\pi, 1) \ \middle|\ n, m \in \mathbb{Z} \ \right\}$$

Then

$$\Gamma^\# = \left\{ \ n(0, 2\pi) + m(2\pi, -2\pi^2) \ \middle|\ n, m \in \mathbb{Z} \ \right\}$$

Since

$$\big[n'(1,0) + m'(\pi,1)\big] \cdot \big[n(0,2\pi) + m(2\pi, -2\pi^2)\big] = 2\pi\big(n'm + m'n\big)$$

every vector of the form $n(0,2\pi) + m(2\pi, -2\pi^2)$ with $m, n \in \mathbb{Z}$ is indeed in $\Gamma^\#$. To verify that only vectors of this form are in $\Gamma^\#$, let $\mathbf{z} = x(0, 2\pi) + y(2\pi, -2\pi^2)$ be any vector in $\mathbb{R}^2$. (Certainly, $(0, 2\pi)$ and $(2\pi, -2\pi^2)$ form a basis for $\mathbb{R}^2$.) For $\mathbf{z}$ to be in $\Gamma^\#$ it is necessary that

$$\mathbf{z} \cdot (1,0) = 2\pi y \in 2\pi\mathbb{Z}$$
$$\mathbf{z} \cdot (\pi,1) = 2\pi x \in 2\pi\mathbb{Z}$$

which forces $x, y \in \mathbb{Z}$.

Problem L.3 Let Γ be generated by $\boldsymbol{\gamma}_1, \ \cdots, \ \boldsymbol{\gamma}_d \in \mathbb{R}^d$ (assumed linearly independent) and let

$$[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d] = \left\{ \ \sum_{j=1}^d t_j \boldsymbol{\gamma}_j \ \middle|\ 0 \le t_j \le 1 \text{ for all } 1 \le j \le d \ \right\}$$

be the parallelepiped with the $\boldsymbol{\gamma}_j$'s as edges. Prove that if $\mathbf{b} \in \Gamma^\#$, then

$$\int_{[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]} d^d\mathbf{x} \ e^{i\mathbf{b}\cdot\mathbf{x}} = \begin{cases} \big|[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]\big| & \text{if } \mathbf{b} = \mathbf{0} \\ 0 & \text{if } \mathbf{b} \ne \mathbf{0} \end{cases}$$

where $\big|[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]\big|$ is the volume of $\big|[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]\big|$. By Problem L.1, the volume $\big|[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]\big|$ is independent of the choice of generators. That is, if Γ is also generated by $\boldsymbol{\gamma}'_1, \ \cdots, \ \boldsymbol{\gamma}'_d \in \mathbb{R}^d$, then $\big|[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]\big| = \big|[\boldsymbol{\gamma}'_1, \cdots \boldsymbol{\gamma}'_d]\big|$. Consequently, it is legitimate to define $|\Gamma| = \big|[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]\big|$. Hence

$$\tfrac{1}{|\Gamma|} \int_{[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]} d^d\mathbf{x} \ e^{i\mathbf{b}\cdot\mathbf{x}} = \begin{cases} 1 & \text{if } \mathbf{b} = \mathbf{0} \\ 0 & \text{if } \mathbf{b} \ne \mathbf{0} \end{cases}$$

Proposition L.8 *If $\boldsymbol{\gamma}_1,\ \cdots,\ \boldsymbol{\gamma}_d \in \mathbb{R}^d$ are linearly independent and*

$$\Gamma = \left\{\ n_1\boldsymbol{\gamma}_1 + \cdots + n_d\boldsymbol{\gamma}_d\ \middle|\ n_1, \cdots, n_d \in \mathbb{Z}\ \right\}$$

then there exist d linearly independent vectors $\mathbf{b}_1,\ \cdots,\ \mathbf{b}_d \in \mathbb{R}^d$ such that

$$\Gamma^\# = \left\{\ n_1\mathbf{b}_1 + \cdots + n_d\mathbf{b}_d\ \middle|\ n_1, \cdots, n_d \in \mathbb{Z}\ \right\}$$

Proof: For each $1 \le i \le d$

$$V_i = \left\{\ x_1\boldsymbol{\gamma}_1 + \cdots + x_d\boldsymbol{\gamma}_d\ \middle|\ x_1, \cdots, x_d \in \mathbb{R},\ x_i = 0\ \right\}$$

is a $d-1$ dimensional subspace of $\mathbb{R}^d$. So $V_1^\perp$ is a one dimensional subspace of $\mathbb{R}^d$. Let $\mathbf{B}_i$ be any nonzero element of $V_i^\perp$ and define

$$\mathbf{b}_i = \tfrac{2\pi}{\boldsymbol{\gamma}_i\cdot\mathbf{B}_i}\mathbf{B}_i$$

Note that $\boldsymbol{\gamma}_i \cdot \mathbf{B}_i$ cannot vanish because then $\boldsymbol{\gamma}_i$ would have to be in V_i, i.e. would have to be a linear combination of $\boldsymbol{\gamma}_j,\ j \ne i$. Denote

$$B = \left\{\ n_1\mathbf{b}_1 + \cdots + n_d\mathbf{b}_d\ \middle|\ n_1, \cdots, n_d \in \mathbb{Z}\ \right\}$$

As

$$\mathbf{b}_i \cdot \boldsymbol{\gamma}_j = \begin{cases} 2\pi & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

we have that $\mathbf{b}_i \in \Gamma^\#$ and hence $B \subset \Gamma^\#$.

If $x_1\mathbf{b}_1 + \cdots + x_d\mathbf{b}_d = \mathbf{0}$ then $(x_1\mathbf{b}_1 + \cdots + x_d\mathbf{b}_d) \cdot \boldsymbol{\gamma}_j = 2\pi x_j = 0$ for every $1 \le j \le d$. So the $\mathbf{b}_i$'s are linearly independent and every vector in $\mathbb{R}^d$ may be written in the form $x_1\mathbf{b}_1 + \cdots + x_d\mathbf{b}_d$. If $x_1\mathbf{b}_1 + \cdots + x_d\mathbf{b}_d \in \Gamma^\#$, then

$$(x_1\mathbf{b}_1 + \cdots + x_d\mathbf{b}_d) \cdot \boldsymbol{\gamma}_j = 2\pi x_j \in 2\pi\mathbb{Z}$$

so that $x_j \in \mathbb{Z}$ for every $1 \le j \le d$. Hence $\boldsymbol{\gamma}^\# \subset B$. ■

From now on, we fix d linearly independent vectors $\boldsymbol{\gamma}_1,\ \cdots,\ \boldsymbol{\gamma}_d \in \mathbb{R}^d$, set

$$\Gamma = \left\{\ n_1\boldsymbol{\gamma}_1 + \cdots + n_d\boldsymbol{\gamma}_d\ \middle|\ n_1, \cdots, n_d \in \mathbb{Z}\ \right\}$$

The set of all C^∞ functions on $\mathbb{R}^d$ that are periodic with respect to Γ is denoted $C^\infty\big(\mathbb{R}^d/\Gamma\big)$. We have already observed that $f(\mathbf{x}) = e^{i\mathbf{b}\cdot\mathbf{x}}$ is in $C^\infty\big(\mathbb{R}^d/\Gamma\big)$ if and only in $\mathbf{b} \in \Gamma^\#$.

Remark L.9 Here is the story (at least in short form) behind the notation $C^\infty\big(\mathbb{R}^d/\Gamma\big)$. We have already observed that $\mathbb{R}^d$ is a group (under addition) and that Γ is a subgroup of of $\mathbb{R}^d$. As $\mathbb{R}^d$ is abelian, all subgroups are normal and the set of equivalence classes under the equivalence relation

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} - \mathbf{y} \in \Gamma$$

is itself a group, denoted, as usual $\mathbb{R}^d/\Gamma$. Precisely, the equivalence class of $\mathbf{x} \in \mathbb{R}^d$ is $[\mathbf{x}] = \big\{\ \mathbf{y} \in \mathbb{R}^d \ \big|\ \mathbf{x} \sim \mathbf{y}\ \big\} \subset \mathbb{R}^d$ and $\mathbb{R}^d/\Gamma = \big\{\ [\mathbf{x}]\ \big|\ \mathbf{x} \in \mathbb{R}^d\ \big\}$. The group operation in $\mathbb{R}^d/\Gamma$ is

$$[\mathbf{x}] + [\mathbf{y}] = [\mathbf{x} + \mathbf{y}]$$

As well as being a group, $\mathbb{R}^d/\Gamma$ can also be turned into a smooth manifold, called a d–dimensional torus. If $\mathcal{O}$ is any open subset of $\mathbb{R}^d$ with the property that no two points of $\mathcal{O}$ are equivalent under $\sim$, then the map

$$\begin{aligned} \xi_{\mathcal{O}}\ &: \mathcal{O} \to \mathbb{R}^d/\Gamma \\ &\mathbf{x} \mapsto [\mathbf{x}] \end{aligned}$$

is one–to–one. Its inverse is a coordinate map for $\mathbb{R}^d/\Gamma$. If Γ is generated by $\boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_d$ and X is any point in $\mathbb{R}^d$, $\big\{\ X + t_1\boldsymbol{\gamma}_1 + \cdots + t_d\boldsymbol{\gamma}_d\ \big|\ 0 < t_j < 1 \text{ for all } 1 \le j \le d\ \big\}$ is one possible choice of $\mathcal{O}$. The notation $C^\infty\big(\mathbb{R}^d/\Gamma\big)$ designates the set of smooth (that is, C^∞) functions on the manifold $\mathbb{R}^d/\Gamma$.

Theorem L.10 (Fourier Series) *A function $f : \mathbb{R}^d \to \mathbb{C}$ is in $C^\infty\big(\mathbb{R}^d/\Gamma\big)$ if and only if*

$$\begin{aligned} f(\mathbf{x}) &= \tfrac{1}{|\Gamma|} \sum_{\mathbf{b}\in\Gamma^\#} \hat{f}_{\mathbf{b}}\, e^{i\mathbf{b}\cdot\mathbf{x}} \\ \textit{with} &\sum_{\mathbf{b}\in\Gamma^\#} |\mathbf{b}|^{2n}\big|\hat{f}_{\mathbf{b}}\big| < \infty \qquad \textit{for all } n \in \mathbb{N} \end{aligned}$$

Furthermore, in this case,

$$\hat{f}_{\mathbf{b}} = \int_{[\boldsymbol{\gamma}_1,\cdots\boldsymbol{\gamma}_d]} d^d\mathbf{x}\ \ e^{-i\mathbf{b}\cdot\mathbf{x}}\, f(\mathbf{x})$$

Proof of "if": Suppose that we are given $\hat{f}_{\mathbf{b}}$, $\mathbf{b} \in \Gamma^\#$ obeying $\sum_{\mathbf{b}\in\Gamma^\#} |\mathbf{b}|^{2n}\big|\hat{f}_{\mathbf{b}}\big| < \infty$ for all $n \in \mathbb{N}$. In particular $\sum_{\mathbf{b}\in\Gamma^\#} \big|\hat{f}_{\mathbf{b}}\big| < \infty$ so the series $\frac{1}{|\Gamma|}\sum_{\mathbf{b}\in\Gamma^\#} \hat{f}_{\mathbf{b}}\, e^{i\mathbf{b}\cdot\mathbf{x}}$ converges absolutely and uniformly to some continuous function that is periodic with respect to Γ. Call the function $f(\mathbf{x})$. Furthermore for any $i_1, \cdots i_d \in \mathbb{N}$

$$\Big|\Big(\prod_{j=1}^d \tfrac{\partial^{i_j}}{\partial x_j^{i_j}}\Big)\hat{f}_{\mathbf{b}}\, e^{i\mathbf{b}\cdot\mathbf{x}}\Big| = \Big|\Big(\prod_{j=1}^d b_{i_j}^{i_j}\Big)\hat{f}_{\mathbf{b}}\, e^{i\mathbf{b}\cdot\mathbf{x}}\Big| \le |\mathbf{b}|^{\Sigma i_j}\,\big|\hat{f}_{\mathbf{b}}\big|$$

so the series $\frac{1}{|\Gamma|}\sum\limits_{\mathbf{b}\in\Gamma^\#}\Big(\prod\limits_{j=1}^{d}\frac{\partial^{i_j}}{\partial x_j^{i_j}}\Big)\hat f_{\mathbf b}\, e^{i\mathbf{b}\cdot\mathbf{x}}$ also converges absolutely and uniformly. This implies that $f(\mathbf{x})$ is C^∞. Furthermore

$$\begin{aligned}
\int_{[\boldsymbol\gamma_1,\cdots\boldsymbol\gamma_d]} d^d\mathbf{x}\ e^{-i\mathbf{b}\cdot\mathbf{x}}\, f(\mathbf{x}) &= \int_{[\boldsymbol\gamma_1,\cdots\boldsymbol\gamma_d]} d^d\mathbf{x}\ e^{-i\mathbf{b}\cdot\mathbf{x}}\Big[\tfrac{1}{|\Gamma|}\sum_{\mathbf{c}\in\Gamma^\#}\hat f_{\mathbf c}\, e^{i\mathbf{c}\cdot\mathbf{x}}\Big]\\
&= \tfrac{1}{|\Gamma|}\sum_{\mathbf{c}\in\Gamma^\#}\int_{[\boldsymbol\gamma_1,\cdots\boldsymbol\gamma_d]} d^d\mathbf{x}\ e^{i(\mathbf{c}-\mathbf{b})\cdot\mathbf{x}}\hat f_{\mathbf c}\\
&= \tfrac{1}{|\Gamma|}\int_{[\boldsymbol\gamma_1,\cdots\boldsymbol\gamma_d]} d^d\mathbf{x}\ \hat f_{\mathbf b} + \tfrac{1}{|\Gamma|}\sum_{\substack{\mathbf{c}\in\Gamma^\#\\ \mathbf{c}\ne\mathbf{b}}}\int_{[\boldsymbol\gamma_1,\cdots\boldsymbol\gamma_d]} d^d\mathbf{x}\ e^{i(\mathbf{c}-\mathbf{b})\cdot\mathbf{x}}\hat f_{c}\\
&= \hat f_{\mathbf b}
\end{aligned}$$

by Problem L.3.

Proof of "only if": Now suppose that we are given $f\in C^\infty\big(\mathbb{R}^d/\Gamma\big)$. Define

$$\hat f_{\mathbf b} = \int_{[\boldsymbol\gamma_1,\cdots\boldsymbol\gamma_d]} d^d\mathbf{x}\ e^{-i\mathbf{b}\cdot\mathbf{x}}\, f(\mathbf{x})$$

Then for any $i_1,\cdots i_d\in\mathbb{N}$

$$\begin{aligned}
\Big|\Big(\prod_{j=1}^{d} b_{i_j}^{i_j}\Big)\hat f_{\mathbf b}\Big| &= \Big|\int_{[\boldsymbol\gamma_1,\cdots\boldsymbol\gamma_d]} d^d\mathbf{x}\ \Big(\prod_{j=1}^{d} b_{i_j}^{i_j}\Big)e^{-i\mathbf{b}\cdot\mathbf{x}} f(\mathbf{x})\Big|\\
&= \Big|\int_{[\boldsymbol\gamma_1,\cdots\boldsymbol\gamma_d]} d^d\mathbf{x}\ \Big(\prod_{j=1}^{d}\frac{\partial^{i_j}}{\partial x_j^{i_j}} e^{-i\mathbf{b}\cdot\mathbf{x}}\Big) f(\mathbf{x})\Big|\\
&= \Big|\int_{[\boldsymbol\gamma_1,\cdots\boldsymbol\gamma_d]} d^d\mathbf{x}\ e^{-i\mathbf{b}\cdot\mathbf{x}}\Big(\prod_{j=1}^{d}\frac{\partial^{i_j}}{\partial x_j^{i_j}} f(\mathbf{x})\Big)\Big|\\
&\le |\Gamma|\sup_{\mathbf x}\Big(\prod_{j=1}^{d}\frac{\partial^{i_j}}{\partial x_j^{i_j}} f(\mathbf{x})\Big)\Big| < \infty
\end{aligned}$$

so that, by Problem L.2,

$$\begin{aligned}
\sum_{\mathbf{b}\in\Gamma^\#}\Big|\Big(\prod_{j=1}^{d} b_{i_j}^{i_j}\Big)\hat f_{\mathbf b}\Big| &= \sum_{\mathbf{b}\in\Gamma^\#}\tfrac{1+|\mathbf{b}|^{d+1}}{1+|\mathbf{b}|^{d+1}}\Big|\Big(\prod_{j=1}^{d} b_{i_j}^{i_j}\Big)\hat f_{\mathbf b}\Big|\\
&\le \Big[\sup_{\mathbf{b}\in\Gamma^\#}(1+|\mathbf{b}|^{d+1})\Big|\Big(\prod_{j=1}^{d} b_{i_j}^{i_j}\Big)\hat f_{\mathbf b}\Big|\Big]\ \sum_{\mathbf{b}\in\Gamma^\#}\tfrac{1}{1+|\mathbf{b}|^{d+1}}\\
&\le \Big[\sup_{\mathbf{b}\in\Gamma^\#}(1+|\mathbf{b}|^{d+1})\Big|\Big(\prod_{j=1}^{d} b_{i_j}^{i_j}\Big)\hat f_{\mathbf b}\Big|\Big]\ \sum_{\mathbf{n}\in\mathbb{Z}^d}\tfrac{1}{1+(c|\mathbf{n}|)^{d+1}} < \infty
\end{aligned}$$

Hence, by the "only if" part of this Theorem, that we have already proven,

$$g(\mathbf{x}) = \tfrac{1}{|\Gamma|} \sum_{\mathbf{b}\in\Gamma^\#} \hat{f}_{\mathbf{b}}\, e^{i\mathbf{b}\cdot\mathbf{x}}$$

is a C^∞ function and

$$\hat{f}_{\mathbf{b}} = \int_{[\boldsymbol{\gamma}_1,\cdots\boldsymbol{\gamma}_d]} d^d\mathbf{x}\;\; e^{-i\mathbf{b}\cdot\mathbf{x}}\, g(\mathbf{x}) \tag{L.1}$$

We just have to show that $g(\mathbf{x}) = f(\mathbf{x})$.

Here is one proof that $g(\mathbf{x}) = f(\mathbf{x})$. By (L.1)

$$\int_{[\boldsymbol{\gamma}_1,\cdots\boldsymbol{\gamma}_d]} d^d\mathbf{x}\;\; e^{-i\mathbf{b}\cdot\mathbf{x}}\, [g(\mathbf{x}) - f(\mathbf{x})] = 0$$

for all $\mathbf{b} \in \Gamma^\#$. Consequently,

$$\int_{[\boldsymbol{\gamma}_1,\cdots\boldsymbol{\gamma}_d]} d^d\mathbf{x}\;\; \varphi(\mathbf{x})\, [g(\mathbf{x}) - f(\mathbf{x})] = 0$$

for any function $\varphi \in \mathcal{P}(\Gamma^\#)$ where $\mathcal{P}(\Gamma^\#)$ is the set of all functions that are finite linear combinations of the $e^{-i\mathbf{b}\cdot\mathbf{x}}$'s with $\mathbf{b} \in \Gamma^\#$. Consequently,

$$\int_{[\boldsymbol{\gamma}_1,\cdots\boldsymbol{\gamma}_d]} d^d\mathbf{x}\;\; \varphi(\mathbf{x})\, [g(\mathbf{x}) - f(\mathbf{x})] = 0$$

for any function $\varphi \in \overline{\mathcal{P}(\Gamma^\#)}$ where $\overline{\mathcal{P}(\Gamma^\#)}$ is the set of all functions that are uniform limits of sequences of functions in $\mathcal{P}(\Gamma^\#)$. But by the Stone–Weierstrass Theorem [Walter Rudin, Principles of Mathematical Analysis, Theorem 7.33], $\overline{\mathcal{P}(\Gamma^\#)}$ is the set of all continuous functions that are periodic with respect to Γ. In particular, the complex conjugate of $g(\mathbf{x}) - f(\mathbf{x})$ is in $\overline{\mathcal{P}(\Gamma^\#)}$. Hence

$$\int_{[\boldsymbol{\gamma}_1,\cdots\boldsymbol{\gamma}_d]} d^d\mathbf{x}\;\; |g(\mathbf{x}) - f(\mathbf{x})|^2 = 0$$

so that $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}$.

One may also build Problem L.5, below, into a second proof that $g(\mathbf{x}) = f(\mathbf{x})$. Just make a change of variables so that Γ is replaced by $2\pi\mathbb{Z}^d$ and apply Problem L.5.b, once in each dimension. ∎

Problem L.4 Let Γ be generated by $\boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_d \in \mathbb{R}^d$ (assumed linearly independent) and also by $\boldsymbol{\gamma}'_1, \cdots, \boldsymbol{\gamma}'_d \in \mathbb{R}^d$ (also assumed linearly independent). Recall that

$$[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d] = \Big\{ \sum_{j=1}^d t_j\boldsymbol{\gamma}_j \;\Big|\; 0 \le t_j \le 1 \text{ for all } 1 \le j \le d \Big\}$$

is the parallelepiped with the $\boldsymbol{\gamma}_j$'s as edges. Let, for $\mathbf{y} \in \mathbb{R}^d$,

$$\mathbf{y} + [\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d] = \left\{ \ \mathbf{y} + \sum_{j=1}^{d} t_j \boldsymbol{\gamma}_j \ \middle| \ 0 \le t_j \le 1 \text{ for all } 1 \le j \le d \ \right\}$$

be the translate of $[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]$ by $\mathbf{y}$. Let $f(\mathbf{x})$ be periodic with respect to Γ. Prove that

$$\int_{[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]} d^d\mathbf{x} \ f(\mathbf{x}) = \int_{\mathbf{y}+[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]} d^d\mathbf{x} \ f(\mathbf{x}) = \int_{[\boldsymbol{\gamma}'_1, \cdots \boldsymbol{\gamma}'_d]} d^d\mathbf{x} \ f(\mathbf{x})$$

We denote

$$\int_{\mathbb{R}^d/\Gamma} d^d\mathbf{x} \ f(\mathbf{x}) = \int_{[\boldsymbol{\gamma}_1, \cdots \boldsymbol{\gamma}_d]} d^d\mathbf{x} \ f(\mathbf{x})$$

Problem L.5 Let $f \in C^1(\mathbb{R})$ be periodic of period 2π. Set

$$c_n = \int_0^{2\pi} e^{-inx} f(x) \ dx$$

and

$$(S_M f)(\theta) = \tfrac{1}{2\pi} \sum_{n=-M}^{M} c_n e^{in\theta}$$

a) Prove that $S_M f(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta + x) \frac{\sin(M+1/2)x}{\sin(x/2)} dx$.
b) Prove that $S_M f(\theta)$ converges to $f(\theta)$ as $M \to \infty$.