Compact operators I

filename: `compact1.tex`

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Additional references

- ○ B. Simon, *Trace ideals and their applications*

Preliminaries

We assume that all operators act on a separable (infinite dimensional) Hilbert space $\mathcal{H}$. An operator A is called invertible if there is a *bounded* operator A^{-1} such that $AA^{-1} = A^{-1}A = I$.

The polar decomposition of a bounded operator:

Lemma 1.1 *[RS VI.10] Every operator A can be written as a product $A = U|A|$ where $|A| = (A^*A)^{1/2}$ and U is a partial isometry with* $\mathrm{Ker}U = \mathrm{KerA}$ *and* $\mathrm{Ran}U = \overline{\mathrm{RanA}}$.

Operator valued analytic functions: A bounded operator valued function $F(z)$ is called analytic if the complex derivative exists, i.e., for every z there is an operator $F'(z)$ with

$$\lim_{w \to 0} \|w^{-1}(F(z+w) - F(z)) - F'(z)\| = 0$$

Here $\|\cdot\|$ denotes the operator norm.

Problem 1.1: Suppose that $F(z)$ is a continuous family of bounded operators. Show that $F(z)$ is analytic if $\langle \phi, F(z)\psi \rangle$ is an analytic function for every choice of ϕ, ψ

Definitions and basic propeties

A bounded operator F has *finite rank* if its range is a finite dimensional subspace of $\mathcal{H}$. A operator of finite rank is essentially an $n \times n$ matrix.

Problem 1.2: Show that every finite rank operator can be written

$$F = \sum_{i=1}^{n} \psi_i \langle \phi_i, \cdot \rangle$$

Is the adjoint F^* also finite rank?

A bounded operator K is *compact* if it is the norm limit of finite rank operators. (An alternative definition is that K is compact if it maps the unit ball in $\mathcal{H}$ to a set with compact closure. For a Hilbert space, these two definitions are equivalent, but not in a Banach space, where the theory of compact operators is more difficult.)

The compact operators form an ideal.

Theorem 1.2 *If K is compact and A is bounded then K^*, AK and KA are compact.*

Theorem 1.3 *A compact operator maps weakly convergent sequences into norm convergent sequences.*

Proof: Let K be a compact operator and suppose $f_n \rightharpoonup f$ is a weakly convergent sequence. Then $g_n = f_n - f$ converges weakly to zero. Every weakly convergent sequence is bounded, so $\sup_n \|g_n\| < C$. Given $\epsilon > 0$ find a finite rank operator F with $\|K - F\| < \epsilon/C$. Then

$$\begin{aligned}\|Kf_n - Kf\| &= \|Kg_n\| = \|(K - F + F)g_n\| \\ &\leq \|(K-F)\|\|g_n\| + \|Fg_n\| \\ &\leq \epsilon + \|Fg_n\|\end{aligned}$$

But $Fg_n = \sum_{i=1}^N \langle \phi_i, g_n \rangle \psi_n$. This tends to zero in norm since each $\langle \phi_i, g_n \rangle \to 0$ by weak convergence, and the sum is finite. Thus

$$\lim_{n\to\infty} \|Kf_n - Kf\| \leq \epsilon$$

for every ϵ. □

Example: This theorem can be used together with a Mourre estimate and the Virial theorem to show that eigenvalues of a Schrödinger operator H cannot accumulate in an interval I. A Mourre estimate is an inequality of the form

$$E_I[H,A]E_I \geq \alpha E_I^2 + K$$

Where $\alpha > 0$ and E_I is a spectral projection for H corresponding to the interval I. If ψ is an eigenfunction of H, i.e., $H\psi = \lambda\psi$ with eigenvalue λ contained in the interval I, then $E_I\psi = \psi$.

The Virial theorem is the statement that $\langle \psi, [H,A]\psi \rangle = 0$. Formally, this is obviously true (by expanding the commutator). However, in applications, H and A are both unbounded operators, and ψ need not lie in the domain of A. In this situation one the commutator $[H,A]$ is defined using a limiting process, and the Virial theorem may be false (see Georgescu and Gérard []).

Suppose, though, that both the Mourre estimate and the Virial theorem hold. Then there cannot be an infinite sequence of eigenvalues λ_j all contained in I. For suppose there was such a sequence. Then the corresponding orthonormal eigenvectors ψ_j converge weakly to zero. Moreover $E_I\psi_j = \psi_j$ so by the Virial theorem and the Mourre estimate

$$0 = \langle\psi_j, [H,A]\psi_j\rangle = \langle\psi_j, E_I[H,A]E_I\psi_j\rangle \geq \alpha\|E_I\psi_j\|^2 + \langle\psi_j, K\psi_j\rangle = \alpha + \langle\psi_j, K\psi_j\rangle$$

But ψ_j converge weakly to zero, so $K\psi_j$ tends to zero in norm. Thus $\langle\psi_j, K\psi_j\rangle \to 0$ which gives rise to the contradiction $0 \geq \alpha$.

The Analytic Fredholm Theorem

In many situations one wants to find a solution ϕ to an equation of the form

$$(I-K)\phi = f$$

If the operator $(I-K)$ is invertible then there is a unique solution given by $\phi = (I-K)^{-1}f$. Otherwise, for a general opeator K, the analysis of this equation is delicate.

Problem 1.3: Find a bounded operator A such that $I-A$ is not invertible, but A does not have 1 as an eigenvalue (i.e., the kernel of $I-A$ is zero).

There are two situations where this equation is easy to analyze. The first is when $\|K\| < 1$. In this case the inverse $(I-K)^{-1}$ exists and is given by the convergent Neumann expansion

$$(I-K)^{-1} = \sum_{n=0}^{\infty} K^n$$

The other situation where the equation is easy to understand is when K has finite rank. In this case $(I-K)$ is invertible if and only if K does not have eigenvalue 1. (If K does have 1 as an eigenvalue, then the equation has either no solutions or infinitely many solutions, depending on whether f is in the range of $I-K$). This situation can be generalized to compact operartors K.

Notice that in the second situation, if $f=0$, then either $I-K$ is invertible, or the equation has a non-trivial solution (any element in the kernel of $(I-K)$). This dichotomy is known as the Fredholm alternative.

In fact it is very fruitful to consider not a single compact operator K but an analytic family of compact operators $K(z)$ defined on some domain D in the complex plane.

Suppose for a moment that $K(z)$ is a matrix with eigenvalues $\lambda_1(z), \ldots, \lambda_n(z)$. Let S denote the values of z for which $I-K(z)$ is not invertible. Then S is the union of the set of zeros of the functions $1-\lambda_1(z)$, . . ., $1-\lambda_n(z)$. This is the same as the set of zeros of $\prod_k(1-\lambda_k(z)) = \det(I-K(z))$. Since $\det(I-K(z))$ is analytic, S is the set of zeros of an analytic function: either all of D (in the case that $\det(I-K(z))$ is identically equal to 0) or a discrete set, i.e., a set with no accumulation points in D.

Theorem 1.4 *[RS VI.16] Let $K(z)$ be a compact operator valued analytic function of z, defined for z in some domain D in the complex plane. Then either*

(i) $I - K(z)$ is never invertible, or

(ii) $I - K(z)$ is invertible for all z in $D \backslash S$ where S is a discrete set in D. In this case $(I - K(z))^{-1}$ is meromorphic in D with finite rank residues at each point in S. For each point in S, the equation $(I - K(z))\psi = 0$ has non-trivial solutions.

Proof: The main step in the proof is this local result. Fix $z_0 \in D$. There is a disk about z_0 such $\|K(z) - K(z_0)\| < 1/2$ for all z in this disk. There is a finite rank operator $F = \sum_{i=1}^n \psi_i \langle \phi_i, \cdot \rangle$ with $\|K(z_0) - F\| < 1/2$. Let $A(z) = K(z) - F$. Then

$$\|A(z)\| = \|K(z) - K(z_0) + K(z_0) - F\| \leq \|K(z) - K(z_0)\| + \|K(z_0) - F\| < 1$$

for z in the disk. So for z in the disk, $I - A(z)$ is invertible and

$$I - K(z) = I - A(z) - F = (I - F(I - A(z))^{-1})(I - A(z))$$

This shows that $I - K(z)$ is invertible if and only if the finite rank operator $(I - F(I - A(z))^{-1})$ is. But $(I - F(I - A(z))^{-1})$ is invertible unless $F(I - A(z))^{-1})$ has eigenvalue 1. At these points z there is a vector ψ such that

$$F(I - A(z))^{-1})\psi = \sum_{i=1}^n \psi_i \langle \phi_i, (I - A(z))^{-1}\psi \rangle = \psi$$

Since ψ lies in the range of F we may expand $\psi = \sum \beta_j \psi_j$ and find that

$$\sum_{i,j=1}^n \beta_j \langle \phi_i, (I - A(z))^{-1}\psi_j \rangle \psi_i = \sum \beta_i \psi_i$$

From this we conclude that for these z, the vector $\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$ lies in the kernel of the $n \times n$ matrix $I - [\langle \phi_i, (I - A(z))^{-1}\psi_j \rangle]$, so

$$\det(I - [\langle \phi_i, (I - A(z))^{-1}\psi_j \rangle]) = 0$$

In other words, the points of non-invertibility for $I - K(z)$ in the disk are the zeros of an analytic function.

At points of invertibility we have

$$(I - K(z))^{-1} = (I - A(z))^{-1}(I - F(I - A(z))^{-1})^{-1}$$

In the disk about z_0, $(I - A(z))^{-1}$ is analytic. The inverse of $(I - F(I - A(z))^{-1})$ can be written in terms of cofactors. This leads to a proof of the second part of the theorem. $\square$

The Fredholm alternative for compact operators

Theorem 1.5 *If K is compact, then either $I - K$ is invertible or there is a non-trivial solution to $K\psi = \psi$.*

Proof: Apply the analytic Fredholm theorem with $K(z) = zK$ at $z = 1$. □

Riesz-Schauder Theorem

Theorem 1.6 *If K is compact, the $\sigma(K)$ is a discrete set with except for a possible accumulation point at 0. Every non-zero $\lambda \in \sigma(K)$ is an eigenvalue of finite multiplicity.*

Proof: We have $K - \lambda I = -\lambda(I - \lambda^{-1}K)$, so we may use the analytic Fredholm theorem with $z = \lambda^{-1}$. □

Hilbert-Schmidt Theorem

Theorem 1.7 *If K is compact and self-adjoint then there is an orthonormal basis of eigenvectors $\{\psi_n\}$ with $K\psi_n = \lambda_n\psi_n$ and $\lambda_n \to 0$.*

The main point here is that a self-adjoint operator is zero if its spectral radius is zero (see Reed-Simon).

Canonical form for compact operators

Theorem *If K is compact, then there exist orthonormal sets $\{\psi_i\}$ and $\{\phi_i\}$ and positive numbers μ_i so that*

$$K = \sum_i \mu_i \langle \psi_i, \cdot \rangle \phi_i$$

The positive numbers μ_i are eigenvalues of $|K|$ and are called the singular values of K.

This is proven using the polar decomposition $K = U|K|$ and the Hilbert-Schmidt theorem for $|K|$. The vectors ψ_i are the eigenvectors of K and $\phi_i = U\psi_i$.