Chain Rule Derivation

Suppose that the function $F(s,t)$ is defined as the composition

$$F(s,t) = f\big(x(s,t), y(s,t)\big)$$

The chain rule says that

$$\begin{aligned}
\frac{\partial F}{\partial s}(s,t) &= \frac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\,\frac{\partial x}{\partial s}(s,t) \ + \ \frac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)\,\frac{\partial y}{\partial s}(s,t) \\
\frac{\partial F}{\partial t}(s,t) &= \frac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\,\frac{\partial x}{\partial t}(s,t) \ + \ \frac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)\,\frac{\partial y}{\partial t}(s,t)
\end{aligned}$$

We'll now see why the upper formula is true. By definition,

$$\begin{aligned}
\frac{\partial F}{\partial s}(s,t) &= \lim_{h\to 0} \frac{F(s+h,t) - F(s,t)}{h} \\
&= \lim_{h\to 0} \frac{f\Big(\overbrace{x(s+h,t)}^{x_0+\Delta x(h)}\,,\,\overbrace{y(s+h,t)}^{y_0+\Delta y(h)}\Big) - f\Big(\overbrace{x(s,t)}^{x_0}\,,\,\overbrace{y(s,t)}^{y_0}\Big)}{h} \\
&= \lim_{h\to 0} \frac{f\big(x_0+\Delta x(h)\,,\,y_0+\Delta y(h)\big) - f\big(x_0\,,\,y_0\big)}{h}
\end{aligned}$$

where

$$\begin{aligned}
x_0 &= x(s,t) \quad \Delta x(h) = x(s+h,t) - x(s,t) \\
y_0 &= y(s,t) \quad \Delta y(h) = y(s+h,t) - y(s,t)
\end{aligned}$$

Now using the linear approximation (which becomes perfect in the limit $h \to 0$)

$$\begin{aligned}
\frac{\partial F}{\partial s}(s,t) &= \lim_{h\to 0} \frac{f_x(x_0,y_0)\,\Delta x(h) + f_y(x_0,y_0)\,\Delta y(h)}{h} \\
&= f_x(x_0,y_0)\lim_{h\to 0}\frac{\Delta x(h)}{h} + f_y(x_0,y_0)\lim_{h\to 0}\frac{\Delta y(h)}{h} \\
&= f_x(x_0,y_0)\lim_{h\to 0}\frac{x(s+h,t)-x(s,t)}{h} + f_y(x_0,y_0)\lim_{h\to 0}\frac{y(s+h,t)-y(s,t)}{h} \\
&= \frac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\,\frac{\partial x}{\partial s}(s,t) \ + \ \frac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)\,\frac{\partial y}{\partial s}(s,t)
\end{aligned}$$