Error Formulae for Taylor Polynomial Approximations

Let

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \tfrac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

be the Taylor polynomial of degree n for the function $f(x)$ and expansion point x_0. Using this polynomial to approximate $f(x)$ introduces an error

$$E_n(x) = f(x) - P_n(x)$$

We shall now prove that

$$E_n(x) = \tfrac{1}{(n+1)!} f^{(n+1)}(z)(x - x_0)^{n+1} \tag{1_n}$$

for some z between x_0 and x and that

$$E_n(x) = \tfrac{1}{n!} \int_{x_0}^{x} (x - z)^n f^{(n+1)}(z)\ dz \tag{2_n}$$

These proofs are not part of the official course. It rarely necessary, or even possible, to evaluate $E_n(x)$ exactly. It is usually sufficient to find a number M such that $\big|f^{(n+1)}(z)\big| \le M$ for all z between x_0 and the x of interest. Both (1_n) and (2_n) then imply that $|E_n(x)| \le \tfrac{1}{(n+1)!} M |x - x_0|^{n+1}$.

Both (1_n) and (2_n) are easily proven in the special case $n = 0$. When $n = 0$, (1_n) and (2_n) are the statements that

$$f(x) - f(x_0) = f'(z)(x - x_0) \tag{1_0}$$

for some z between x_0 and x and that

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(z)\ dz \tag{2_0}$$

So (1_0) is just a restatement of the mean–value theorem and (2_0) is just a restatement of part of the fundamental theorem of calculus.

To prove (1_n) with $n \ge 1$, we need the following small generalization of the mean–value theorem.

Theorem (Generalized Mean–Value Theorem) *Let the functions $F(x)$ and $G(x)$ both be defined and continuous on $a \le x \le b$ and both be differentiable on $a < x < b$. Furthermore, suppose that $G'(x) \ne 0$ for all $a < x < b$. Then, there is a number c obeying $a < c < b$ such that*

$$\tfrac{F(b) - F(a)}{G(b) - G(a)} = \tfrac{F'(c)}{G'(c)}$$

Proof: Define

$$h(x) = \big[F(b) - F(a)\big]\big[G(x) - G(a)\big] - \big[F(x) - F(a)\big]\big[G(b) - G(a)\big]$$

Observe that $h(a) = h(b) = 0$. So, by the mean–value theorem, there is a number c obeying $a < c < b$ such that

$$0 = h'(c) = \big[F(b) - F(a)\big]G'(c) - F'(c)\big[G(b) - G(a)\big]$$

As $G(a) \ne G(b)$ (otherwise the mean–value theorem would imply the existence of an $a < x < b$ obeying $G'(x) = 0$), we may divide by $G'(c)\big[G(b) - G(a)\big]$ which gives the desired result. ■

Proof of (1_n): To prove (1_1), that is (1_n) for $n = 1$, simply apply the generalized mean–value theorem with $F(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$, $G(x) = (x - x_0)^2$, $a = x_0$ and $b = x$. Then $F(a) = G(a) = 0$, so that

$$\tfrac{F(b)}{G(b)} = \tfrac{F'(c)}{G'(c)} \ \Rightarrow\ \tfrac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2} = \tfrac{f'(c)-f'(x_0)}{2(c-x_0)}$$

for some c between x_0 and x. By the mean–value theorem (the standard one, but with $f(x)$ replaced by $f'(x)$), $\frac{f'(c)-f'(x_0)}{c-x_0} = f''(z)$, for some z between x_0 and c (which forces z to also be between x_0 and x). Hence

$$\tfrac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2} = \tfrac{1}{2}f''(z)$$

which is exactly (1_1).

At this stage, we know that (1_n) applies to all (sufficiently differentiable) functions for $n = 0$ and $n = 1$. To prove it for general n, we proceed by induction. That is, we assume that we already know that (1_n) applies to $n = k - 1$ for some k (as is the case for $k = 1, 2$) and that we wish to prove that it also applies to $n = k$. We apply the generalized mean–value theorem with $F(x) = E_k(x)$, $G(x) = (x - x_0)^{k+1}$, $a = x_0$ and $b = x$. Then $F(a) = G(a) = 0$, so that

$$\tfrac{F(b)}{G(b)} = \tfrac{F'(c)}{G'(c)} \ \Rightarrow\ \tfrac{E_k(x)}{(x-x_0)^{k+1}} = \tfrac{E_k'(c)}{(k+1)(c-x_0)^k}$$

But

$$\begin{aligned} E_k'(c) &= \tfrac{d}{dx}\Big[f(x) - f(x_0) - f'(x_0) - \cdots - \tfrac{1}{k!}f^{(k)}(x_0)(x-x_0)^k\Big]_{x=c} \\ &= \Big[f'(x) - f'(x_0) - \cdots - \tfrac{1}{(k-1)!}f^{(k)}(x_0)(x-x_0)^{k-1}\Big]_{x=c} \\ &= f'(c) - f'(x_0) - \cdots - \tfrac{1}{(k-1)!}f^{(k)}(x_0)(c-x_0)^{k-1} \end{aligned}$$

The last expression is exactly the definition of $E_{k-1}(c)$, but for the function $f'(x)$, instead of the function $f(x)$. But we already know that (1_{k-1}) is true, so we already know that the last expression equals

$$\tfrac{1}{(k-1+1)!}\big(f'\big)^{(k-1+1)}(z)(c-x_0)^{k-1+1} = \tfrac{1}{k!}f^{(k+1)}(z)(c-x_0)^k$$

for some z between x_0 and c. Subbing this in

$$\tfrac{E_k(x)}{(x-x_0)^{k+1}} = \tfrac{E_k'(c)}{(k+1)(c-x_0)^k} = \tfrac{1}{(k+1)!}f^{(k+1)}(z)$$

which is exactly (1_k). Repeating this for $k = 2, 3, 4, \cdots$ gives (1_k) for all k. ■

Proof of (2_n): We again proceed by induction. That is, we assume that we already know that (2_n) applies to $n = k - 1$ for some k (as is the case for $k = 1$) and we then prove that it also applies to $n = k$. So we are assuming that

$$E_{k-1}(x) = \tfrac{1}{(k-1)!}\int_{x_0}^{x}(x-z)^{k-1}f^{(k)}(z)\ dz$$

Integrate by parts with $u(z) = f^{(k)}(z)$ and $v'(z)\,dz = \frac{(x-z)^{k-1}}{(k-1)!}\,dz$. Note that z is now the integration variable and x is just some constant. So $u'(z)\,dz = f^{(k+1)}(z)\,dz$ and we may take $v(z) = -\frac{1}{k!}(x-z)^k$. This gives

$$\begin{aligned} E_{k-1}(x) &= -\tfrac{1}{k!}(x-z)^k f^{(k)}(z)\Big|_{z=x_0}^{z=x} + \tfrac{1}{k!}\int_{x_0}^{x}(x-z)^k f^{(k+1)}(z)\ dz \\ &= \tfrac{1}{k!}(x-x_0)^k f^{(k)}(x_0) + \tfrac{1}{k!}\int_{x_0}^{x}(x-z)^k f^{(k+1)}(z)\ dz \end{aligned}$$

Since

$$E_k(x) = f(x) - P_k(x) = f(x) - P_{k-1}(x) - \tfrac{1}{k!}(x-x_0)^k f^{(k)}(x_0) = E_{k-1}(x) - \tfrac{1}{k!}(x-x_0)^k f^{(k)}(x_0)$$

we have

$$E_k(x) = \tfrac{1}{k!}\int_{x_0}^{x}(x-z)^k f^{(k+1)}(z)\ dz$$

which is exactly (2_k). Repeating this for $k = 2, 3, 4, \cdots$ gives (2_k) for all k. ■