Richardson Extrapolation and Romberg Integration

There are many approximation procedures in which one first picks a step size h and then generates an approximation $A(h)$ to some desired quantity A. Often the order of the error generated by the procedure is known. This means

$$A = A(h) + Kh^k + K'h^{k+1} + K''h^{k+2} + \ \cdots$$

with k being some known constant, called the order of the error, and $K,\ K',\ K'',\ \cdots$ being some other (usually unknown) constants. For example, A might be the value of some integral $\int_a^b f(x)\,dx$. For the Trapezoidal Rule with n steps, $\Delta x = \frac{b-a}{n}$ plays the role of the step size. If $A(h)$ is the approximation to A produced by Trapezoidal Rule with $\Delta x = h$, then $k = 2$. If Simpson's Rule is used, $k = 4$.

The notation $O(h^{k+1})$ is conventionally used to stand for "a sum of terms of order h^{k+1} and higher". So the above equation may be written

$$A = A(h) + Kh^k + O(h^{k+1}) \tag{1}$$

If we were to drop the, hopefully tiny, term $O(h^{k+1})$ from this equation, we would have one linear equation, $A = A(h) + Kh^k$, in the two unknowns A, K. But this really gives a different equation for each different value of h. We can get two different equations for A and K by just using two different step sizes. Then the two equations may be solved, yielding approximate values of A and K. We do this now, using step sizes h and $h/2$, for any h. Taking 2^k times

$$A = A(h/2) + K(h/2)^k + O(h^{k+1}) \tag{2}$$

(note that, in equations (1) and (2), the symbol "$O(h^{k+1})$" is used to stand for two **different** sums of terms of order h^{k+1} and higher) and subtracting equation (1) gives

$$\begin{aligned} \left(2^k - 1\right) A &= 2^k A(h/2) - A(h) + O(h^{k+1}) \\ A &= \frac{2^k A(h/2) - A(h)}{2^k - 1} + O(h^{k+1}) \end{aligned}$$

Hence if we define

$$B(h) = \frac{2^k A(h/2) - A(h)}{2^k - 1} \tag{3}$$

then

$$A = B(h) + O(h^{k+1}) \tag{4}$$

and $B(h)$ is an approximation whose error is of order $k+1$, one better than $A(h)$'s. The generation of a "new improved" approximation for A from two $A(h)$'s with different values of h is called Richardson Extrapolation.

If $A(h)$ has been computed for three values of h, we can generate $B(h)$ for two values of h. If the order of the error in $B(h)$ is known, the above procedure can be repeated with a new value of k. And so on. One widely used numerical integration algorithm, called Romberg integration, applies this procecdure repeatedly to the Trapezoidal Rule. It is known that the Trapezoidal Rule approximation $T(h)$ to an integral I has error behaviour (assuming that the integrand $f(x)$ is smooth)

$$I = T(h) + K_1h^2 + K_2h^4 + K_3h^3 + \cdots$$

Hence

$$T_1(h) = \frac{4T(h/2) - T(h)}{3} \quad \text{has error of order 4, so that}$$
$$T_2(h) = \frac{16T_1(h/2) - T_1(h)}{15} \quad \text{has error of order 6, so that}$$
$$T_3(h) = \frac{64T_2(h/2) - T_2(h)}{63} \quad \text{has error of order 8 and so on}$$

We know another method which produces an error of order 4 – Simpson's Rule. In fact, $T_1(h)$ is exactly Simpson's Rule (for step size $\frac{h}{2}$).

Example

$A = \int_0^\pi \sin x \; dx = 2$

h	$T(h)$	%	$T_1(h)$	%	$T_2(h)$	%	$T_3(h)$	%
$\pi/4$	1.896	5.2	2.0002692	-1.3×10^{-2}	1.999999752	1.2×10^{-5}	2.000000000060	-2.9×10^{-9}
$\pi/8$	1.974	1.3	2.0000166	-8.3×10^{-4}	1.999999996	1.9×10^{-7}		
$\pi/16$	1.993	.32	2.0000010	-5.2×10^{-5}				
$\pi/32$	1.998	.08						

The "%" column gives the percentage error.