# Error Control for the Polar Area Formula

Suppose that we wish to derive a formula for finding the area of the region

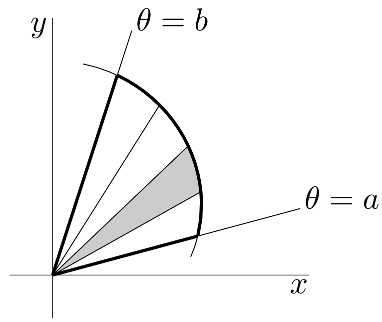
$$0 \le r \le f(\theta) \qquad a \le \theta \le b$$

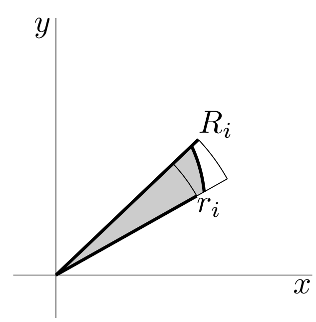
Call this area $A$. Suppose further that

$$f(\theta) \le M \qquad |f'(\theta)| \le L$$

for all $a \le \theta \le b$.

Divide the interval $a \le \theta \le b$ into $n$ equal subintervals, each of length $\Delta\theta = \frac{b-a}{n}$. Let $\theta_i^*$ be the midpoint of the $i^{\rm th}$ interval. On the $i^{\rm th}$ interval, $\theta$ runs from $\theta_i^* - \frac{1}{2}\Delta\theta$ to $\theta_i^* + \frac{1}{2}\Delta\theta$ and the radius runs over all values of $f(\theta)$ with $\theta_i^* - \frac{1}{2}\Delta\theta \le \theta \le \theta_i^* + \frac{1}{2}\Delta\theta$. Because $|f'(\theta)| \le L$ all of these values of $f(\theta)$ lie between $r_i = f(\theta_i^*) - \frac{1}{2}L\Delta\theta$ and $R_i = f(\theta_i^*) + \frac{1}{2}L\Delta\theta$.




So the area of the region $0 \le r \le f(\theta),\ \theta_i^* - \frac{1}{2}\Delta\theta \le \theta \le \theta_i^* + \frac{1}{2}\Delta\theta$ must lie between

$$\tfrac{1}{2}\Delta\theta\, r_i^2 = \tfrac{1}{2}\Delta\theta\big[f(\theta_i^*) - \tfrac{1}{2}L\Delta\theta\big]^2 \quad \text{and} \quad \tfrac{1}{2}\Delta\theta\, R_i^2 = \tfrac{1}{2}\Delta\theta\big[f(\theta_i^*) + \tfrac{1}{2}L\Delta\theta\big]^2$$

Observe that

$$\big[f(\theta_i^*) \pm \tfrac{1}{2}L\Delta\theta\big]^2 = f(\theta_i^*)^2 \pm Lf(\theta_i^*)\Delta\theta + \tfrac{1}{4}L^2\Delta\theta^2$$

implies that, since $f(\theta) \le M$,

$$f(\theta_i^*)^2 - LM\Delta\theta + \tfrac{1}{4}L^2\Delta\theta^2 \le \big[f(\theta_i^*) \pm \tfrac{1}{2}L\Delta\theta\big]^2 \le f(\theta_i^*)^2 + LM\Delta\theta + \tfrac{1}{4}L^2\Delta\theta^2$$

Hence

$$\tfrac{1}{2}f(\theta_i^*)^2\Delta\theta - \tfrac{1}{2}LM\Delta\theta^2 + \tfrac{1}{8}L^2\Delta\theta^3 \le \text{area of sector \#i} \le \tfrac{1}{2}f(\theta_i^*)^2\Delta\theta + \tfrac{1}{2}LM\Delta\theta^2 + \tfrac{1}{8}L^2\Delta\theta^3$$

and the total area $A$ obeys

$$\sum_{i=1}^n \Big[\tfrac{1}{2}f(\theta_i^*)^2\Delta\theta - \tfrac{1}{2}LM\Delta\theta^2 + \tfrac{1}{8}L^2\Delta\theta^3\Big] \le A \le \sum_{i=1}^n \Big[\tfrac{1}{2}f(\theta_i^*)^2\Delta\theta + \tfrac{1}{2}LM\Delta\theta^2 + \tfrac{1}{8}L^2\Delta\theta^3\Big]$$

$$\tfrac{1}{2}\sum_{i=1}^n f(\theta_i^*)^2\Delta\theta - \tfrac{1}{2}nLM\Delta\theta^2 + \tfrac{1}{8}nL^2\Delta\theta^3 \le A \le \sum_{i=1}^n \tfrac{1}{2}f(\theta_i^*)^2\Delta\theta + \tfrac{1}{2}nLM\Delta\theta^2 + \tfrac{1}{8}nL^2\Delta\theta^3$$

Since $\Delta\theta = \frac{b-a}{n}$,

$$\frac{1}{2}\sum_{i=1}^{n} f(\theta_i^*)^2\Delta\theta - \tfrac{LM}{2}\tfrac{(b-a)^2}{n} + \tfrac{L^2}{8}\tfrac{(b-a)^3}{n^2} \le A \le \sum_{i=1}^{n} \tfrac{1}{2} f(\theta_i^*)^2\Delta\theta + \tfrac{LM}{2}\tfrac{(b-a)^2}{n} + \tfrac{L^2}{8}\tfrac{(b-a)^3}{n^2}$$

Now take the limit as $n \to \infty$. Since

$$\begin{aligned}
\lim_{n\to\infty} \Big[ \sum_{i=1}^{n} \tfrac{1}{2} f(\theta_i^*)^2\Delta\theta \pm \tfrac{LM}{2}\tfrac{(b-a)^2}{n} + \tfrac{L^2}{8}\tfrac{(b-a)^3}{n^2} \Big]
&= \tfrac{1}{2}\int_a^b f(\theta)^2\,d\theta \pm \lim_{n\to\infty} \tfrac{LM}{2}\tfrac{(b-a)^2}{n} + \lim_{n\to\infty} \tfrac{L^2}{8}\tfrac{(b-a)^3}{n^2} \\
&= \tfrac{1}{2}\int_a^b f(\theta)^2\,d\theta
\end{aligned}$$

we have that

$$A = \tfrac{1}{2}\int_a^b f(\theta)^2\,d\theta$$

exactly.