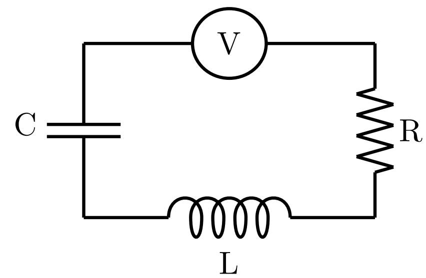
The RLC Circuit

The RLC circuit is the electrical circuit consisting of a resistor of resistance R, a coil of inductance L, a capacitor of capacitance C and a voltage source arranged in series.



If the charge on the capacitor is Q and the current flowing in the circuit is I, the voltage across R, L and C are RI, $L\frac{dI}{dt}$ and $\frac{Q}{C}$ respectively. By the Kirchhoff's law that says that the voltage between any two points has to be independent of the path used to travel between the two points,

$$LI'(t) + RI(t) + \tfrac{1}{C}Q(t) = V(t)$$

Assuming that R, L, C and V are known, this is still one differential equation in two unknowns, I and Q. However the two unknowns are related by $I(t) = \frac{dQ}{dt}(t)$ so that

$$LQ''(t) + RQ'(t) + \tfrac{1}{C}Q(t) = V(t)$$

or, differentiating with respect to t before subbing in $\frac{dQ}{dt}(t) = I(t)$,

$$LI''(t) + RI'(t) + \tfrac{1}{C}I(t) = V'(t)$$

For an ac voltage source, choosing the origin of time so that $V(0) = 0$, $V(t) = E_0 \sin(\omega t)$ and the differential equation becomes

$$LI''(t) + RI'(t) + \tfrac{1}{C}I(t) = \omega E_0 \cos(\omega t) \tag{1}$$

The General Solution

Let us look for a particular solution of (1) of the form $I(t) = A\sin(\omega t - \varphi)$ with the amplitude A and phase φ to be determined. Any such particular solution must obey

$$\begin{aligned} -L\omega^2 A\sin(\omega t-\varphi) + R\omega A\cos(\omega t-\varphi) + \tfrac{1}{C}A\sin(\omega t-\varphi) &= \omega E_0\cos(\omega t) \\ &= \omega E_0\cos(\omega t-\varphi+\varphi) \end{aligned}$$

and hence

$$\big(\tfrac{1}{C} - L\omega^2\big)A\sin(\omega t-\varphi) + R\omega A\cos(\omega t-\varphi) = \omega E_0\cos(\varphi)\cos(\omega t-\varphi) - \omega E_0\sin(\varphi)\sin(\omega t-\varphi)$$

Matching coefficients of $\sin(\omega t - \varphi)$ and $\cos(\omega t - \varphi)$ on the left and right hand sides gives

$$\big(L\omega^2 - \tfrac{1}{C}\big)A = \omega E_0\sin(\varphi) \tag{2}$$

$$R\omega A = \omega E_0\cos(\varphi) \tag{3}$$

It is now easy to solve for A and φ

$$\begin{aligned} \frac{(2)}{(3)} &\Longrightarrow \tan(\varphi) = \frac{L\omega^2 - \frac{1}{C}}{R\omega} \qquad \Longrightarrow \varphi = \tan^{-1}\Big(\frac{L\omega}{R} - \frac{1}{RC\omega}\Big) \\ \sqrt{(2)^2 + (3)^2} &\Longrightarrow \sqrt{\big(L\omega^2 - \tfrac{1}{C}\big)^2 + R^2\omega^2}\; A = \omega E_0 \Longrightarrow A = \frac{\omega E_0}{\sqrt{\big(L\omega^2 - \frac{1}{C}\big)^2 + R^2\omega^2}} \end{aligned} \tag{4}$$

Assuming that $R^2 \neq 4L/C$, the complementary solution for (1) is

$$c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where

$$r_{1,2} = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L} \tag{5}$$

are the two roots of

$$Lr^2 + Rr + \tfrac{1}{C} = 0$$

Hence, the general solution of (1) is

$$I(t) \;=\; c_1 e^{r_1 t} + c_2 e^{r_2 t} + A\sin(\omega t - \varphi)$$

with $r_1,\ r_2$ given in (5) and $A,\ \varphi$ given in (4). The arbitrary constants c_1 and c_2 are determined by initial conditions. However, when $e^{r_1 t}$ and $e^{r_2 t}$ damp out quickly, as is often the case, their values don't matter.