Eigenvector Review

Problems pulled from archived UBC final exams
`https://www.math.ubc.ca/Ugrad/pastExams/`

Note: many eigenvector problems involve differential equations (our next topic)

If you're not aware, some solutions are on the wiki,
`http://wiki.ubc.ca/Science:Math_Exam_Resources/Courses/MATH152`

- Find an eigenvector corresponding to eigenvalue 2 for the matrix below:
$$\begin{bmatrix} 5 & -2 \\ 6 & -4 \end{bmatrix}$$
- Find an eigenvector of eigenvalue 2 for the matrix
$$\begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$$
- The matrix
$$A = \begin{bmatrix} 7 & 0 & 10 \\ 5 & -3 & -5 \\ 5 & 0 & -8 \end{bmatrix}$$
has eigenvalue -3. Find all eigenvectors corresponding to this eigenvalue.

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- Standard question type: given eigenvalue, find associated eigenvector.
- Nonstandard wording: find ALL eigenvectors (instead of one representative of the bunch)
- Strategy: use the definition of an eigenvector (or the shortcut: set up the homogeneous system of equations with matrix of coefficients $A - \lambda I$.)

$$Ax = \lambda x \qquad \text{Find } x \text{ for given } A \text{ and } \lambda = -3$$

$$\begin{bmatrix} 7 & 0 & -10 \\ 5 & -3 & -5 \\ 5 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 7x + 0y - 10z \\ 5x - 3y - 5z \\ 5x + 0y - 8z \end{bmatrix} = \begin{bmatrix} -3x \\ -3y \\ -3z \end{bmatrix} \qquad \text{make the right side constants}$$

$$\begin{bmatrix} 10x + 0y - 10z \\ 5x + 0y - 5z \\ 5x + 0y - 5z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -10 & 0 \\ 5 & 0 & -5 & 0 \\ 5 & 0 & -5 & 0 \end{array}\right] \qquad \text{row-reduce}$$

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$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Parametrized solutions: let $y = r$ and $z = s$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

These vectors are the eigenvectors associated with eigenvalue -3.

- Let A be a 2×2 matrix with real entries. Suppose that

$$\begin{bmatrix} 1 \\ i \end{bmatrix}$$

is an eigenvector of A with eigenvalue $2 + 3i$. What is another eigenvalue of A and its associated eigenvector?

- Consider

$$A = \begin{bmatrix} 1 & y \\ 2 & z \end{bmatrix}$$

Find y and z so that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of A with eigenvalue 2.

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If $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector associated with eigenvalue 2:

$$\begin{bmatrix} 1 & y \\ 2 & z \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 1 + 3y \\ 2 + 3z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$y = \frac{2-1}{3} = \frac{1}{3} \qquad z = \frac{6-2}{3} = \frac{4}{3}$$

- Let A be a 2×2 matrix which represents a reflection across a line in $\mathbb{R}^2$. Suppose that $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 1 and $\begin{bmatrix} \alpha \\ 6 \end{bmatrix}$ is an eigenvector with eigenvalue $\beta \neq 1$. What are the values of α and β?

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Strategy: interpret geometrically. Sometimes a picture helps.

If a vector corresponds to eigenvalue 1, that means it's not changed by multiplication with the matrix. For a reflection, that means it's on the like of reflection. So, our line of reflection is in the direction $[2, 3]$.

Another line whose reflection is parallel to itself is a line perpendicular to this. Its reflection is its negative, so $\beta = -1$. Also, $[\alpha, 6] \cdot [2, 3] = 0$, so $\alpha = -9$.

- The matrix below represents rotation in 3D about a line through the origin.

$$\begin{bmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix}$$

  Find a vector in the direction of the line of rotation.

- The matrix

$$\begin{bmatrix} 1/2 & -\sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 1/2 & \sqrt{2}/2 & 1/2 \end{bmatrix}$$

  represents a rotation in 3D relative to some axis.

2 marks Find all the eigenvalues of A.

1. Find the direction vector of the axis of rotation. *Hint:* this vector remains unchanged after the rotation.
2. Find the angle of rotation around the axis in the previous part. *Hint:* rotate a vector that is perpendicular to the rotation axis.

Consider the probability transition matrix

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/3 \\ 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2/3 \end{bmatrix} \text{ and initial probability } x_0 = \begin{bmatrix} 1/11 \\ 3/11 \\ 3/11 \\ 4/11 \end{bmatrix}$$

What is $\lim_{n\to\infty} P^n x_0$? *Hint:* this can be done without extensive calculations.

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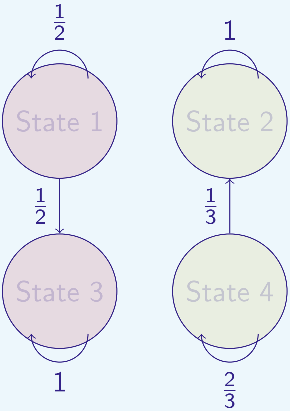
Our theorem about the equilibrium probability being an eigenvector associated to eigenvalue 1 DOES NOT APPLY because our matrix has zeroes in it.
Let's think about the system using a sketch

$$P = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/3 \\ 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2/3 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 1/11 \\ 3/11 \\ 3/11 \\ 4/11 \end{bmatrix}$$



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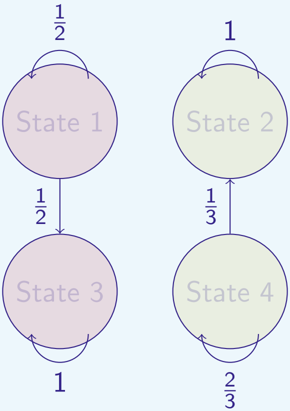

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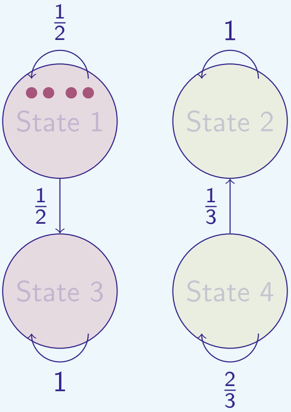

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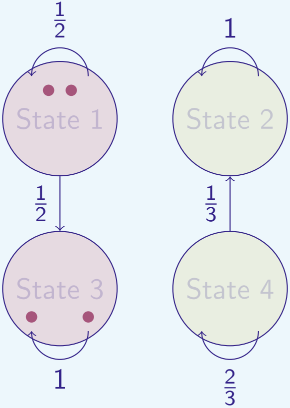

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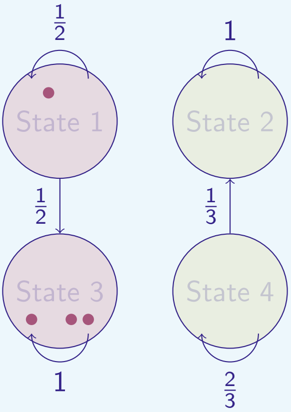

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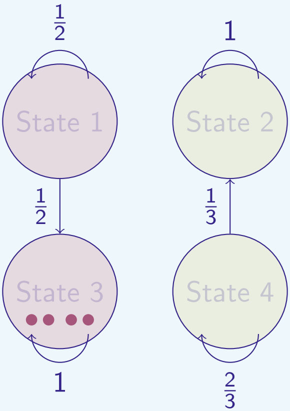

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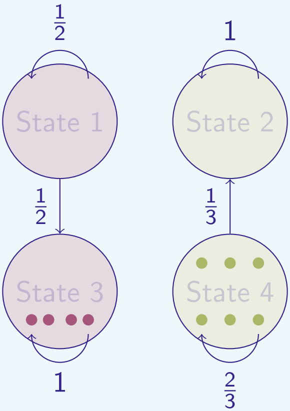

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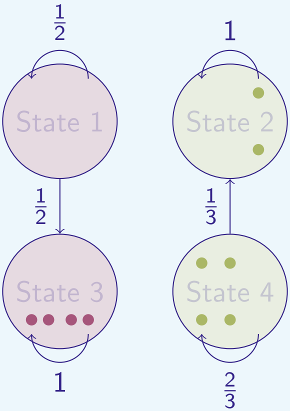

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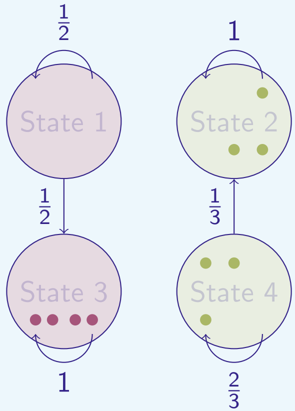

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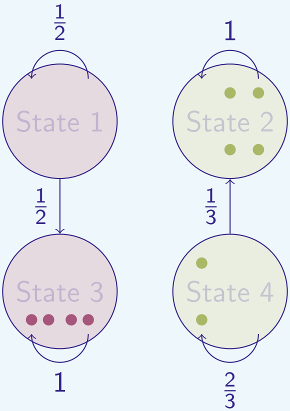

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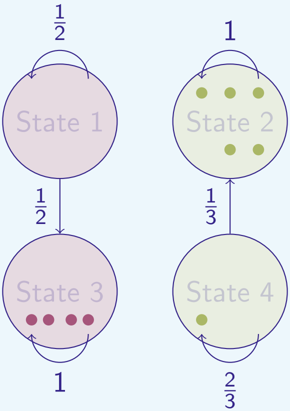

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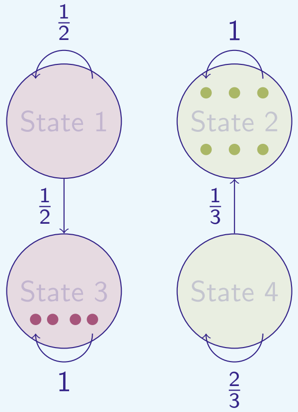

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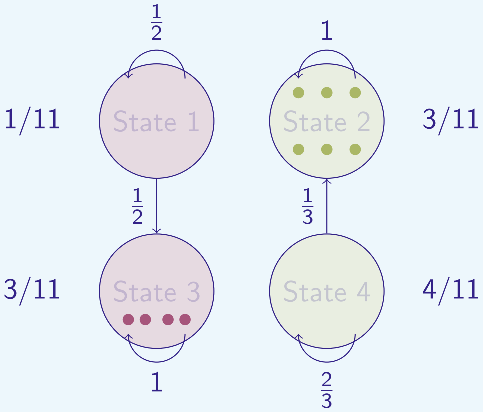

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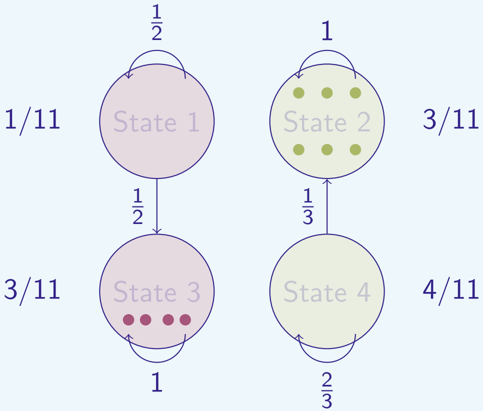

Let

$$P = \begin{bmatrix} 0.6 & 0.6 & 0.2 \\ .2 & .2 & .4 \\ .2 & .2 & .4 \end{bmatrix}$$

be the transition matrix of a random walk. It is known that P has three distinct eigenvalues. Two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 0.2$ with corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \text{ and } v_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

2 marks Find the third eigenvalue λ_3 and its associated eigenvector v_3.

2 If the initial state is

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

find $x_{10} = P^{10}x$. *Hint:* x_0 is a linear combination of v_2 and v_3.

1 What is $\lim_{n\to\infty} x_n$?

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Second question: use result from first question

Suppose in the year 2020, 50 million people live in cities and 50 million in the suburbs. Every year, 10% of city residents move to teh suburbs and 20% of the residents of the suburbs move to cities.

1 mark Write down the 2×2 probability transition matrix P for this problem, using the ordering (1) city and (2) suburbs.

1 What fraction of residents will be living in cities in 2022?

2 Find the eigenvalues of P and a basis of eigenvectors.

1 Assuming the overall population does not change (i.e. remains at 100 million), how many people will be living in the suburbs far in the future?

The percentage of people with the disease, March Madness, is recorded every week. Note that is it possible to recover from March Madness one week and catch it again the following week. Records indicate that the disease can be modelled by a random walk and that if 50% of the population is infected with March Madness one week, then 60% of the population will be infected the next week. Records also indicate that if 100% of the population is infected one week, then 90% of the population will be infected the next week. It is known that 10% of the population has March Madness this week.

2 marks What is the 2×2 probability transition matrix for this system?

1 What percentage of the population will have March Madness two weeks from now?

1 What percentage of the population had March Madness last week?

1 Approximately what will be the percentage of people with March Madness many weeks from now?

- Consider the 3×3 matrix

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & -4 & 5 \end{bmatrix}.$$

1 mark Find an eigenvector of A corresponding to the eigenvalue $\lambda = 1$

2 Find all other eigenvalues of A.

2 Find a basis of eigenvectors of A.

- Let A be a 2×2 matrix in the form

$$\begin{bmatrix} 2 & 4 \\ 1 & a \end{bmatrix}$$

1 mark Find a such that A is not invertible

1 For the value of a above, find the eigenvalues of A.

2 Find the eigenvectors associated with the eigenvalues found above.

1 Do the eigenvectors found above form a basis of $\mathbb{R}^2$? Justify briefly.

Let A be the matrix

$$\begin{bmatrix} 4 & -1 & 7 \\ 0 & 3 & 0 \\ 1 & 2 & -2 \end{bmatrix}$$

2 marks Find an eigenvector of A corresponding to eigenvalue $\lambda_1 = 3$.

2 Find all the other eigenvalues of A.

1 How many linearly independent eigenvectors does A have? Justify briefly.

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2 Find all the other eigenvalues of A. $\lambda_2 = -3$ and $\lambda_3 = 5$

1 How many linearly independent eigenvectors does A have? Justify briefly.

Option 1: find eigenvectors for the remaining two eigenvalues, show they are linearly independent.

$$\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are all eigenvectors of A, and they are linearly independent. Since there can't be any MORE than three linearly independent vectors in $\mathbb{R}^3$, we see A has three linearly independent eigenvectors.

Let A be the matrix $\begin{bmatrix} 4 & -1 & 7 \\ 0 & 3 & 0 \\ 1 & 2 & -2 \end{bmatrix}$

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Option 2: show that the three eigenvectors are linearly independent WITHOUT actually finding them.

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2 Find all the other eigenvalues of A. $\lambda_2 = -3$ and $\lambda_3 = 5$

1 How many linearly independent eigenvectors does A have? Justify briefly.

Option 2: show that the three eigenvectors are linearly independent WITHOUT actually finding them.It's a fact that distinct eigenvalues have linearly independent eigenvectors, but I don't think it's proved in your book. We can prove it on our own in this simple case.

- ANY collection of linearly independent vectors in $\mathbb{R}^3$ has size AT MOST three. So, there are at most three linearly independent eigenvectors.
- For every eigenvalue of A, there exists a (nonzero) eigenvector. Say these eigenvectors are x_1, x_2, and x_3.
- If these eigenvectors are linearly independent, then we have

We want to show that eigenvectors x_1, x_2, x_3 are linearly independent. Suppose they aren't (we'll show this leads to trouble). Then we can write one as a linear combination of the others, say $x_1 = c_2 x_2 + c_3 x_3$. Since x_1 is an eigenvector, it's not allowed to be a zero vector, so it's not the case that c_2 and c_3 are both zero. So, let's say $c_2 \neq 0$. Now we'll use the definition of eigenvector twice.

$$
\begin{aligned}
Ax_1 &= \lambda_1 x_1 = \lambda_1(c_2 x_2 + c_3 x_3) \\
Ax_1 &= A(c_2 x_2 + c_3 x_3) = Ac_2 x_2 + Ac_3 x_3 = \lambda_2 c_2 x_2 + \lambda_3 c_3 x_3 \\
\text{So, } \lambda_1(c_2 x_2 &+ c_3 x_3) = \lambda_2 c_2 x_2 + \lambda_3 c_3 x_3 \\
0 &= (\lambda_1 - \lambda_2) c_2 x_2 + (\lambda_1 - \lambda_3) c_3 x_3
\end{aligned}
$$

Since $\lambda_1 \neq \lambda_2$ and $c_2 \neq 0$:

$$x_2 = \frac{(\lambda_1 - \lambda_3) c_3}{(\lambda_1 - \lambda_2) c_2} x_3$$

But if x_1 and x_2 are parallel, they correspond to the same eigenvalue, which contradicts the way we chose them.